TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 338, Number 1, July 1993

CHARACTERIZATION OF AUTOMORPHISMS ON THE BARRETT AND THE DIEDERICH-FORNAESS WORM DOMAINS

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ABSTRACT. In this paper we show that every automorphism on either the Barrett or the Diederich-Fornaess worm domains is given by a rotation in w-variable. In particular, any automorphism on either one of these two domains can be extended smoothly up to the boundary.

I. INTRODUCTION

In several complex variables extending a biholomorphism or an automorphism smoothly up to the boundary is always a very important and fundamental problem which is closely related to the classification problem of domains in $\mathbf{C}^n$. The extension phenomenon in general is false as shown in Barrett [3] if the domains are sitting in some general complex manifolds. However, it is still widely believed that such extension phenomena should hold if the domains are contained in $\mathbf{C}^n$, namely, we conjecture the following two statements,

(1.1) Any biholomorphism between two smoothly bounded domains D_1 and D_2 in $\mathbf{C}^n$, $n \geq 2$, can be extended smoothly to a CR-diffeomorphism between $\overline{D_1}$ and $\overline{D_2}$,

and its weaker counterpart

(1.2) Any automorphism of a smoothly bounded domain D in $\mathbf{C}^n$, $n \geq 2$, can be extended smoothly up to the boundary, i.e., $\operatorname{Aut}(D) = \operatorname{Aut}(\overline{D})$.

Indeed, it has been shown in Bell and Ligocka [6] that if condition R holds on both D_1 and D_2, then (1.1) is valid. Here condition R means that the Bergman projection associated with the domain D maps $C^\infty(\overline{D})$ continuously into itself. Condition R was shown to hold on a large class of (pseudoconvex or nonpseudoconvex) domains. But surprisingly Barrett constructed in [1] a smoothly bounded nonpseudoconvex domain Ω in $\mathbf{C}^2$ which fails to satisfy condition R.

On the other hand, Diederich and Fornaess in [8] constructed a smoothly bounded pseudoconvex domain Ω_r in $\mathbf{C}^2$ which possesses many pathological properties that include a nontrivial Nebenhulle and the nonexistence of a C^3

Received by the editors April 29, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32A07, 32M05.
Key words and phrases. Automorphism group, Barrett's domain, Diederich-Fornaess domain.
Research partially supported by a FRAP grant from SUNY at Albany.

plurisubharmonic defining function for Ω_r. Very recently Barrett showed in [4] that the Bergman projection associated with Ω_r does not preserve the Sobolev space $W^k(\Omega_r)$ if $k \in \mathbf{R}$ is large enough. It is still not clear whether condition R holds on Ω_r or not.

In this article we want to show that despite these pathological properties found on Ω and Ω_r, statement (1.2) is still valid on both Ω and Ω_r. In fact we can prove more, namely,

Main Theorem. *Any automorphism f of either the Barrett or the Diederich-Fornaess worm domains is given by a rotation in w-variable, i.e., $f(z, w) = (z, e^{i\phi}w)$ for some constant $\phi \in \mathbf{R}$. In particular, f can be extended smoothly up to the boundary.*

We make a remark here that although Ω does not satisfy condition R, it still enjoys an a priori estimate on Sobolev space $W^k(\Omega)$ as shown in Boas and Straube [7].

II. Proof on the Barrett's domain

We first recall the definition of Ω. The domain Ω is a smoothly bounded domain defined in $\mathbf{C}^2$ as follows,

$$\Omega = \{(z, w) \in \mathbf{C}^2 | 1 < |w| < 6,\ |z - c(|w|)| > r_1(|w|) \text{ and } |z| < r_2(|w|)\},$$

where the functions $r_1(|w|)$, $r_2(|w|)$, and $c(|w|)$ are chosen to meet the following conditions: Let k be a positive integer. Define

$$r_1(x) = \begin{cases} 3 - \sqrt{x-1} & \text{for } x \text{ near } 1, \\ \text{decreasing} & \text{for } x \in [1, 2], \\ 1 & \text{for } x \in [2, 5], \\ \text{increasing} & \text{for } x \in [5, 6], \\ 3 - \sqrt{6-x} & \text{for } x \text{ near } 6. \end{cases}$$

