TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 338, Number 2, August 1993

EXTENDING THE t-DESIGN CONCEPT

A. R. CALDERBANK AND P. DELSARTE

ABSTRACT. Let $\mathfrak{B}$ be a family of k-subsets of a v-set V, with $1 \le k \le v/2$. Given only the inner distribution of $\mathfrak{B}$, i.e., the number of pairs of blocks that meet in j points (with $j = 0, 1, \ldots, k$), we are able to completely describe the regularity with which $\mathfrak{B}$ meets an arbitrary t-subset of V, for each order t (with $1 \le t \le v/2$). This description makes use of a linear transform based on a system of dual Hahn polynomials with parameters v, k, t. The main regularity parameter is the dimension of a well-defined subspace of $\mathbb{R}^{t+1}$, called the t-form space of $\mathfrak{B}$. (This subspace coincides with $\mathbb{R}^{t+1}$ if and only if $\mathfrak{B}$ is a t-design.) We show that the t-form space has the structure of an ideal, and we explain how to compute its canonical generator.

1. INTRODUCTION

Consider a set (a *design*) $\mathfrak{B}$ whose elements, called blocks, are k-subsets of a given v-set V of points, with $1 \le k \le v/2$. For a given integer t, in the range $0 \le t \le k$, we are interested in some regularity properties of $\mathfrak{B}$, of order t, which can be defined as follows. With any t-subset x^t of V and any integer $i \in [0, t]$ we associate the number $D_i(x^t)$ that counts the blocks in $\mathfrak{B}$ meeting x^t in $t - i$ points. Suppose that we have a linear relation

$$f_0 D_0(x^t) + f_1 D_1(x^t) + \cdots + f_t D_t(x^t) = c \ ,$$

where f_0, $f_1, \ldots, f_t$ and c are fixed real numbers. Then we say that the $(t+1)$-tuple $(f_i)_{i=0}^t$ is a t-form for $\mathfrak{B}$. The set of t-forms clearly is a vector space, which will be called the *t-form space* of $\mathfrak{B}$. We propose to take the dimension of that space as a measure of the regularity of $\mathfrak{B}$ with respect to the t-subsets of V.

The idea originates mainly from a recent paper [2] in which linear relations as above, with the restrictive property $f_1 \neq 0$, $f_2 = \cdots = f_t = 0$, are deduced from a strengthening of the Assmus-Mattson theorem in coding theory. This investigation has been continued in a companion paper [1]. Recall that $\mathfrak{B}$ is said to be a t-design, of index c, when $D_0(x^t) = c$ for all t-subsets x^t of V. (For a thorough algebraic study, see especially [13].) This implies that every $(t+1)$-tuple $(f_i)_{i=0}^t$ is a t-form for $\mathfrak{B}$. So, our definition of regularity can be viewed as an extension of the classical t-design concept.

Received by the editors November 3, 1989 and, in revised form, May 20, 1991.
1991 *Mathematics Subject Classification*. Primary 05B20, 05B30; Secondary 33A63, 20C30.
Key words and phrases. t-design, t-form space, t-distribution matrix, dual Hahn polynomials.

We shall see how the theory of t-form spaces provides us with a satisfactory "combinatorial interpretation" of the algebraic notion of a *T-design* [3, 4], with T a subset of $[0, t]$. One of the main results of this paper is an explicit description of the t-form space in terms of the *inner distribution* of $\mathfrak{B}$. In fact, this space is proved to be an ideal of a well-defined $(t+1)$-dimensional commutative algebra, and to admit a canonical generator which can be computed from the inner distribution. Furthermore, the theory gives us an efficient method to determine the *t-distribution matrix* of $\mathfrak{B}$, having the numbers $D_i(x^t)$ as its entries, from a minimum set of data. The methods of proof are based on a suitable matrix transform, involving a system of *dual Hahn polynomials* (with parameters v, k, t) [6, 7, 10]. Some combinatorial applications of the Hahn polynomials (quite different from those considered in what follows) can be found in [4, 12, and 13].

For several reasons, it is useful to replace the "combinatorial" notion of a design $\mathfrak{B}$ by the more general "algebraic" notion of a *design vector* $\boldsymbol{\beta}$, which is defined simply as a real row vector, of size $\binom{v}{k}$, with components labelled by the k-subsets of V (see [4], for example). The concept of the t-form space can be extended naturally to this general definition of a design vector. (The situation described above corresponds to the special case where $\boldsymbol{\beta}$ is the characteristic vector of a combinatorial design $\mathfrak{B}$.) All our theorems are valid for general design vectors. Note that the theory would be essentially the same if we would consider only rational vectors $\boldsymbol{\beta}$ or, equivalently, integer vectors $\boldsymbol{\beta}$, i.e., *signed designs* [8, 9, 11].

