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APPLYING COORDINATE PRODUCTS TO THE TOPOLOGICAL IDENTIFICATION OF NORMED SPACES

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ABSTRACT. Using the l^2-products we find pre-Hilbert spaces that are absorbing sets for all Borelian classes of order $\alpha \geq 1$. We also show that the following spaces are homeomorphic to Σ^∞, the countable product of the space $\Sigma = \{(x_n) \in R^\infty : (x_n) \text{ is bounded}\}$:

(1) every coordinate product $\prod_C H_n$ of normed spaces H_n in the sense of a Banach space C, where each H_n is an absolute $F_{\sigma\delta}$-set and infinitely many of the H_n 's are Z_σ-spaces,
(2) every function space $\widetilde{L}^p = \bigcap_{p'<p} L^{p'}$ with the L^q-topology, $0 < q < p \leq \infty$,
(3) every sequence space $\widetilde{l}^p = \bigcap_{p<p'} l^{p'}$ with the l^q-topology, $0 \leq p < q < \infty$.

We also note that each additive and multiplicative Borelian class of order $\alpha \geq 2$, each projective class, and the class of nonprojective spaces contain uncountably many topologically different pre-Hilbert spaces which are Z_σ-spaces.

1. INTRODUCTION

We are interested in the topological classification of noncomplete normed linear spaces. The main tool in this area is the method of absorbing sets discovered and applied in the σ-compact case by Anderson and Bessaga and Pełczyński (see [2]). Absorbing sets which are not necessarily σ-closed in a considered copy s of l^2 were developed by Bestvina and Mogilski [4]. A disadvantage of the approach presented in [4] was that two homeomorphic absorbing sets in s might not have been relatively homeomorphic. The difficulty was overcome in [7] due to replacing the strong universality property by its relative version (see Theorem 2.2). We construct linear subspaces F_α, $\alpha \geq 1$ (respectively, G_α, $\alpha \geq 2$) of l^2 that are absorbing sets for the additive Borelian class $\mathscr{A}_\alpha$ (respectively, the multiplicative Borelian class $\mathscr{M}_\alpha$) and such that the pair (l^2, F_α) (respectively, (l^2, G_α)) is strongly $(\mathscr{M}_1, \mathscr{A}_\alpha)$-universal (respectively, $(\mathscr{M}_1, \mathscr{M}_\alpha)$-universal). Applying Theorem 2.2, (l^2, F_α) and (l^2, G_α) are homeomorphic to (s, Λ_α) and (s, Ω_α), respectively, where Λ_α and Ω_α are absorbing sets in s constructed in [4].

One may guess that F_α (respectively, G_α) is the weak l^2-product $\sum_{l^2} H_n$ (respectively, the l^2-product $\prod_{l^2} H_n$) of pre-Hilbert spaces H_n that contain a

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closed copy of Λ_α (respectively, Ω_α). The crucial step is to show that $\sum_{l^2} H_n$ and $\prod_{l^2} H_n$ are strongly $\mathscr{A}_\alpha$- and $\mathscr{M}_\alpha$-universal, respectively. Actually, we are able to verify the strong $(\mathscr{K}, \mathscr{L})$-universality property of an arbitrary normed coordinate product pair $(\prod_C E_n, \sum_C H_n)$ provided each element of $(\mathscr{K}, \mathscr{L})$ admits a relative closed embedding into every (E_n, H_n) (see Proposition 3.1). A version of 3.1 for cartesian products was earlier applied [11, 13, 12] in order to identify some function and sequence spaces that are homeomorphic to $\Omega_2 = \Sigma^\infty$. Applying 3.1 (and its variations), we show that several absolute $F_{\sigma\delta}$-spaces that underlie a "product" structure are homeomorphic to Ω_2. In particular, we prove that every normed coordinate product $\prod_C H_n$, C being a Banach space, is homeomorphic to Ω_2 provided each $H_n \in \mathscr{M}_2$ and infinitely many of the H_n's are Z_σ-spaces. Another application concerns the function space $\widetilde{L}^p = \bigcap_{p'<p} L^{p'}$ in the L^q-topology $(q < p)$ and the sequence space $\widetilde{l}^p = \bigcap_{p<p'} l^{p'}$ in the l^q-topology $(p < q)$. We prove that $\widetilde{L}^p$, $0 < q < p \leq \infty$, and $\widetilde{l}^p$, $0 \leq p < q < \infty$, are homeomorphic to Ω_2. Actually, we show that the pairs $(L^q, \widetilde{L}^p)$, $(l^q, \widetilde{l}^p)$, and (s, Ω_2) are homeomorphic. The fact that the space $\widetilde{L}^p$ considered as a subspace of L^0 (of all measurable functions with the topology of convergence in measure) and the space $\widetilde{l}^p$ as a subspace of R^∞ are homeomorphic to Ω_2 was previously obtained in [13]. Let us note that dealing with these different topologies on $\widetilde{L}^p$ (same for $\widetilde{l}^p$) the natural linear map $\Psi : L^0 \to (L^0)^\infty$ is employed. In the present paper Ψ is considered as a linear isomorphism of L^1 onto $\prod_{l^1} L^1$ with the following key property:

$$\Psi(\widetilde{L}^p) \cap \sum_{l^1} L^1 = \sum_{l^1} \widetilde{L}^p.$$

In the last section we provide some examples of pre-Hilbert spaces with rather mysterious topological structure. They all are of the form

$$Y(A) \times F_\alpha \quad \text{and} \quad Y(A) \times G_\alpha,$$

where $Y(A)$ is the linear span of a linearly independent subset A in l^2. In particular, we show that every projective class $\mathscr{P}_n \setminus \bigcup_{k<n} \mathscr{P}_k$, $n \geq 1$, contains uncountably many nonhomeomorphic pre-Hilbert spaces. The same is true for the class of spaces which are nonprojective. We observe that the argument of Henderson and Pełczyński [2] showing that there are uncountably many σ-compact pre-Hilbert spaces applies (after a minor change) to produce uncountably many nonhomeomorphic pre-Hilbert spaces in each class $\mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$ and $\mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$ for $\alpha \geq 2$.

The results of §3 will be applied to construct absorbing sets for all projective classes in a forthcoming paper by the first named author.

The authors wish to note that J. Dijkstra and J. Mogilski have recently obtained the same results concerning $\widetilde{L}^p$- and $\widetilde{l}^p$-spaces [10].

Convention. All spaces considered are separable and metrizable. Maps are continuous functions.

2. Preliminaries

Let us recall that a closed subset A of a space X is a Z-set (respectively, a strong Z-set) if for every open cover $\mathscr{U}$ of X there exists a $\mathscr{U}$-close to the identity map $f : X \to X$ such that $f(X) \cap A = \varnothing$ (respectively, $\mathrm{cl}(f(X)) \cap A = \varnothing$). A space which is a countable union of Z-sets is called a Z_σ-space. Note that every Z_σ-space is of the first category. In the case where X is an absolute neighborhood retract a closed set A is a Z-set iff given n every map of the n-dimensional cube I^n into X can be approximated by maps into $X \setminus A$. Every not necessarily closed set A satisfying the above condition is called locally homotopy negligible in X (see [17]).

Fix a pair of spaces (K, L), i.e., $L \subseteq K$. We say that a pair of spaces (X, Y) is strongly (K, L)-universal if, for every closed subset D of K, every map $f : K \to X$ whose restriction to D is a Z-embedding (i.e., $f|D$ is an embedding and $f(D)$ is a Z-set in X) and satisfies the condition

$$(f|D)^{-1}(Y) = D \cap L,$$

and every open cover $\mathscr{U}$ of X, there exists a Z-embedding $g : K \to X$ which is $\mathscr{U}$-close to f and satisfies the conditions

$$g|D = f|D \quad \text{and} \quad g^{-1}(Y) = L.$$

We find it convenient to formulate the following technical fact concerning the strong (K, L)-universality (cf. [4, Proposition 2.2]).

Proposition 2.1. *Let an absolute neighborhood retract X, its subsets $Y \subseteq Y'$, and a pair of spaces (K, L) satisfy the following conditions*:

(i) *every Z-set in X is a strong Z-set,*
(ii) *$X \setminus Y$ is locally homotopy negligible in X,*
(iii) *Y' is locally homotopy negligible in X,*
(iv) *given open subsets U of K and V of X, a map $f : K \to X$ with $f(U) \subset V \cap Y$ and $f(K \setminus U) \subset X \setminus V$, and an open cover $\mathscr{V}$ of V, there exists a closed embedding $g : U \to V$ which is $\mathscr{V}$-close to $f|U$ and satisfies $g(U) \subset Y'$ and $g^{-1}(V \cap Y) = L \cap U$.*

Then for every $Z \subseteq X$ with $Z \cap Y' = Y$, the pair (X, Z) is strongly (K, L)-universal.

Before we give a proof of 2.1 we recall that $f : K \to X$ is closed over a set $A \subset X$ if for every $a \in A$ and every neighborhood U of $f^{-1}(\{a\})$ there exists a neighborhood V of a such that $f^{-1}(V) \subset U$ (see [4]).

Proof of 2.1. Let D be a closed subset of K and let $\bar{f} : K \to X$ be a map such that $\bar{f}|D$ is a Z-embedding satisfying $(\bar{f}|D)^{-1}(Z) = D \cap L$. Since $\bar{f}(D)$ is a strong Z-set in X and $X \setminus Y$ is locally homotopy negligible in X, we can apply [4, Lemma 1.1; 17, Theorem 2.4] to approximate $\bar{f}$ by f such that

(1) $\bar{f}|D = f|D$,
(2) f is closed over $\bar{f}(D)$,
(3) $f(K \setminus D) \subset Y \setminus f(D)$.

Set $U = K \setminus D$ and $V = X \setminus f(D)$. Let $\mathscr{U}$ be an open cover of X. Fix a metric d on X and choose an open cover $\mathscr{V}$ of V which is inscribed in $\mathscr{U}$ and such that

(4) for every element W of $\mathscr{V}$, $\mathrm{diam}(W) < \mathrm{dist}(W, X \setminus V)$.

By our assumption, there exists a closed embedding $g : U \to V$ which is $\mathscr{V}$-close to $f|U$ and such that $g^{-1}(V \cap Y) = U \cap L$ and $g(U) \subset Y'$. By (4), g

can be continuously extended by $\overline{f} = f$ over D to a one-to-one map which is $\mathscr{U}$-close to f. Denote this extension also by g. We have $g^{-1}(V \cap Y) = U \cap L$ and consequently $g(L) \subset Z$. Moreover, if $g(x) \in Z$ and $x \notin D$ then $g(x) \in Z \cap Y' = Y$. This, together with $(\overline{f}|D)^{-1}(Z) = D \cap L$, yields $g^{-1}(Z) = L$. To show that $g : K \to X$ is a closed embedding, let $\{g(x_n)\}_{n=1}^{\infty}$ converge to $y \in X$. If $y \in \overline{f}(D)$ then, by (4), $\{f(x_n)\}_{n=1}^{\infty}$ converges to y and consequently, by (2), $\{x_n\}_{n=1}^{\infty}$ converges to $f^{-1}(y)$. Otherwise, $y \in V$ and $\{x_n\}_{n=1}^{\infty}$ converges to $g^{-1}(y)$. Since $g(K) \subset \overline{f}(D) \cup Y'$, the union of a Z-set and a locally homotopy negligible set, $g(K)$, is a Z-set in X.

Let $\mathscr{K}$ and $\mathscr{L}$ be classes of spaces. We write $(K, L) \in (\mathscr{K}, \mathscr{L})$ provided $K \in \mathscr{K}$ and $L \in \mathscr{L}$. A pair of spaces (X, Y) is said to be strongly $(\mathscr{K}, \mathscr{L})$-universal if (X, Y) is strongly (K, L)-universal for every pair $(K, L) \in (\mathscr{K}, \mathscr{L})$. This concept was introduced in [6]. If the pair (Y, Y) is strongly (L, L)-universal for every $L \in \mathscr{L}$ then, according to [4], Y is strongly $\mathscr{L}$-universal.

In what follows, $\mathscr{L}$ will satisfy the following conditions:

(a) if L and L' are homeomorphic and $L \in \mathscr{L}$, then $L' \in \mathscr{L}$,
(b) if a space L is a union of its two closed subspaces which belong to $\mathscr{L}$, then $L \in \mathscr{L}$,
(c) every closed subset of an element of $\mathscr{L}$ belongs to $\mathscr{L}$.

The following fact proved in [7] extends the uniqueness theorem for absorbing sets discovered by Anderson and Bessaga and Pełczyński (see [2]).

Theorem 2.2 [7, Theorem 2.1]. *Let X be a topological copy of l^2 and let Y_1 and Y_2 be two subsets of X. Assume that both $Y = Y_1$ and Y_2 satisfy the following conditions:*

(i) *$X \setminus Y$ is locally homotopy negligible in X,*
(ii) *Y is a Z_σ-space,*
(iii) *Y is a countable union of closed sets that are elements of $\mathscr{L}$,*
(iv) *(X, Y) is strongly $(\mathscr{M}, \mathscr{L})$-universal, where $\mathscr{M}$ is the class of completely metrizable spaces.*

Then, for every open cover $\mathscr{U}$ of X, there exists a $\mathscr{U}$-close to the identity homeomorphism of (X, Y_1) onto (X, Y_2).