$$r_2(x) = \begin{cases} 3 + \sqrt{x-1} & \text{for } x \text{ near } 1, \\ \text{increasing} & \text{for } x \in [1, 2], \\ 4 & \text{for } x \in [2, 5], \\ \text{decreasing} & \text{for } x \in [5, 6], \\ 3 - \sqrt{6-x} & \text{for } x \text{ near } 6. \end{cases}$$

and

$$c(x) = \begin{cases} 0 & \text{for } x \in [1, 2], \\ \text{decreasing} & \text{for } x \in [2, 3], \\ (x-3)^{2k} - 1 & \text{for } x \text{ near } 3, \\ \text{increasing} & \text{for } x \in [3, 4], \\ -(x-4)^{2k} + 1 & \text{for } x \text{ near } 4, \\ \text{decreasing} & \text{for } x \in [4, 5], \\ 0 & \text{for } x \in [5, 6]. \end{cases}$$

Then the following theorem that shows Ω fails to satisfy condition R was proved in Barrett [1].

Theorem. *$P(C_0^\infty(\Omega))$ is not contained in $L^p(\Omega)$ for $p \geq 2 + \frac{1}{k}$, where P is the Bergman projection associated with Ω.*

Now we proceed to prove our main theorem on this domain. Let $f = (f_1, f_2)$ be an automorphism of Ω. We first show the following lemma. A similar statement was proved in Boas and Straube [7].

Lemma 2.1. *Let g be a bounded holomorphic function on Ω. Then g can be extended holomorphically to D, where*

$$D = \{(z, w) \in \mathbf{C}^2 \mid |z| < r_2(|w|) \text{ and } 1 < |w| < 6\}.$$

Proof. By Laurent series expansion one can write

$$g(z, w) = \sum_{n=-\infty}^{\infty} a_n(z) w^n, \tag{2.2}$$

where

$$a_n(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{g(z, w)}{w^{n+1}}\, dw, \tag{2.3}$$

and $|w| = R$ is any circle contained in the slice corresponding to z. Then one can see easily from (2.3) that $a_n(z)$ is locally uniformly bounded and can be extended to a holomorphic function on $\Delta(0; 4)$; i.e.,

$$a_n(z) \in H(\Delta(0; 4)) \quad \text{for all } n \in \mathbf{Z}.$$

Next we show as in [7] that the series (2.2) in fact converges on $\{(z, w) \in \mathbf{C}^2 \mid |z| < 3 \text{ and } 1 < |w| < 6\}$ to a holomorphic function. Consider first the nonnegative indices, i.e.,

$$\sum_{n=0}^{\infty} a_n(z) w^n. \tag{2.4}$$

Put $u(z) = \overline{\lim}_{n\to\infty} |a_n(z)|^{1/n}$, and let $u^*(z)$ be the upper semicontinuous regularization of $u(z)$, i.e.,

$$u^*(z) = \overline{\lim_{z' \to z}}\, u(z').$$

Then by the fact that $a_n(z)$ is locally uniformly bounded, we see that $u^*(z)$ is subharmonic on the disk $\Delta(0; 4)$, and it is easy to see that $u^*(z) \leq \frac{1}{6}$ for $|z| = 3$. Hence by maximum principle we obtain that $u^*(z) \leq \frac{1}{6}$ for $|z| \leq 3$ and $u(z) \leq \frac{1}{6}$ for $|z| \leq 3$. It follows that the series (2.4) is holomorphic on $\{(z, w) \in \mathbf{C}^2 \mid |z| < 3 \text{ and } |w| < 6\}$. For the negative indices part we simply replace w by $\frac{1}{w}$, then an analogous argument will go through as well. This completes the proof of the lemma.

It follows thus from Lemma 2.1 that $f_k(z, w) \in H(D)$ for $k = 1, 2$. We claim that in fact we have $f = (f_1, f_2) \in \operatorname{Aut}(D)$.