2. t-FORMS AND THE t-DISTRIBUTION MATRIX

Let V be a v-set (i.e., a set of finite cardinality v), and let $\mathscr{P}^i$ be the set of i-subsets of V, for $i = 0, 1, \ldots, v$. Given an integer k, with $0 \le k \le \lfloor v/2 \rfloor$, we shall be interested in the real $\binom{v}{k}$-dimensional vector space $\mathbb{R}\mathscr{P}^k$, with entries indexed by the elements x^k of $\mathscr{P}^k$. (In this paper, we always use **row** vectors.) Consider a vector $\boldsymbol{\beta}$ in $\mathbb{R}\mathscr{P}^k$. If $\beta(x^k) = 0$ or 1 for every x^k, then we can identify $\boldsymbol{\beta}$ with the set $\mathfrak{B} = \{x^k \in \mathscr{P}^k : \beta(x^k) = 1\}$ that has $\boldsymbol{\beta}$ as its characteristic vector. Such a subset $\mathfrak{B}$ of $\mathscr{P}^k$ is a "design" for the Johnson scheme $J(v, k)$; the elements of $\mathfrak{B}$ are often called blocks. By extension, we shall say that any vector $\boldsymbol{\beta}$ in $\mathbb{R}\mathscr{P}^k$ is a *design vector* for $J(v, k)$.

Let t be a fixed integer, with $t \in [0, k]$. We shall express the "regularity properties of order t" of a given design vector $\boldsymbol{\beta}$ through the notions of the t-distribution matrix and the t-form space.

Definition 2.1. For $i \in [0, t]$, the ith *adjacency matrix* of $\mathscr{P}^k$ with respect to $\mathscr{P}^t$ is the $\{0, 1\}$ matrix $C_i^{k,t}$, of size $\binom{v}{k} \times \binom{v}{t}$, with rows indexed by $x^k \in \mathscr{P}^k$ and columns indexed by $x^t \in \mathscr{P}^t$, defined by

$$C_i^{k,t}(x^k, x^t) = 1 \quad \text{if and only if} \quad |x^k \cap x^t| = t - i .$$

The *t-distribution matrix* of the design vector $\boldsymbol{\beta} \in \mathbb{R}\mathscr{P}^k$ is the real matrix D^t, of size $(t+1) \times \binom{v}{t}$, with ith row

$$D_i^t = \boldsymbol{\beta} C_i^{k,t}, \qquad \text{for } i \in [0, t] .$$

The combinatorial meaning of this definition is the following. If $\boldsymbol{\beta}$ is the characteristic vector of a subset $\mathfrak{B}$ of $\mathscr{P}^k$, then the (i, x^t) entry $D_i^t(x^t)$ of the t-distribution matrix of $\boldsymbol{\beta}$ *counts the blocks in* $\mathfrak{B}$ *that have* $t-i$ *points in common with* x^t. (Note that the k-distribution matrix D^k coincides with the "outer distribution matrix" in [3].)

Definition 2.2. A vector $\mathbf{f}^t = (f_0^t, f_1^t, \ldots, f_t^t)$ in $\mathbb{R}^{t+1}$ is a *t-form* for the design vector $\boldsymbol{\beta} \in \mathbb{R}\mathscr{P}^k$ if and only if it satisfies

$$\mathbf{f}^t D^t = c\mathbf{1}, \text{ with } c \in \mathbb{R},$$

where $\mathbf{1}$ denotes the all-one vector. The subspace of $\mathbb{R}^{t+1}$ formed by all the t-forms for $\boldsymbol{\beta}$ will be denoted by $\mathscr{F}^t$ and will be referred to as the *t-form space* of $\boldsymbol{\beta}$. Finally, the *t-degree* of $\boldsymbol{\beta}$, denoted by r^t (or simply by r), is defined as the codimension of the t-form space, that is,

$$r^t = r = t + 1 - \dim(\mathscr{F}^t).$$

By definition, $\mathbf{f}^t$ is a t-form iff the linear combination $\sum_{i=0}^t f_i^t D_i^t(x^t)$ is constant over the set $\mathscr{P}^t$. It seems therefore natural to consider that a design vector $\boldsymbol{\beta}$ is highly regular with respect to $\mathscr{P}^t$ when it admits a large number of linearly independent t-forms, i.e., when its t-degree r^t is small. One may say that r^t is a measure of the "t-irregularity" of $\boldsymbol{\beta}$. The most regular designs $\mathfrak{B}$ are the *t-designs* (in the usual sense), defined by the fact that $D_0^t(x^t)$ is a constant. This implies that $D_i^t(x^t)$ is a constant, for each $i \in [0, t]$. Therefore, a t-design $\mathfrak{B}$ is characterized by the fact that its t-form space coincides with $\mathbb{R}^{t+1}$, i.e., by the property $r^t = 0$.