Every subset Y of X which is strongly $\mathscr{L}$-universal and fulfils (i)–(iii) is called an $\mathscr{L}$-absorbing set in X. In [4], it was shown that two $\mathscr{L}$-absorbing sets in a copy of l^2 are homeomorphic. Theorem 2.2 may be rephrased in its weaker form as follows: two $\mathscr{L}$-absorbing sets in a copy of l^2 are relatively homeomorphic provided they are strongly $(\mathscr{M}, \mathscr{L})$-universal.

3. Strong universality in products

Let C be a normed countable coordinate space (briefly, a normed coordinate space), i.e., $C = (C, \|\cdot\|_C)$ is a normed linear space of real sequences such that

(c_1) for every bounded sequence $\lambda = (\lambda_n)$ and every $c = (c_n) \in C$, we have $\lambda \cdot c = (\lambda_n c_n) \in C$ and $\|\lambda \cdot c\|_C \leq \|\lambda\|_\infty \|c\|_C$, where $\|\lambda\|_\infty = \sup_{n \geq 1} |\lambda_n|$,
(c_2) for every $\varepsilon > 0$ and every $(c_n) \in C$ there exists k such that

$$\|(0, \dots, 0, c_k, c_{k+1}, \dots)\|_C < \varepsilon,$$

(c_3) each unit vector $u_n = (\delta_n^i)$ belongs to C.

We took the notion of a normed coordinate space from [1] (see also [16]) where the following equivalent condition replaces (c_3): C is contained in no hyperplane $\{(c_n): c_k = 0\}$, $k \geq 1$. (For examples of normed coordinate spaces, see [1].) Note that C contains all eventually zero sequences C_0. Later on we have to assume that $C \setminus C_0 \neq \varnothing$. This, of course, is the case if C is a Banach space.

Let $\{(E_n, \|\cdot\|_n)\}_{n=1}^{\infty}$ be a sequence of normed linear spaces. We consider the linear spaces

$$\prod\nolimits_C E_n = \left\{ (y_n) \in \prod_{n=1}^{\infty} E_n : (\|y_n\|_n) \in C \right\}$$

and

$$\sum\nolimits_C E_n = \left\{ (y_n) \in \prod_{n=1}^{\infty} E_n : \ y_n = 0 \text{ for almost all } n \right\}$$

which are both equipped with the norm $|||(y_n)||| = \left\| (\|y_n\|_n) \right\|_C$. These spaces are called, respectively, the normed coordinate product (of the E_n's in the sense of C) and the weak normed coordinate product (briefly, C-product and weak C-product of the E_n's). For $y = (y_n) \in \prod_C E_n$ and $k \geq 1$, we write $r_k(y) = (0, \dots, y_k, y_{k+1}, \dots)$, $s_k(y) = y - r_k(y)$, and $\pi_k(y) = y_k$. Identifying E_n with the natural subspace of $\prod_C E_n$, we have

(A) $|||s_k(y)||| \leq |||y|||$,
(B) $|||\pi_k(y)||| \leq |||r_k(y)||| \leq |||y|||$,
(C) $\lim |||r_k(y)||| = 0$,

for every $k \geq 1$ and $y \in \prod_C E_n$.

We now give the main result of this section.

Proposition 3.1. *Let $\{(E_n, H_n)\}_{n=1}^{\infty}$ be a sequence of pairs of nontrivial normed linear spaces with each H_n dense in E_n and let C be a normed coordinate space that contains an element with infinitely many nonzero terms. Fix a pair of spaces (K, L) and assume that for every $n \geq 1$ there exists a bounded closed embedding $\psi_n : K \to E_n$ with $\psi^{-1}(H_n) = L$. Then, for every $Z \subseteq E = \prod_C E_n$ with $Z \cap \sum_C E_n = \sum_C H_n$, the pair (E, Z) is strongly (K, L)-universal.*

We shall make use of the next two lemmas.

Lemma 3.2. *There exists a homotopy*

$$\Phi = (\Phi_n) : \left(E \times [0, 1], \sum\nolimits_C H_n \times [0, 1]\right) \to \left(E, \sum\nolimits_C H_n\right)$$

satisfying the following conditions:

(i) $\Phi(\cdot, 0) = \mathrm{id}$,
(ii) *if* $n \geq \frac{1}{t} + 2$, *then* $\Phi_n(y, t) = 0$,
(iii) *if for some sequence* $\{(y(i), t_i)\}_{i=1}^{\infty} \subset E \times [0, 1]$ *with* $\lim t_i = 0$ *there exists* $y \in E$ *such that* $\lim \Phi(y(i), t_i) = y$, *then* $\lim y(i) = y$.

Lemma 3.3. *There exists a one-to-one map*

$$\varphi : K \times (0, 1] \to E$$

satisfying the following conditions:

(iv) $\varphi^{-1}\left(\sum_C H_n\right) = L \times (0, 1]$,

(v) $|||\varphi(x, t)||| \le t$ *for all* $(x, t) \in K \times (0, 1]$,

(vi) *if* $x \in K$ *and* $\frac{1}{n+1} < t \le \frac{1}{n}$, *then* $\pi_{n+2}\varphi(x, t) \neq 0$ *while* $\pi_k \varphi(x, t) = 0$ *for all* $k < n$ *and* $k \ge n+4$,

(vii) *if the sequence* $\{\varphi(x_i, t_i)\}_{i=1}^{\infty}$ *converges in* E, $\{(x_i, t_i)\}_{i=1}^{\infty} \subset K \times (0, 1]$, *and* $\lim t_i = t_0 > 0$, *then* $\{x_i\}_{i=1}^{\infty}$ *converges in* K.

First, we derive Proposition 3.1 from Lemmas 3.2 and 3.3.

Proof of 3.1. We make use of 2.1 with the following data: $X = E$, $Y = \sum_C H_n$, $Y' = \sum_C E_n$, and Z. It is known that Z-sets in E are strong Z-sets (see [5, Lemma 2.6; 13, Lemma 2.1]). It is also clear that $E \setminus \sum_C H_n$ and $\sum_C E_n$ are locally homotopy negligible in E (see, e.g., [17]). Fix a map $\overline{f} : K \to E$ and open sets $U \subset K$ and $V \subset E$ such that $f = \overline{f}|U$ maps U into $V \cap \sum_C H_n$ and $\overline{f}(x) \notin V$ for all $x \notin U$. Let $\mathscr{V}$ be an open cover of V. Pick a map $\omega : V \to (0, 1]$ such that

(1) whenever $y \in V$ and $z \in E$ satisfy $|||y - z||| < 2\omega(y)$ then there exists an element $\mathscr{V}$ containing both y and z.

Let Φ be a homotopy of 3.2. Pick a map $\varepsilon : E \to [0, 1]$ such that

(2) $\varepsilon^{-1}(\{0\}) = E \setminus V$,

(3) $|||\Phi(y, \varepsilon(y)) - y||| < \omega(y)$ for all $y \in V$,

(4) $(\frac{1}{\varepsilon(y)} + 4)^{-1} < \omega(y)$ for all $y \in V$.

Write $\varepsilon(x) = \varepsilon(f(x))$ and $\lambda(x) = (\frac{1}{\varepsilon(x)} + 4)^{-1}$. Pick a homotopy φ from 3.3 and define $g : U \to E$ by the formula

$$g(x) = \Phi(f(x), \varepsilon(x)) + \varphi(x, \lambda(x)).$$

Applying (3)–(4) and (v), we get

$$|||f(x) - g(x)||| < \omega(f(x)) + \lambda(x) < 2\omega(f(x))$$

for every $x \in U$. The property (1) of ω assures that the range of g is V and that g is $\mathscr{V}$-close to f. Clearly, g takes values in $\sum_C E_n$ and, by (iv), $g^{-1}(V \cap \sum_C H_n) = L \cap U$.

To finish the proof, it remains to show that $g : U \to V$ is a closed embedding. First we check that g is one-to-one. If $\frac{1}{n+1} < \varepsilon(x) \le \frac{1}{n}$ then, by (ii), $\Phi_p(f(x), \varepsilon(x)) = 0$ for all $p \ge n+3$. Since $\frac{1}{n+5} < \lambda(x) \le \frac{1}{n+4}$ we have, by (vi), $\varphi_{n+6}(x, \lambda(x)) \neq 0$ and $\varphi_k(x, \lambda(x)) = 0$ for $k \neq n+4, n+5, n+6, n+7$. Assume that $g(x) = g(x')$ and $\varepsilon(x') \le \varepsilon(x)$. Letting $\frac{1}{n'+1} < \varepsilon(x') \le \frac{1}{n'}$, we see that $n' \ge n$ and, by (vi), $\varphi_{n'+6}(x', \lambda(x')) \neq 0$. It follows that

$$\varphi_{n'+6}(x', \lambda(x')) = g_{n'+6}(x') = g_{n'+6}(x) = \varphi_{n'+6}(x, \lambda(x)) \neq 0;$$

hence, $n' = n$ or $n+1$. Then, for every $p \ge n+4$, we have $\Phi_p(f(x'), \varepsilon(x')) = \Phi_p(f(x), \varepsilon(x)) = 0$ and consequently $\varphi_p(x, \lambda(x)) = g_p(x) = g_p(x') = \varphi_p(x', \lambda(x'))$. Since, by (vi), $\varphi_p(x, \lambda(x)) = \varphi_p(x', \lambda(x')) = 0$ for all $p \le n+3$, we conclude that

$$\varphi(x, \lambda(x)) = \varphi(x', \lambda(x')).$$

The latter yields $x = x'$ because φ is one-to-one. Now, suppose $\{g(x_i)\}_{i=1}^{\infty}$ converges to $y = (y_n) \in V$ for some sequence $\{x_i\}_{i=1}^{\infty} \subset U$. Write $\varepsilon_i = \varepsilon(x_i)$ and $\lambda_i = \lambda(x_i)$. We can assume that $\{\varepsilon_i\}_{i=1}^{\infty}$ converges to $\varepsilon_0 \in [0, 1]$. If $\varepsilon_0 = 0$, then $\lim \lambda_i = 0$. Using (v), we get $\lim \Phi(f(x_i), \varepsilon_i) = y$. Then, by (iii), $\{f(x_i)\}_{i=1}^{\infty}$ converges to y. By the continuity of ε, we get $\varepsilon(y) = 0$ which contradicts (2). Therefore, we can assume that $\varepsilon_0 > 0$. Let

$$\varepsilon_0 = s_0 \frac{1}{n} + (1 - s_0)\frac{1}{n+1}$$

for some $0 < s_0 \leq 1$. We can further assume that

$$\varepsilon_i = s_i \frac{1}{n} + (1 - s_i)\frac{1}{n \pm 1}.$$

Then, we have $\Phi_{n+j}(x_i, \varepsilon_i) = 0$ for all i and $j \geq 3$; and consequently the sequence $\{\varphi_{n+j}(x_i, \lambda_i)\}_{i=1}^{\infty} = \{g_{n+j}(x_i)\}_{i=1}^{\infty}$ converges to y_{n+j} for all $j \geq 3$. For $p \neq n + j$, we have, by (vi), $\varphi_p(x_i, \lambda_i) = 0$. It follows that $\{\varphi(x_i, \lambda_i)\}_{i=1}^{\infty}$ converges in E. Since $\lim \lambda_i = (\frac{1}{\varepsilon_0} + 4)^{-1} > 0$, by (vii), the sequence $\{x_i\}_{i=1}^{\infty}$ is convergent in K. If $\lim x_i = x \in K \setminus U$, then $\lim f(x_i) = f(x) \in E \setminus V$ and $\lim \varepsilon(x_i) = \varepsilon(f(x)) = 0$, contradicting (2). We have shown that g is a closed embedding.

Proof of 3.2. Pick a vector $e_n \in H_n$ with $|||e_n||| = 1$. Define $\Phi : E \times [0, 1] \to E$ by $\Phi(y, 0) = y$,

$$\Phi\left(y, \frac{1}{n}\right) = (y_1, \dots, y_{n-1}, 0, |||r_n(y)||| \cdot e_{n+1}, 0, 0, \dots)$$

and

$$\Phi\left(y, s\frac{1}{n} + (1-s)\frac{1}{n+1}\right) = s\Phi\left(y, \frac{1}{n}\right) + (1-s)\Phi\left(y, \frac{1}{n+1}\right)$$

for every $n \geq 1$, $0 \leq s \leq 1$, and $y = (y_n) \in E$. It is clear that Φ transforms $\sum_C H_n \times [0, 1]$ into $\sum_C H_n$, is continuous on $E \times (0, 1]$, and satisfies (i) and (ii). The continuity of Φ at the points $(y, 0)$ will follow from the auxiliary estimations.

