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# A GENERALIZATION OF THE AIRY INTEGRAL FOR $f'' - z^n f = 0$

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ABSTRACT. It is well known that the Airy integral is a solution of the Airy differential equation $f'' - zf = 0$ and that the Airy integral is a contour integral function with special properties. We show that there exist analogous special contour integral solutions of the more general equation $f'' - z^n f = 0$ where $n$ is any positive integer. Related results are given.

## 1. INTRODUCTION

Consider the second order linear differential equation

$$f'' - z^n f = 0 \tag{1.1}$$

where $n = 1, 2, 3, \ldots$. Solutions of equation (1.1) arise in many applications, such as in the theory of propagation of waves in varying media [B2] and in the theory of reflection of light from a medium of varying refractive index [FW]. When $n = 1$, equation (1.1) is of course the classical Airy differential equation, which also has well-known applications in the theory of diffraction, the dispersion of water waves, and the turning point problem (see, e.g., [JJ] and [SH]).

It is well known [BL] that any solution $f \not\equiv 0$ of (1.1) is an entire function of order $(n+2)/2$. We note that if $f(z)$ is a solution of (1.1) and if $\alpha$ is a constant that satisfies $\alpha^{n+2} = 1$, then $g(z) = f(\alpha z)$ is also a solution of (1.1).

Associated with equation (1.1) are the $n+2$ *critical rays*

$$\arg z = \frac{2k-1}{n+2}\pi, \qquad k = 1, 2, \ldots, n+2.$$

For $0 < \varepsilon < \pi/(n+2)$ and $k = 1, 2, \ldots, n+2$, we let $U_k(\varepsilon)$ denote the sector

$$\left|\arg z - \frac{2k-1}{n+2}\pi\right| < \varepsilon.$$

The following propositions hold:

(a) [H3, pp. 340–342] If $f \not\equiv 0$ is any solution of equation (1.1), then for any $\varepsilon$, all but at most finitely many zeros of $f$ must lie in the $n+2$ sectors $U_k(\varepsilon)$, $k = 1, 2, \ldots, n+2$. Furthermore, if $f$ possesses an infinite number

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of zeros $z_1, z_2, \ldots$ that lie in some particular sector $U_k(\varepsilon)$, then the exponent of convergence of these zeros $z_1, z_2, \ldots$ is equal to $(n+2)/2$.

(b) [B1] Suppose that $f \not\equiv 0$ is a solution of (1.1) that possesses an infinite number of zeros $z_1, z_2, \ldots$ which lie in some particular sector $U_k(\varepsilon)$. Then these zeros $z_1, z_2, \ldots$ will approach the critical ray $\arg z = (2k-1)\pi/(n+2)$.

Next we make several definitions. Let $f \not\equiv 0$ be a solution of (1.1). If $f$ possesses only a finite number of zeros in some particular sector $U_k(\varepsilon)$, then we say that the critical ray $\arg z = (2k-1)\pi/(n+2)$ is a *shortage ray* of $f$. We define the *shortage* of $f$ to be the total number of shortage rays of $f$, and we denote the shortage of $f$ by $s(f)$. The following properties hold (see [G, HR]):

(I) For each solution $f \not\equiv 0$ of (1.1), $s(f)$ is an even integer and $0 \le s(f) \le n+2$.

(II) There exist $m \ge 3$ pairwise linearly independent solutions $f_1, f_2, \ldots, f_m$ of (1.1) such that $s(f_k) \ge 2$ for each $k = 1, 2, \ldots, m$, and where

$$\sum_{k=1}^{m} s(f_k) = 2(n+2). \tag{1.2}$$

Furthermore, if $f \not\equiv 0$ is any solution of (1.1) that is not a constant multiple of some $f_j$, $j = 1, 2, \ldots, m$, then $s(f) = 0$.

If $f \not\equiv 0$ is a solution of (1.1) that satisfies $s(f) > 0$, then we call $f$ a *shortage solution.* The above properties illustrate that the shortage solutions of (1.1) are exceptional solutions. A phrase that is sometimes used in the literature is “subdominant solution” (see, e.g., [S; H3, p. 342]), and for equation (1.1) the subdominant solutions and the shortage solutions are the same solutions. The word *shortage* puts the emphasis on the frequency of zeros, while the word *subdominant* puts the emphasis on the growth.

If $\mathscr{B}$ is a collection of pairwise linearly independent shortage solutions of (1.1) such that any shortage solution of (1.1) is a constant multiple of some solution in $\mathscr{B}$, then we call $\mathscr{B}$ a *basis of shortage solutions* of (1.1).

We give a classical example. The Airy differential equation (see [O; JJ; IM; G, p. 288] for this discussion)

$$f'' - zf = 0 \tag{1.3}$$

possesses a special contour integral solution called the Airy integral, which is denoted by $Ai(z)$ and which has the form

$$Ai(z) = \frac{1}{2\pi i} \int_C \exp\left\{\frac{1}{3}w^3 - zw\right\} dw \tag{1.4}$$

where the contour $C$ runs from $\infty$ to $0$ along $\arg w = -\pi/3$ and then from $0$ to $\infty$ along $\arg w = \pi/3$. The exponent of convergence of the sequence of zeros of $Ai(z)$ is equal to $3/2$, and all the zeros of $Ai(z)$ are real and negative. The three critical rays of equation (1.3) are $\arg z = \pi/3$, $\pi$, $5\pi/3$, and we have $s(Ai) = 2$. If $\alpha_1, \alpha_2, \alpha_3$ are the three distinct cube roots of unity, and if we set

$$\phi_j(z) = Ai(\alpha_j z), \qquad j = 1, 2, 3, \tag{1.5}$$

then $\phi_1, \phi_2, \phi_3$ are three pairwise linearly independent solutions of (1.3) that satisfy $s(\phi_1) = s(\phi_2) = s(\phi_3) = 2$. If $f \not\equiv 0$ is any solution of (1.3) that is not a constant multiple of $\phi_1$, $\phi_2$, or $\phi_3$, then $s(f) = 0$. Hence the three solutions $\phi_1$, $\phi_2$, $\phi_3$ form a basis of shortage solutions of (1.3). Thus from (1.4) and (1.5), all the shortage solutions of (1.3) are special contour integral functions.

This turns out to be the case for all $n$ in equation (1.1). More specifically, we will show that for each $n = 1, 2, \dots$, all the shortage solutions of (1.1) are constant multiples of special contour integral functions of the form

$$\psi(z) = \int_C e^{P(z,w)}\,dw \tag{1.6}$$

where $P(z, w)$ is a polynomial in $z$ and $w$, and where $C$ is a suitably chosen contour.

For the special situation when $z$ is restricted to the real domain, Hardy [H1] (see also [W, pp. 320–324]) has given other forms of contour integral solutions of the form (1.6) for equation (1.1) when $n$ is even. Also, Heading [H2] has given Laplace-type contour integral solutions of the form (1.6) for equation (1.1).

## 2. The main results

Our main results are summarized in Theorems 1 and 2 below.