Let us now emphasize the simple relationship between the t-degree r^t and the rank of D^t. (In what follows, we denote by $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle$ the inner product $\boldsymbol{\xi}\boldsymbol{\eta}^T$ of two vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in $\mathbb{R}\mathscr{P}^k$.) We need a preliminary result.

Lemma 2.1. *The constant* c *in the equation* $\mathbf{f}^t D^t = c\mathbf{1}$ *is determined from* $\mathbf{f}^t$ *and* $\langle \boldsymbol{\beta}, \mathbf{1} \rangle$ *by*

$$c\binom{v}{t} = \langle \boldsymbol{\beta}, \mathbf{1} \rangle \sum_{i=0}^{t} f_i^t \binom{k}{t-i}\binom{v-k}{i}.$$

Proof. We have $D_i^t \mathbf{1}^T = \langle \boldsymbol{\beta}, \mathbf{1} \rangle \binom{k}{t-i}\binom{v-k}{i}$, by simple counting of t-sets relative to a fixed k-set. Using this identity, we prove the lemma by computing the inner product of both sides of $\mathbf{f}^t D^t = c\mathbf{1}$ with $\mathbf{1}$. □

Proposition 2.1. *The t-form space* $\mathscr{F}^t$ *of any design vector* $\boldsymbol{\beta}$ *contains the all-one vector* $\mathbf{1}$, *and the t-distribution matrix* D^t *of* $\boldsymbol{\beta}$ *has rank* r^t *or* $r^t + 1$ *according as* $\langle \boldsymbol{\beta}, \mathbf{1} \rangle = 0$ *or* $\langle \boldsymbol{\beta}, \mathbf{1} \rangle \neq 0$.

Proof. The first part is obvious, for $\mathbf{1}D^t = \langle \boldsymbol{\beta}, \mathbf{1} \rangle \mathbf{1}$. The second part is proved as follows. If $\langle \boldsymbol{\beta}, \mathbf{1} \rangle = 0$, then any t-form equation $\mathbf{f}^t D^t = c\mathbf{1}$ must satisfy $c = 0$, by Lemma 2.1, which means that $\mathscr{F}^t$ equals the orthogonal complement of the column space of D^t. On the other hand, if $\langle \boldsymbol{\beta}, \mathbf{1} \rangle \neq 0$, then we see that the t-form $\mathbf{1}$ is not orthogonal to the column space of D^t. □

Note that the distinction between the cases $\langle \boldsymbol{\beta}, \mathbf{1} \rangle = 0$ and $\langle \boldsymbol{\beta}, \mathbf{1} \rangle \neq 0$ is inconsequential, since replacing $\boldsymbol{\beta}$ by $\boldsymbol{\beta} + a\mathbf{1}$, with any $a \in \mathbb{R}$, does not affect the t-form space.

3. Harmonic analysis and dual Hahn polynomials

To determine the t-form space of a design vector we shall use a transform method involving dual Hahn polynomials. We need some preliminaries about the harmonic analysis of the symmetric group S_v (see [5, 6, 7]). Under the natural action of S_v, the space $\mathbb{R}\mathscr{P}^k$ decomposes as the orthogonal sum of $k+1$ irreducible S_v-invariant subspaces, called the harmonic subspaces of $\mathbb{R}\mathscr{P}^k$. A "constructive definition" is given below.

Definition 3.1. For $s \in [0, k]$, the *homogeneous subspace* of degree s of $\mathbb{R}\mathscr{P}^k$, denoted by Hom_s^k, is defined as the row space of $(C_0^{k,s})^T$ (which is the characteristic matrix of the inclusion relation between s-subsets and k-subsets of V). The *harmonic subspace* of degree s of $\mathbb{R}\mathscr{P}^k$ is denoted by Harm_s^k and is defined as the orthogonal complement of Hom_{s-1}^k in Hom_s^k.

The latter definition makes sense since Hom_{s-1}^k is a subspace of Hom_s^k, as a consequence of $C_0^{k,s}C_0^{s,s-1} = (k-s+1)C_0^{k,s-1}$. Note the properties $\mathrm{Hom}_k^k = \mathbb{R}\mathscr{P}^k$ and $\mathrm{Hom}_0^0 = \mathrm{Harm}_0^0 = \mathbb{R}\mathbf{1}$. As the columns of $C_0^{k,s}$ are linearly independent, Hom_s^k has dimension $\binom{v}{s}$ and Harm_s^k has dimension $\binom{v}{s} - \binom{v}{s-1}$.

Definition 3.2. Let $\mathscr{A}^{k,t}$ be the real $(t+1)$-dimensional vector space generated by the adjacency matrices $C_i^{k,t}$ with $i \in [0, t]$. In particular, $\mathscr{A}^{k,k}$ is the *Bose-Mesner algebra* of the Johnson scheme $J(v, k)$. For $s \in [0, k]$, let E_s^k be the orthogonal projection matrix from $\mathbb{R}\mathscr{P}^k$ onto Harm_s^k. The matrices E_s^k are the *irreducible idempotents* of $\mathscr{A}^{k,k}$.