Given $y = (y_n) \in E$, we have

$$\left|\left|\left|y - \Phi\left(y, \frac{1}{n}\right)\right|\right|\right| \leq |||y - s_n(y)||| + \left|\left|\left|s_n(y) - \Phi\left(y, \frac{1}{n}\right)\right|\right|\right|$$
$$= |||r_n(y)||| + |||\,|||r_n(y)|||\cdot e_{n+1}||| = 2\,|||r_n(y)|||\,.$$

For $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$, $0 \leq s \leq 1$, and $y \in E$, we have

(1) $|||y - \Phi(y, t)||| = |||s\,(y - \Phi\,(y, \frac{1}{n})) + (1 - s)(y - \Phi(y, \frac{1}{n+1}))||| \leq s|||y - \Phi(y, \frac{1}{n})||| + (1 - s)|||y - \Phi(y, \frac{1}{n+1})||| \leq 2s|||r_n(y)||| + 2(1-s)|||r_{n+1}(y)||| \leq 2|||r_n(y)|||$.

Let $\{(y(i), t_i)\}_{i=1}^{\infty}$ be a sequence of $E \times (0, 1]$ that is convergent to $(y, 0) \in E \times \{0\}$ and let

(2) $t_i = s_i\frac{1}{n_i} + (1-s_i)\frac{1}{n_i+1}$ for some $0 \leq s_i \leq 1$ and $n_i \to \infty$.

Using (1), we get

(3) $|||y - \Phi(y(i), t_i)||| \leq |||y - y(i)||| + |||y(i) - \Phi(y(i), t_i)||| \leq |||y - y(i)||| + 2|||r_{n_i}(y(i))|||$.

On the other hand, applying (B), we obtain

$$(4)\quad |||r_n(y(i))||| \le |||r_n(y)||| + |||r_n(y - y(i))||| \le |||r_n(y)||| + |||y - y(i)|||.$$

Combining (3) and (4), we get

$$|||y - \Phi(y(i), t_i)||| \le 3|||y - y(i)||| + 2|||r_{n_i}(y)|||.$$

The latter inequality together with (C) yields the continuity of Φ at $(y, 0)$.

Now, let $\{(y(i), t_i)\}_{i=1}^{\infty} \subset E \times (0, 1]$ be such that $\lim \Phi(y(i), t_i) = y \in E$ and $\lim t_i = 0$. Express t_i in the form of (2). We have

$$|||y - y(i)||| \le |||s_{n_i}(y) - s_{n_i}(y(i))||| + |||r_{n_i}(y)||| + |||r_{n_i}(y(i))|||.$$

We see that $s_{n_i}(y(i)) = s_{n_i}(\Phi(y(i), t_i))$. Therefore, after using (A), we get

$$|||s_{n_i}(y) - s_{n_i}(y(i))||| = |||s_{n_i}(y - \Phi(y(i), t_i))||| \le |||y - \Phi(y(i), t_i)|||.$$

This implies

$$|||y - y(i)||| \le |||y - \Phi(y(i), t_i)||| + |||r_{n_i}(y)||| + |||r_{n_i}(y(i))|||.$$

Note that the first two terms tend to 0 if $i \to \infty$. To show (iii), it remains to verify that the last term also tends to 0. It is clear that it is enough to consider the case where $s_i \ge \frac{1}{2}$ for all i and the case where $s_i < \frac{1}{2}$ for all i. In the first case, we apply (B) to the $(n_i + 1)$-coordinate and get

$$|||y - \Phi(y(i), t_i)||| \ge |||y_{n_i+1} - s_i|||r_{n_i}(y(i))||| \cdot e_{n_i+1}|||.$$

Then, since $|||e_{n_i+1}||| = 1$, we estimate

$$\begin{aligned} |||r_{n_i}(y(i))||| &\le \frac{1}{s_i}(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+1}|||) \\ &\le 2(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+1}|||). \end{aligned}$$

Finally, according to (B) and (C), $\lim_{i\to\infty} r_{n_i}(y(i)) = 0$. In the case where $s_i < \frac{1}{2}$, we apply (B) to the n_i-coordinate and obtain

$$|||y - \Phi(y(i), t_i)||| \ge |||y_{n_i} - (1 - s_i)y_{n_i}(i)|||$$

and hence

$$\begin{aligned} |||y_{n_i}(i)||| &\le \frac{1}{1 - s_i}(|||y - \Phi(y(i), t_i)||| + |||y_{n_i}|||) \\ &< 2(|||y - \Phi(y(i), t_i)||| + |||y_{n_i}|||). \end{aligned}$$

The same argument applied to the $(n_i + 2)$-coordinate yields

$$|||y - \Phi(y(i), t_i)||| \ge |||y_{n_i+2} - (1 - s_i)|||r_{n_i+1}(y(i))||| \cdot e_{n_i+2}|||.$$

As before, we get

$$\begin{aligned} |||r_{n_i+1}(y(i))||| &\le \frac{1}{1 - s_i}(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+2}|||) \\ &\le 2(|||y - \Phi(y(i), t_i)||| + |||y_{n_i+2}|||). \end{aligned}$$

The latter, in turn, implies

$$\begin{aligned} |||r_{n_i}(y(i))||| &\le |||y_{n_i}(i)||| + |||r_{n_i+1}(y(i))||| \\ &\le 4|||y - \Phi(y(i), t_i)||| + 2(|||y_{n_i}||| + |||y_{n_i+2}|||). \end{aligned}$$

Finally, according to (B) and (C), the last two terms of the above inequality tend to 0 if $i \to \infty$.

Note. Condition (iii) is equivalent to the fact that the map $(y, t) \to (\Phi(y, t), t)$ from $E \times [0, 1]$ into $E \times [0, 1]$ is closed over $E \times \{0\}$.

Proof of 3.3. By our assumption there exists a closed embedding $\psi_n : K \to E_n$ such that

$$\psi^{-1}(H_n) = L \quad \text{and} \quad |||\psi_n(x)||| \leq \frac{1}{2n}.$$

Pick a vector $e_n \in H_n$ with $|||e_n||| = \frac{1}{2n}$. Define $\varphi = (\varphi_p)$ as follows:

$$\varphi_p(x, \frac{1}{n}) = \begin{cases} 0 & \text{if } p \neq n, n+2, \\ \psi_n(x) & \text{if } p = n, \\ e_{n+2} & \text{if } p = n+2. \end{cases}$$

and

$$\varphi\left(x, s\frac{1}{n} + (1-s)\frac{1}{n+1}\right) = s\varphi\left(x, \frac{1}{n}\right) + (1-s)\varphi\left(x, \frac{1}{n+1}\right)$$

for $n \geq 1$ and $0 \leq s \leq 1$. It is clear that φ is continuous and satisfies (vi) and (iv). We have

$$\left|\left|\left|\varphi\left(x, \frac{1}{n}\right)\right|\right|\right| \leq |||\psi_n(x)||| + |||e_{n+2}||| \leq \frac{1}{n}.$$

Consequently, we estimate

$$\begin{aligned} |||\varphi(x, t)||| &\leq s\left|\left|\left|\varphi\left(x, \frac{1}{n}\right)\right|\right|\right| + (1-s)\left|\left|\left|\varphi\left(x, \frac{1}{n+1}\right)\right|\right|\right| \\ &\leq s\frac{1}{n} + (1-s)\frac{1}{n+1} = t. \end{aligned}$$

To show that φ is one-to-one, let $\varphi(x, t) = \varphi(x', t')$ for some $(x, t), (x', t') \in K \times (0, 1]$. If $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$ with $n \geq 1$ and $0 \leq s < 1$ (respectively, $t = 1$), then the last nonvanishing coordinate of $\varphi(x, t)$ is the $(n+3)$-coordinate (respectively, the third coordinate) and it equals $(1-s)e_{n+3}$ (respectively, e_3). This shows that $t = t'$. Clearly, we have $\varphi_{n+1}(x, t) = (1-s)\psi_{n+1}(x)$ (respectively, $\varphi_1(x, t) = \psi_1(x)$). Since ψ_n (respectively, ψ_1) is an embedding, we get $x = x'$.

Let $\{(x_i, t_i)\}_{i=1}^{\infty}$ be a sequence of $K \times (0, 1]$ such that $\{\varphi(x_i, t_i)\}_{i=1}^{\infty}$ converges in E and $\lim t_i = t_0 > 0$. Assume that $t_0 = s_0\frac{1}{n} + (1-s_0)\frac{1}{n+1}$ for some $n \geq 1$ and $0 < s_0 < 1$ (the case where $t_0 = \frac{1}{n}$, $n \geq 1$, can be treated similarly). We may suppose that $t_i = s_i\frac{1}{n} + (1-s_i)\frac{1}{n+1}$ for all i, where $0 < s_i < 1$ and $\lim s_i = s_0$. Since $\varphi_n(x_i, t_i) = s_i\psi_n(x_i)$, $\{\psi_n(x_i)\}_{i=1}^{\infty}$ converges in E_n. Finally, $\{x_i\}_{i=1}^{\infty}$ converges in K because ψ_n is a closed embedding.

In §§4 and 5, we will employ the following variation of Proposition 3.1.

Proposition 3.4. *Let $\{(E_n, H_n)\}_{n=1}^{\infty}$ be a sequence of pairs of nontrivial normed linear spaces with each H_n dense in E_n and let C be a normed coordinate space. Fix a pair of spaces (K, L). Assume there are pairwise disjoint infinite subsets $N_1, N_2, \ldots$ of the set of integers N such that $N_k \cap \{1, 2, \ldots, k-1\} = \varnothing$ and, writing*

$$C_k = \{(c_p)_{p \in N_k} : \exists_{c \in C} \forall_{p \in N_k}\ \pi_p(c) = c_p\}$$

and identifying C_k with the natural subspace of C, each C_k contains an element with infinitely many nonzero terms and there exists a bounded closed embedding $\psi_k : K \to \prod_{C_k} E_p$ with $\psi_k^{-1}(\prod_{C_k} H_p) = L$ for $k \geq 1$. Then, the pair $(\prod_C E_n, \prod_C H_n)$ is strongly (K, L)-universal.

A proof of 3.4 will be omitted. Let us indicate that to get it one has to follow the proof of 3.1 and replace 3.3 by the lemma below.

Lemma 3.5. *There exists a one-to-one map*

$$\varphi : K \times (0, 1] \to \prod\nolimits_C E_n$$

satisfying conditions (v) *and* (vii) *of* 3.3 *together with*

(iv′) $\varphi^{-1}(\prod_C H_n) = L \times (0, 1]$,

(vi′) *given $n \geq 1$ there exists an integer $k_n > k_{n-1}$ $(k_1 \geq 1)$ such that, if $x \in K$ and $\frac{1}{n+1} < t \leq \frac{1}{n}$ then $\pi_{k_n}\varphi(x, t) \neq 0$ while $\pi_k\varphi(x, t) = 0$ for all $k \in N \setminus (N_{n+1} \cup N_{n+2} \cup \{k_n, k_{n+1}\})$.*

Proof. Pick $k_n \in N_1$ with $k_n > k_{n-1}$ ($k_1 \geq 1$). By our assumption, there exists a closed embedding $\psi_n : K \to \prod_{C_{n+1}} E_p$, $n \geq 1$, such that

$$\psi_n^{-1}\left(\prod\nolimits_{C_{n+1}} H_p\right) = L \quad \text{and} \quad |||\psi_n(x)||| \leq \frac{1}{n}$$

for all $x \in K$. Pick a vector $e_{k_n} \in H_{k_n}$ with $|||e_{k_n}||| = \frac{1}{2n}$. Define $\varphi = (\varphi_k)$ as

$$\varphi_k\left(x, \frac{1}{n}\right) = \begin{cases} 0 & \text{if } k \in N \setminus (N_{n+1} \cup \{k_n\}), \\ e_{k_n} & \text{if } k = k_n, \\ \pi_k \psi_n(x) & \text{if } k \neq k_n, \end{cases}$$

and

$$\varphi\left(x, s\frac{1}{n} + (1-s)\frac{1}{n+1}\right) = s\varphi\left(x, \frac{1}{n}\right) + (1-s)\varphi\left(x, \frac{1}{n+1}\right)$$

for $n \geq 1$ and $0 \leq s \leq 1$. Conditions (iv′) and (vi′) follow easily. To verify (v) and (vii), repeat a reasoning of the proof of 3.3.

The next result is a counterpart of Proposition 3.1 for cartesian products and can be viewed as a relative version of [4, Proposition 2.5]. We need to recall that by the weak product of X_i's with the basepoints $*_i \in X_i$ we mean

$$W(X_i, *_i) = \left\{(x_i) \in \prod_{i=1}^{\infty} X_i : \; x_i = *_i \text{ for almost all } i\right\}$$

(endowed with the subspace topology).

Proposition 3.6. *Let X_i be a noncompact absolute retract and let Y_i be a subset of X_i such that $X_i \setminus Y_i$ is locally homotopy negligible in X_i for $i = 1, 2, \ldots$. Fix a pair of spaces (K, L) and assume that for every $i \geq 1$ there exists a closed embedding*

$$h_i : K \to X_i \quad \text{with} \quad h_i^{-1}(Y_i) = L.$$

Then, for every choice of basepoints $*_i \in Y_i$ and every set $Z \subseteq X = \prod_{i=1}^{\infty} X_i$ with $Z \cap W(X_i, *_i) = W(Y_i, *_i)$, the pair (X, Z) is strongly (K, L)-universal.