Proof of the claim. Put

$$D_0 = \{(z, w) \in \mathbf{C}^2 \mid |z| < 4 \text{ and } 1 < |w| < 6\},$$

and let $\widetilde{D}$ be the envelope of holomorphy of D. We have that $\Omega \subseteq D \subseteq \widetilde{D} \subseteq D_0$. Let $p = (z_0, w_0)$ be a point in $D - \Omega$. Consider the circle

$$C_p = \{(z, w_0) \in \Omega \,|\, |z| = 3\}.$$

By maximum modulus principle it is easy to see that $f(p) \in D_0$. We wish to show that $f(p) \in \widetilde{D}$, and hence $f(D) \subseteq \widetilde{D}$.

Suppose that $q = f(p) \notin \widetilde{D}$ with $5 \leq |f_2(p)| \leq 6$. Set

$$E(\widetilde{D}) = \widetilde{D} \cup \{(z, w) \in \mathbf{C}^2 \,|\, |z| < 4 \text{ and } |w| < 4\},$$

and

$$E_+(\widetilde{D}) = \{(|z|, |w|) \in \mathbf{R}^2 \,|\, (z, w) \in E(\widetilde{D})\}.$$

It is well known that the logarithmic image of $E_+(\widetilde{D})$ is geometrically convex. Since the point $q^* = (\ln|f_1(p)|, \ln|f_2(p)|)$ is not in $\ln(E_+(\widetilde{D}))$ and the rational number is dense in $\mathbf{R}$, one can find two positive integers m and n such that the straight line L,

$$L = \{(x, y) \in \mathbf{R}^2 \,|\, mx + ny = c_1 \text{ for some constant } c_1\},$$

go through q^* and such that the whole set $\ln(|f(C_p)|_+)$ lies in the half plane defined by $\{(x, y) \in \mathbf{R}^2 \,|\, mx + ny - c_1 < 0\}$, where

$$|f(C_p)|_+ = \{(x, y) \in \mathbf{R}^2 \,|\, (x, y) = (|f_1(z, w_0)|, |f_2(z, w_0)|) \text{ for some } |z| = 3\}.$$

Now consider the entire holomorphic function $h(z, w) = e^{-c_1} z^m w^n$. We see that the restriction of $h \circ f$ to Δ_p, where $\Delta_p = \{(z, w) \in D \,|\, |z| < 3$ and $w = w_0\}$, will violate the maximum modulus principle. A similar argument via the mapping $w \mapsto \frac{1}{w}$ can be applied to show that no point of $D - \Omega$ can be mapped to $D_0 - \widetilde{D}$ with $1 \leq |w| \leq 2$. This shows that $f(p) \in \widetilde{D}$, thus we have $f(D) \subseteq \widetilde{D}$.

Let g be the inverse mapping of f. Since g can be extended holomorphically to $\widetilde{D}$, it is legitimate to consider $g \circ f \colon D \to \mathbf{C}^2$. Then by identity theorem and the fact $g \circ f|_\Omega =$ identity mapping, we get $g \circ f =$ identity mapping on D. Similarly $f \circ g$ is also the identity mapping on D. This shows that $f \in \operatorname{Aut}(D)$, and the proof of the claim is now completed.

Next we observe that the domain D is Reinhardt. Therefore f can be extended holomorphically to a small open neighborhood of $\overline{D}$. In particular, we have $f \in \operatorname{Aut}(\overline{D})$. For instance see Barrett [2]. However, we want to show more that f in fact is given by a rotation in w-variable. So we next characterize a Reinhardt hypersurface in $\mathbf{C}^2$ that contains a Riemann surface in it. By a Reinhardt hypersurface we mean that the hypersurface is invariant under the rotations in all directions. The result might have some interest of itself. If H is a Reinhardt hypersurface in $\mathbf{C}^2$, we denote by H_+ the corresponding curve in $\mathbf{R}^2$. Then we have

Lemma 2.5. *Let H be a Reinhardt hypersurface in $\mathbf{C}^2$ such that H_+ is decreasing in ρr-space with $\rho = |w|$ and $r = |z|$. Then H contains a Riemann surface near $p_0 \in H$ if and only if H is either flat in one of the coordinates or H_+ is defined near p_0 by a hyperbola, namely,*

$$H_+ = \{(\rho, r) \in \mathbf{R}^2 \,|\, r\rho^c = \text{constant, for some } c \geq 0\}.$$