Let us now give two useful results concerning, first, the relation between the spaces $\mathscr{A}^{k,t}$ and the algebra $\mathscr{A}^{k,k}$ and, next, the relation between the eigenspaces of both algebras $\mathscr{A}^{k,k}$ and $\mathscr{A}^{t,t}$.

Lemma 3.1. *The space $\mathscr{A}^{k,t}$ is generated by the $t+1$ matrices $E_s^k C_0^{k,t}$ with $s \in [0, t]$.*

Proof. A straightforward argument shows that if X belongs to $\mathscr{A}^{k,k}$, then $XC_0^{k,t}$ belongs to $\mathscr{A}^{k,t}$. It remains to show that the matrices $E_s^k C_0^{k,t}$ are linearly independent. This follows from the well-known fact (easily deduced from the information above) that $C_0^{k,t}(C_0^{k,t})^T$ is a linear combination of E_0^k, $E_1^k, \ldots, E_t^k$ with positive coefficients. More precisely, as shown in [3], the coefficient of E_i^k is $\binom{k-i}{k-t}\binom{v-t-i}{k-t}$ for $i = 0, 1, \ldots, t$. □

Lemma 3.2. *For $s \in [0, t]$, the harmonic subspaces of degree s of $\mathbb{R}\mathscr{P}^k$ and $\mathbb{R}\mathscr{P}^t$ are related by*

$$\mathrm{Harm}_s^k = \mathrm{Harm}_s^t (C_0^{k,t})^T, \quad \mathrm{Harm}_s^t = \mathrm{Harm}_s^k C_0^{k,t}.$$

Hence, the irreducible idempotents of the Bose-Mesner algebras $\mathscr{A}^{k,k}$ and $\mathscr{A}^{t,t}$ are related by $C_0^{k,t}E_s^t = E_s^k C_0^{k,t}$.

Proof. We start from $C_0^{k,t}C_0^{t,s} = \binom{k-s}{t-s}C_0^{k,s}$. This shows that the first statement of the lemma is satisfied when Harm is replaced by Hom (in view of the fact that $C_0^{k,t}(C_0^{k,t})^T$ preserves Hom_s^k). The statement about harmonic spaces can then be deduced (by use of Definition 3.1) from the fact that Harm_s^k is an eigenspace of $C_0^{k,t}(C_0^{k,t})^T$, with a positive eigenvalue. The second statement readily follows from the first one. Further details are omitted. □

Next, we introduce a three-parameter family of dual Hahn polynomials [10] (with a normalization adapted to our subject), and recall some of their well-known properties.

Definition 3.3. For $i \in [0, t]$, the *dual Hahn polynomial* $Q_i^{k,t}(z)$, with $0 \le t \le k \le \lfloor v/2 \rfloor$, is defined by

$$\binom{k}{t} Q_i^{k,t}(z) = \binom{k}{t-i}\binom{v-k}{i}\sum_{l=0}^{i}\frac{(-i)_l(-z)_l(z-v-1)_l}{(1)_l(-t)_l(k-v)_l},$$

where $(a)_l = a(a+1)\cdots(a+l-1)$. Thus, $Q_i^{k,t}(z)$ has degree $2i$ in z and degree i in the variable $\lambda^t(z) = Q_1^{t,t}(z) = t(v-t) - z(v+1-z)$.

Theorem 3.1. *The dual Hahn polynomials satisfy the following orthogonality relation and three-term recurrence relation:*

$$\sum_{s=0}^{t}\frac{v-2s+1}{v-s+1}\binom{v}{s}\binom{k-s}{k-t}\binom{v-t-s}{k-t}Q_i^{k,t}(s)Q_j^{k,t}(s) = \binom{v}{k}\binom{k}{t-i}\binom{v-k}{i}\delta_{i,j},$$

$$\lambda^t(z)Q_i^{k,t}(z) = \sum_{j=i-1}^{i+1} w_{i,j}^{k,t}Q_j^{k,t}(z),$$

with $w_{i,i-1}^{k,t} = (t+1-i)(v+1-k-i)$, $w_{i,i}^{k,t} = (t-i)(k-t+i) + i(v-k-i)$, $w_{i,i+1}^{k,t} = (i+1)(k+1-t+i)$, *and* $\lambda^t(z) = t(v-t) - z(v+1-z)$.

The fundamental reason why the dual Hahn polynomials occur in our investigation can be explained in a group theoretic setting by the fact that these polynomials give the spherical and intertwining functions involved in the harmonic analysis relative to the symmetric group [6, 7]. Not surprisingly, the *spherical case,* $t = k$, plays a special role in our theory (see [5]). The following theorem shows, in explicit terms, how the dual Hahn polynomials $Q_i^{k,t}(z)$ appear on the scene.