Proof. We apply Proposition 2.1 with the following data: X, $Y = W(Y_i, *_i)$, $Y' = W(X_i, *_i)$, and Z. It is clear that X is an absolute retract and both $X \setminus Y$

and Y' are locally homotopy negligible in X. By [13, Lemma 2.2] Z-sets are strong Z-sets in X. Let $\overline{f}: K \to X$ be such that $f = (f_i) = \overline{f}|U$ maps U into $V \cap Y$ and $\overline{f}(x) \notin V$ for all $x \notin U$, where $U \subset K$ and $V \subset X$ are open sets. Let $\mathscr{V}$ be an open cover of V. We pick a map $\mu : X_i \times X_i \times [0, 1] \to X_i$ such that for all $i \geq 1$

(1) $\mu_i(x, y, 0) = x$ and $\mu_i(x, y, 1) = y$ for every $x, y \in X_i$,
(2) $\mu_i(Y_i \times Y_i \times [0, 1]) \subset Y_i$.

To construct μ_i, choose any map $\lambda_i : X_i \times X_i \times [0, 1] \to X_i$ satisfying (1) and a homotopy $(\varphi_t^i) : X_i \times [0, 1] \to X_i$ with $\varphi_0^i = \mathrm{id}_{X_i}$ and $\varphi_t^i(X_i) \subseteq Y_i$ for all $t > 0$ and define

$$\mu_i(x, y, t) = \varphi_{t(1-t)}^i\big(\lambda_i(x, y, t)\big).$$

To produce φ_t^i, use the fact that $X_i \setminus Y_i$ is locally homotopy negligible in X_i and apply [17, Theorem 2.4]. The same property implies that Y_i is an absolute retract [17, Theorem 3.1]; moreover, since X_i is noncompact, Y_i is nontrivial. As a consequence, there exists an embedding $\alpha_i : [0, 1] \to Y_i$ with

(3) $\alpha_i(0) = *_i$ for $i \geq 1$.

Fix $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$, $n \geq 1$ and $0 \leq s \leq 1$, and let $\Phi(x, t) = (y_i)$, where

(4) $y_i = f_i(x)$ for $i \leq n$,
(5) $y_i = *_i$ for $i = n+6$ and $i \geq n+9$,
(6) $y_{n+1} = \mu_{n+1}\big(f_{n+1}(x), *_{n+1}, s\big)$,
(7) $y_{n+2} = \mu_{n+2}\big(*_{n+2}, h_{n+2}(x), s\big)$,
(8) $y_{n+i} = h_{n+i}(x)$ for $i = 3$ and 4,
(9) $y_{n+5} = \mu_{n+5}\big(h_{n+5}(x), *_{n+5}, s\big)$,
(10) $y_{n+7} = \alpha_{n+7}(s)$ and $y_{n+8} = \alpha_{n+8}(1-s)$.

Letting $\Phi(x, 0) = f(x)$, we easily check that $\Phi : K \times [0, 1] \to X$ is well defined and continuous. Notice that, by (2),

(11) $\Phi(K \times (0, 1]) \subset W(X_i, *_i)$,
(12) $\Phi^{-1}\big(W(Y_i, *_i)\big) \cap \big(K \times (0, 1]\big) = L \times (0, 1]$.

We claim that $\Phi|K \times (0, 1]$ is one-to-one. In fact, let (x, t) and (x', t') be such that $\Phi(x, t) = \Phi(x', t')$. If $t = s\frac{1}{n} + (1-s)\frac{1}{n+1}$, $n \geq 1$ and $0 \leq s < 1$ (respectively, $t = 1$), the last pth coordinate of $\Phi(x, t)$, different from $*_p$, occurs when $p = n+8$ (respectively, $p = 8$) and it is equal to $\alpha_{n+8}(1-s)$ (respectively, $\alpha_8(1)$). Since α is an embedding, $\Phi(x, t)$ determines t and hence $t = t'$. According to (8), $\Phi_{n+3}(x, t) = h_{n+3}(x)$ (respectively, $\Phi_4(x, t) = h_4(x)$). Therefore, $h_{n+3}(x) = h_{n+3}(x')$ (respectively, $h_4(x) = h_4(x')$) and consequently we get $x = x'$.

Choose a map $\varepsilon : X \to [0, 1]$ such that

(13) $\varepsilon^{-1}(\{0\}) = X \setminus V$,
(14) whenever $y \in V$ and $y' \in X$ satisfy $d(y, y') < \varepsilon(y)$ then there is an element of $\mathscr{V}$ containing both y and y',

where d is a metric on $X = \prod_{i=1}^{\infty} X_i$ chosen so that $d(y, y') < \frac{1}{n+1}$ if y and y' agree on the first n coordinates. By the choice of d and (4), we get

(15) $d\big(\Phi(x, \varepsilon(f(x))), f(x)\big) < \varepsilon(f(x))$.

To see (15), observe that if $\frac{1}{n+1} < \varepsilon(f(x)) \leq \frac{1}{n}$, then $d\big(\Phi(x, \varepsilon(f(x))), f(x)\big) \leq$

$\frac{1}{n+1} < \varepsilon(f(x))$. Define $g : U \to X$ by

$$g(x) = \Phi\bigl(x, \varepsilon(f(x))\bigr).$$

By (14) and (15), g is $\mathscr{V}$-close to f and takes values in V. In turn, (5) and (12) imply that g takes values in $W(X_i, *_i)$ and satisfies $g^{-1}\bigl(V \cap W(Y_i, *_i)\bigr) = L \cap U$.

It remains to verify that $g: U \to V$ is a closed embedding. For some sequence $\{x_k\}_{k=1}^{\infty} \subset U$ let $\lim g(x_k) = y = (y_i) \in V$. We may assume that $\{\varepsilon(f(x_k))\}_{k=1}^{\infty}$ converges to some $\varepsilon_0 \in [0, 1]$. If $\varepsilon_0 = 0$ then, by (15), $\lim f(x_k) = y$, contradicting the fact that $\varepsilon(y) > 0$. If $\varepsilon_0 > 0$, then we may assume that that $\varepsilon(f(x_k)) \in \bigl(\frac{1}{n+1}, \frac{1}{n-1}\bigr)$ for some n and all k. According to (8), $g_{n+3}(x_k) = h_{n+3}(x_k)$ and consequently $\lim h_{n+3}(x_k) = y_{n+3}$. Since h_{n+3} is a closed embedding, $\{x_k\}_{k=1}^{\infty}$ converges in K. If $\lim x_k = x \in K \setminus U$, then $\lim f(x_k) = f(x) \notin V$ and, by (13), $\lim \varepsilon(f(x_k)) = \varepsilon(f(x)) = 0$, a contradiction.

Note 3.7. Our proof of Proposition 3.6 requires that at least infinitely many of the X_n's are noncompact. Otherwise, it may happen that not all Z-sets are strong Z-sets in X (see [13]). However, the proof (after minor modifications) still works if one assumes that all the X_n's are nontrivial local compacta.

4. Borelian absorbing sets can be linearly represented in l^2

For every countable ordinal $\alpha \geq 0$, by $\mathscr{A}_\alpha$ and $\mathscr{M}_\alpha$ we denote the additive and multiplicative classes of all absolute Borelian sets of order α, respectively. To be more specific, $\mathscr{M}_0$ consists of all compacta, $\mathscr{A}_1$ consists of all σ-compact spaces, $\mathscr{M}_1 = \mathscr{M}$ consists of all completely metrizable spaces, and $\mathscr{M}_2$ consists of all absolute $F_{\sigma\delta}$-sets. By $\mathscr{P}_n$, $n \geq 1$, we denote the class of all projective sets of order n; $\mathscr{P}_0 = \bigcup_\alpha \mathscr{A}_\alpha$. A set that does not belong to $\bigcup_{n=1}^{\infty} \mathscr{P}_n$ is called nonprojective.

The aim of this section is to find a linear representation of an $\mathscr{A}_\alpha$-absorbing set F_α (respectively, $\mathscr{M}_\alpha$-absorbing set G_α) in l^2. To perform this, we will make use of $\mathscr{A}_\alpha$-absorbing sets Λ_α and $\mathscr{M}_\alpha$-absorbing sets Ω_α constructed in copies s of l^2 in [4]. By the uniqueness theorem for absorbing sets [4], F_α is homeomorphic to Λ_α and G_α is homeomorphic to Ω_α. Actually, we show that the pairs (l^2, F_α) and (s, Λ_α), $\alpha \geq 1$ (respectively, (l^2, G_α) and (s, Ω_α), $\alpha \geq 2$), are homeomorphic. The last is achieved by proving the strong $(\mathscr{M}, \mathscr{A}_\alpha)$- and $(\mathscr{M}, \mathscr{M}_\alpha)$-universality of suitable pairs. The multiplicative case of order $\alpha = 1$ differs from the others and is treated separately in [8]; we include a description of G_1 in our text in order to formulate the result in full generality.

We briefly recall the definition of Λ_α and Ω_α. Set $\Lambda_1 = \Sigma \subset R^\infty = s$ and $\Omega_1 = W(R^\infty, 0) \subset (R^\infty)^\infty = s$. Inductively, if $\alpha = \beta + 1$ let $\Omega_\alpha = \Lambda_\beta^\infty \subset s_\beta^\infty = s$, where Λ_β is represented in s_β. If α is a limit ordinal let $\Omega_\alpha = \prod_{\xi<\alpha} \Lambda_\xi^\infty \subset \prod_{\xi<\alpha} s_\xi^\infty = s$, where Λ_ξ is represented in s_ξ. Finally, let $\Lambda_\alpha = W(s_\alpha \setminus \Omega_\alpha, *) \subset s_\alpha^\infty = s$, where Ω_α is represented in s_α and $*$ is an arbitrary basepoint of $s_\alpha \setminus \Omega_\alpha$. By the Kadec-Anderson theorem [2], the spaces s (in which Λ_α and Ω_α are represented) are copies of l^2.

Proposition 4.1 (cf. [4]). *For every $\alpha \geq 2$, the pairs (s, Λ_α) and (s, Ω_α) are strongly $(\mathscr{M}, \mathscr{A}_\alpha)$- and $(\mathscr{M}, \mathscr{M}_\alpha)$-universal, respectively. The pair $(s, \Lambda_1) = (R^\infty, \Sigma)$ is strongly $(\mathscr{M}, \mathscr{A}_1)$-universal.*

Proof. We only present a proof of the multiplicative case. We repeat the argument of [4, Lemma 6.3] and use Proposition 3.6.

To show that (R^∞, Σ) is strongly $(\mathscr{M}, \mathscr{A}_1)$-universal and fix a pair $(K, L) \in (\mathscr{M}, \mathscr{A}_1)$. There exists a closed embedding $h : K \to R^\infty$ with $h^{-1}(\Sigma) = L$. (Take any closed embedding of K into R^∞ and compose it with a homeomorphism of R^∞ sending $h(L) \cup \Sigma$ onto Σ; see [2].) Now, by 3.6, $((R^\infty)^\infty, W(\Sigma, 0))$ is strongly $(\mathscr{M}, \mathscr{A}_1)$-universal. Since the latter pair is homeomorphic to (R^∞, Σ) [2, p. 275], the strong $(\mathscr{M}, \mathscr{A}_1)$-universality of (s, Λ_1) follows.

We assume that $\alpha \geq 2$ and $\alpha = \beta + 1$ (the case of a limit ordinal is analogous). Given $(K, L) \in (\mathscr{M}, \mathscr{M}_\alpha)$ there exists $L_i \subset K$, $L_i \in \mathscr{A}_\beta$ $(i \geq 1)$, such that $L = \bigcap_{i=1}^\infty L_i$. By the inductive assumption, we find a closed embedding $h_i : K \to s_\beta$ with $h^{-1}(\Lambda_\beta) = L_i$. Writing $h = (h_i)$, we see that h is a closed embedding of K into $s_\beta^\infty = s$ with $h^{-1}(\Omega_\alpha) = \bigcap_{i=1}^\infty L_i = L$. Finally, Proposition 3.6 yields the strong $(\mathscr{M}, \mathscr{M}_\alpha)$-universality of $(s_\beta^\infty, \Lambda_\beta^\infty) = (s, \Omega_\alpha)$.