**Theorem 1.** *Let $n \geq 1$ be a fixed integer. Let $m_1$ and $m_2$ be two integers such that $m_1 - m_2$ is not an integer multiple of $n+2$. Let $G(z)$ denote the contour integral function defined by*

$$G(z) = \frac{1}{2\pi i}\int_C e^{P(z,w)}\,dw \tag{2.1}$$

*where*

$$P(z, w) = \sum_{k=0}^{[\![(n+2)/2]\!]} \frac{(-1)^k}{n+2-k}\binom{n+2-k}{k} z^k w^{n+2-2k}, \tag{2.2}$$

*$[\![(n+2)/2]\!]$ is the greatest integer that is $\leq (n+2)/2$, and $C = C(m_1, m_2)$ is the contour that runs from $\infty$ to $0$ along $\arg w = (2m_1+1)\pi/(n+2)$ and then from $0$ to $\infty$ along $\arg w = (2m_2+1)\pi/(n+2)$.*

*Then $G(z)$ is a shortage solution of equation (1.1), $s(G) = 2$, and the two shortage rays of $G(z)$ are*

$$\arg z = \frac{2m_1 + 2m_2 + 1}{n+2}\pi \quad \text{and} \quad \arg z = \frac{2m_1 + 2m_2 + 3}{n+2}\pi. \tag{2.3}$$

We include the $1/2\pi i$ factor in (2.1) to follow the lead of the classical Airy integral (1.4). There are cosmetic reasons for doing this.

From Theorem 1 it follows that there are many different representations of a basis of shortage solutions of (1.1) such that each solution in each basis will be a function of the form (2.1). We give one such representation in the following theorem.

**Theorem 2.** *For $j$ an integer, let $G_j(z)$ denote the contour integral function defined by*

$$G_j(z) = \frac{1}{2\pi i}\int_{C_j} e^{P(z,w)}\,dw \tag{2.4}$$

*where the polynomial* $P(z, w)$ *is given by* (2.2), *and where the contour* $C_j$ *runs from* $\infty$ *to* 0 *along* $\arg w = (2[\![j/2]\!]-1)\pi/(n+2)$ *and then from* 0 *to* $\infty$ *along* $\arg w = (2[\![(j+1)/2]\!]+1)\pi/(n+2)$ *where* $[\![\cdot]\!]$ *denotes the greatest integer function.*

*Then the following statements hold*:

(i) *For each integer* $j$, $G_j(z)$ *is a shortage solution of* (1.1), $s(G_j) = 2$, *and the two shortage rays of* $G_j(z)$ *are*

$$\arg z = \frac{2j-1}{n+2}\pi \quad \textit{and} \quad \arg z = \frac{2j+1}{n+2}\pi. \tag{2.5}$$

(ii) *The functions* $G_j(z)$, $j = 0, 1, \dots, n+1$, *form a basis of shortage solutions of* (1.1).

We now make several comments and observations about Theorems 1 and 2. From Theorem 2 it follows that if $f \not\equiv 0$ is any shortage solution of (1.1), then $s(f) = 2$.

If

$$\alpha_j = \exp\left\{-j\frac{2\pi i}{n+2}\right\}$$

where $j$ is an integer, then $G_0(\alpha_j z)$ is a solution of (1.1) where $G_0(z)$ is defined as in (2.4). From Theorem 2 we see that $G_j(z)$ and $G_0(\alpha_j z)$ have the same two shortage rays (2.5), and so it follows from Theorem 2 and Lemma 7 in §5 that there exists a constant $K_j$ such that

$$G_j(z) \equiv K_j G_0(\alpha_j z). \tag{2.6}$$

The constant $K_j$ can be readily computed.

In the case of the Airy equation (1.3), the basis of shortage solutions $G_0$, $G_1$, $G_2$ of (1.3) that is given in Theorem 2 is in agreement with the well-known basis of shortage solutions $\phi_1$, $\phi_2$, $\phi_3$ in (1.5). Specifically, we have $G_0(z) \equiv Ai(z)$ where $Ai(z)$ is the Airy integral (1.4), and then from (2.6) and (1.5) we have that $G_0$, $G_1$, $G_2$ are constant multiples of $\phi_1$, $\phi_2$, $\phi_3$.

In the case when $n = 2$, it is well known that all the shortage solutions of (1.1) can be expressed in terms of the parabolic cylinder function $D_\nu(z)$ for $\nu = -1/2$. In §4 we will illustrate the relationships of these well-known expressions with the basis of shortage solutions $G_0$, $G_1$, $G_2$, $G_3$ that is given in Theorem 2.

Theorem 2(ii) illustrates that there exists a basis of shortage solutions for equation (1.1) which consists of $n+2$ contour integral functions of the form (1.6) with $n+2$ different contours $C$ but where the polynomial $P(z, w)$ is the same for all of the functions. On the other hand, it follows from (2.6) and Theorem 2(ii) that a basis of shortage solutions of (1.1) can also be expressed as $n+2$ functions of the form (1.6) with $n+2$ different polynomials $P(z, w)$ but where the contour $C$ is the same for all of the functions.

If $G(z)$, $m_1$, $m_2$ are given as in Theorem 1, and if $q$ is the unique integer such that $j = q(n+2) + m_1 + m_2 + 1$ satisfies $0 \le j \le n+1$, then $G(z)$ in (2.1) and $G_j(z)$ in (2.4) will have the same two shortage rays from (2.3) and (2.5). Hence from Theorem 2 and Lemma 7 in §5, there exists a constant $K = K(j, q, m_1, m_2)$ such that

$$G(z) \equiv K G_j(z), \qquad j = 0, 1, \dots, n+1, \tag{2.7}$$

and the constant $K$ can be readily computed. Theorem 2(ii) and (2.7) show that the collection of shortage solutions $G(z)$ of (1.1) that is given by (2.1) for various values of $m_1$ and $m_2$ can be divided into $n+2$ disjoint classes of solutions where any two solutions in the same class will be linearly dependent and where any two solutions in two different classes will be linearly independent.

## 3. Properties of the shortage solutions of (1.1)

Let $n \geq 1$ be a fixed integer. Set

$$p = \frac{1}{n+2} \quad \text{and} \quad \mu = \frac{n+2}{2}, \tag{3.1}$$

and let $J_p(z)$ and $J_{-p}(z)$ denote the standard Bessel functions. It is well known that

$$f_1(z) = z^{\frac{1}{2}} J_p\left(\frac{i}{\mu} z^\mu\right) \quad \text{and} \quad f_2(z) = z^{\frac{1}{2}} J_{-p}\left(\frac{i}{\mu} z^\mu\right) \tag{3.2}$$

are linearly independent solutions of equation (1.1). It is also known [SH] that

$$A_n(z) = p z^{\frac{1}{2}} \left\{ I_{-p}\left(\frac{z^\mu}{\mu}\right) - I_p\left(\frac{z^\mu}{\mu}\right) \right\} \tag{3.3}$$

is a shortage solution of (1.1), where $I_p(z) = i^{-p} J_p(iz)$ and $I_{-p}(z) = i^p J_{-p}(iz)$ are the modified Bessel functions. If $G_0(z)$ is defined as in (2.4), then we will show that

$$G_0(z) \equiv A_n(z) \tag{3.4}$$

for all $n \geq 1$. In §8 we will show that

$$G_0(0) = \frac{1}{(n+2)^{\frac{n+1}{n+2}} \Gamma\left(\frac{n+1}{n+2}\right)} \quad \text{and} \quad G_0'(0) = -\frac{1}{(n+2)^{\frac{1}{n+2}} \Gamma\left(\frac{1}{n+2}\right)}. \tag{3.5}$$

Then from (3.5) and [SH, p. 1402] we have $G_0(0) = A_n(0)$ and $G_0'(0) = A_n'(0)$, and so by the uniqueness of solutions of (1.1), we obtain (3.4). We mention here that for all $j$ in (2.4), the values of $G_j(0)$ and $G_j'(0)$ can be found by the same method as in §8.