Theorem 3.2. *The adjacency matrices $C_i^{k,t}$ are related to the irreducible idempotents E_s^k of the Bose-Mesner algebra of $J(v, k)$ by*

$$C_i^{k,t} = \left(\sum_{s=0}^{t} Q_i^{k,t}(s)E_s^k\right) C_0^{k,t}, \quad \text{for } i \in [0, t].$$

Proof. In view of Lemma 3.1, we may write the relation above if we replace $Q_i^{k,t}(s)$ by an unknown real number $X_i^{k,t}(s)$. To prove the desired coincidence, we start from the matrix identity

$$C_i^{k,t}C_1^{t,t} = \sum_{j=0}^{t} w_{i,j}^{k,t} C_j^{k,t} .$$

Here, $w_{i,j}^{k,t}$ counts the elements y^t of $\mathscr{P}^t$ that satisfy $|x^k \cap y^t| = t - i$ and $|y^t \cap x^t| = t - 1$, for a given pair $(x^k, x^t) \in \mathscr{P}^k \times \mathscr{P}^t$ with $|x^k \cap x^t| = t - j$. In particular, $w_{i,j}^{k,t} = 0$ if $|i - j| \geq 2$. A counting argument shows that the intersection numbers $w_{i,j}^{k,t}$ thus defined (for $|i - j| \leq 1$) coincide with the recurrence parameters for the dual Hahn polynomials (see Theorem 3.1). Post-multiplying the identity above by E_s^t (with $s \in [0, t]$) and using the last statement in Lemma 3.2, together with the well-known formula $C_1^{t,t}E_s^t = \lambda^t(s)E_s^t$, we readily obtain the three-term relation

$$\lambda^t(s)X_i^{k,t}(s) = \sum_{j=i-1}^{i+1} w_{i,j}^{k,t} X_j^{k,t}(s) .$$

This uniquely determines the unknowns $X_i^{k,t}(s)$, from the initial values $X_0^{k,t}(s) = 1$ and $X_{-1}^{k,t}(s) = 0$. A comparison with the second part of Theorem 3.1 yields $X_i^{k,t}(s) = Q_i^{k,t}(s)$ as was to be proved. □

Definition 3.4. Let $Q^{k,t}$ be the rational square matrix of order $t + 1$ with (i, s) entry $Q_i^{k,t}(s)$ for $i, s \in [0, t]$. Given any vector $\mathbf{f}^t = (f_0^t, f_1^t, \ldots, f_t^t)$ in $\mathbb{R}^{t+1}$, the *dual Hahn transform* of $\mathbf{f}^t$, of *type* (k, t), is defined as the vector $\hat{\mathbf{f}}^t = \mathbf{f}^t Q^{k,t}$. In other terms, $\hat{\mathbf{f}}^t = (\hat{f}^t(0), \hat{f}^t(1), \ldots, \hat{f}^t(t))$ where $\hat{f}^t(s)$ is the value at point s of the *dual Hahn transform polynomial*

$$\hat{f}^t(z) = \sum_{i=0}^{t} f_i^t Q_i^{k,t}(z) .$$

4. $\boldsymbol{\beta}$-REGULARITY AND T-DESIGNS

This section contains some further results about T-designs, introduced in [3]. In particular, we shall discover a close connection between the T-design concept and the t-form concept.

Definition 4.1. Given a design vector $\boldsymbol{\beta}$ in $\mathbb{R}\mathscr{P}^k$, a subspace Φ of $\mathbb{R}\mathscr{P}^k$ is $\boldsymbol{\beta}$*-regular* if and only if

$$\langle \boldsymbol{\beta}, \phi \rangle = \frac{\langle \boldsymbol{\beta}, \mathbf{1} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle} \langle \mathbf{1}, \phi \rangle \quad \text{for all } \phi \in \Phi .$$

Let T be a subset of the integer interval $[0, k]$. A vector $\boldsymbol{\beta} \in \mathbb{R}\mathscr{P}^k$ is a *T-design* vector for the Johnson scheme $J(v, k)$ if and only if

$$\langle \boldsymbol{\beta}, \phi \rangle = 0 \quad \text{for all } \phi \in \mathrm{Harm}_s^k \text{ with } 0 \neq s \in T .$$

Let Φ be an S_v-invariant subspace of $\mathbb{R}\mathscr{P}^k$. Then Φ decomposes as an orthogonal sum $\Phi = \bigoplus_{s\in T} \mathrm{Harm}_s^k$ for a well-determined subset T of $[0, k]$. This set T will be referred to as the *harmonic support* of Φ.