Theorem 4.2. *For every $\alpha \geq 1$, there exists a linear subspace F_α of l^2 which is an $\mathscr{A}_\alpha$-absorbing set and such that the pair (l^2, F_α) is strongly $(\mathscr{M}, \mathscr{A}_\alpha)$-universal. In particular, (l^2, F_α) and (s, Λ_α) are homeomorphic.*

Proof. *Construction of F_α*. Let (A, B) be a copy of (s, Λ_α). Consider a closed embedding h of A onto a linearly independent subset of the unit sphere in l^2 satisfying the following condition:

(∗) *for every $a \in A$ and every closed subset $F \subset A$ with $a \notin F$ there exists a continuous linear functional $x^* : l^2 \to R$ such that $x^*(h(F)) \subseteq \{0\}$ while $x^*(h(a)) \neq 0$.*

Condition (∗) is taken from [3] where it was checked that the embedding described by Bessaga and Pełczyński [2, p. 193] fulfils (∗). Denote by H the linear span of $h(B)$ in l^2 and by $\overline{H}$ the closure of H. Since B is dense in A, $\overline{H}$ contains $h(A)$ as a (closed) subset. Write $(E_n, H_n) = (\overline{H}, H)$ and set

$$E = \prod\nolimits_{l^2} E_n \quad \text{and} \quad F_\alpha = \sum\nolimits_{l^2} H_n.$$

Clearly, E is isomorphic to l^2 and F_α is dense in E.

According to 4.1 and the fact that Λ_α is an $\mathscr{A}_\alpha$-absorbing set [4], it suffices to prove that the pair (E, F_α) fulfils the requirements (i)–(iv) of 2.2 for the class $\mathscr{L} = \mathscr{A}_\alpha$. Condition (i) is a consequence of the fact that F_α is linear and dense in E (see, e.g., [17, Remark 2.9]). Since each set $A_k = \{(y_n) \in F_\alpha : y_i = 0 \text{ for } i \geq k+1\}$ is a Z-set in F_α and $F_\alpha = \bigcup_{k=1}^\infty A_k$, F_α is a Z_σ-space. Condition (iv) follows directly from 3.1 and 4.1 because A is closed in E_n. The remaining condition (iii) can be concluded from (ii) and the lemma below.

Lemma 4.3. *The space H is in $\mathscr{A}_\alpha$.*

Proof. We shall make use of the cross-section argument due to Klee [2, p. 271]. The n-fold product B^n admits a σ-closed cross-section, i.e., there exists a subset F of B^n that is a countable union of closed sets F_k such that

(1) if $(b_1, b_2, \dots, b_n) \in F$ then $b_i \neq b_j$ for $i \neq j$,
(2) whenever $\{y_i\}_{i=1}^n$ are n distinct points of B then there exists exactly one permutation of $y_1, y_2, \dots, y_n$ that belongs to F.

By (1) and (2) the linear combination map χ given by

$$((b_1, b_2, \dots, b_n), (\lambda_1, \lambda_2, \dots, \lambda_n)) \to \lambda_1 h(b_1) + \lambda_2 h(b_2) + \cdots + \lambda_n h(b_n)$$

transforms in a one-to-one way the product $F_k \times D_k^p$ onto $N_k^p \subset H$, where

$$D_k^p = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) : \frac{1}{p} \le |\lambda_i| \le p \text{ for all } i \right\}$$

and $p = 1, 2, \dots$. Employing, as in [3, Lemma 3.3], the condition $(*)$ one shows that $\chi | F_k \times D_k^p$ is a homeomorphism. It shows that each N_k^p, and hence $H^n = \bigcup_{k,p=1}^{\infty} N_k^p$, belongs to $\mathscr{A}_\alpha$. Since $H = \bigcup_{n=1}^{\infty} H^n$, we get $H \in \mathscr{A}_\alpha$.

Theorem 4.4. *For every $\alpha \ge 1$, there exists a linear subspace G_α of l^2 which is an $\mathscr{M}_\alpha$-absorbing set and such that the pair (l^2, G_α) is strongly $(\mathscr{M}, \mathscr{M}_\alpha)$-universal. In particular, (l^2, G_α) and (s, Ω_α) are homeomorphic for $\alpha \ge 2$.*

Proof. The case for $\alpha = 1$ differs from the others. In [8] it was shown that the space

$$G_1 = \left\{ (x_n) \in l^2 ; \sum_{n=1}^{\infty} |x_n| < \infty \text{ and } \sum_{n=1}^{\infty} x_n = 0 \right\}$$

is an $\mathscr{M}$-absorbing set and, moreover, the pair (l^2, G_1) is strongly $(\mathscr{M}, \mathscr{M})$-universal.

Construction of G_α. Let $\alpha \ge 2$. If α is a limit ordinal, then choose an increasing sequence of ordinals $\{\beta_n\}_{n=1}^{\infty}$ convergent to α; otherwise, $\alpha = \beta + 1$ and let $\beta_n = \beta$. Pick, by 4.2, a pair $(E_n, F_n) = (l^2, F_{\beta_n})$ which is $(\mathscr{M}, \mathscr{A}_{\beta_n})$-universal. Set

$$E = \prod_{l^2} E_n \quad \text{and} \quad G_\alpha = \prod_{l^2} F_n.$$

The space E is isomorphic to l^2 and G_α is its linear dense subspace.

Using 4.1 and the fact that Ω_α is an $\mathscr{M}_\alpha$-absorbing set [4], it is enough to verify conditions (i)–(iv) of 2.2 for the pair (E, G_α) and $\mathscr{L} = \mathscr{M}_\alpha$. Condition (i) follows as in the proof of 4.2. A simple argument shows that $G_\alpha \in \mathscr{M}_\alpha$. Since F_1 is a Z_σ-space, G_α is also a Z_σ-space. It remains to verify that (E, G_α) is strongly $(\mathscr{M}, \mathscr{M}_\alpha)$-universal. We will apply 3.4. Let $N_1, N_2, \dots$ be any decomposition of the set of integers N into pairwise disjoint infinite sets. Then, the space C_k defined in 3.4 is $l^2(N_k)$, the space of all square summable sequences indexed by the integers of N_k. As a result $(\prod_{C_k} E_p, \prod_{C_k} H_p) = (\prod_{l^2(N_k)} E_p, \prod_{l^2(N_k)} F_p)$. By the choice of β_n, it is clear that the following lemma will finish the proof of 4.4.

Lemma 4.5. *For every $(K, L) \in (\mathscr{M}, \mathscr{M}_\alpha)$ there exists a bounded closed embedding $\psi : K \to E$ with $\psi^{-1}(G_\alpha) = L$.*

Proof. Let $L = \bigcap_{k=1}^{\infty} L_k$, where $L_k \subset K$, $L_{k+1} \subset L_k$, and $L_k \in \mathscr{A}_{\beta_{n_k}}$ for some n_k with $n_{k+1} > n_k$. Write $\beta(k) = \beta_{n_k}$. Since (E_{n_k}, F_{n_k}) is strongly $(\mathscr{M}, \mathscr{A}_{\beta(k)})$-universal, there exists a closed embedding $\psi_{n_k} : K \to E_{n_k}$ such that

(1) $\|\psi_{n_k}(x)\| \le (\frac{1}{2})^{n_k}$ for all $x \in K$,
(2) $\psi_{n_k}^{-1}(F_{n_k}) = L_k$.

Write $\psi_k \equiv 0$ for all $n \neq n_k$ ($k \geq 1$) and set $\psi = (\psi_n)$. By (1), ψ is continuous and bounded. It is easy to see that $\psi : K \to E$ is a closed embedding with $\psi^{-1}(G_\alpha) = \bigcap_{k=1}^{\infty} L_k = L$.

Remark 4.6. As pointed out in [8], the pair $(s, \Omega_1) = ((R^\infty)^\infty, W(R^\infty, 0))$ is not strongly $(\mathscr{M}, \mathscr{M})$-universal. (If it were, then by 2.2, (s, Ω_1) and (l^2, G_1) would be homeomorphic; consequently, G_1 would be σ-closed in l^2, which contradicts a result of [15].)

Remark 4.7. The spaces Λ_α and Ω_α can be realized as linear subspaces in other normed coordinate products $\prod_C E_n$. The only restriction is the condition $(*)$.

Remark 4.8. The result of 4.1 can be readily generalized to the triple case. Representing Λ_α ($\alpha \geq 1$) and Ω_α ($\alpha \geq 2$) in R^∞, we could consider the triples

$$(\overline{R}^\infty, R^\infty, \Lambda_\alpha) \quad \text{and} \quad (\overline{R}^\infty, R^\infty, \Omega_\alpha),$$

where $\overline{R} = [-\infty, +\infty]$. These triples are strongly $(\mathscr{M}_0, \mathscr{M}, \mathscr{A}_\alpha)$- and $(\mathscr{M}_0, \mathscr{M}, \mathscr{M}_\alpha)$-universal, respectively (with an obvious meaning of the triple strong universality).

In §6 we shall need the following fact concerning the complements of F_α and G_α.

Corollary 4.9. *The space $l^2 \setminus F_\alpha$ (respectively, $l^2 \setminus G_\alpha$) has the following properties*:

(i) $l^2 \setminus F_\alpha$ *(respectively, $l^2 \setminus G_\alpha$) is a Baire space,*
(ii) $l^2 \setminus F_\alpha \in \mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$ *(respectively, $l^2 \setminus G_\alpha \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$),*
(iii) $l^2 \setminus F_\alpha$ *(respectively, $l^2 \setminus G_\alpha$) is homogeneous,*
(iv) *for every (closed) ball $B \subset l^2$, $B \setminus F_\alpha$ (respectively, $B \setminus G_\alpha$) is an absolute retract.*

Proof. We will only deal with the F_α-case, the G_α-case is analogous. Conditions (i) and (ii) follow from the fact that F_α is of the first category and that $F_\alpha \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$. To show (iii), we produce a homeomorphism of l^2 that preserves F_α and carries $y \in l^2 \setminus F_\alpha$ onto $y' \in l^2 \setminus F_\alpha$. Let h be any homeomorphism of l^2 with $h(y) = y'$, e.g., h is the translation. Then, the pairs $(l^2, h(F_\alpha))$ and (l^2, F_α) are strongly $(\mathscr{M}, \mathscr{A}_\alpha)$-universal. The proof of [7, Theorem 2.1] can be easily modified to achieve a homeomorphism k of l^2 that carries $h(F_\alpha)$ onto F_α and preserves $\{y'\}$ (set $X_0 = Y_0 = \{y'\} \subset X_1 \cap Y_1$). We see that $k \circ h$ preserves F_α and sends y onto y'. Condition (iv) is a consequence of the fact that $B \cap F_\alpha$ is locally homotopy negligible in B and [17, Theorem 3.1]. Assume $0 \in \operatorname{int} B$ and pick $y_0 \in B \setminus F_\alpha$. Since the homotopy $f_t(y) = (1-t)y + ty_0$ ($0 \leq t \leq 1$) takes its values in $B \setminus F_\alpha$ for $t > 0$ and $f_0 = \mathrm{id}$, the local homotopy negligibility of $B \cap F_\alpha$ in B follows.

5. Application to $F_{\sigma\delta}$-spaces

In this section we identify various absolute $F_{\sigma\delta}$-sets carrying product structures to be homeomorphic to $\Omega_2 = \Sigma^\infty$. The spaces we deal with will be considered with both normed and cartesian product topologies. We start with a direct application of Proposition 3.4 to coordinate products of normed $F_{\sigma\delta}$-spaces. A counterpart of 5.1 for cartesian products was previously obtained in [13].

Theorem 5.1. *Let $\prod_C H_n$ be a normed coordinate product of absolute $F_{\sigma\delta}$-spaces H_n in the sense of a Banach space C. Assume that infinitely many of the H_n's are Z_σ-spaces. Then $\prod_C H_n$ is homeomorphic to Ω_2. Moreover, writing E_n for the linear completion of H_n, the pairs $\left(\prod_C E_n, \prod_C H_n\right)$ and (s, Ω_2) are homeomorphic.*

We shall make use of the two lemmas below. A proof of the first one is implicitly contained in [12, Lemma 5.4] and therefore it will be omitted.