Proof. Put $z = (x, y) = re^{i\theta}$ and $w = (u, v) = \rho e^{i\phi}$. Let the defining function for H_+ (hence for H) by $\xi(\rho, r)$. Rewrite $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in terms of polar coordinates, we obtain

$$\frac{\partial}{\partial x} = -\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r}, \qquad \frac{\partial}{\partial y} = \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r}.$$

Next the tangential type-$(1, 0)$ vector field is generated by

$$L = \frac{\partial\xi}{\partial z}\frac{\partial}{\partial w} - \frac{\partial\xi}{\partial w}\frac{\partial}{\partial z} = \frac{1}{2}e^{-i\theta}\frac{\partial\xi}{\partial r}\frac{\partial}{\partial w} - \frac{1}{2}e^{-i\phi}\frac{\partial\xi}{\partial\rho}\frac{\partial}{\partial z}.$$

Suppose that H contains a Riemann surface $\mathscr{R}$ near p_0 with $\frac{\partial\xi}{\partial r}(p_0) \neq 0$. Then one can choose L to be

$$L = \frac{\partial}{\partial w} - e^{i(\theta-\phi)}\cdot\frac{\partial\xi}{\partial\rho}\left(\frac{\partial\xi}{\partial r}\right)^{-1}\frac{\partial}{\partial z}.$$

Put $g(z, w) = e^{i(\theta-\phi)}\cdot\frac{\partial\xi}{\partial\rho}\left(\frac{\partial\xi}{\partial r}\right)^{-1}$. Since $[L, \overline{L}]|_{\mathscr{R}} \equiv 0 \bmod (L \oplus \overline{L})$, we see that the restriction $g|_{\mathscr{R}}$ is holomorphic. Then consider

$$\frac{gw}{z}\Big|_{\mathscr{R}} = \frac{\partial\xi/\partial\rho}{\partial\xi/\partial r}\cdot\frac{\rho}{r}.$$

It shows that the function $\frac{gw}{z}|_{\mathscr{R}}$ is holomorphic and real valued. Hence it must be a real constant function, namely,

$$\frac{gw}{z}\Big|_{\mathscr{R}} \equiv c, \quad \text{for some } c \in \mathbf{R}.$$

Also locally one can express r as a function of ρ, and the slope of H_+ near p_0 is given by

$$\frac{dr}{d\rho} = -\frac{\partial\xi/\partial\rho}{\partial\xi/\partial r} = -c\frac{r}{\rho}.$$

Therefore by solving this first order differential equation we get $r\rho^c = e^{c_0}$, for some constant $c_0 \in \mathbf{R}$. Since H_+ is decreasing, the constant c is nonnegative.

On the other hand, if H_+ is defined locally near some p_0 by $r\rho^c = c_1$ with $c, c_1 > 0$, then by direct computation we get

$$L = w\frac{\partial}{\partial w} - cz\frac{\partial}{\partial z}. \tag{2.6}$$

Hence we have $[L, \overline{L}] \equiv 0$. It follows that there exists a Riemann surface in H near p_0. This completes the proof of the lemma.

It is interesting to note that the vector field $-2\operatorname{Im} L$, where L is given in (2.6), is generated by the following S^1-action,

$$\Lambda\colon S^1 \times \mathscr{R} \to \mathscr{R},$$
$$(\theta, (z, w)) \mapsto (e^{-ic\theta}z, e^{i\theta}w).$$

Now we go back to the automorphism f of D. We see that f will map biholomorphically a Riemann surface in the boundary onto another Riemann surface in the boundary. In particular, if we set

$$\mathscr{R}_{a,b,\theta} = \{(z, w) \in bD | z = 4e^{i\theta} \text{ and } a < |w| < b \text{ with } a \leq 2 \text{ and } b \geq 5\}$$

to be the largest annulus sitting in the boundary with $|z| = 4$, and set

$$C_{a,\theta} = \{(4e^{i\theta}, w) \in bD \mid |w| = a\}$$

to be the inner boundary of $\mathscr{R}_{a,b,\theta}$, and similarly let $C_{b,\theta}$ be the outer boundary of $\mathscr{R}_{a,b,\theta}$. Then f will map $\mathscr{R}_{a,b,\theta}$ to a Riemann surface in the boundary. We claim that $\mathscr{R}_{a,b,\theta}$ cannot be mapped to any Riemann surface contained in the boundary with $1 < |w| < 2$ or $5 < |w| < 6$. First it is not hard to see that there are only three different types of Riemann surfaces in these regions, they are