The $\boldsymbol{\beta}$-regularity property means that $\boldsymbol{\beta}$ behaves like the all-one vector $\mathbf{1}$ with respect to the considered "test space" Φ. When $\boldsymbol{\beta}$ is the characteristic vector of a subset $\mathfrak{B}$ of $\mathbb{R}\mathscr{P}^k$, our $\boldsymbol{\beta}$-regularity concept reduces to the $\mathfrak{B}$-regularity introduced in [2]. Note that $\mathfrak{B}$ is a t-design if and only if the space Hom_t^k is $\mathfrak{B}$-regular.

Let us now introduce the important notion of the inner distribution [3, 4], with a suitable normalization.

Definition 4.2. The *inner distribution* of a design vector $\boldsymbol{\beta} \in \mathbb{R}\mathscr{P}^k$ is the $(k+1)$-vector $\mathbf{b}^k = (b_0^k, b_1^k, \dots, b_k^k)$ with

$$\binom{k}{i}\binom{v-k}{i} b_i^k = \langle \boldsymbol{\beta}, C_i^{k,k} \boldsymbol{\beta} \rangle .$$

(If $\boldsymbol{\beta}$ is the characteristic vector of a set $\mathfrak{B}$, then the right-hand side counts the pairs of blocks in $\mathfrak{B}$ that have $k-i$ points in common.)

Consider the dual Hahn transform $\hat{\mathbf{b}}^k = \mathbf{b}^k Q^{k,k}$, of spherical type (k, k), of the inner distribution $\mathbf{b}^k$ (see Definition 3.4). By specializing Theorem 3.2 to the spherical case $(C_0^{k,k} = I)$, and by using the orthogonality relation in Theorem 3.1, we obtain the formula

$$\frac{v-2s+1}{v-s+1}\binom{v}{s}\hat{b}^k(s) = \binom{v}{k}\langle \boldsymbol{\beta}, E_s^k \boldsymbol{\beta} \rangle .$$

We now are in a position to give two characterizations of the T-design property, in terms of the inner distribution [3] and in terms of the $\boldsymbol{\beta}$-regularity concept [2], respectively.

Theorem 4.1. (i) *The vector $\boldsymbol{\beta}$ is a T-design vector if and only if the dual Hahn transform of its inner distribution satisfies $\hat{b}^k(s) = 0$ for all $s \in T \setminus \{0\}$.* (ii) *The vector $\boldsymbol{\beta}$ is a T-design vector if and only if the S_v-invariant subspace of $\mathbb{R}\mathscr{P}^k$ with harmonic support T is $\boldsymbol{\beta}$-regular.*

Proof. Note first that $\langle \boldsymbol{\beta}, E_s^k \boldsymbol{\beta} \rangle = ||E_s^k \boldsymbol{\beta}||^2$, with $E_s^k \boldsymbol{\beta}$ the orthogonal projection of $\boldsymbol{\beta}$ in Harm_s^k. Hence the formula above yields $\hat{b}^k(s) \geq 0$, for all $s \in [0, k]$, and $\hat{b}^k(s) = 0$ if and only if $\boldsymbol{\beta}$ is orthogonal to Harm_s^k. This proves the first statement. The second part is immediate (from Definition 4.1), since $\langle \mathbf{1}, \phi \rangle = 0$ for every $\phi \in \mathrm{Harm}_s^k$ with $s \geq 1$. □

We now point out an interesting application of Lemma 3.2 concerning the relationship between T-design vectors for two different Johnson schemes (with the same point set).

Proposition 4.1. *If $\boldsymbol{\beta}$ is a T-design vector for $J(v, k)$, then $\boldsymbol{\beta} C_0^{k,t}$ is a $(T \cap [0, t])$-design vector for $J(v, t)$. Conversely, if γ is a T-design vector for $J(v, t)$, then $\gamma (C_0^{k,t})^T$ is a $(T \cup [t+1, k])$-design vector for $J(v, k)$.*

We now come back to the t-form concept (Definition 2.2). Let $\mathbf{f}^t \in \mathbb{R}^{t+1}$. With each t-subset x^t of V we associate the vector $\boldsymbol{\xi}^t$ (in $\mathbb{R}\mathscr{P}^k$) defined by

$$\binom{v-t}{k-t}\boldsymbol{\xi}^t = \sum_{i=0}^{t} f_i^t (C_i^{k,t}(\cdot, x^t))^T ,$$

where $C_i^{k,t}(\cdot, x^t)$ denotes the x^t-column of the adjacency matrix $C_i^{k,t}$. It follows from Lemma 3.1 that $\boldsymbol{\xi}^t$ belongs to the homogeneous space Hom_t^k of degree t. Let $\Phi(\mathbf{f}^t)$ be the subspace of Hom_t^k generated by the vectors $\boldsymbol{\xi}^t$ with $x^t \in \mathscr{P}^t$. It is obvious that $\Phi(\mathbf{f}^t)$ is S_v-invariant.