We have the following result.

**Theorem A.** *For each* $n \geq 1$, *let* $G_0(z)$ *be defined as in* (2.4).

(i) *The nonreal zeros of* $G_0(z)$ *are all contained in the region* $S$ *given by* $S = \bigcup_k S_k$, *where* $S_k$ *denotes the double sector*

$$S_k = \left\{ z : \frac{2k}{n+1}\pi < |\arg z| < \frac{2k+1}{n+2}\pi \right\}$$

*and where* $k$ *runs over all integers that satisfy* $0 < 2k < n+1$.

(ii) *If* $n$ *is even,* $G_0(z)$ *has no real zeros.*

(iii) *If* $n$ *is odd,* $G_0(z)$ *has an infinite sequence of real negative zeros, but has no real nonnegative zeros.*

Theorem A follows from (3.4) and Theorem 1 in [SH], except for the statement that $G_0(z)$ has no real nonnegative zeros. To see why this statement holds, suppose that $G_0(a) = 0$ for some $a \geq 0$. Since $G_0(z)$ is a solution of (1.1) with the real initial values (3.5), $G_0(x)$ is real for real $x$. Since $G_0(0) > 0$

from (3.5), we can assume that $G_0(x) > 0$ for $0 < x < a$. Then it follows from (1.1) that $G_0(x)$ is concave upward on $0 < x < a$, concave downward on $a < x < \infty$, and $G_0(x) \to -\infty$ as $x \to +\infty$. But this is impossible because from Theorem 2, $s(G_0) = 2$ and the two shortage rays of $G_0(z)$ are $\arg z = \pm\pi/(n+2)$, which means that $G_0(x) \to 0$ as $x \to +\infty$ from Lemma 7 in §5.

From Theorem A and (2.6) it follows that if $n$ is odd and $G_j(z)$ is defined as in (2.4), then there is the one special critical ray

$$(3.6) \qquad L_j\colon \arg z = \frac{n+2+2j}{n+2}\pi ,$$

which has the following properties: (i) $G_j(z)$ possesses an infinite number of zeros that lie on $L_j$, (ii) all the zeros of $G_j(z)$ that lie "near" $L_j$ must actually lie on $L_j$, (iii) $G_j(z)$ does not have any zeros that lie on any other critical ray besides $L_j$. In the special case of the Airy equation (1.3), all the zeros of $G_j(z)$ lie on $L_j$.

On the other hand, when $n$ is even, it follows from Theorem A, (2.6), and Theorem 2(ii), that if $f = H(z)$ is any shortage solution of (1.1), then $H(z)$ does not possess a zero on any critical ray.

We will prove the following result.

**Theorem 3.** *Let $a$ and $b$ be two distinct zeros of $G_j(z)$ in* (2.4).

(i) *If $n$ is even, then* $\arg a \neq \arg b$.

(ii) *If $n$ is odd, then either* $\arg a \neq \arg b$ *or else $a$ and $b$ must both lie on the ray $L_j$ in* (3.6).

We see from (3.4), (3.3), (2.6), and Theorem 2(ii), that any shortage solution of (1.1) has both a Bessel function representation and a contour integral representation. Actually, any solution of (1.1) has both a Bessel function representation and a contour integral representation from (3.2) and Theorem 2(ii). Although the Bessel function representation was used in the proof of Theorem 1 in [SH] (see the sentence that follows the statement of Theorem A above), it can be observed that the contour integral representation works equally as well as the Bessel function representation in this proof because all that was used in this proof was Lommel's method and the two initial values in (3.5).

There have been several recent investigations of the properties of the solutions of the differential equation

$$(3.7) \qquad f'' + R(z)f = 0$$

where $R(z)$ is a nonconstant polynomial, of which (1.1) is a special case. It seems to us that one might have a better chance of finding useful contour integral representations for the shortage solutions of some given equation of the form (3.7) than of finding useful Bessel function representations for these shortage solutions. Heading [H2] says the Bessel function representations of the solutions of (1.1) "must be regarded more or less as a coincidence," and he and some others have argued that it is best to eliminate from the discussion of solutions of (1.1) all reference to Bessel functions.

We also note that the Bessel function representations of the solutions of (1.1) that comes from (3.2) are completely out of character with equation (1.1), because all solutions of (1.1) are entire functions, and yet the Bessel function expressions

$$J_{\pm p}\left(\frac{i}{\mu} z^{\mu}\right)$$

in (3.2) are not entire and are only "restored" to be entire by the factor $z^{1/2}$. On the other hand, the contour integral representations of the solutions of (1.1) that come from Theorems 1 and 2 are entire functions which are more in keeping with the character of equation (1.1). For more discussion on the contour integral representations versus the Bessel function representations of solutions of (1.1), see [H2 and SH].

Last, we mention the following. As in [SH], set

$$B_n(z) = (pz)^{\frac{1}{2}} \left\{ I_{-p}\left(\frac{z^{\mu}}{\mu}\right) + I_p\left(\frac{z^{\mu}}{\mu}\right) \right\} \tag{3.8}$$

where $p$ and $\mu$ are given in (3.1) and where $I_{\pm p}(z)$ are the modified Bessel functions; see (3.3). Then $B_n(z)$ is a solution of (1.1), and with $A_n(z)$ as in (3.3), we have [SH, p. 1402]

$$B_n(0) = \frac{A_n(0)}{p^{\frac{1}{2}}} \quad \text{and} \quad B_n'(0) = -\frac{A_n'(0)}{p^{\frac{1}{2}}}. \tag{3.9}$$

So $A_n(z)$ and $B_n(z)$ are linearly independent. When $n = 1$, $A_n(z)$ and $B_n(z)$ are the standard Airy functions $Ai(z)$ and $Bi(z)$.