Lemma 5.2. *Let $X \in \mathscr{M}$ be an absolute retract and $Y \subset X$ be a Z_σ-space such that $X \setminus Y$ is locally homotopy negligible in X. Then, for every $L \in \mathscr{A}_1$ of the Hilbert cube I^∞, there exists a map $\varphi : I^\infty \to X$ with $\varphi^{-1}(Y) = L$.*

Lemma 5.3. *Let $(H_n, \|\cdot\|_{H_n})$ be a normed linear space that is noncompactly embedded into a Banach space $(E_n, \|\cdot\|_{E_n})$, i.e., $H_n \subseteq E_n$, $\|\cdot\|_{E_n} \le \|\cdot\|_{H_n}$, and the E_n-closures of H_n-balls are noncompact. Then, for every coordinate Banach space C and every $K \in \mathscr{M}$, there exists a bounded closed embedding $\psi : K \to \prod_C E_n$ such that $\psi(K) \subset \prod_C H_n$.*

Proof. Since C is complete, there is $c = (c_n) \in C$ with all the c_n's strictly positive. It is enough to construct a closed embedding $\psi = (\psi_n) : K \to \prod_C E_n$ with $\psi_n(x) \in H_n$ and $\|\psi_n(x)\|_{H_n} \le c_n$ for all $x \in K$ and $n \ge 1$. Fix a metric d on the Hilbert cube $I^\infty = [0, 1]^\infty$. Embed K into I^∞ and write $I^\infty \setminus K = \bigcup_{n=1}^\infty F_n$, where each F_n is a closed subset of I^∞. Define $d_n(q) = \mathrm{dist}_d(q, F_n)$, $q \in I^\infty$. We claim that there exists a closed embedding $\alpha_n : [0, \infty) \to E_n$ such that $\alpha_n(t) \in H_n$ and $\|\alpha_n(t)\|_{H_n} \le 1$ for all $t \ge 0$ and $n \ge 1$. This follows from the noncompactness of the E_n-closure of the H_n-unit ball and Klee's result [14] that every noncompact closed convex subset F of E_n contains a copy of $[0, \infty)$. (The piecewise linear embedding α constructed by Klee can be improved to get the nodes of α contained in any dense linear subset of F.) Pick a vector $e_{2n} \in H_{2n}$ with $\|e_{2n}\|_{H_{2n}} = 1$. Define $\psi = (\psi_n)$ by

$$\psi_{2n}(q) = c_{2n} q_n e_{2n} \quad \text{and} \quad \psi_{2n-1}(q) = c_{2n-1}\alpha_{2n-1}\left((d_n(q))^{-1}\right)$$

for $q=(q_n)\in K$. It is clear that $\psi : K \to \prod_C E_n$ is one-to-one and $\|\psi_n(q)\|_{H_n} \le c_n$, $n \ge 1$, $q \in K$. By [16, Lemma 1.4], ψ is continuous. If $\{\psi(q(i))\}_{i=1}^\infty$ is convergent in $\prod_C E_n$, then there exists $q_0 \in I^\infty$ with $\lim q(i) = q_0$. Assume that $q_0 \in F_k$ for some k. Then $\lim_{i\to\infty} d_k(q(i)) = 0$, contradicting the fact that $\{(d_k(q(i)))^{-1}\}_{i=1}^\infty$ converges. The latter is a consequence of the facts that the sequence $\{\alpha_{2k-1}((d_k(q(i)))^{-1})\}_{i=1}^\infty$ is convergent in E_k and that α is a closed embedding.

Proof of 5.1. We show that the pair $(E, H) = \left(\prod_C E_n, \prod_C H_n\right)$ satisfies (i)–(iv) of 2.2 with $\mathscr{L} = \mathscr{M}_2$. By the Kadec-Anderson theorem [2], E is a copy of l^2. A standard argument yields (i)–(iii). The strong $(\mathscr{M}, \mathscr{M}_2)$-universality of (E, H) will be derived from Proposition 3.4. Fix a pair $(K, L) \in (\mathscr{M}, \mathscr{M}_2)$. Find pairwise disjoint infinite subsets $N_1, N_2, \ldots$ of N so that H_p is a Z_σ-space for every $p \in \bigcup_{k=1}^\infty N_k$. Let C_k be the subspace of C that corresponds to N_k (see 3.4). Write $E^k = \prod_{C_k} E_p$ and $H^k = \prod_{C_k} H_p$. To fulfil the hypothesis of 3.4 we have to produce a bounded closed embedding $\psi_k : K \to E^k$ with $\psi_k^{-1}(H^k) = L$. To this end we split $C_k = C_k^1 \oplus C_k^2$ into two coordinate spaces and find a bounded closed embedding $\psi_k^1 : K \to \prod_{C_k^1} E_p$ with $\psi_k^1(K) \subset \prod_{C_k^1} H_p$

and a bounded map $\psi_k^2 : K \to \prod_{C_k^2} E_p$ with $(\psi_k^2)^{-1}(\prod_{C_k^2} H_p) = L$. Finally, letting $\psi_k = (\psi_k^1, \psi_k^2)$ we get a required embedding (we identify $\prod_{C_k} E_p$ with $\prod_{C_k^1} E_p \oplus \prod_{C_k^1} E_p)$.

To find ψ_k^1 we apply Lemma 5.3. In this case H_p is a genuine subspace of an infinite-dimensional Banach space E_p; hence the balls in E_p are noncompact. Let $c = (c_p) \in C_k^2$ be such that all c_p are strictly positive. Embed K into I^∞ and represent $L = \bigcap_p L_p$ so that each L_p is σ-compact and $\{L_p\}$ is descending. Use Lemma 5.2 with $X = B(c_p)$, the closed ball in E_p centered at 0 with radius c_p, $Y = B(c_p) \cap H_p$, and $L = L_p$. We get maps $\varphi_p : I^\infty \to B(c_p)$ with $\varphi_p^{-1}(Y) = L_p$. Finally, we set $\psi_k^2(x) = (\varphi_p(x))$, $x \in K$. The continuity of ψ_k^2 follows from [16, Lemma 4.1]. To verify the hypothesis of 5.2, notice that Y is convex and dense in X. Let $H_p = \bigcup_{m=1}^\infty A_m$, where each A_m is a Z-set in H_p. Then $\text{int}(B(c_p)) \cap A_m$ is a Z-set in $\text{int}(B(c_p)) \cap H_p$. Since $\text{int}(B(c_p)) \cap H_p$ is convex and dense in $B(c_p) \cap H_p$, $B(c_p) \cap A_m$ is a Z-set in $B(c_p) \cap H_p$. It shows that Y is a Z_σ-space.

A direct consequence of 5.1 is

Note 5.4. The simplest pre-Hilbert space representation of Ω_2 is the space $\prod_{l^2} l_f^2$, where $l_f^2 = \{(x_i) \in l^2 : \ x_i = 0 \text{ for almost all } i\}$. Moreover, the pairs $(\prod_{l^2} l^2, \prod_{l^2} l_f^2)$ and (s, Ω_2) are homeomorphic.

Consider the set $\prod_C H_n = H$ as a subspace of the cartesian product $\prod_{n=1}^\infty E_n = E$. By the Kadec-Anderson theorem [2], E is a copy of l^2. Easily, $E \setminus H$ is locally homotopy negligible in E. We claim that C is an $F_{\sigma\delta}$-subset of R^∞. This is a consequence of the equality

$$C = \left\{ (x_n) \in R^\infty : \ \forall_{\varepsilon>0} \exists_k \forall_{m>k} \left\| \sum_{i=k}^m x_i u_i \right\|_C \le \varepsilon \right\}$$

(u_i is the ith unit vector). Consider the map $f(x) = (\|x_n\|)$, $x = (x_n) \in E$, and notice that $f^{-1}(C) = \prod_C E_n$. This shows that $\prod_C E_n \in \mathscr{M}_2$. Since $\prod_C H_n = \prod_C E_n \cap \prod_{n=1}^\infty H_n$, H is an absolute $F_{\sigma\delta}$-set. Repeating (with obvious changes) the remaining part of the proof of 5.1, we get the following generalization of a result [13].

Theorem 5.5. *Let $\{H_n\}_{n=1}^\infty$ be a sequence of normed linear spaces such that each H_n is an absolute $F_{\sigma\delta}$-set and infinitely many of the H_n's are Z_σ-spaces. Then, for every coordinate Banach space C, the space $\prod_C H_n$ considered in the product topology, is homeomorphic to Ω_2. Moreover, if E_n is the linear completion of H_n then the pairs $(\prod_{n=1}^\infty E_n, \prod_C H_n)$ and (s, Ω_2) are homeomorphic.*

Remark 5.6. The hypothesis that infinitely many of the H_n's are Z_σ-spaces is essential. Consider the coordinate space $c_0 = \{(x_i) \in R^\infty : \ \lim x_i = 0\}$ with the $\|\cdot\|_\infty$-norm. Note that $c_0 \subset \bigcup_{k=1}^\infty B_\infty(k)$, where $B_\infty(k) = \{x \in R^\infty : \|x\|_\infty \le k\}$. This shows that c_0 is contained in a σ-compact subset of R^∞. On the other hand Ω_2 contains a copy of R^∞ closed in s. This shows that $(R^\infty, \prod_{c_0} R)$ and (s, Ω_2) are not homeomorphic, contrary to the expectation expressed in [13]. Proposition 3.6 and Lemma 5.2 yield the strong $(\mathscr{M}_0, \mathscr{M}_2)$-universality of the pair (R^∞, c_0).

For the c_0-products we have the following generalization of [13, Theorem 4.2].

Theorem 5.7. *Let, for* $n \geq 1$, X_n *be a subset of a Banach space* $(E_n, \|\cdot\|_n)$ *with* $0 \in X_n$ *so that* $\inf_{n\geq 1} \operatorname{diam}(X_n) = \alpha > 0$. *Assume that each* X_n *is an absolute retract that is an absolute* $F_{\sigma\delta}$*-set. Then the space*

$$X = \prod_{c_0} X_n = \left\{ (x_n) \in \prod_{n=1}^{\infty} X_n ; \ \|x_n\|_n \to 0 \right\}$$

(*endowed with the product topology*) *is homeomorphic to* Ω_2.

Proof. We will show that X is an $\mathscr{M}_2$-absorbing set in some copy of l^2, i.e., we will verify conditions (i)–(iii) of 2.2 and the strong $\mathscr{M}_2$-universality of X. Then, the uniqueness theorem for absorbing sets [4] yields our assertion. Since $\prod_{n=1}^{\infty} X_n$ is an absolute retract and $\prod_{n=1}^{\infty} X_n \setminus X$ is locally homotopy negligible, X is also an absolute retract [17, Theorem 3.1]. Decompose the set of integers N into pairwise disjoint infinite sets $N_1, N_2, \ldots$. Write $X^i = \prod_{p\in N_i} X_p$ and $Y^i = \{(x_p) \in X^i : \|x_p\|_p \to 0 \}$ and let $\Psi : \prod_{n=1}^{\infty} X_n \to \prod_{i=1}^{\infty} X^i$ be the natural isomorphism. We have

(1) $\Psi(X) \supset W(Y^i, 0)$.

Note that each Y^i is noncompact. (If it were compact then, because Y^i is dense in X^i, we would get $X^i = Y^i$, contradicting the fact that $\alpha > 0$.) Now, each Y^i has a completion Z^i with $Z^i \setminus Y^i$ locally homotopy negligible in Z^i [17, Proposition 4.1]. Since Y^i is noncompact, we can assume that Z^i is also noncompact (if it were compact take $Z^i \setminus \{*\}$, where $* \in Z^i \setminus Y^i$). Then the product $s = \prod_{i=1}^{\infty} Z^i$ is a copy of l^2 [18, Theorem 5.1] with $s \setminus X$ locally homotopy negligible in s; this shows (i). An argument preceding Theorem 5.5 applies to show that X is an absolute $F_{\sigma\delta}$-set. Writing $A_m = \{(x_n) \in X : \|x_j\|_j \leq \frac{\alpha}{2}$ for all $j \geq m+1\}$, we see that $\bigcup_{m=1}^{\infty} A_m = X$ and that each A_m is a Z-set in X. In proving the strong $\mathscr{M}_2$-universality of X we employ 3.6 with $Y_i = Y^i$, $K = L \in \mathscr{M}_2$, and $Z = \Psi(X)$. To produce a closed embedding of L into Y^i, we may assume that $Y^i = X$.

Write $B^i(\varepsilon) = \{(x_p) \in X^i : \|x_p\|_p \leq \varepsilon$ for all $p \in N_i \}$ and notice that

(2) $\prod_{i=1}^{\infty} Y^i \cap B^i(2^{-i})$ is a closed subset of $\Psi(X)$.

By (2) and the fact that a countable product of Z_σ-spaces that are absolute retracts contains a closed copy of Ω_2 [13, Corollary 2.5], it suffices to check that each $Y^i \cap B^i(2^{-i})$ contains a closed Z_σ-space that is an absolute retract. Assuming $\alpha \geq 2^{-i}$, we choose in each X_p an arc T_p joining 0 with some x_p with $\|x_p\|_p = 2^{-i}$. Write $T^i = Y^i \cap \prod_{p\in N_i} T_p$. Then T^i is a closed subset of Y^i. The argument showing that X is a Z_σ-space applies also to verify that T^i is a Z_σ-space. The proof is completed.

Let us note a relative version of [13, Corollary 2.7].

Remark 5.8. Let $X_n \in \mathscr{M}$ be a noncompact absolute retract and let Y_n be a subset of X_n such that $X_n \setminus Y_n$ is locally homotopy negligible in X_n, $n = 1, 2, \ldots$. If each Y_n is an absolute $F_{\sigma\delta}$-set and infinitely many of the Y_n's are Z_σ-spaces, then the pairs $\left(\prod_{n=1}^{\infty} X_n, \prod_{n=1}^{\infty} Y_n\right)$ and (s, Ω_2) are homeomorphic. Apply 2.2 together with 3.6. To produce a closed embedding $h : K \to \prod_{n=1}^{\infty} X_n$

with $h^{-1}(\prod_{n=1}^{\infty} Y_n) = L$ employ Lemma 5.2 and the fact that $\prod_{n=1}^{\infty} X_n$ contains a closed copy of $[0, \infty)$ that lives in $\prod_{n=1}^{\infty} Y_n$. Moreover, adopting Theorem 2.2 and Lemma 5.2 to the triple case one can get a homeomorphism of suitable triples (see [9]); 4.8.