By definition, $\mathbf{f}^t$ is a t-form for $\boldsymbol{\beta}$ if and only if the inner product $\langle \boldsymbol{\beta}, \boldsymbol{\xi}^t\rangle$ is independent of the choice of x^t. This means that $\Phi(\mathbf{f}^t)$ is a $\boldsymbol{\beta}$-regular space (see Definition 4.1). It follows from Theorem 4.1 that $\mathbf{f}^t$ is a t-form for $\boldsymbol{\beta}$ if and only if $\boldsymbol{\beta}$ is an $S(\mathbf{f}^t)$-design vector, where $S(\mathbf{f}^t)$ denotes the harmonic support of $\Phi(\mathbf{f}^t)$.

To identify this set $S(\mathbf{f}^t)$ we shall obtain another "T-design characterization" of the t-form property. We argue as follows. From a given $\boldsymbol{\beta}$ we construct the vectors $\gamma_s^t = \boldsymbol{\beta} E_s^k C_0^{k,t}$ in $\mathbb{R}\mathscr{P}^t$, for $s \in [0, t]$. We have $\boldsymbol{\beta} E_s^k \in \mathrm{Harm}_s^k$, whence $\gamma_s^t \in \mathrm{Harm}_s^t$ by Lemma 3.2. Since $D_i^t = \sum_{s=0}^t Q_i^{k,t}(s)\gamma_s^t$, by Theorem 3.2, we observe that $\mathbf{f}^t$ is a t-form for $\boldsymbol{\beta}$ if and only if its dual Hahn transform $\hat{\mathbf{f}}^t = \mathbf{f}^t Q^{k,t}$ satisfies the equation $\sum_{s=0}^t \hat{f}^t(s)\gamma_s^t = c\mathbf{1}$, with $c \in \mathbb{R}$. This amounts to $\hat{f}^t(s)\gamma_s^t = \mathbf{0}$ for each $s \in [1, t]$. Since $\gamma_s^t = \mathbf{0}$ if and only if $\boldsymbol{\beta} E_s^k = \mathbf{0}$, this means that $\boldsymbol{\beta}$ is orthogonal to Harm_s^k for every s in $[1, t]$ that satisfies $\hat{f}^t(s) \neq 0$. As a conclusion we obtain the following result.

Theorem 4.2. *For any vector* $\mathbf{f}^t \in \mathbb{R}^{t+1}$, *let* $T(\mathbf{f}^t)$ *be the set containing the indices of the nonzero components of its dual Hahn transform* $\hat{\mathbf{f}}^t = \mathbf{f}^t Q^{k,t}$, *that is,*

$$T(\mathbf{f}^t) := \{s \in [0, t]\, \hat{f}^t(s) \neq 0\} .$$

(i) *The set* $T(\mathbf{f}^t)$ *equals the harmonic support* $S(\mathbf{f}^t)$ *of the* S_v*-invariant subspace* $\Phi(\mathbf{f}^t)$ *of* Hom_t^k *generated by the vectors* $\boldsymbol{\xi}^t$ *with* $x^t \in \mathscr{P}^t$. (ii) *The vector* $\mathbf{f}^t$ *is a t-form for a given* $\boldsymbol{\beta}$ *in* $\mathbb{R}\mathscr{P}^k$ *if and only if* $\boldsymbol{\beta}$ *is a* $T(\mathbf{f}^t)$*-design vector or, equivalently, if and only if the space* $\Phi(\mathbf{f}^t)$ *is* $\boldsymbol{\beta}$*-regular.*

Proof. The essential part of the argument was given above. It remains only to show that we have $0 \in S(\mathbf{f}^t)$ if and only if $0 \in T(\mathbf{f}^t)$. This follows from the identity $\langle \mathbf{1}, \boldsymbol{\xi}^t\rangle = \hat{f}^t(0)$, which is easily proved from the definition of $\boldsymbol{\xi}^t$. □

5. Structure of the t-form space

In this section, it is shown how the t-form space of a design vector $\boldsymbol{\beta}$ can be characterized in terms of the inner distribution $\mathbf{b}^k$ of $\boldsymbol{\beta}$ (Theorems 5.1 and 5.3), and how the t-distribution matrix of $\boldsymbol{\beta}$ can be determined from its first r rows, with r denoting the t-degree of $\boldsymbol{\beta}$ (Theorem 5.2).