(i) $\mathscr{R}_c \subseteq \{(z, w) \in bD \mid |z|\,|w|^c = A$ for some constants A and $c > 0$, and $\alpha < |w| < \beta$ with $5 < \alpha < \beta < 6\}$, or an equivalent counterpart in the region $1 < |w| < 2$.

(ii) $\mathscr{R}_z = \{(z, w_0) \in bD \mid w_0 = \rho e^{i\phi}$ for some ρ and ϕ with $5 < \rho < 6$ or $1 < \rho < 2$, and $s < |z| < t$ for some $3 < s < t < 4\}$.

(iii) $\mathscr{R}_w = \{(z_0, w) \in bD \mid z_0 = re^{i\theta}$ for some r and θ with $3 < r < 4$, and $s < |w| < t$ for some $1 < s < t < 2$ or $5 < s < t < 6\}$.

We can rule out $\mathscr{R}_z$ and $\mathscr{R}_w$ immediately by considering the ratio of the radii of the boundaries of these annuli. To knock out $\mathscr{R}_c$ we first observe that if f maps $\mathscr{R}_{a,b,\theta}$ onto some $\mathscr{R}_c$, then by continuity f must map $c_{b,\theta}$ for any θ into exactly one of $\{(z, w) \in bD \mid |w| = \alpha\}$ or $\{(z, w) \in bD \mid |w| = \beta\}$. Suppose that $C_{b,\theta}$ is mapped to $\{(z, w) \in bD \mid |w| = \alpha\}$. Then by maximum modulus principal we see that f will map $\{(z, w) \in \overline{D} \mid |z| \le 4$ and $|w| = b\}$ biholomorphically onto

$$\{(z, w) \in \overline{D} \mid |z| \le A/\alpha^c \text{ and } |w| = \alpha\}.$$

In particular, f must map a disk

$$\Delta_w = \{(z, w) \in D \mid |z| < 4 \text{ and } w = be^{i\phi} \text{ for some } \phi\}$$

biholomorphically onto another disk

$$\Delta_{w'} = \{(z, w') \in D \mid |z| < A/\alpha^c \text{ and } w' = \alpha e^{i\phi'} \text{ for some } \phi'\}.$$

Since the restriction $f|_\Omega$ is an automorphism, we also have the following biholomorphic equivalence between two annuli induced by $f|_\Omega$, namely, $f|_\Omega : \Delta_w \cap \Omega \xrightarrow{\sim} \Delta_{w'} \cap \Omega$. However, this cannot happen simply by examining the ratio of the radii of boundaries of these two annuli. Thus we have shown that $f(\mathscr{R}_{a,b,\theta}) = \mathscr{R}_{a,b,\eta(\theta)}$, for some real-valued function $\eta(\theta)$ that maps S^1 bijectively onto itself.

Next we divide our arguments into two subcases.

Case 1. If f maps $C_{b,\theta}$ to $C_{b,\eta(\theta)}$ for some θ. Then by continuity f will map $C_{b,\theta}$ to $C_{b,\eta(\theta)}$ for all $\theta \in [0, 2\pi]$. For each fixed $\theta \in [0, 2\pi]$, we see by reflection principle that f_2 can be extended to an entire function which preserves the modulus on $|w| = a$ and $|w| = b$. Hence f_2 must take the following form

$$f_2(4e^{i\theta}, w) = e^{i\delta(\theta)} \cdot w, \tag{2.7}$$

for some real-valued function $\delta(\theta)$. Also we have

$$f_1(4e^{i\theta}, w) = 4e^{i\eta(\theta)}, \tag{2.8}$$

independent of w for $a < |w| < b$.