Let us recall that by $L^p[a, b]$ we denote the space of equivalence classes of Lebesgue measurable functions $x : [a, b] \to R$ with

$$\|x\|_p = \left(\int_a^b |x(t)|^p \, dt\right)^{\min(1, \frac{1}{p})} < \infty$$

with the topology induced by the F-norm $\|\cdot\|_p$, $0 < p < \infty$. Write $\widetilde{L}^p[a, b] = \bigcap_{p'<p} L_{p'}[a, b]$, $0 < p \leq \infty$, and by $\widetilde{L}^p_q[a, b]$ denote the set $\widetilde{L}^p[a, b]$ with the $\|\cdot\|_q$-topology, $q < p$. Note that $\widetilde{L}^p_q[a, b]$ is dense in $L^q[a, b]$. We skip the symbol $[a, b]$ if $[a, b] = [0, 1]$.

Theorem 5.9. *The pairs* $(L^q, \widetilde{L}^p_q)$ *and* (s, Ω_2) *are homeomorphic for* $0 < q < p \leq \infty$.

Proof. Mazur's homeomorphism [2, p. 207] of L^1 onto L^q transforms $\widetilde{L}^p$ onto $\widetilde{L}^{pq}$. Therefore, it sufficies to consider the case of $q = 1$ (and arbitrary $p > 1$).

We write $\widetilde{L}^p = \widetilde{L}^p_1$. Since L^1 is a copy of l^2 [2], it is enough to verify conditions (i)–(iv) of 2.2. The local homotopy negligibility of $L^1 \setminus \widetilde{L}^p$ follows in a standard way. Note that each L^p is an F_σ-subspace of L^q for $p > q$. (This is a consequence of the facts that L^p is an F_σ-subspace of L^0, the space of measurable functions with the convergence in measure topology (see [13]), and that the L^0-topology is weaker than the $\|\cdot\|_q$-topology.) Select an increasing sequence $\{p_n\}_{n=1}^{\infty} \subset (1, p)$ that converges to p. Since $\widetilde{L}^p = \bigcap_{n=1}^{\infty} L^{p_n}$, we get $\widetilde{L}^p \in \mathscr{M}_2$.

To prove that $\widetilde{L}^p$ is a Z_σ-space, we choose $1 < p' < p$ and write

$$B_{p'}(\varepsilon) = \{x \in L^{p'} : \|x\|_{p'} \leq \varepsilon\}.$$

Since $\widetilde{L}^p = \bigcup_{k=1}^{\infty} B_{p'}(k) \cap \widetilde{L}^p$ it suffices to check that each $A = B_{p'}(k) \cap \widetilde{L}^p$ is a Z-set in $\widetilde{L}^p$. First of all, note that $B_{p'}(k)$ is a Z-set in L^1 because it is a closed subset of a locally homotopy negligible set $L^{p'}$ in L^1. Then, using the fact that $L^1 \setminus \widetilde{L}^p$ is locally homotopy negligible in L^1, we infer that A is a Z-set in $\widetilde{L}^p$ (see [5, Lemma 2.6]).

We make use of 3.1 to verify the strong $(\mathscr{M}, \mathscr{M}_2)$-universality of $(L^1, \widetilde{L}^p)$. The map Ψ given by

$$\Psi(x) = (x|[2^{-n}, 2^{-n+1}])_{n=1}^{\infty},$$

$x \in L^1$, is a linear isomorphism of L^1 onto $\prod_{l^1} L^1[2^{-n}, 2^{-n+1}]$. Writing $Z = \Psi(\widetilde{L}^p)$, we have

$$Z \cap \sum\nolimits_{l^1} L^1[2^{-n}, 2^{-n+1}] = \sum\nolimits_{l^1} \widetilde{L}^p[2^{-n}, 2^{-n+1}].$$

Since the pair $(E_n, H_n) = (L^1[2^{-n}, 2^{-n+1}], \widetilde{L}^p[2^{-n}, 2^{-n+1}])$ is (naturally) isomorphic to $(L^1, \widetilde{L}^p)$, the lemma below verifies the hypothesis of 3.1 and thus finishes the proof of 5.9.

Lemma 5.10. *Let* $(K, L) \in (\mathscr{M}, \mathscr{M}_2)$. *There exists a bounded closed embedding* $\psi : K \to L^1$ *with* $\psi^{-1}(\widetilde{L}^p) = L$.

Proof. We repeat a reasoning from the proof of 5.1. First, we find a bounded closed embedding $\psi^1 : K \to L^1[\frac{1}{2}, 1]$ with $\psi^1(K) \subset \widetilde{L}^p$. Then, we produce a bounded map $\psi^2 : K \to L^1[0, \frac{1}{2}]$ with $(\psi^2)^{-1}(\widetilde{L}^p[0, \frac{1}{2}]) = L$. Finally, we let $\psi = (\psi^1, \psi^2)$. To get ψ^1, we apply 5.3 with $H_n = L^p[\frac{1}{2}, 1]$, $E_n = L^1[\frac{1}{2}, 1]$, and $C = l^1$. It is clear that H_n is noncompactly embedded in E_n. Consequently there exists a bounded closed embedding $\psi^1 : K \to \prod_{l^1} L^1[\frac{1}{2}, 1] = L^1[\frac{1}{2}, 1]$ such that $\psi^1(K) \subset \prod_{l^1} L^p[\frac{1}{2}, 1] \subset \prod_{l^p} L^p[\frac{1}{2}, 1] = L^p[\frac{1}{2}, 1] \subset \widetilde{L}^p[\frac{1}{2}, 1]$. To obtain ψ^2, embed K into I^∞ and represent $L = \bigcap_{n=2}^\infty L_n$ with each L_n σ-compact and $L_{n+1} \subset L_n$ for $n \geq 2$. Recall that $\{p_n\} \subset (1, p)$ converges to p. If we find maps $\varphi_n : K \to L^{p_n}[2^{-n}, 2^{-n+1}]$ such that $\varphi_n^{-1}(\widetilde{L}^p[2^{-n}, 2^{-n+1}]) = L_n$ and $\|\varphi_n(x)\|_{p_n} \leq 2^{-n}$ for all $x \in K$ and $n \geq 2$, then ψ^2 defined by

$$\psi^2(x)|[2^{-n}, 2^{-n+1}] = \varphi_n(x),$$

$x \in K$, $n \geq 2$, is as required.

To produce φ_n, we apply 5.2 for $(X, Y) = (B_{p_n}(2^{-n}), B_{p_n}(2^{-n}) \cap \widetilde{L}^p)$. Since Y is convex and dense in X, $X \setminus Y$ is locally homotopy negligible in X. Pick $p_n < p' < p$. We have

$$Y = B_{p_n}(2^{-n}) \cap \widetilde{L}^p = \bigcup_{k=1}^\infty B_{p'}(k) \cap B_{p_n}(2^{-n}) \cap \widetilde{L}^p.$$

We claim that each $A = B_{p'}(k) \cap B_{p_n}(2^{-n}) \cap \widetilde{L}^p$ is a Z-set in Y. Since $B_{p'}(k)$ is a Z-set in L^{p_n}, it easily follows that $B_{p'}(k) \cap B_{p_n}(2^{-n})$ is a Z-set in $B_{p_n}(2^{-n})$. Now the local homotopy negligibility of $X \setminus Y$ in X implies, via [5, Lemma 2.6], that A is a Z-set in Y. This finishes the proof.

Remark 5.11. One could likely elaborate an abstract scheme of identifying some normed coordinate products that are homeomorphic to Ω_2, as done for cartesian products in [13]. Due to replacing the convex structure by a suitable equiconnected structure on $L^0([0, 1], G)$, the space of measurable G-valued functions on $[0, 1]$, it was proved in [13] that $\widetilde{L}^p([0, 1], G)$ (with the L^0-topology) is homeomorphic to Ω_2, provided G is a closed unbounded subset of a Banach space. Using 3.6 and 5.2, one can show that the pair $(L^0([0, 1], G), \widetilde{L}^p([0, 1], G))$ is homeomorphic to (s, Ω_2) for $0 < p \leq \infty$. To produce a closed embedding of R^∞ in $L^0([0, 1], G)$ with values in $B = \{x \in L^1 : |x(t)| \leq \varepsilon$ almost everywhere $\}$, we use the argument of 5.3 and the fact that $B \cap L^0([0, 1], G)$ is a copy of l^2 [2]. It is likely that the pairs $(L^q([0, 1], G), \widetilde{L}^p_q([0, 1], G))$ and (s, Ω_2) are also homeomorphic.

By l^p we denote the space of real-valued sequences $x = (x_n)$ such that

$$\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{\min(1, \frac{1}{p})} < \infty$$

with the topology induced by the F-norm $\|\cdot\|_p$, $0 < p < \infty$. Write $\widetilde{l}^p =$

$\bigcap_{p'>p} l^{p'}$, $0 \le p < \infty$, and denote by $\widetilde{l^p_q}$ the space $\widetilde{l^p}$ with the $\|\cdot\|_q$-topology, $q > p$. Note that $\widetilde{l^p_q}$ is a dense linear subspace of l^q.

Theorem 5.12. *The pairs* $(l^q, \widetilde{l^p_q})$ *and* (s, Ω_2) *are homeomorphic for* $0 \le p < q < \infty$.

Proof. As in the proof of 5.9, we only need to check that $(l^1, \widetilde{l^p})$, $0 < p < 1$, fulfils conditions (i)–(iv) of 2.2; we write $\widetilde{l^p} = \widetilde{l^p_1}$. A verification of (i) and (iii) is almost the same as in 5.9 and uses the observation that

$$B_{p'}(\varepsilon) = \{x \in l^{p'} : \|x\| \le \varepsilon\}$$

is closed in the $\|\cdot\|_p$-topology ($p > p'$). Also, every set $B_{p'}(k) \cap \widetilde{l^p}$ is a Z-set in $\widetilde{l^p}$, yielding (ii). To verify (iv) we make use of 3.1. Decompose N into pairwise disjoint infinite sets $N_1, N_2, \ldots$. Consider the linear isomorphism $\Psi : l^1 \to \prod_{l^1} l^1(N_n)$, where $l^1(N_n)$ is an isomorphic copy of l^1 of sequences indexed by integers of N_n, given by

$$\Psi(x) = ((x_k)_{k\in N_1}, (x_k)_{k\in N_2}, \ldots),$$

$x \in l^1$. Writing $Z = \Psi(\widetilde{l^p})$, we see that

$$Z \cap \sum\nolimits_{l^1} l^1(N_n) = \sum\nolimits_{l^1} \widetilde{l^p}(N_n).$$

The following lemma enables us to apply 3.1 and hence to finish the proof of 5.12.

Lemma 5.13. *Let* $(K, L) \in (\mathscr{M}, \mathscr{M}_2)$. *There exists a bounded closed embedding* $\psi : K \to l^1$ *with* $\psi^{-1}(\widetilde{l^p}) = L$.

Proof. We follow the proof of 5.10. As in 5.10 we embed K into I^∞, represent $L = \bigcap_{n=1}^\infty L_n$, and pick a sequence $(p_n) \subset (p, 1)$ convergent to p. A bounded closed embedding $\psi^1 : K \to l^1(N_1)$ with $\psi^1(K) \subset \widetilde{l^p}$ is obtained via Lemma 5.3. We take $H_n = (l^p(N_1), \|\cdot\|_p)$, $E_n = l^1(N_1)$, and $C = l^1$. (Formally, we are not eligible to apply 5.3 because H_n is not a normed space. This assumption was only used to construct the closed embedding of $[0, \infty)$. In our case the unit closed ball B in $l^p(N_1)$ is homeomorphic, via Mazur's homeomorphism [2, p. 207], to the closed unit ball in the Hilbert space which, in turn, is homeomorphic to R^∞. Therefore B being closed in $l^1(N_1)$ admits a required embedding.) Hence, we get a bounded closed embedding $\psi^1 : K \to \prod_{l^1} l^1(N_1) = l^1(N_1)$ with $\psi^1(K) \subset \prod_{l^1} l^p = l^p \subset \widetilde{l^p}$. To produce ψ^2, we apply 5.2 to the pair $(X, Y) = (B_{p_n}(2^{-n}), B_{p_n}(2^{-n}) \cap \widetilde{l^p})$ and find maps $\varphi_n : K \to l^{p_n}(N_n)$, $n \ge 2$, with $\varphi_n^{-1}(\widetilde{l^p}(N_n)) = L_n$ and $\|\varphi_n(x)\|_{p_n} \le 2^{-n}$ for all $x \in K$. It is easy to see that the map $\psi^2(x) = (\varphi_n(x))_{n=2}^\infty$ satisfies $(\psi^2)^{-1}(\widetilde{l^p}(N \setminus N_1)) = L$. We let $\psi = (\psi^1, \psi^2)$.

Let us formulate a more specific result concerning $\widetilde{l^p}$-products whose proof is a modification of the proof of 5.7 (and therefore will be omitted).

