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THE NOETHERIAN PROPERTY IN RINGS OF INTEGER-VALUED POLYNOMIALS

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ABSTRACT. Let D be a Noetherian domain, D' its integral closure, and $\operatorname{Int}(D)$ its ring of integer-valued polynomials in a single variable. It is shown that, if D' has a maximal ideal M' of height one for which D'/M' is a finite field, then $\operatorname{Int}(D)$ is not Noetherian; indeed, if M' is the only maximal ideal of D' lying over $M' \cap D$, then not even $\operatorname{Spec}(\operatorname{Int}(D))$ is Noetherian. On the other hand, if every height-one maximal ideal of D' has infinite residue field, then a sufficient condition for $\operatorname{Int}(D)$ to be Noetherian is that the global transform of D is a finitely generated D-module.

1. INTRODUCTION

Throughout this paper D denotes an integral domain (commutative, with unity), K denotes its field of fractions, and X denotes an indeterminate. The *ring of integer-valued polynomials* of D (in the single indeterminate X) is

$$\operatorname{Int}(D) = \{f(X) \in K[X] \colon \forall a \in D,\ f(a) \in D\}.$$

The rings of integer-valued polynomials on various domains have been studied at least since the 1919 articles of Ostrowski [O] and Pólya [P]. Even at that time it was known that *the* ring of integer-valued polynomials, $\operatorname{Int}(\mathbb{Z})$, is generated as a $\mathbb{Z}$-module by the binomial coefficient polynomials

$$\binom{X}{n} = X(X-1)\cdots(X-n+1)/n!$$

for $n = 0, 1, 2, \ldots$. The ring $R = \operatorname{Int}(\mathbb{Z})$ is an interesting example of a non-Noetherian ring that "occurs in nature": It follows from [Ca1, Corollary 1.3] that $\dim(R) = 2$, and Brizolis [B] showed that R is a Prüfer domain, so R is not Noetherian. (There are perhaps more direct ways of reaching the same conclusion—cf. [Ch2, p. 1]—but we use a similar method below.) In the present paper we investigate the Noetherian property of the rings $\operatorname{Int}(D)$, and of their spectra. We will need the following well-known result.

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Fact 1.1 [SSY, Theorem 2; Ch1, Proposition 2.2]. *If D is a Noetherian domain, then $\operatorname{Int}(D) \neq D[X]$ iff there is a maximal ideal M of D that is an associated prime of a principal ideal aD (i.e., for some b in D, $M = aD : b$) and such that the residue field D/M is finite.*

We prove the following statements for a Noetherian domain D with integral closure D': If D' has a maximal ideal M' of height one for which D'/M' is finite, then $\operatorname{Int}(D)$ is not Noetherian. (In fact, if M' is the only maximal ideal of D' lying over $M' \cap D$, then not even $\operatorname{Spec}(\operatorname{Int}(D))$ is Noetherian.) In particular, if $\dim(D) = 1$, then $\operatorname{Int}(D)$ is Noetherian iff $\operatorname{Int}(D) = D[X]$. On the other hand, if D' has no height-one maximal ideal with finite residue field, then a sufficient condition for $\operatorname{Int}(D)$ to be Noetherian is that the global transform of D (in the sense of [Mj]) is a finitely generated module over D. We also provide an example of a two-dimensional Noetherian domain D for which, for each maximal ideal M of D, $\operatorname{Int}(D_M)$ is Noetherian, but $\operatorname{Int}(D)$ is not Noetherian.

In terms of the spectrum of $\operatorname{Int}(D)$, we prove in Theorem 3.1 that for a one-dimensional local domain (D, M) with finite residue field, $\operatorname{Spec}(\operatorname{Int}(D))$ is Noetherian if and only if D' has more than one maximal ideal.

We consistently use $'$ to denote integral closure of a domain (in its field of fractions). When no confusion is likely, we write E in place of $\operatorname{Int}(D)$. The spectrum (i.e., the set of proper prime ideals) and the maximal spectrum of a domain D are denoted $\operatorname{Spec}(D)$ and $\operatorname{Mspec}(D)$, respectively. The terms "local" and "semilocal" include "Noetherian". We use $<$ to denote proper set inclusion.

Let a be any element of D. Then the evaluation map $\varphi: E \to D$ defined by $\varphi(p) = p(a)$ is a retraction of D-algebras. Its kernel is the prime ideal $(X-a)K[X] \cap E$. Since D is an algebraic retract of E, D is "ideally closed" in E—i.e., for each ideal I of D, $IE \cap D = I$. In particular, if E is Noetherian, then D is Noetherian. Thus, most of our results lose nothing with the addition of a Noetherian hypothesis on D. Moreover, if S is a multiplicatively closed subset of D, then the equality $(\operatorname{Int}(D))_S = \operatorname{Int}(D_S)$ holds if D is Noetherian [CC, p. 303], but not in general, as is shown in [G] by the construction of an almost Dedekind domain D for which $\operatorname{Int}(D)$ is not a Prüfer domain.