Then by the maximum modulus principle we have

$$|f_2(z, w)| = |w| \quad \text{for } |z| \le 4 \quad \text{and} \quad a < |w| < b.$$

This implies that f will map S_c biholomorphically onto S_c, where $S_c = \{(z, w) \in \overline{D} \mid |z| \le 4, \ |w| = c \text{ with } a < c < b\}$. Therefore, we conclude that f must map a disk $\Delta_{c,\phi}$ onto another disk $\Delta_{c,\phi'}$ i.e.,

$$f: \Delta_{c,\phi} \xrightarrow{\sim} \Delta_{c,\phi'}, \tag{2.9}$$

where $\Delta_{c,\phi} = \{(z, w) \in \overline{D} \mid |z| \le 4, \ w = ce^{i\phi}\}$. Thus if we combine equations (2.7) and (2.9), we obtain for fixed ϕ that $f_2(4e^{i\theta}, ce^{i\phi}) = ce^{i\delta(\theta)} \cdot e^{i\phi} = ce^{i\phi'}$, for all $\theta \in [0, 2\pi]$. This implies that $\delta(\theta)$ is a constant function, namely, $\delta(\theta) = \phi_0$. Hence we obtain that $f_2(z, w) = e^{i\phi_0}w$.

Next equation (2.8) shows that the restriction $f_1|_{\Delta_{c,\phi}}$ of f_1 to every disk $\Delta_{c,\phi}$ with $a < c < b$ and all ϕ has the same boundary value. So we conclude that $f_1(z, w) = f_1(z)$ is independent of w. Then by the facts that $|f_1(4e^{i\theta})| = 4$ and $|f_1(3e^{i\theta})| = 3$, we get

$$f_1(z, w) = f_1(z) = e^{i\theta_0}z \quad \text{for some constant } \theta_0 \in \mathbf{R}.$$

Since $f|_\Omega$ is an automorphism of Ω, it is easy to see that $\theta_0 = 0$. This shows that

$$f = (f_1, f_2) = (z, e^{i\phi_0}w), \tag{2.10}$$

and the proof for Case 1 is now completed.

Finally we show that f cannot map $C_{b,\theta}$ to $C_{a,\eta(\theta)}$. This will also complete the proof of our main theorem on the Barrett's domain.

Case 2. If f maps $C_{b,\theta}$ to $C_{a,\eta(\theta)}$ for some θ. Then again by continuity f will map $C_{b,\theta}$ to $C_{a,\eta(\theta)}$ for all $\theta \in [0, 2\pi]$. Consider the map

$$g(z, w): D \to D',$$
$$(z, w) \mapsto \left(f_1(z, w), \frac{ab}{f_2(z, w)}\right),$$

where D' is biholomorphic to D via the map $(z, w) \mapsto (z, \frac{ab}{w})$.

So one can repeat the above argument and obtain that

$$f_2(z, w) = \frac{ab}{w}e^{i\phi_0} \quad \text{for some constant } \phi_0 \in \mathbf{R}. \tag{2.11}$$

Since $5 \le ab \le 12$, in order to preserve the boundaries at two ends, we must have $ab = 6$. We may also conclude that

$$f_1(z, w) = z. \tag{2.12}$$

Next consider the point $p_0 = (\frac{1}{2}, 3)$. We see that $p_0 \in \Omega$. Since $f \in \mathrm{Aut}(\Omega)$, we must have $f(p_0) \in \Omega$. However, equations (2.11) and (2.12) show that $f(p_0) = (\frac{1}{2}, 2e^{i\phi_0})$, and this point is clearly not in Ω. This gives the desired contradiction.

Hence any automorphism on the Barrett's domain is given by a rotation in w-variable.

III. Proof on the Diederich-Fornaess domains

We first recall briefly the definition of the Diederich-Fornaess domain here. Fix a smooth function $\lambda: \mathbf{R} \to \mathbf{R}$ satisfying

(a) $\lambda(x) = 0$ if $x \leq 0$,
(b) $\lambda(x) > 1$ if $x > 1$,
(c) $\lambda''(x) \geq 100\lambda'(x)$ for all x,
(d) $\lambda''(x) > 0$ if $x > 0$,
(e) $\lambda'(x) > 100$ if $\lambda(x) > \frac{1}{2}$.