Theorem 5.14. *Let, for* $n \ge 1$, X_n *be a subset of a Banach space* $(E_n, \|\cdot\|_n)$ *with* $0 \in X_n$ *so that* $\inf_{n\ge1} \operatorname{diam}(X_n) = \alpha > 0$. *Assume that each* X_n *is an* *absolute retract that is an absolute* $F_{\sigma\delta}$*-set. Then the space*

$$\widetilde{l}^p(X_n) = \left\{ (x_n) \in \prod_{n=1}^{\infty} X_n : \forall_{p'>p} \sum_{n=1}^{\infty} \|x_n\|_n^{p'} < \infty \right\}$$

(as a subspace of $\prod_{n=1}^{\infty} X_n$*) is homeomorphic to* Ω_2 *for every* $0 \le p < \infty$.

Remark 5.15. Assume that each $X_n \in \mathscr{M}$. We may ask whether $(\prod_{n=1}^{\infty} X_n, \widetilde{l}^p(X_n))$ is homeomorphic to (s, Ω_2). This, in general, is not necessarily the case. The space $\widetilde{l}^p(R) = \widetilde{l}^p$ is contained in a σ-compact subset of R^∞ (cf. Remark 5.6). Let us notice that the pair $(R^\infty, \widetilde{l}^p)$ is strongly $(\mathscr{M}_0, \mathscr{M}_2)$-universal. The assertion of Theorem 2.2 holds if one replaces $\mathscr{M}$ by $\mathscr{M}_0$ and add in the hypothesis that both Y_1 and Y_2 are contained in σ-compact subsets of X. As a consequence, the pairs (R^∞, c_0) and $(R^\infty, \widetilde{l}^p)$ are homeomorphic for $0 \le p < \infty$. This shows that two $\mathscr{L}$-absorbing sets Y_1 and Y_2 can be relatively homeomorphic in a copy X of l^2 while none of the pairs (X, Y_1) and (X, Y_2) are strongly $(\mathscr{M}, \mathscr{L})$-universal.

6. The spaces F_α and G_α as factors of exotic pre-Hilbert spaces

In this section we present some examples concerning the topological classification of pre-Hilbert spaces. Examples we deal with are of the form $Y(A) \times F_\alpha$ and $Y(A) \times G_\alpha$, where $Y(A) =$ span(A) and A is a linearly independent subset of l^2.

Fix a linearly independent arc $T = [0, 1]$ in l^2 such that $Y(A)$ is dense in l^2 for every infinite set $A \subseteq T$ (see [2, p. 267]). Since $Y(A)$ is contained in a σ-compact subspace of l^2, $Y(A)$ is a Z_σ-space provided it is infinite-dimensional (i.e., A is infinite).

Proposition 6.1. *Let* A *be any subset of* T *and* $\alpha \ge 1$. *Then:*

(a) $Y(A) \times F_\alpha$ *(respectively,* $Y(A) \times G_\alpha$*) contains no closed copy of* $l^2 \setminus G_{\alpha+1}$ *(respectively,* $l^2 \setminus F_{\alpha+1}$*),*

(b) $Y(A) \times F_\alpha$ *(respectively,* $Y(A) \times G_\alpha$*) contains no closed copy of* $l^2 \setminus F_\alpha$ *(respectively,* $l^2 \setminus G_\alpha$*).*

First, in full detail, we consider the following particular case of part (b) with $\alpha = 1$ (as $l^2 \setminus F_1$ is a copy of l^2, see [2]).

Lemma 6.2. *For every subset* A *of* T, *the space* $Y(A) \times \Sigma$ *contains no closed copy of* l^2. *In particular,* $Y(A) \times \Sigma$ *is homeomorphic neither to* Λ_α, $\alpha \ge 2$, *nor to* Ω_α, $\alpha \ge 1$.

Proof. We apply the cross-section argument described in 4.3. For k and $p \ge 1$, we write

$$C_k^p = \left\{ (t_1, t_2, \dots, t_k) \in T^k : t_1 \le t_2 \le \dots \le t_k, \ \|t_i - t_j\| \ge \frac{1}{p} \right\}$$

and

$$D_k^p = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_k) \in R^k : \frac{1}{p} \le |\lambda_i| \le p \ \text{ for all } i \right\}.$$

(The union $\bigcup_{k,p=1}^{\infty} C_k^p$ is a particular σ-compact cross-section for T^k.) The map χ_k given by $\chi_k((t_1, t_2, \dots, t_k), (\lambda_1, \lambda_2, \dots, \lambda_k)) = \lambda_1 t_1 + \lambda_2 t_2 + \dots + \lambda_k t_k$

is a homeomorphism of $C_k^p \times D_k^p$ onto $M_k^p \subset Y(T)$. Clearly,

$$M_k^p \cap Y(A) = \chi_k((C_k^p \cap A^k) \times D_k^p) = N_k^p$$

is a closed subset in $Y(A)$. Since $Y(A) = \{0\} \cup \bigcup_{k,p=1}^{\infty} N_k^p$, we get

$$Y(A) \times \Sigma = (\{0\} \times \Sigma) \cup \bigcup_{k,p=1}^{\infty} (N_k^p \times \Sigma).$$

Assume that X being a copy of l^2 is contained as a closed subset of $Y(A) \times \Sigma$. Using a Baire category argument and the fact that no open set in l^2 is σ-compact we find indices k and p such that $N_k^p \times \Sigma$ contains an open subset U of X. It follows that a copy B of a closed ball in l^2 inscribed in U is closed in $N_k^p \times \Sigma$. It is easy to see that there exists a connected set $K \subset A^k$ such that

$$B \subset \chi_k((K \cap C_k^p) \times D_k^p) \times \Sigma.$$

Each connected subset K of A^k is of the form $I_1 \times I_2 \times \cdots \times I_k$, where every I_j is a connected component of A (i.e., I_j is an interval). Hence, $K \cap C_k^p$ is locally compact and so is $\chi_k\left((K \cap C_k^p) \times D_k^p\right)$. Finally, B, being a closed subset of a σ-compact space $\chi_k\left((K \cap C_k^p) \times D_k^p\right) \times \Sigma$, is itself σ-compact, a contradiction.

Proof of 6.1. Assume $Y(A) \times F_\alpha$ contains a closed copy X of $l^2 \setminus G_{\alpha+1}$. Using the notation of the proof of 6.2, we have

$$Y(A) \times F_\alpha = (\{0\} \times F_\alpha) \cup \bigcup_{k,p=1}^{\infty} (N_k^p \times F_\alpha).$$

Since X is a Baire space (see 4.9), there exist k and p and a closed set $P \subset N_k^p \times F_\alpha$ such that P has nonempty interior in X and P is a copy of $B \setminus G_{\alpha+1}$ for some closed ball in l^2. According to 4.9, P is connected. As in the proof of 6.2, we get

$$P \subset \chi_k((K \cap C_k^p) \times D_k^p) \times F_\alpha,$$

where $K \cap C_k^p$ is locally compact. Now, it follows that $\chi_k\left((K \cap C_k^p) \times D_k^p\right) \times F_\alpha \in \mathscr{A}_\alpha$; consequently $P \in \mathscr{A}_\alpha$. Since X is homogeneous (see 4.9) and the interior of P in X is nonempty, X is locally in the class $\mathscr{A}_\alpha$. The latter yields $X \in \mathscr{A}_\alpha$, contradicting $G_{\alpha+1} \in \mathscr{M}_{\alpha+1} \setminus \mathscr{A}_{\alpha+1}$.

All the remaining cases can be proved in the same way. (A minor change is needed for G_1; namely, G_1 must be represented as a countable union of complete metrizable spaces.)

Corollary 6.3. *For every subset A of T the spaces $Y(A) \times F_\alpha$ and $Y(A) \times G_\alpha$, $\alpha \geq 1$, are topologically distinct.*

Proof. By 6.1, $Y(A) \times F_\alpha$ contains no closed copy of $l^2 \setminus F_\alpha \in \mathscr{M}_\alpha$. Since $\mathscr{M}_\alpha \subset \mathscr{A}_\beta \cap \mathscr{M}_\beta$ for $\beta > \alpha$, the spaces $Y(A) \times F_\beta$ and $Y(A) \times G_\beta$ do contain such a copy. Also $Y(A) \times G_\alpha$ contains a closed copy of $l^2 \setminus F_\alpha$. As a consequence, we conclude that $Y(A) \times F_\alpha$ is not homeomorphic to $Y(A) \times F_\beta$ for $\alpha \neq \beta$ and that $Y(A) \times F_\alpha$ is not homeomorphic to $Y(A) \times G_\beta$ for $\beta \geq \alpha$. Analogously, we prove that $Y(A) \times G_\alpha$ is not homeomorphic to $Y(A) \times G_\beta$ for $\beta \neq \alpha$ and that $Y(A) \times G_\beta$ is not homeomorphic to $Y(A) \times F_\alpha$ for $\beta \leq \alpha$.

The same argument applies in the following

Corollary 6.4. *For every subset A of T, the spaces $Y(A)\times F_\alpha$ and $Y(A)\times G_\alpha$ are homeomorphic neither to F_β nor to G_β for $\beta \neq \alpha$.*

Corollary 6.5. *We have:*

(a) *if $A \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$, then the spaces F_α, $Y(A)\times F_\beta$, and $Y(A)\times G_\beta$ belong to $\mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$ and are topologically distinct for $\beta < \alpha$,*
(b) *if $A \in \mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$, then the spaces G_α, $Y(A)\times G_\beta$, and $Y(A)\times F_\beta$ belong to $\mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$ and are topologically distinct for $\beta < \alpha$,*
(c) *if $A \in \mathscr{P}_n \setminus \bigcup_{k<n} \mathscr{P}_k$, then the spaces $Y(A)\times F_\alpha$ and $Y(A)\times G_\alpha$ belong to $\mathscr{P}_n \setminus \bigcup_{k<n} \mathscr{P}_k$ and are topologically distinct,*
(d) *if $A \notin \bigcup_{n=1}^\infty \mathscr{P}_n$, then the spaces $Y(A)\times F_\alpha$ and $Y(A)\times G_\alpha$ do not belong to $\bigcup_{n=1}^\infty \mathscr{P}_n$ and are topologically distinct.*

Corollary 6.5 is a direct consequence of 6.3 and 6.4 and the following fact, which seems to be well known; however we could not find it formulated in such a generality in literature.

Lemma 6.6. *We have:*

(a) *if $A \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$, then $Y(A) \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$, $\alpha \geq 1$,*
(b) *if $A \in \mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$, then $Y(A) \in \mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$, $\alpha \geq 2$,*
(c) *if $A \in \mathscr{P}_n \setminus \bigcup_{k<n} \mathscr{P}_k$, then $Y(A) \in \mathscr{P}_n \setminus \bigcup_{k<n} \mathscr{P}_k$, $n \geq 1$,*
(d) *if $A \notin \bigcup_{n=1}^\infty \mathscr{P}_n$, then $Y(A) \notin \bigcup_{n=1}^\infty \mathscr{P}_n$.*

Proof. Since A is closed in $Y(A)$, $A \notin \mathscr{L}$ implies $Y(A) \notin \mathscr{L}$ provided $\mathscr{L}$ is closed with respect to closed subsets. Therefore, it suffices to show that $A \in \mathscr{L}$ implies $Y(A) \in \mathscr{L}$, where $\mathscr{L} = \mathscr{A}_\alpha, \mathscr{M}_\alpha$ and $\bigcup_{k<n} \mathscr{P}_k$. The case $\mathscr{A}_\alpha$ and $\bigcup_{k<n} \mathscr{P}_n$ is a result of Klee [2, p. 272]. Let $A \in \mathscr{M}_\alpha$ and $\alpha \geq 2$. Represent $A = \bigcap_{n=1}^\infty A_n$, $A_n \in \mathscr{A}_{\beta_n}$ for $\beta_n < \alpha$, and employ $Y(A_n) \in \mathscr{A}_{\beta_n}$ to conclude that $Y(A) = \bigcap_{n=1}^\infty Y(A_n) \in \mathscr{M}_\alpha$.

Remark 6.7. Corollary 6.5(a) and (b) (see also 6.2) provide a negative answer to the question of whether a pre-Hilbert space that contains a Hilbert cube and is of the exact Borelian class of order α must be homeomorphic to either F_α or G_α, $\alpha \geq 2$. The answer to this question is "yes" for $\mathscr{A}_1$.

Remark 6.8. From 6.5(c) it follows that each class $\mathscr{P}_n \setminus \bigcup_{k<n} \mathscr{P}_k$ contains uncountably many topologically distinct pre-Hilbert spaces that are Z_σ-spaces.

Remark 6.9. Each class $\mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$, $\alpha \geq 1$, and $\mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$, $\alpha \geq 2$, contains uncountably many topologically distinct pre-Hilbert spaces that are Z_σ-spaces. To show this, take $A \in \mathscr{A}_\alpha \setminus \mathscr{M}_\alpha$ (respectively, $A \in \mathscr{M}_\alpha \setminus \mathscr{A}_\alpha$) and repeat Henderson and Pełczyński's argument to the spaces $Y(A) \times X$, $X \in \mathscr{X}$, where $\mathscr{X}$ is that of [2, p. 282].

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