2. The Noetherian property in $\operatorname{Int}(D)$

It turns out that the domain $\operatorname{Int}(D')$ plays a significant role in our work on the problem of determining conditions under which $\operatorname{Int}(D)$ is Noetherian, particularly in the case where D is one-dimensional. In general, if D_1 is a subring of an integral domain D_2, it need not be true that $\operatorname{Int}(D_1)$ is contained in $\operatorname{Int}(D_2)$. (They are, of course, both subrings of $K[X]$, where K is the field of fractions of D_2.) For example, if k is the field of two elements, t is an indeterminate, $D_1 = k[[t^2, t^3]]$, and $D_2 = k[[t]] = (D_1)'$, then $[(1/t^2)X(X+1)]^2$ is in $\operatorname{Int}(D_1)$ but not in $\operatorname{Int}(D_2)$. This phenomenon forced us to exercise some care, especially in our notation: If $E = \operatorname{Int}(D)$, we need not have $E' = \operatorname{Int}(D')$. (In fact, in Proposition 6.1, we give necessary and sufficient conditions on a one-dimensional Noetherian domain D in order that $\operatorname{Int}(D) \subseteq \operatorname{Int}(D')$.) Thus Proposition 2.2 below is reassuring. Recall that a *transversal* of a family $\mathcal{N}$ of sets is a set $\{s_N : N \in \mathcal{N}\}$ such that $s_N \in N$ for each N in $\mathcal{N}$.

Remark 2.1. Let R and S be subrings of a larger ring (commutative, with unity), and let $\mathcal{N}$ be a family of multiplicatively closed subsets of $R \cap S$ such that every transversal of $\mathcal{N}$ generates the unit ideal in R. Then an element s of S is integral over R iff, for each N in $\mathcal{N}$, the image of s in S_N is integral over R_N.

Proof. If s in S is integral over R, it is clear that the image of s in S_N is integral over R_N. Conversely, if the local condition is satisfied by s, then for each N in $\mathcal{N}$, s is a root of a polynomial $p_N(X)$ in $R[X]$ with leading coefficient in N. Since a finite set of these leading coefficients generates the unit ideal in R, the ideal in $R[X]$ generated by the corresponding polynomials $p_N(X)$ contains a monic polynomial, of which s is a root. □

Proposition 2.2. *Let D be a Noetherian domain, with field of fractions K and integral closure D'. Then the ring* $\mathrm{Int}(D, D')$ *of polynomials $f(X)$ in $K[X]$ for which $f(D) \subseteq D'$ is the integral closure of* $\mathrm{Int}(D)$.

Proof. Since $\mathrm{Int}(D, D')$ is contained in the field of fractions $K(X)$ of $\mathrm{Int}(D)$, it is enough to show that $\mathrm{Int}(D, D')$ is integrally closed and integral over $\mathrm{Int}(D)$. But it is immediate that, for any subset S of an integrally closed domain R, the ring $\mathrm{Int}(S, R)$ of polynomials $f(X)$ with coefficients in the field of fractions of R and for which $f(S) \subseteq R$ is integrally closed (since its integral closure is contained in the ring of polynomials over the field of fractions of R, and if $g(X)$ is a polynomial over the field of fractions of R integral over $\mathrm{Int}(S, R)$, then for each s in S, $g(s)$ is integral over R and hence in R).

To see that $\mathrm{Int}(D, D')$ is integral over $\mathrm{Int}(D)$, let $f(X)$ be an element of $\mathrm{Int}(D, D')$. It suffices to show that, for each maximal ideal M' of D', there is an element s of $D' - M'$ for which sf is integral over $\mathrm{Int}(D_{M' \cap D})$ (for then, possibly with a different s, sf will be integral over $\mathrm{Int}(D)$, and since the integral closure $\mathrm{Int}(D)'$ of $\mathrm{Int}(D)$ is a module over D', it will follow that $f \in \mathrm{Int}(D)'$). So take the maximal ideal M' of D', and assume D is local with maximal ideal $M' \cap D$.

Suppose first that M' has height greater than one. Then by Proposition 4.1, $f \in \bigcap\{D'_{P'}[X] : P' \subseteq M', \mathrm{ht}(P') = 1\} = D'_{M'}[X]$, so there is an element s of $D' - M'$ for which $sf \in D'[X]$, i.e., sf is integral over $D[X] \subseteq \mathrm{Int}(D)$. Similarly, if M' has height one and D'/M' is infinite, then again $f \in D'_{M'}[X]$, so there is an $s \in D' - M'$ such that sf is integral over $\mathrm{Int}(D)$.

Now suppose M' has height one and D'/M' is finite. Since $f(D) \subseteq D' \cap (1/d)D$, where d is a common denominator for the coefficients of f, there is a ring R contained in D', finitely generated over D, for which $f(D) \subseteq R$ and M' is the only maximal ideal of D' lying over $M' \cap R = N$. Let $C = D :_R R$, s be an element of $R - N$ for which $s(CR_N \cap R) \subseteq C$ (such an element exists because R is Noetherian, so that $CR_N \cap R$ is finitely generated), q be the cardinality of R/N, and n be a positive integer for which $N^n \subseteq CR_N \cap R$ (such an integer exists because $\mathrm{ht}(N) = 1$, so that $CR_N \cap R$ is N-primary). Then for each a in D, $(sf(a))^q - sf(a) \in N$, so $((sf(a))^q - sf(a))^n \in CR_N \cap R$, so

$$((sf(a))^q - sf(a))^{n+1} = s(s^{q-1}f(a)^q - f(a))((sf(a))^q - sf(a))^n \in s(CR_N \cap R) \subseteq C \subseteq D,$$

so $g = ((sf)^q - sf)^{n+1} \in \mathrm{Int}(D)$. Thus sf is a root of the monic polynomial $(Y^q - Y)^{n+1} - g$ over $\mathrm{Int}(D)$. □

Theorem 2.3. *Let D be a Noetherian domain for which $\mathrm{Int}(D)$ is also Noetherian. Then for every maximal ideal M' of the integral closure D' of D, either M' has height greater than one, or else the residue field D'/M' is infinite.*