Then for any $r > 1$ we define $\Omega_r = \{(z, w) \in \mathbf{C}^2 | \rho_r(z, w) < 0\}$ where $\rho_r(z, w) = |z + e^{i \ln |w|^2}|^2 - 1 + \lambda(1/|w|^2 - 1) + \lambda(|w|^2 - r^2)$.

Theorem [8]. *Ω_r is a smoothly bounded pseudoconvex domain in $\mathbf{C}^2$. The boundary is strictly pseudoconvex everywhere except on the following annulus,*

$$M_r = \{(z, w) \in b\Omega_r | z = 0 \text{ and } 1 \leq |w| \leq r\}.$$

Now let $f = (f_1, f_2)$ be an automorphism of Ω_r. Then f can be extended smoothly up to the boundary on $b\Omega_r - M_r$. For instance, see Bell [5]. Therefore, if we consider the deleted torus

$$T_a = \{(z, w) \in b\Omega_r | 1 < |w| = a < r \text{ and } z \neq 0\},$$

we see that $\eta_r = \rho_r \circ f$ is a defining function for T_a, $1 < a < r$, namely, the equation $|f_1(z, w) + e^{i \ln |f_2(z, w)|^2}| = 1$ defines T_a. This implies that $|f_2(z, w)|^2 = |w|^2 \cdot e^{2k\pi}$, for some fixed integer k. Hence by considering the points $(z, w) \in T_a$ with a close to either 1 or r, we conclude that $k = 0$ and $|f_2(z, w)| = |w|$ for $(z, w) \in T_a$ with $1 < a < r$.

Next fix the constant a with $1 < a < r$, and a point z_0 with $|z_0 + e^{i \ln |a|^2}| < 1$ such that z_0 lies in a small open neighborhood of $-2e^{i \ln |a|^2}$. Then we consider the annulus defined by

$$A_{z_0} = \{(z_0, w) \in \mathbf{C}^2\} \cap \Omega_r$$

with the inner boundary $C_\alpha = \{(z_0, w) \in b\Omega_r | |w| = \alpha\}$ and the outer boundary C_β such that $\alpha < a < \beta$. A_{z_0} can be identified with $A = \{w \in \mathbf{C} | \alpha < |w| < \beta\}$. Hence via this identification we obtain that

$$f_2(z_0, C_\alpha) = C_\alpha \quad \text{and} \quad f_2(z_0, C_\beta) = C_\beta, \tag{3.1}$$

and $f_2(z_0, \cdot)$ can be extended to an entire function by reflection principle. Then by (3.1) we must have that

$$f_2(z, w) = e^{i\phi(z)} \cdot w$$

for some real-valued function $\phi(z)$. Since $f_2(z, w)$ is also holomorphic in z, we conclude that $\phi(z) = \phi_0$ for some constant $\phi_0 \in \mathbf{R}$, and

$$f_2(z, w) = e^{i\phi_0} \cdot w \quad \text{for } (z, w) \in \Omega_r. \tag{3.2}$$

Then we consider the open solid torus π_a defined by

$$\pi_a = \{(z, w) \in \Omega_r | 1 < |w| = a < r \text{ and } |z + e^{i \ln |w|^2}| < 1\}.$$

Put

$$\Delta_{a,\phi} = \{(z, ae^{i\phi}) \in \Omega_r \mid |z + e^{i\ln|a|^2}| < 1\}.$$

It follows that the restriction of f_1 to Δ_{a,ϕ_1} must map Δ_{a,ϕ_1} biholomorphically onto Δ_{a,ϕ_2} for some ϕ_2. This implies that the restriction of f_1 to Δ_{a,ϕ_1} can be extended at least smoothly up to $\overline{\Delta_{a,\phi_1}}$. Since $f_1(0, ae^{i\phi_1}) = 0$, it follows that $f_1(z, w)$ can be expressed via the automorphisms on the unit disk as