When $n$ is even we have [SH, p. 1402]

$$B_n(-z) = \frac{A_n(z)}{p^{\frac{1}{2}}}. \tag{3.10}$$

Since $A_n(z)$ is a shortage solution of (1.1) for any $n$, it follows from (3.10) that $B_n(z)$ is a shortage solution of (1.1) when $n$ is even. Furthermore, when $n$ is even and $G_{(n+2)/2}(z)$ is defined as in (2.4), then it follows from Theorem 2, (3.10), and (3.4) that $G_{(n+2)/2}(z)$ and $B_n(z)$ both have the same two shortage rays

$$\arg z = \frac{n+1}{n+2}\pi \quad \text{and} \quad \arg z = \frac{n+3}{n+2}\pi.$$

It follows from Theorem 2 and Lemma 7 in §5 that $B_n(z)$ and $G_{(n+2)/2}(z)$ are linearly dependent.

When $n$ is odd, then $B_n(z)$ in (3.8) is not a shortage solution. This can be seen, for example, by first noting that from (2.6), (3.4), and (3.9), we obtain that

$$\frac{B_n'(0)}{B_n(0)} \neq \frac{G_j'(0)}{G_j(0)} \quad \text{for all odd } n \text{ and for all } j. \tag{3.11}$$

Combining (3.11) with Theorem 2(ii) shows that $B_n(z)$ is not a shortage solution when $n$ is odd.

## 4. The case when $n = 2$ and the parabolic cylinder function

It is well known that the shortage solutions of the equation

$$f'' - z^2 f = 0 \tag{4.1}$$

can all be expressed in terms of the parabolic cylinder function $D_{\nu}(z)$ for $\nu = -1/2$. We will now illustrate the relationships between these well-known expressions and our results on (4.1) in §2. For the discussion below on the parabolic cylinder function, see, e.g., [AS, BO, EMOT, and WW].

To simplify the notation, set $U(z) = D_{-\frac{1}{2}}(z)$. Then $f = U(z)$ satisfies the equation

$$f'' - \frac{1}{4}z^2 f = 0 \tag{4.2}$$

and

$$U(x) \to 0 \quad \text{as } x \to +\infty \tag{4.3}$$

along the positive real axis. For $j$ an integer, set

$$W_j(z) = U(\beta_j z) \quad \text{where } \beta_j = \sqrt{2}\exp\left\{-j\frac{\pi i}{2}\right\}. \tag{4.4}$$

Then from (4.2) and (4.3), we obtain that $W_j(z)$ is a solution of (4.1) and $W_j(z) \to 0$ as $z \to \infty$ along $\arg z = j\pi/2$. Hence from Lemma 7 in §5 it follows that $W_j(z)$ is a shortage solution of (4.1) and the two shortage rays of $W_j(z)$ are

$$\arg z = \frac{2j-1}{4}\pi \quad \text{and} \quad \arg z = \frac{2j+1}{4}\pi. \tag{4.5}$$

If $G_j(z)$ is defined as in Theorem 2 when $n = 2$, then from Theorem 2, (4.5), and Lemma 7 in §5, we obtain that $G_j(z)$ and $W_j(z)$ are linearly dependent for all $j$.

A second way to obtain this observation is as follows. From [AS, p. 687] we have

$$U(0) = \frac{\sqrt{\pi}}{2^{\frac{1}{4}}\Gamma(\frac{3}{4})} \quad \text{and} \quad U'(0) = -\frac{2^{\frac{1}{4}}\sqrt{\pi}}{\Gamma(\frac{1}{4})}, \tag{4.6}$$

and then from (4.6) and (4.4), together with (3.5) and (2.6) when $n = 2$, we deduce that

$$\frac{W_j'(0)}{W_j(0)} = \frac{G_j'(0)}{G_j(0)} \quad \text{for all } j. \tag{4.7}$$

Since for each $j$, $G_j(z)$ and $W_j(z)$ are both solutions of (4.1), it follows from (4.7) that $G_j(z)$ and $W_j(z)$ are linearly dependent for all $j$.

## 5. Lemmas

We will use Lemma 4 below to show that the function $G(z)$ in (2.1) is a solution of equation (1.1). We will use Lemmas 1, 2, and 3 in the proof of Lemma 4. The proofs of Lemmas 1, 2, and 3 contain calculations with combinatorial sums which are similar to some calculations in the book by Egorychev [E].

**Lemma 1.** *If $m$ and $l$ are positive integers that satisfy $l + 1 \le m \le 2l$, then*

$$\sum_{k=m-l-1}^{l} \binom{2l-k}{k}\binom{2l+2-m+k}{m-k} = \sum_{k=m-l}^{l} \binom{2l+1-k}{k}\binom{2l+1-m+k}{m-k}. \tag{5.1}$$

*Proof*. Set

$$\lambda = \sum_{k=m-l-1}^{l} \binom{2l-k}{k}\binom{2l+2-m+k}{m-k}. \tag{5.2}$$

Then $\lambda$ can be written in the following form:

$$\lambda = \sum_{k=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_1} \frac{(1+w)^{2l-k}}{w^{k+1}}\,dw \cdot \frac{1}{2\pi i}\int_{\gamma_2} \frac{(1+z)^{2l+2-m+k}}{z^{m-k+1}}\,dz \tag{5.3}$$

where $\gamma_1\colon |w| = 1/2$ and $\gamma_2\colon |z| = \varepsilon$ where $\varepsilon > 0$ is a small fixed constant. From (5.3) we obtain

$$\begin{aligned}
\lambda &= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_2}\int_{\gamma_1} \frac{(1+w)^{2l}(1+z)^{2l+2-m}}{w z^{m+1}} \left[\frac{z(1+z)}{w(1+w)}\right]^k dw\,dz \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_2}\int_{\gamma_1} \frac{(1+w)^{2l}(1+z)^{2l+2-m}}{w z^{m+1}} \sum_{k=0}^{\infty}\left[\frac{z(1+z)}{w(1+w)}\right]^k dw\,dz \\
&= \frac{1}{(2\pi i)^2}\int_{\gamma_2}\int_{\gamma_1} \frac{(1+w)^{2l+1}(1+z)^{2l+2-m}}{(w-z)(w+z+1)z^{m+1}}\,dw\,dz.
\end{aligned}$$

Integration over $\gamma_1$ gives

$$\lambda = \frac{1}{2\pi i}\int_{\gamma_2} \frac{(1+z)^{4l+3-m}}{(2z+1)z^{m+1}}\,dz. \tag{5.4}$$

Now set

$$\mu = \sum_{k=m-l}^{l} \binom{2l+1-k}{k}\binom{2l+1-m+k}{m-k}. \tag{5.5}$$

Using similar reasoning to the above, we obtain

$$\begin{aligned}
\mu &= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_2}\int_{\gamma_1} \frac{(1+w)^{2l+1}(1+z)^{2l+1-m}}{w z^{m+1}} \left[\frac{z(1+z)}{w(1+w)}\right]^k dw\,dz \\
&= \frac{1}{(2\pi i)^2}\int_{\gamma_2}\int_{\gamma_1} \frac{(1+w)^{2l+2}(1+z)^{2l+1-m}}{(w-z)(w+z+1)z^{m+1}}\,dw\,dz \\
&= \frac{1}{2\pi i}\int_{\gamma_2} \frac{(1+z)^{4l+3-m}}{(2z+1)z^{m+1}}\,dz.
\end{aligned}$$

Thus $\mu = \lambda$ from (5.4), and (5.1) follows from (5.2) and (5.5). This proves Lemma 1.