Proof. Assume by way of contradiction that a maximal ideal M' of D' has height one and finite residue field; set $M = M' \cap D$. Then by localizing D and D' at $D \backslash M$, we preserve the hypothesis, so we may assume (D, M) is local. Next, adjoining to D an element of M' not in any of the other maximal ideals of D' (these are finite in number [Na, (33.10)]), we form a ring R, finitely generated as a D-module, for which $D' = R'$ and M' is the only maximal ideal of D' lying over $M' \cap R = N$. Since the nonzero conductor of R into D also multiplies $\mathrm{Int}(R)$ into $\mathrm{Int}(D)$, the compositum $E[\mathrm{Int}(R)]$ of $E = \mathrm{Int}(D)$ and $\mathrm{Int}(R)$ is (contained in) a finitely generated module over $\mathrm{Int}(D)$ and hence is also Noetherian. Localizing at the multiplicative set $R \backslash N$, we see that $(E[\mathrm{Int}(R)])_{R\backslash N} = E[(\mathrm{Int}(R))_{R\backslash N}] = E[\mathrm{Int}(R_N)]$ is also Noetherian, and that the integral closure of R_N is $D'_{M'}$, a discrete rank-one valuation domain. By Proposition 2.2, $\mathrm{Int}(D'_{M'})$ is contained in the integral closure $(\mathrm{Int}(R_N))'$ of $\mathrm{Int}(R_N)$ and hence in $(E[\mathrm{Int}(R_N)])'$. On the one hand, as the integral closure of a Noetherian domain, $(E[\mathrm{Int}(R_N)])'$ is Krull. On the other hand, as an overring of $\mathrm{Int}(D'_{M'})$, which is a Prüfer domain by, for example, [Ch2, Corollaire 6.5], $(E[\mathrm{Int}(R_N)])'$ is also a Prüfer domain; so it is Dedekind and hence one dimensional. However, since both $E = \mathrm{Int}(D)$ and $\mathrm{Int}(R_N)$ are contained in the rank-two valuation domain $V = \{f \in K[X]_{(X)}; f(0) \in D'_{M'}\}$, $(E[\mathrm{Int}(R_N)])' \subseteq V$, and hence the dimension of the Prüfer domain $(E[\mathrm{Int}(R_N)])'$, being equal to its valuative dimension, is at least 2, the desired contradiction. □

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The construction of the famous examples of Nagata of Noetherian domains D that are not catenary (cf. [Na, pp. 204–205; Mu1, pp. 87–88; ZS, pp. 327–329]) start with a ground field k, which can be taken to be a finite field. If k is finite, then such a domain D has dimension greater than 1, the integral closure D' of D fails to satisfy the conclusion of Theorem 2.3, and hence $\mathrm{Int}(D)$ is not Noetherian. These domains have nonzero conductors with respect to their integral closures, so they provide counterexamples to Proposition 3.1 of [Ch1]; the error in the proof of Proposition 3.1 of [Ch1] is its tacit assumption that each maximal ideal of the integral closure has height greater than one. We note that (3.1) of [Ch1] is used in the appendix of that paper to show that $\dim(\mathrm{Int}(D)) = \dim(D) + 1$ for a Noetherian domain D; a proof for this latter statement independent of [Ch1, (3.1)] is given in [Ca1], where Corollary 1.3 asserts validity of the equality for a Jaffard domain D.

From Theorem 2.3 we have the following corollaries for one-dimensional domains and normal domains:

Corollary 2.4. *For D a one-dimensional Noetherian domain, the following statements are equivalent*:

(1) $\mathrm{Int}(D)$ *is Noetherian.*

(2) $\mathrm{Int}(D) = D[X]$.

(3) *The residue field at each maximal ideal of D is infinite.*

Proof. (3) $\Rightarrow$ (2): Apply Fact 1.1.

(2) $\Rightarrow$ (1): Clear.

(1) $\Rightarrow$ (3): Apply Theorem 2.3. $\square$

Corollary 2.5. *For D an integrally closed Noetherian domain, the following statements are equivalent*:

(1) $\mathrm{Int}(D)$ *is Noetherian.*

(2) $\mathrm{Int}(D) = D[X]$.

(3) *D/P is infinite for each height-one prime ideal P of D.*

3. The Noetherian property in $\mathrm{Spec}(\mathrm{Int}(D))$

Suppose that (D, M) is a one-dimensional local domain for which D/M is finite, and let $E = \mathrm{Int}(D)$. It is natural to ask for conditions under which $\mathrm{Spec}(E)$ is Noetherian. In [Ch1] it was shown that for such a D, $\mathrm{Spec}(E)$ consists of height-one primes $f(X)K[X] \cap E$, where $f(X)$ is irreducible in $K[X]$, and maximal ideals of the form $M_a = \{f(X) \in E : f(a) \in \widehat{M}\}$, where a varies over the completion $\widehat{D}$ of D and $\widehat{M} = M\widehat{D}$ is the maximal ideal of $\widehat{D}$ (the ideals M_a are distinct only if $\widehat{D}$ is an integral domain [CGH; Ch3]). The primes contracted from $K[X]$ form a Noetherian space, so $\mathrm{Spec}(E)$ is Noetherian iff the maximal ideals M_a form a Noetherian space; moreover, since the ideals M_a are precisely the minimal primes of ME, this is true iff they are finite in number—i.e., iff $\mathrm{Spec}(E/ME)$ is finite. Suppose D' is a finitely generated D-module. Since $\widehat{D}$ is an integral domain iff D' is local and a finitely generated D-module [Na, Exercise 1, p. 122], if D' is local, then it follows from the result in [CGH or Ch3] cited above that $\mathrm{Spec}(E/ME)$ is infinite and hence $\mathrm{Spec}(E)$ is not Noetherian. On the other hand, if D' has more than one maximal ideal, then Cahen [Ca2, Lemme 1.2] shows that for a, b in D, the ideals M_a, M_b can be distinct only if a, b represent different cosets in D of the nonzero conductor C of D' into D. If $f \in M_a$, then $f(a+C) \subseteq M$, and hence, by continuity, $f(a+C\widehat{D}) \subseteq \widehat{M}$. Since $\widehat{D} = D + C\widehat{D}$, it follows that if the elements u, v of $\widehat{D}$ belong to the same coset of $C\widehat{D}$, then $M_u = M_v$. Because $\widehat{D}/C\widehat{D} \cong D/C$ is finite, then $\mathrm{Spec}(E/ME)$ is finite in this case and $\mathrm{Spec}(E)$ is Noetherian. The following result modifies the proofs of these results only slightly to determine whether $\mathrm{Spec}(E)$ is Noetherian in the case where D' is not a finitely generated D-module.