(3.3)
$$f_1(z, w) = e^{i\ln|w|^2}\left(\frac{1-\overline{b(w)}e^{i\ln|w|^2}}{e^{i\ln|w|^2}-b(w)}\right)\left(\frac{z+e^{i\ln|w|^2}-b(w)}{1-\overline{b(w)}(z+e^{i\ln|w|^2})}\right) - e^{i\ln|w|^2},$$

for some real analytic function $b(w)$ satisfying $|b(w)| < 1$ for $1 < |w| < r$. Equation (3.3) shows that there exists a small number $\varepsilon > 0$ such that $f_1(z, w)$ is real analytic on $\Delta(0; \varepsilon) \times A_\delta$, where $A_\delta = \{w \in \mathbf{C} \mid 1 + \delta < |w| < r - \delta$ for some small $\delta > 0\}$. This in turn implies that $f_1(z, w)$ is holomorphic on $\Delta(0; \varepsilon) \times A_\delta$. Therefore, one can write

$$f_1(z, w) = \sum_{k=1}^{\infty} a_k(w) z^k,$$

with $a_k(w) \in H(A_\delta)$ for all $k \geq 1$. By direct computation we get

$$a_1(w) = \frac{\partial f_1}{\partial z}(0, w) = \frac{1-|b(w)|^2}{|1-\overline{b(w)}e^{i\ln|w|^2}|^2}.$$

It shows that $a_1(w)$ is a positive real constant, i.e., $a_1(w) = c > 0$. Next the computation of $a_2(w)$ shows that

(3.4)
$$a_2(w) = \frac{1}{2}\frac{\partial^2 f_1}{\partial z^2}(0, w) = c \cdot \frac{\overline{b(w)}}{1-\overline{b(w)}e^{i\ln|w|^2}}.$$

We claim that $a_2(w) \equiv 0$. Set

$$g(w) = \frac{a_2(w)}{c} = \frac{\overline{b(w)}}{1-\overline{b(w)}e^{i\ln|w|^2}} \in H(A_\delta),$$

we have

$$\begin{aligned} c &= \frac{1-|b(w)|^2}{|1-\overline{b(w)}e^{i\ln|w|^2}|^2} = |1+g(w)e^{i\ln|w|^2}|^2 - |g(w)|^2 \\ &= 1 + 2\,\mathrm{Re}\,(g(w)e^{i\ln|w|^2}). \end{aligned}$$

Therefore, one can write

$$g(w)e^{i\ln|w|^2} = c_0 + iI(w),$$

with $c_0 = \frac{1}{2}(c-1)$ and $I(w)$ is a smooth real-valued function on A_δ. Hence we obtain

(3.5)
$$g(w) = c_0 e^{-i\ln|w|^2} + iI(w)e^{-i\ln|w|^2} \in H(A_\delta).$$

Locally one can multiply equation (3.5) by $e^{2i\ln w}$ to get a new well-defined holomorphic function, and get

(3.6)
$$g(w)e^{2i\ln w} = c_0 e^{-2\operatorname{Arg} w} + iI(w)e^{-2\operatorname{Arg} w}.$$

The real part of $g(w)e^{2i\ln w}$ is a harmonic function. So let $w = u + iv$, by direct computation we get

$$\Delta_w(c_0 e^{-2\operatorname{Arg} w}) = c_0\Delta_w(e^{-2\tan^{-1} v/u}) = \frac{4c_0}{u^2+v^2}e^{-2\tan^{-1} v/u} \equiv 0.$$

It follows that $c_0 = 0$, and hence $c = 1$. This reduces (3.5) to

$$-ig(w) = I(w)e^{-i\ln|w|^2} \in H(A_\delta). \tag{3.7}$$

Then repeat the same argument, we see that

$$-ig(w)e^{2i\ln w} = I(w)e^{-2\operatorname{Arg} w} = c_1,$$

where c_1 is a global constant. Hence

$$I(w) = c_1 e^{2\operatorname{Arg} w}$$

is a well-defined function on A_δ. It forces $c_1 = 0$. This shows $g(w) \equiv 0$, and the proof of our claim is now completed.

It follows then from (3.4) that we have $b(w) \equiv 0$ on A_δ, and equation (3.3) can be simplified to

$$f_1(z, w) = z \quad \text{on } \Omega_r. \tag{3.8}$$

Our main theorem now follows from (3.2) and (3.8). So we are done.

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