**Lemma 2.** *If $m$ and $l$ are positive integers that satisfy $l \le m \le 2l-1$, then*

$$\sum_{k=m-l}^{l-1} \binom{2l-1-k}{k}\binom{2l+1-m+k}{m-k} = \sum_{k=m-l}^{l} \binom{2l-k}{k}\binom{2l-m+k}{m-k}. \tag{5.6}$$

*Proof*. Let $\gamma_1\colon |w| = 1/2$ and $\gamma_2\colon |z| = \varepsilon$ where $\varepsilon > 0$ is a small fixed constant. Set

$$\lambda = \sum_{k=m-l}^{l-1} \binom{2l-1-k}{k} \binom{2l+1-m+k}{m-k}. \tag{5.7}$$

Then with the same reasoning as in Lemma 1, we obtain

$$\begin{aligned} \lambda &= \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} \int_{\gamma_2} \int_{\gamma_1} \frac{(1+w)^{2l-1}(1+z)^{2l+1-m}}{w z^{m+1}} \left[\frac{z(1+z)}{w(1+w)}\right]^k dw\, dz \\ &= \frac{1}{(2\pi i)^2} \int_{\gamma_2} \int_{\gamma_1} \frac{(1+w)^{2l}(1+z)^{2l+1-m}}{(w-z)(w+z+1)z^{m+1}}\, dw\, dz \\ &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{(1+z)^{4l+1-m}}{(2z+1)z^{m+1}}\, dz. \end{aligned} \tag{5.8}$$

Now set

$$\mu = \sum_{k=m-l}^{l} \binom{2l-k}{k} \binom{2l-m+k}{m-k}. \tag{5.9}$$

Then with the same reasoning as above,

$$\begin{aligned} \mu &= \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} \int_{\gamma_2} \int_{\gamma_1} \frac{(1+w)^{2l}(1+z)^{2l-m}}{w z^{m+1}} \left[\frac{z(1+z)}{w(1+w)}\right]^k dw\, dz \\ &= \frac{1}{(2\pi i)^2} \int_{\gamma_2} \int_{\gamma_1} \frac{(1+w)^{2l+1}(1+z)^{2l-m}}{(w-z)(w+z+1)z^{m+1}}\, dw\, dz \\ &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{(1+z)^{4l+1-m}}{(2z+1)z^{m+1}}\, dz. \end{aligned}$$

Thus $\mu = \lambda$ from (5.8), and (5.6) follows from (5.7) and (5.9).

**Lemma 3.** *Let $m$ and $n \geq 1$ be integers that satisfy $0 \leq m \leq [\![(n-1)/2]\!]$ where $[\![(n-1)/2]\!]$ is the greatest integer that is $\leq (n-1)/2$. Then*

$$\sum_{k=0}^{m} \binom{n-1-k}{k} \binom{n+1-m+k}{m-k} = \sum_{k=0}^{m} \binom{n-k}{k} \binom{n-m+k}{m-k}. \tag{5.10}$$

*Proof*. Let $\gamma_1\colon |w| = 1/2$ and $\gamma_2\colon |z| = \varepsilon$ where $\varepsilon > 0$ is a small fixed constant. If $\lambda$ is the left side of (5.10) and $\mu$ is the right side of (5.10), then with the same reasoning as in Lemma 1, we obtain

$$\begin{aligned} \lambda &= \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} \int_{\gamma_2} \int_{\gamma_1} \frac{(1+w)^{n-1}(1+z)^{n+1-m}}{w z^{m+1}} \left[\frac{z(1+z)}{w(1+w)}\right]^k dw\, dz \\ &= \frac{1}{(2\pi i)^2} \int_{\gamma_2} \int_{\gamma_1} \frac{(1+w)^{n}(1+z)^{n+1-m}}{(w-z)(w+z+1)z^{m+1}}\, dw\, dz \\ &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{(1+z)^{2n+1-m}}{(2z+1)z^{m+1}}\, dz, \end{aligned}$$

and

$$\begin{aligned}\mu &= \frac{1}{(2\pi i)^2}\sum_{k=0}^{\infty}\int_{\gamma_2}\int_{\gamma_1}\frac{(1+w)^n(1+z)^{n-m}}{wz^{m+1}}\left[\frac{z(1+z)}{w(1+w)}\right]^k dw\,dz\\ &= \frac{1}{(2\pi i)^2}\int_{\gamma_2}\int_{\gamma_1}\frac{(1+w)^{n+1}(1+z)^{n-m}}{(w-z)(w+z+1)z^{m+1}}\,dw\,dz\\ &= \frac{1}{2\pi i}\int_{\gamma_2}\frac{(1+z)^{2n+1-m}}{(2z+1)z^{m+1}}\,dz.\end{aligned}$$

Thus $\mu = \lambda$, and this proves Lemma 3.

**Lemma 4.** *Let $n \geq 1$ be an integer, and let $P(z, w)$ be the polynomial in* (2.2). *Set*

(5.11) $$Q(z, w) = \sum_{k=0}^{[\![(n-1)/2]\!]} (-1)^k \binom{n-1-k}{k} z^k w^{n-1-2k}$$

*where $[\![(n-1)/2]\!]$ is the greatest integer that is $\leq (n-1)/2$.*

*Then the following identity holds*:

(5.12) $$\frac{\partial^2}{\partial z^2}\{e^{P(z,w)}\} - z^n e^{P(z,w)} \equiv \frac{\partial}{\partial w}\{Q(z, w)e^{P(z,w)}\}.$$

*Proof*. Corresponding to (5.11) and (2.2), for convenience we set

(5.13) $$q = [\![(n-1)/2]\!],$$

(5.14) $$a_k = (-1)^k\binom{n-1-k}{k} \quad \text{for } 0 \leq k \leq q,$$

(5.15) $$p = [\![(n+2)/2]\!],$$

and

(5.16) $$b_k = \frac{(-1)^k}{n+2-k}\binom{n+2-k}{k} \quad \text{for } 0 \leq k \leq p.$$

From (2.2), (5.11), (5.13), (5.14), (5.15), and (5.16), we obtain that the identity (5.12) is equivalent to the following identity:

(5.17) $$\begin{aligned}&\sum_{k=2}^{p} k(k-1)b_k z^{k-2}w^{n+2-2k} + \left(\sum_{k=1}^{p} kb_k z^{k-1}w^{n+2-2k}\right)^2 - z^n\\ &\quad = \alpha + \sum_{k=0}^{q-1}(n-1-2k)a_k z^k w^{n-2-2k}\\ &\quad + \left(\beta + \sum_{k=0}^{p-1}(n+2-2k)b_k z^k w^{n+1-2k}\right)\sum_{k=0}^{q} a_k z^k w^{n-1-2k}\end{aligned}$$

where

(5.18) $$\alpha = \begin{cases} 0 & \text{if } n \text{ is odd,}\\ a_q z^q & \text{if } n \text{ is even,}\end{cases}$$

and

$$\beta = \begin{cases} b_p z^p & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{5.19}$$

From (5.13), (5.14), (5.15), (5.16), and (5.18), it can be deduced that the first term on the left side of (5.17) is equal to the sum of the first two terms on the right side of (5.17). Therefore, Lemma 4 will be proven if we can show that the following identity holds:

$$\begin{aligned} &\left(\sum_{k=0}^{p-1}(k+1)b_{k+1}z^k w^{n-2k}\right)^2 - z^n \\ &\quad = \left(\beta + \sum_{k=0}^{p-1}(n+2-2k)b_k z^k w^{n+1-2k}\right)\sum_{k=0}^{q} a_k z^k w^{n-1-2k}. \end{aligned} \tag{5.20}$$

We will now prove that (5.20) holds by showing that for each $m$, $0 \le m \le n$, the coefficients of

$$z^m w^{2n-2m} \tag{5.21}$$

on both sides of (5.20) are equal. We will use (5.13), (5.14), (5.15), (5.16), and (5.19) to show this.