Theorem 3.1. *Let (D, M) be a one-dimensional local domain with D/M finite.*
(1) *Suppose D' is local (and hence a discrete rank-one valuation domain). If a, b are distinct elements of D, then there is an $f(X) \in \mathrm{Int}(D)$ for which $f(a) \notin M$ but $f(b) = 0$. In particular, $M_a \neq M_b$.*

(2) *Suppose D' is not local. Let R be a finite extension of D in D' such that the maximal ideals of D' have distinct contractions $N_1, \dots, N_k$ to R, and suppose the positive integer α is such that $(N_1 \cap \cdots \cap N_k)^\alpha$ is contained in the*

(nonzero) conductor of R into D. If a, b in $\widehat{D}$ are such that $a-b \in (N_1^\alpha \cap D)\widehat{D}$, then $M_a = M_b$.

Therefore Spec(Int(D)) *is Noetherian iff D' has more than one maximal ideal.*

Proof. (1) Suppose we have found an element $g(X)$ of Int(D) for which $g(a-b) \notin M$ and $g(0) = 0$; then $f(X) = g(X-b)$ has the desired property. So we may assume $b = 0$. Let T be a set of coset representatives of the maximal ideal M' in D' for which $0 \in T$, and let d be a generator of M'. Set $f_1(X) = (1/d)\prod\{(X-t): t \in T\}$ and for $k > 1$, set $f_k(X) = f_1(f_{k-1}(X))$. Then it is easy to see that, for any positive integer n and any element c of D', $f_1(d^n c)$ is associate in D' to $d^{n-1}c$, so if $n \geq k$, then $f_k(d^n c)$ is associate to $d^{n-k}c$. It follows that if a is associate in D' to d^k, then $f_k(a)$ is a unit in D', but $f_k(0) = 0$. The same statements hold for any power of $f_k(X)$. Now since $f_k(D)$ is contained in both D' and a finitely generated D-submodule of K, it is contained in a subring R of D' that is finitely generated over D; and R is also local, say with maximal ideal N. Some power N^m of N is contained in the nonzero conductor of R into D. Let q, p denote the (finite) cardinality and characteristic, respectively, of D'/M'. Then the $(q-1)$st power of any element of R differs from either 0 or 1 (which are in D) by an element of N; and since $p \in N$, it follows from a basic property of binomial coefficients that if $n = (q-1)p^m$, then the nth power of any element of R is in D (cf. [Na, (31.3)]). Thus $f_k^n(X) \in$ Int(D).

(2) Note first that since D' has only finitely many maximal ideals by the Krull-Akizuki Theorem [Na, (33.2)], one can always find an R with the desired property by adjoining to D one element from each maximal ideal of D' that is not in any other maximal ideal.

Assume by way of contradiction that $M_a \neq M_b$; then there is an element $g(X) \in$ Int(D) for which $g(a) - g(b) \notin \widehat{M}$. Write $I = N_1^\alpha \cap D$. Since $g(X)$ is continuous, we can replace a, b with approximating elements a_1, b_1 of D such that $a_1 - b_1 \in I$ and $g(a_1) - g(b_1) \notin M$; so we may assume $a, b \in D$. Now suppose the maximal ideals of D' are $M_1', \dots, M_k'$, lying over $N_1, \dots, N_k$ respectively, and for $i = 1, \dots, k$, let v_i denote the valuation associated to the discrete rank-one valuation domain $D'_{M_i'}$, and let γ_i be a positive integer such that for x, y in K,

$$v_i(x-y) > \gamma_i \quad \text{implies} \quad v_i(g(x) - g(y)) > 0.$$

(For instance, γ_i could be greater than the negative of the smallest v_i-value among those of the coefficients of $g(X)$.) By the Chinese Remainder Theorem, choose z in R such that

$$z \equiv b \mod N_1^{\max\{\alpha, \gamma_1\}}, \qquad z \equiv a \mod N_2^{\max\{\alpha, \gamma_2\}},$$

and

$$z \equiv a \mod N_i^\alpha, \qquad i = 3, \dots, k.$$

Then $z - a = (z-b) - (a-b) \in N_1^\alpha$ and $z - a \in N_i^\alpha$ for $i > 1$. Since the N_i^α are comaximal, their intersection is their product, so $z - a$ is in the conductor of R into D, and hence $z \in D$. Thus, $g(z) - g(b) \in M_1' D'_{M_1'} \cap D = M$ and $g(z) - g(a) \in M_2' D'_{M_2'} \cap D = M$; but this yields $g(a) - g(b) = (g(z) - g(b)) - (g(z) - g(a)) \in M$, the desired contradiction.

It remains to prove the final statement in the theorem: If D' is local, then by (1), $\operatorname{Spec}(E/ME)$ has at least the cardinality of D itself, so $\operatorname{Spec}(E/ME)$ is infinite. But if D' is not local, then by (2) the cardinality of $\operatorname{Spec}(E/ME)$ is bounded above by the cardinality of $\widehat{D}/I\widehat{D} = D/I$ [Na, (17.9)], a finite ring. □

The question of when $\operatorname{Spec}(\operatorname{Int}(D))$ is Noetherian for D not local or of higher dimension than one remains open. But Corollary 4.4 gives an affirmative answer in many cases of higher dimension, since a finitely generated algebra over a ring with Noetherian spectrum has Noetherian spectrum.

Another consequence of (2) above (or, in the case where D' is a finitely generated D-module, of Lemme 1.2 of [Ca2]) is:

Remark 3.2. Let (D, M) be a one-dimensional local domain with D/M finite and with integral closure D' not local. Then the integral closure E' of $E = \operatorname{Int}(D)$ is not a finitely generated E-module.

Proof. By (2), $\operatorname{Spec}(E/ME)$ is finite; so it is enough to show that $\operatorname{Spec}(E'/ME')$ is infinite, since if E' were finitely generated over E, then there would be only finitely many prime ideals of E' having the same contraction to E. (Indeed, for any P in $\operatorname{Spec}(E)$, $E' \otimes_E (E_P/PE_P)$ is a finite-dimensional vector space over E_P/PE_P, so the dimension of $E' \otimes_E (E_P/PE_P)$ is an upper bound for the number of primes of E' lying over P. Cf. [Mu2, Exercise 9.3, p. 69].) Note first that, because $\operatorname{Int}(D')$ is a Prüfer domain, all rings between it and its field of fractions $K(X)$ are integrally closed; so $E' = E[\operatorname{Int}(D')]$. Now suppose $a \in D$, and fix a maximal ideal M' of D'. Then $a \in \widehat{D}$ and $a \in \widehat{D'_{M'}}$. Thus if v denotes the discrete rank-one valuation associated with $\widehat{D'_{M'}}$, then the rank-two valuation domain $V_a = \{f \in K[X]_{(X-a)} \colon v(f(a)) \geq 0\}$ contains E and $\operatorname{Int}(D'_{M'})$; so $E' \subseteq E[\operatorname{Int}(D'_{M'})] \subseteq V_a$. Moreover, if $a, b \in D$ with $a \neq b$, then V_a, V_b have distinct centers on $\operatorname{Int}(D'_{M'})$ and hence on $\operatorname{Int}(D')$ and hence on E'. Since these centers contain M, $\operatorname{Spec}(E'/ME')$ has at least the cardinality of D. □

Question 3.3. If D is a one-dimensional local domain with finite residue field for which $D < D'$, is it possible for $(\operatorname{Int}(D))'$ to be a finitely generated $\operatorname{Int}(D)$-module? For example, if $D = k[[t^2, t^3]]$, where k is a finite field and t is an indeterminate, is $(\operatorname{Int}(D))'$ a finitely generated $\operatorname{Int}(D)$-module?

4. The inclusion of $\operatorname{Int}(D)$ in $D'[X]$ and the Noetherian property revisited

The methods used to prove Fact 1.1 yield a slightly stronger result, which will be useful to us in proving a partial converse to Theorem 2.3.

Proposition 4.1. *Let S be a subset of a domain R, P be a prime ideal of R, and $f(X)$ be a polynomial with coefficients in the field of fractions of R for which $f(S) \subseteq R$. If the number of cosets of P in R met by S is greater than $\deg(f)$, then $f(X) \in R_P[X]$.*

Proof. Write $f(X) = f_0 + f_1 X + \cdots + f_n X^n$, and take elements $t_0, t_1, \ldots, t_n$ of S in distinct cosets of P. If T denotes the $(n+1) \times (n+1)$ Vandermonde matrix whose jth column is the $(j-1)$st powers of the elements t_i in order, and if F is the column matrix of the coefficients f_i, then TF is the column matrix with entries $f(t_i)$, all in R. The determinant of T is $d = \prod_{i<j}(t_i - t_j)$, which by our choice of the elements t_i is not in P. But for each $i = 0, 1, \ldots, n$, Cramer's rule shows that df_i is the determinant of a matrix with entries in R, and hence in R_P, and since d is a unit in R_P, $f_i \in R_P$. □

Definition 4.2. For a domain D that is not a finite field, we set

$$D^a = \bigcap\{D_P \colon P \in \operatorname{Spec}(D) \text{ such that } D/P \text{ is infinite}\}.$$

We note that, for a prime P of D, D/P is infinite iff either P is not maximal or P is maximal but has infinite residue field. The proof of part (1) of the following result is immediate from Proposition 4.1, and part (2) follows from part (1).

Corollary 4.3. *Suppose D is a subring of a domain R.* (1) *For each P in* $\operatorname{Spec}(R)$ *such that $D/(P \cap D)$ is infinite,* $\operatorname{Int}(D) \subseteq R_P[X]$. *In particular,* $\operatorname{Int}(D) \subseteq D^a[X]$.