First we note that when $n$ is odd, the coefficients of $z^n$ on both sides of (5.20) are equal to $-1$, while when $n$ is even, the coefficients of $z^n$ on both sides of (5.20) are equal to zero. Thus when $m = n$, the coefficients of (5.21) on both sides of (5.20) are equal.

Next we consider the coefficients of (5.21) when $0 \le m \le q$ on both sides of (5.20). On the left side of (5.20), this coefficient $\lambda_1$ has the value

$$\lambda_1 = \sum_{k=0}^{m}(k+1)(m-k+1)b_{k+1}b_{m-k+1} = (-1)^m \sum_{k=0}^{m}\binom{n-k}{k}\binom{n-m+k}{m-k},$$

while on the right side of (5.20), this coefficient $\mu_1$ has the value

$$\mu_1 = \sum_{k=0}^{m}(n+2-2m+2k)a_k b_{m-k} = (-1)^m \sum_{k=0}^{m}\binom{n-1-k}{k}\binom{n+1-m+k}{m-k}.$$

Thus $\mu_1 = \lambda_1$ from Lemma 3, and so the coefficients of (5.21) when $0 \le m \le q$ on both sides of (5.20) are equal.

The remaining case to consider is the coefficients of (5.21) when $q+1 \le m \le n-1$ and $n \ge 2$ on both sides of (5.20). On the left side of (5.20), this coefficient $\lambda_2$ has the value

$$\lambda_2 = \sum_{k=m-p+1}^{p-1}(k+1)(m-k+1)b_{k+1}b_{m-k+1} = (-1)^m \sum_{k=m-p+1}^{p-1}\binom{n-k}{k}\binom{n-m+k}{m-k},$$

while on the right side of (5.20), this coefficient $\mu_2$ separates into the following two cases:

*Case* 1. If $n \ge 3$ is odd, then

$$\mu_2 = \sum_{k=m-p}^{q}(n+2-2m+2k)a_k b_{m-k} = (-1)^m \sum_{k=m-p}^{q}\binom{n-1-k}{k}\binom{n+1-m+k}{m-k}.$$

Setting $n = 2l + 1$, we see from Lemma 1 that $\mu_2 = \lambda_2$ when $n$ is odd.

*Case* 2. If $n$ is even, then

$$\mu_2 = \sum_{k=m-p+1}^{q} (n + 2 - 2m + 2k) a_k b_{m-k}$$
$$= (-1)^m \sum_{k=m-p+1}^{q} \binom{n-1-k}{k}\binom{n+1-m+k}{m-k}.$$

Setting $n = 2l$, we see from Lemma 2 that $\mu_2 = \lambda_2$ when $n$ is even.

We have now shown that for all $n \geq 2$, the coefficients of (5.21) when $q + 1 \leq m \leq n - 1$ on both sides of (5.20) are equal.

Thus we have now shown that for each $n \geq 1$ and for each $m$ satisfying $0 \leq m \leq n$, the coefficients of (5.21) on both sides of (5.20) are equal. Therefore, (5.20) holds, and this proves Lemma 4.

The next lemma will be used in the proofs of Theorem 1 and (3.5).

**Lemma 5.** *For each* $k = 0, 1, 2, \dots$, *and for each* $m = 1, 2, 3, \dots$, *we have*

$$\int_0^\infty x^k \exp\left\{-\frac{x^m}{m}\right\} dx = m^\beta \Gamma\left(\frac{k+1}{m}\right)$$

*where* $\beta = (k + 1 - m)/m$.

*Proof.* By making the substitution $mt = x^m$ on the right side of the equation

$$\Gamma\left(\frac{k+1}{m}\right) = \int_0^\infty e^{-t} t^\beta \, dt,$$

we obtain

$$\Gamma\left(\frac{k+1}{m}\right) = \int_0^\infty x^{m-1} \left(\frac{x^m}{m}\right)^\beta \exp\left\{-\frac{x^m}{m}\right\} dx,$$

and this yields the assertion.

The next lemma is used in the proof of Theorem 1.

**Lemma 6.** *Suppose equation* (1.1) *possesses a shortage solution* $F_0(z)$ *such that* $F_0(0) \neq 0$ *and* $F_0'(0) \neq 0$. *Then* $s(F) = 2$ *for any shortage solution* $F(z)$ *of* (1.1).

*Proof.* For $0 \leq k \leq n + 1$, set

$$F_k(z) = F_0(\alpha_k z) \quad \text{where } \alpha_k = \exp\left\{k\frac{2\pi i}{n+2}\right\}. \tag{5.22}$$

Then each $F_k(z)$ is a shortage solution of (1.1). By considering the values of $F_k(0)$ and $F_k'(0)$ for each $k$ and noting that $F_0(0) \neq 0$ and $F_0'(0) \neq 0$, it is easy to see from (5.22) that the shortage solutions $F_0(z), F_1(z), \dots, F_{n+1}(z)$ are pairwise linearly independent. Therefore, it follows from §1(II) that $s(F_k) = 2$ for all $k = 0, 1, \dots, n+1$, and that $s(f) = 0$ for any solution $f \not\equiv 0$ of (1.1) that is not a constant multiple of some $F_k$, $k = 0, 1, \dots, n + 1$. This proves Lemma 6.

We refer to the next well-known result several times throughout the paper.

**Lemma 7** [H3, Chapter 7.4]. *Consider the sector*

$$\frac{2k-1}{n+2}\pi + \varepsilon < \arg z < \frac{2k+1}{n+2}\pi - \varepsilon \tag{5.23}$$

*where $k$ is any fixed integer, $n \geq 1$ is an integer, and $\varepsilon > 0$ is any fixed small constant.*

*There exists a shortage solution $f_0(z)$ of equation* (1.1) *such that $f_0(z) \to 0$ as $z \to \infty$ in the sector* (5.23), *and where the two critical rays*

$$\arg z = \frac{2k-1}{n+2}\pi \quad \text{and} \quad \arg z = \frac{2k+1}{n+2}\pi \tag{5.24}$$

*are both shortage rays of $f_0(z)$.*

*Let $f \not\equiv 0$ be any solution of* (1.1) *that is not a constant multiple of $f_0(z)$. Then the following two statements hold*:

(i) *$f(z) \to \infty$ as $z \to \infty$ in the sector* (5.23).