(2) *If D is Noetherian and $\bigcap\{R_P \colon P \in \operatorname{Spec}(R)$ and $D/(P \cap D)$ is infinite$\}$ is a finitely generated D-module, then* $\operatorname{Int}(D)$ *is Noetherian. In particular, if D^a is a finitely generated D-module, then* $\operatorname{Int}(D)$ *is Noetherian.*

Suppose D is Noetherian. Then it follows from [Na, (33.10)] that for a prime ideal P' of D', D'/P' is infinite iff $D/(P' \cap D)$ is infinite, and hence $D^a \subseteq (D')^a$. If for every height-one prime P' of D', D'/P' is infinite, then the fact that D' is Krull implies that

$$(D')^a \subseteq \bigcap\{D'_{P'} \colon P' \text{ is a height-one prime of } D'\} = D',$$

so that $(D')^a = D'$. These observations and Corollary 4.3 then yield the following result.

Corollary 4.4. *Let D be a Noetherian domain such that for every height-one prime P' of the integral closure D' of D, D'/P' is infinite (for example, such that D' has no height-one maximal ideals). Then* $\operatorname{Int}(D) \subseteq D'[X]$. *In particular, in this case, if D' is a finitely generated D-module, then* $\operatorname{Int}(D)$ *is Noetherian.*

A familiar domain to which this corollary applies is constructed as follows: Let u, v be indeterminates over the finite field k, and set $D = k[[u, v]]$. Write D' in the form $D' = k + M$, where $M = (u, v)k[[u, v]]$ is the unique maximal ideal of D', and set $D = k + M^2$. Then the unique maximal ideal M^2 of D is an associated prime of a principal ideal (for example, $M^2 = (u^2D : u^2v)$), so $\operatorname{Int}(D) > D[X]$. But since the conductor of D' into D is nonzero (in fact, the conductor is M^2), D' is a finitely generated D-module, so $\operatorname{Int}(D)$ is Noetherian. This shows that Corollary 2.4 does not extend to higher dimensional domains D.

In [Mj], Matijevic defined the "global transform" of a Noetherian domain D as follows:

$$D^g = \{x \in K \colon \exists I \text{ an ideal of } D \text{ such that } \dim(D/I) = 0 \text{ and } xI \subseteq D\}.$$

In other words, D^g is the directed union of the transforms of all finite products of maximal ideals of D. For D not a field, this definition is equivalent to

$$D^g = \bigcap\{D_P \colon P \in \operatorname{Spec}(D) \setminus \operatorname{Mspec}(D)\}.$$

Thus, for a Noetherian domain D, D^a is contained in D^g. Hence, by Corollary 4.3, if D^g is a finitely generated D-module, then $\mathrm{Int}(D)$ is Noetherian. Nishimura [Ni, Proposition 1.2] shows that if every maximal ideal of the integral closure $D'_{D\setminus M}$ of D_M has height greater than one, the MD_M-transform $\bigcap\{D_P\colon P \in \mathrm{Spec}(D),\ P < M\}$ of D_M is integral over D_M—i.e., is contained in $D'_{D\setminus M}$. Taking intersection as M varies over $\mathrm{Mspec}(D)$, we conclude that

$$\mathrm{Int}(D) \subseteq D^a[X] \subseteq D^g[X] \subseteq D'[X].$$

Thus, we obtain an alternative proof of Corollary 4.4. Ferrand and Raynaud [FR, Proposition 3.3 and Remarques 3.7] give examples of two-dimensional local domains (D, M) for which the transform D^g of M is not a finitely generated D-module, but in these examples D/M is infinite. It seems possible that there exist examples of the type constructed by Ferrand and Raynaud but with finite residue field. So the question of whether, for a local domain D of dimension greater than 1, $\mathrm{Int}(D)$ is Noetherian iff $\mathrm{Int}(D)$ is a finitely generated D-algebra remains open.

5. Locally Noetherian does not imply Noetherian

Our next example shows that $\mathrm{Int}(D)$ may be locally Noetherian in a strong sense without being Noetherian. The final touch in the example is provided by the following lemma.

Lemma 5.1. *Let (D, M) be a local domain with finite residue field and integral closure D' having no height-one maximal ideals. Assume that M is an associated prime of a principal ideal of D. Then for some maximal ideal M' of D', $(M', X)D'[X] \cap \mathrm{Int}(D) = P$ is an associated prime of $(X)\,\mathrm{Int}(D)$, and $P \cap D = M$.*

Proof. Since M has the form $dD :_D c$ for some elements d, c of D, if we set $b = c/d$, we get $M = D :_D b$. Let T be a set of coset representatives of M in D such that $0 \in T$, and set $f(X) = \prod\{(X - t)\colon t \in T\}$. Then $bf(X) \in \mathrm{Int}(D)\setminus D[X]$ and $bf(X)/X \notin \mathrm{Int}(D)$, so $(X)\,\mathrm{Int}(D) :_{\mathrm{Int}(D)} bf(X)$ is a proper ideal of $\mathrm{Int}(D)$. Since $(M, X)D[X]$ is maximal, it is contained in only maximal ideals of $D'[X]$, of the form $(M', X)D'[X]$. But $(M, X)\,\mathrm{Int}(D) \subseteq (X)\,\mathrm{Int}(D) :_{\mathrm{Int}(D)} bf(X)$, so at least one of the minimal primes P of $(M, X)\,\mathrm{Int}(D)$ is an associated prime of $(X)\,\mathrm{Int}(D)$. □

Example 5.2. A two-dimensional Noetherian domain D such that $\mathrm{Int}(D)$ is not Noetherian but $\mathrm{Int}(D_M)$ is Noetherian for each maximal ideal M of D: Let k be a finite field, p be its characteristic, and u, v be indeterminates. Enumerate the nonzero elements of $k[u, v]$: $g_1, g_2, \ldots$. For each positive integer i in turn: Pick a maximal ideal N_i of $k[u, v]$ such that $b_1 \cdots b_{i-1} g_1 \cdots g_i \notin N_i$. (This is possible because $k[u, v]$ is a Hilbert ring. The elements b_j will be selected below; there are no b_j in the product avoided to select N_1.) Let S_i denote the localization of $k[u, v]$ at N_i, a two-dimensional regular local ring, and pick $a_i, b_i \in k[u, v]$ such that $b_i \in N_i\setminus(N_i^2 \cup N_1 \cup \cdots \cup N_{i-1})$ and a_i, b_i is a minimal generating set for $N_i S_i$. Let R_i denote the localization of $S_i[(a_i/b_i)^2, (a_i/b_i)^3]$ at its maximal ideal $(a_i, b_i, (a_i/b_i)^2, (a_i/b_i)^3)$, so that the integral closure R'_i of R_i is the localization of $S_i[a_i/b_i]$ at its maximal ideal $(a_i/b_i, b_i)$. Set $D = \bigcap_{i=1}^{\infty} R_i$; then since for any f in R'_i, $f^p \in R_i$, we see that $D' = \bigcap_{i=1}^{\infty} R_i'$ is integral over D. Also D' is contained in the field of fractions $k(u, v)$ of D and is integrally closed, so D' is indeed the integral closure of D.

Next we note that $\{R_i\}_{i=1}^{\infty}$ and $\{R_i'\}_{i=1}^{\infty}$ are locally finite families, i.e., that any nonzero element of $k(u, v)$ is a unit in all but finitely many of each family. Since $k[u, v] \subseteq R_i \subseteq R_i'$, it is enough to note that, by our choice of the N_i, no nonzero element g_j of $k[u, v]$ is contained in infinitely many of the centers N_i of R_i and R_i' on $k[u, v]$. The property of local finiteness allows us to "commute localization and intersection," i.e., for any multiplicatively closed subset T of D, $D_T = \bigcap_{i=1}^{\infty}(R_i)_T$ and similarly for D'. In particular, if $T = k[u, v]\backslash N_i$, then for any j distinct from i, since $b_j \notin N_i$, $(R_j)_T = (R_j')_T \supseteq (S_j)_T = \bigcap\{k[u, v]_P : P \in \operatorname{Spec}(k[u, v]),\ P \subseteq N_i \cap N_j\} \supseteq R_i' \supseteq R_i$, while $(R_i)_T = R_i$; so if M_i is the center on D of the maximal ideal of R_i, we get $R_i = D_T \subseteq D_{M_i} \subseteq R_i$, so $D_{M_i} = R_i$. Similarly if M_i' denotes the center on D' of the maximal ideal of R_i', then $(D')_{M_i'} = R_i'$.

Now we want to show that D is Noetherian. Let d be a nonzero element of D; then $dD = \bigcap_{i=1}^{\infty} dR_i$, and $dR_i = R_i$ for all but finitely many positive integers i, so dD is the intersection of finitely many primary ideals (contracted from the remaining R_i). So to see that D is Noetherian, it suffices to show that, for each nonzero primary ideal Q of D, D/Q is Noetherian.

Suppose first that the radical P of Q is a height-one prime of D. Then P does not contain any M_i, so $(R_i)_{D\backslash P}$ is a proper localization of R_i, i.e., one of a_i, b_i, $(a_i/b_i)^2$, $(a_i/b_i)^3$ is a unit in $(R_i)_{D\backslash P}$. It is easy to check in each of these four cases that $a_i/b_i \in (R_i)_{D\backslash P}$, and hence $(R_i)_{D\backslash P} = (R_i')_{D\backslash P}$. Now the height-one primes of R_i' are b_iR_i' and some of the form $(f)k[u, v]_{(f)} \cap R_i'$ for f irreducible in $k[u, v]$. Since b_iR_i' lies over N_i in $k[u, v]$, the localization of R_i' at b_iR_i' is a discrete rank-one valuation domain containing $k[u, v]$, and by our choice of the N_i these localizations form a locally finite family. The localizations of $k[u, v]$ at height-one primes also form a locally finite family, so the intersection of any subfamily of the union of these two families is a Krull domain. In particular, $D_P = \bigcap_{i=1}^{\infty}(R_i)_{D\backslash P} = \bigcap_{i=1}^{\infty}(R_i')_{D\backslash P}$ is Krull, and since D_P is also one-dimensional and local, it is a discrete rank-one valuation domain. Thus Q is a symbolic power of P. Moreover, D/P is Noetherian: If $P = (f)k[u, v]_{(f)} \cap D$, then D/P lies between $k[u, v]/(f)$ and its field of fractions, so by the Krull-Akizuki Theorem D/P is Noetherian. If $P = b_iR_i' \cap D$ for some i, then D/P lies between the field $F = k[u, v]/N_i$ and R_i'/b_iR_i', a polynomial ring in one indeterminate over F; so again D/P is Noetherian. It follows from [HL, Lemma 3.3] that D/Q is Noetherian in this case.

Now suppose the radical M of Q is maximal in D. Then $D/Q = D_M/QD_M$, so it is enough to show that D_M is Noetherian. For $M = M_i$, $D_M = R_i$ is Noetherian. If $M \neq M_i$ for every i, then $D_M = \bigcap_{i=1}^{\infty}(R_i)_{D\backslash M} = \bigcap_{i=1}^{\infty}(R_i')_{D\backslash M}$ is a Krull domain between $k[u, v]$ and its field of fractions; so it is Noetherian by [H, Theorem 9].