(ii) *If $s(f) = 2$, then at least one of the two rays in* (5.24) *will not be a shortage ray of $f(z)$.*

## 6. Proof of Theorem 1

We will first show that the function $G(z)$ in (2.1) is a solution of equation (1.1). From (2.1),

$$G''(z) - z^n G(z) = \frac{1}{2\pi i}\int_C \left(\frac{\partial^2}{\partial z^2}\{e^{P(z,w)}\} - z^n e^{P(z,w)}\right) dw. \tag{6.1}$$

If $Q(z, w)$ is defined as in (5.11), then it follows from (2.2), (5.12), (6.1), and the definition of the contour $C$, that

$$G''(z) - z^n G(z) = \frac{1}{2\pi i}[Q(z, w)e^{P(z,w)}]_C = 0.$$

Thus $G(z)$ is a solution of (1.1).

We will now show that $G(z)$ is bounded on the ray

$$\arg z = \frac{2m_1 + 2m_2 + 2}{n+2}\pi. \tag{6.2}$$

From this fact it will be easy to prove the remaining parts of Theorem 1.

It will be convenient to make the following notations. We set

$$\psi = \frac{2m_1 + 1}{n+2}\pi \quad \text{and} \quad \zeta = \frac{2m_2+1}{n+2}\pi, \tag{6.3}$$

$$\alpha = \exp\left\{-2\pi i \frac{m_1 + m_2 + 1}{n+2}\right\}, \tag{6.4}$$

and

$$c_k = \frac{1}{n+2-k}\binom{n+2-k}{k} \tag{6.5}$$

for all $0 \leq k \leq p$ where

$$p = [\![(n+2)/2]\!] \tag{6.6}$$

is the greatest integer that is $\leq (n+2)/2$. We note that

$$c_k > 0 \quad \text{for all } 0 \leq k \leq p. \tag{6.7}$$

Now from (2.1), (6.3), and Lemma 5, we obtain

$$G(0) = \frac{\Gamma\left(\frac{1}{n+2}\right)\left(e^{i\zeta} - e^{i\psi}\right)}{2\pi i (n+2)^{\frac{n+1}{n+2}}} \tag{6.8}$$

and

$$G'(0) = -\frac{\Gamma\left(\frac{n+1}{n+2}\right)\left(e^{i\zeta(n+1)} - e^{i\psi(n+1)}\right)}{2\pi i (n+2)^{\frac{1}{n+2}}}. \tag{6.9}$$

Since it is assumed that $m_1 - m_2$ is not an integer multiple of $n+2$, we obtain from (6.9), (6.8), and (6.3) that

$$G(0) \neq 0 \quad \text{and} \quad G'(0) \neq 0. \tag{6.10}$$

From (6.9), (6.8), and (6.3), we deduce that

$$\frac{G'(0)}{G(0)} = \alpha\lambda \tag{6.11}$$

where $\alpha$ is given in (6.4) and

$$\lambda = -\frac{\Gamma\left(\frac{n+1}{n+2}\right)(n+2)^{\frac{n}{n+2}}}{\Gamma\left(\frac{1}{n+2}\right)}. \tag{6.12}$$

We will now divide the proof that $G(z)$ is bounded on the ray (6.2) into the two cases when $n$ is odd or even.

*Case* 1. Suppose that $n$ is odd.

We will first let $H_0(z)$ denote the particular function $G(z)$ in (2.1) when

$$m_1 = [\![(n+1)/4]\!] \quad \text{and} \quad m_2 = -1 - [\![(n+1)/4]\!] \tag{6.13}$$

where $[\![\cdot]\!]$ is the greatest integer function. Since $\zeta = -\psi$ from (6.13) and (6.3), we can deduce from (2.1), (2.2), (6.5), and (6.6) that for any $R > 0$,

$$|H_0(R)| \leq \int_0^\infty \exp\left\{\sum_{k=0}^{p}(-1)^k c_k R^k r^{n+2-2k} \cos(n+2-2k)\psi\right\} dr. \tag{6.14}$$

Now from (6.3),

$$\cos(n+2-2k)\psi = -\cos(2k\psi) = (-1)^{k+1}\cos(2k\psi - k\pi),$$

and from (6.13), (6.3), and the assumption that $n$ is odd, we can deduce that

$$2k\psi - k\pi = \pm\frac{k\pi}{n+2}.$$

Hence

$$\cos(n+2-2k)\psi = (-1)^{k+1}\cos\left(\frac{k\pi}{n+2}\right). \tag{6.15}$$

From (6.6) we have

$$\cos\left(\frac{k\pi}{n+2}\right) > 0 \quad \text{for all } 0 \leq k \leq p. \tag{6.16}$$

Hence from (6.14) together with (6.7), (6.15), and (6.16), it follows that for all $R \geq 1$,

$$|H_0(R)| \leq \int_0^\infty \exp\left\{-\sum_{k=0}^{p} c_k R^k r^{n+2-2k} \cos\left(\frac{k\pi}{n+2}\right)\right\} dr$$
$$\leq \int_0^\infty \exp\left\{-\sum_{k=0}^{p} c_k r^{n+2-2k} \cos\left(\frac{k\pi}{n+2}\right)\right\} dr = M < \infty,$$

where the finite constant $M$ does not depend on $R$. It follows that $H_0(z)$ is bounded on the positive real axis.

We now turn to the general case when $n$ is odd. Let $G(z)$ be defined as in (2.1) where $n$ is odd and where $m_1$ and $m_2$ are any given integers such that $m_1 - m_2$ is not an integer multiple of $n+2$. Then from (6.11),

$$\frac{G'(0)}{G(0)} = \alpha\lambda \tag{6.17}$$

where $\alpha$ is given in (6.4) and $\lambda$ is given in (6.12).

Now set

$$H(z) = H_0(\alpha z). \tag{6.18}$$

Since $H_0(z)$ is the particular function $G(z)$ in (2.1) that has the values $m_1$ and $m_2$ in (6.13), it follows from (6.11) and (6.4) that

$$\frac{H_0'(0)}{H_0(0)} = \lambda. \tag{6.19}$$

Hence from (6.19), (6.18), and (6.17), we have

$$\frac{G'(0)}{G(0)} = \frac{H'(0)}{H(0)}. \tag{6.20}$$

Since $H(z)$ and $G(z)$ are both solutions of (1.1), it follows from (6.20) that $G(z)$ must be a constant multiple of $H(z)$. Since we showed that $H_0(z)$ is bounded on the positive real axis, it follows from (6.18) and (6.4) that $G(z)$ must be bounded on the ray (6.2). This proves the assertion for odd $n$.

*Case* 2. Now suppose that $n$ is even.