We can also use the descriptions of the localizations of D at its maximal ideals M in the last paragraph to see that $\operatorname{Int}(D_M)$ is Noetherian: If $M = M_i$, then $\operatorname{Int}(D_M) = \operatorname{Int}(R_i)$, which lies between $R_i[X]$ and $R_i'[X]$; since R_i' has nonzero conductor into R_i, we see that $\operatorname{Int}(D_M)$ is Noetherian. Suppose $M \neq M_i$ for every i; then as we saw above, D_M is integrally closed. Moreover, if M were of height one, then D_M would be a discrete rank-one valuation domain and hence a localization of D', i.e., a localization of one of the R_i' at a height-one prime, i.e., a localization of one of the $R_i = D_{M_i}$, which would imply that $M \subseteq M_i$. It follows that M has height two, and hence by Corollary 2.5 that $\mathrm{Int}(D_M) = D_M[X]$, which is Noetherian.

Finally, assume by way of contradiction that $\mathrm{Int}(D)$ is Noetherian. For each positive integer i, $M_iR_i = (a_i/b_i)^2R_i : (a_i/b_i)^3$ is an associated prime of $(a_i/b_i)^2R_i$; so by Lemma 5.1, $(M_i', X)R_i'[X] \cap \mathrm{Int}(R_i) = P_i$ is an associated prime of (X) in $\mathrm{Int}(R_i)$, and hence $P_i \cap \mathrm{Int}(D)$ is an associated prime of (X) in $\mathrm{Int}(D)$. All the $P_i \cap \mathrm{Int}(D)$ are distinct (since their intersections with D are the distinct M_i), so (X) has infinitely many associated primes in $\mathrm{Int}(D)$, the desired contradiction. Thus, $\mathrm{Int}(D)$ is not Noetherian.

6. The inclusion of $\mathrm{Int}(D)$ in $\mathrm{Int}(D')$

In conclusion we provide a description of the one-dimensional Noetherian domains D for which $\mathrm{Int}(D) \subseteq \mathrm{Int}(D')$.

Proposition 6.1. *Let D be a one-dimensional Noetherian domain with integral closure D'. The following conditions are equivalent:*

(1) $\mathrm{Int}(D) \subseteq \mathrm{Int}(D')$.

(2) *For each maximal ideal M' of D' for which D'/M' is finite, we have $D'/M' = D/(M' \cap D)$ and $(M' \cap D)D'_{M'} = M'D'_{M'}$.*

Proof. It is clear that (2) is a local statement; since D and D' are Noetherian, it follows easily from [SSY, Proposition 5] that (1) is also local—i.e., that $\mathrm{Int}(D) \subseteq \mathrm{Int}(D')$ iff $\mathrm{Int}(D_{M' \cap D}) \subseteq \mathrm{Int}(D'_{M'})$ for every maximal ideal M' of D'. Thus we may assume that D is local with maximal ideal M (so that D' is a semilocal principal ideal domain) and consider a particular maximal ideal M' of D'. Moreover, if D'/M' is infinite, then D/M is also infinite, so $\mathrm{Int}(D) = D[X] \subseteq D'_{M'}[X] = \mathrm{Int}(D'_{M'})$; thus we need only consider those M' for which the residue field is finite.

(2) $\Rightarrow$ (1). Let $R = D'_{M'}$; then the hypotheses of (2) imply that $R = D + MR$. Take $f(X) \in \mathrm{Int}(D)$, and write $f(X) = g(X)/d$, where $g(X) \in D[X]$ and $d \in D \backslash (0)$. Choose a power M^n of M for which $M^n \subseteq dD$. Then

$$R = D + MR = D + M(D + MR) = D + M^2R = \cdots = D + M^nR = D + dR.$$

Thus, for any element r of R there is an element a in D such that $r - a \in dR$; and since $g(X) \in D[X] \subseteq R[X]$, we have $g(r) - g(a)$ is divisible in R by $r - a$ and hence by d. Thus, $g(r) = d(f(a)) + (g(r) - g(a)) \in dR$—i.e., $f(r) \in R$. We conclude that $f(X) \in \mathrm{Int}(R) = \mathrm{Int}(D'_{M'})$.

(1) $\Rightarrow$ (2). Let T be a set of coset representatives of D in M such that $0 \in T$, and let $p(X) = \prod\{X - t : t \in T\}$. If $D/M < D'/M'$, then there exists an element b of D' that is congruent to no element of T modulo M'. Choose m in $M \backslash (0)$ and a positive integer k such that $M^k \subseteq mD$. Then $f(X) = p(X)^k/m \in \mathrm{Int}(D)$; but $p(b)^k/m \notin D'$ because, by the choice of b, $p(b) \notin M'$, and hence $p(b)^k \notin M' \supseteq mD'$. Therefore $f(X) \notin \mathrm{Int}(D')$, and $\mathrm{Int}(D) \not\subseteq \mathrm{Int}(D')$ in this case. On the other hand, if $D/M = D'/M'$ but $MD'_{M'} < M'D'_{M'}$, then $MD'_{M'} \subseteq [M'D'_{M'}]^2 = a^2D'_{M'}$, where a is a generator of M' in D'. Thus if $g(X) = p(X)^2/a^3$, then $g(D)$ is contained in the Jacobson radical of D'. As in the proof of Proposition 2.2, it then follows that

$g(X)^m \in \mathrm{Int}(D)$ for some positive integer m. However, $g(X)^m \notin \mathrm{Int}(D')$ because, for example, $g(a)^m \notin D'_{M'}$ for any positive integer m. This completes the proof. □

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Addendum

We remark that in Theorem 6 and Corollary 7 of [GHLS], the hypothesis that the domain D be countable can be replaced by the hypothesis that the spectrum of D is countable. Thus, for example, Corollary 7 of [GHLS] applies in the case where D is a complete discrete rank-one valuation domain.

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