We first let $H_1(z)$ denote the particular function $G(z)$ in (2.1) when

$$m_1 = -(n+2)/2 \quad \text{and} \quad m_2 = 0. \tag{6.21}$$

Corresponding to (6.2), we will show that $H_1(z)$ is bounded on the ray

$$\arg z = -\frac{n}{n+2}\pi. \tag{6.22}$$

From (6.21), (2.1), (2.2), (6.5), and (6.7), we obtain that for all $R \geq 1$,

$$\left|H_1\left(R\exp\left\{-i\frac{n\pi}{n+2}\right\}\right)\right| \leq \int_0^\infty \exp\left\{-\sum_{k=0}^{p} c_k R^k r^{n+2-2k}\right\} dr$$
$$\leq \int_0^\infty \exp\left\{-\sum_{k=0}^{p} c_k r^{n+2-2k}\right\} dr = M^* < \infty,$$

where the finite constant $M^*$ does not depend on $R$. It follows that $H_1(z)$ is bounded on the ray (6.22).

We now turn to the general case when $n$ is even. Let $G(z)$ be defined as in (2.1) where $n$ is even and where $m_1$ and $m_2$ are any given integers such that $m_1 - m_2$ is not an integer multiple of $n+2$. Then (6.11) holds.

Now set

$$E(z) = H_1(\beta z) \quad \text{where } \beta = \alpha \exp\left\{-\frac{n\pi i}{n+2}\right\} \tag{6.23}$$

and $\alpha$ is given in (6.4). Since $H_1(z)$ is the particular function $G(z)$ in (2.1) where $m_1$ and $m_2$ have the values in (6.21), it follows from (6.11) and (6.4) that

$$\frac{H_1'(0)}{H_1(0)} = \lambda \exp\left\{\frac{n\pi i}{n+2}\right\} \tag{6.24}$$

where $\lambda$ is given in (6.12). Then from (6.11), (6.23), and (6.24), we obtain that

$$\frac{G'(0)}{G(0)} = \frac{E'(0)}{E(0)}. \tag{6.25}$$

Since $G(z)$ and $E(z)$ are both solutions of (1.1), it follows from (6.25) that $G(z)$ must be a constant multiple of $E(z)$. Since we showed that $H_1(z)$ is bounded on the ray (6.22), it follows from (6.23) and (6.4) that $G(z)$ is bounded on the ray (6.2). This proves the assertion for even $n$.

We have now completed the two cases, i.e., we have shown that if $G(z)$ is defined as in (2.1) for any values of $n$, $m_1$, and $m_2$, then $G(z)$ is bounded on the ray (6.2).

It then follows from Lemma 7 that $G(z)$ must be a shortage solution of equation (1.1) and that the two rays in (2.3) must be shortage rays of $G(z)$. From (6.10) and Lemma 6, we obtain that $s(G) = 2$. This completes the proof of Theorem 1.

## 7. Proof of Theorem 2

By combining Theorem 1 with the definition of $G_j(z)$ in (2.4), we obtain that $G_j(z)$ is a shortage solution of (1.1), $s(G_j) = 2$, and the two shortage rays of $G_j(z)$ are given in (2.5). This proves (i).

Now if for two integers $j$ and $k$, $G_j(z)$ and $G_k(z)$ are linearly dependent, then for both $G_j(z)$ and $G_k(z)$ the two shortage rays in (2.5) must be exactly the same pair of rays (mod $2\pi$). It follows that the functions $G_j(z)$, $j = 0, 1, \ldots, n+1$, are pairwise linearly independent. In view of §1(II) we see that the functions $G_j(z)$, $j = 0, 1, \ldots, n+1$, form a basis of shortage solutions of (1.1). This proves (ii).

*Remark.* It is easy to see that if $j_0$ is any fixed integer, then the functions $G_j(z)$, $j = j_0, j_0+1, \ldots, j_0+n+1$, form a basis of shortage solutions of (1.1).

## 8. Proof of (3.5)

Since $G_0(z)$ in (2.4) is $G(z)$ in (2.1) with

$$m_1 = -1 \quad \text{and} \quad m_2 = 0, \tag{8.1}$$

we obtain from (6.8) and (6.3) that

$$G_0(0) = \frac{\sin\left(\frac{\pi}{n+2}\right)\Gamma\left(\frac{1}{n+2}\right)}{\pi(n+2)^{\frac{n+1}{n+2}}}. \tag{8.2}$$

From the well-known identity [A, p. 198]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{8.3}$$

and (8.2), we obtain

$$G_0(0) = \frac{1}{(n+2)^{\frac{n+1}{n+2}}\Gamma\left(\frac{n+1}{n+2}\right)}. \tag{8.4}$$

On the other hand, from (8.1), (6.9), and (6.3), we obtain that

$$G_0'(0) = -\frac{\sin\left(\frac{n+1}{n+2}\pi\right)\Gamma\left(\frac{n+1}{n+2}\right)}{\pi(n+2)^{\frac{1}{n+2}}}. \tag{8.5}$$

Then from (8.5) and (8.3),

$$G_0'(0) = -\frac{1}{(n+2)^{\frac{1}{n+2}}\Gamma\left(\frac{1}{n+2}\right)}. \tag{8.6}$$

Then (3.5) follows from (8.6) and (8.4).

## 9. Proof of Theorem 3

We will first prove Theorem 3 for $G_0(z)$. We will use Lommel's method. We suppose that $G_0(a) = G_0(b) = 0$ where $\arg a = \arg b$ and $|b| > |a|$. Since $G_0(z)$ is a solution of (1.1), it is easy to see that

$$\begin{aligned} &\frac{d}{dz}\{\bar{a}G_0(az)G_0'(\bar{a}z) - aG_0(\bar{a}z)G_0'(az)\} \\ &\quad\equiv (\bar{a}^{n+2} - a^{n+2})z^n G_0(az)G_0(\bar{a}z). \end{aligned} \tag{9.1}$$

Set $\lambda = b/a$ and note that $\lambda$ is real with $\lambda > 1$. Then from (9.1) we obtain

$$\begin{aligned} &(\bar{a}^{n+2} - a^{n+2})\int_1^\lambda x^n G_0(ax)G_0(\bar{a}x)\,dx \\ &\quad= [\bar{a}G_0(ax)G_0'(\bar{a}x) - aG_0(\bar{a}x)G_0'(ax)]_1^\lambda. \end{aligned} \tag{9.2}$$

As noted in §3, $G_0(z)$ is real on the real axis, and so $G_0(\bar{a}) = 0$. We also have $G_0(\bar{a}\lambda) = 0$ since $\overline{G_0(\bar{a}\lambda)} = G_0(a\lambda) = G_0(b) = 0$. Hence we obtain from (9.2) that

$$(\bar{a}^{n+2} - a^{n+2})\int_1^\lambda x^n\,|G_0(ax)|^2\,dx = 0.$$

It follows that $\bar{a}^{n+2} = a^{n+2}$, which means that $a^{n+2}$ is real. Then from Theorem A, we can deduce that $a$ must be real and that Theorem 3 holds for $G_0(z)$.

In light of (2.6) and (3.6), Theorem 3 immediately follows for $G_j(z)$ for any $j$.

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