Definition 5.1. The *t-annihilator set* A^t for $\boldsymbol{\beta}$ is defined from the dual Hahn transform $\hat{\mathbf{b}}^k = \mathbf{b}^k Q^{k,k}$ of $\mathbf{b}^k$ by

$$A^t := \{s \in [1, t]\, \hat{b}^k(s) \neq 0\}\ .$$

The *t-annihilator polynomial* $\hat{g}^t(z)$ for $\boldsymbol{\beta}$, of degree $|A^t|$ in the variable

$$\lambda^t(z) = t(v-t) - z(v+1-z)\,,$$

is defined by

$$\hat{g}^t(z) = \prod_{s \in A^t} (\lambda^t(z) - \lambda^t(s))\ .$$

Theorem 5.1. *The t-degree r of a design vector $\boldsymbol{\beta}$ is equal to the cardinality of the annihilator set A^t for $\boldsymbol{\beta}$. The t-form space $\mathscr{F}^t$ of $\boldsymbol{\beta}$ consists of the vectors $\mathbf{f}^t$ in $\mathbb{R}^{t+1}$ that satisfy $(\mathbf{f}^t Q^{k,t})(s) = 0$ for all $s \in A^t$. Equivalently, $\mathbf{f}^t$ belongs to $\mathscr{F}^t$ if and only if its dual Hahn transform polynomial $\hat{f}^t(z) = \sum_{i=0}^t f_i^t Q_i^{k,t}(z)$ is divisible by the t-annihilator polynomial $\hat{g}^t(z)$ for $\boldsymbol{\beta}$.*

Proof. From Theorems 4.1 and 4.2 it follows that $\mathbf{f}^t$ is a t-form for $\boldsymbol{\beta}$ if and only if $\hat{f}^t(s)\hat{b}^k(s) = 0$ for all $s \in [1, t]$, i.e., if and only if $\hat{f}^t(s) = 0$ for all $s \in A^t$. This means that $\hat{f}^t(z)$ is divisible by $\hat{g}^t(z)$. □

Theorem 5.1 tells us that the dual Hahn transform image of the space $\mathscr{F}^t$ is *an ideal of the algebra of real polynomials in the variable $\lambda^t(z)$, reduced modulo $m^t(z) = \prod_{s=0}^t (\lambda^t(z) - \lambda^t(s))$*. This ideal contains a unique monic polynomial of minimum degree, namely the t-annihilator $\hat{g}^t(z)$. Note that $\hat{g}^t(z)$ divides the modulus polynomial $m^t(z)$. (In that respect, the only restriction is $\hat{g}^t(0) \neq 0$.) Conversely, any ideal of this type is the dual Hahn transform image of every design vector $\boldsymbol{\beta}$ that satisfies the condition $\hat{b}^k(s) = 0$ if and only if $\hat{g}^t(s) \neq 0$ for $s \in [1, t]$.

Definition 5.2. Let $\hat{g}^t(z)$ be the t-annihilator polynomial for a given design vector $\boldsymbol{\beta}$. Consider the dual Hahn expansion $\hat{g}^t(z) = \sum_{i=0}^t g_i^t Q_i^{k,t}(z)$, with $g_r^t \neq 0$ and $g_i^t = 0$ for $i \in [r+1, t]$. The real vector $\mathbf{g}^t = (g_i^t)_{i=0}^t$ thus defined will be called the *generator t-form* for $\boldsymbol{\beta}$ (in agreement with Theorem 5.1).

The following theorem is an extension of a known result concerning the outer distribution matrix [3] to the more general concept of the t-distribution matrix D^t (see Definition 2.1).

Theorem 5.2. *Let r be the t-degree of a given design vector $\boldsymbol{\beta}$. For each $i \in [r, t]$, the row D_i^t of the t-distribution matrix of $\boldsymbol{\beta}$ is a rational linear combination of the $r+1$ vectors $\langle \boldsymbol{\beta}, \mathbf{1}\rangle \mathbf{1}$, $D_0^t, \ldots, D_{r-1}^t$, the coefficients of which can be calculated from the annihilator set for $\boldsymbol{\beta}$.*

Proof. Consider the generator t-form $\mathbf{g}^t$. In view of Definition 2.2 and Lemma 2.1, the row D_r^t of D^t can be expressed as follows:

$$D_r^t = (g_r^t)^{-1} \left[\binom{v}{t}^{-1} \binom{k}{t} \hat{g}^t(0) \langle \boldsymbol{\beta}, \mathbf{1}\rangle \mathbf{1} - \sum_{i=0}^{r-1} g_i^t D_i^t \right]\ .$$

We can obtain a similar result for the remaining rows of D^t by considering "shifted versions" $(\lambda^t(z))^j \hat{g}^t(z)$ of the t-annihilator polynomial, with $j \in [1, t-r]$. Details are omitted. □

Remark 5.1. We briefly mention a combinatorial application. Let $\mathfrak{B}$ be a nonempty subset of $\mathscr{P}^k$. Define the *t-covering radius* of $\mathfrak{B}$ as the maximum for $x^t \in \mathscr{P}^t$ of the minimum for $x^k \in \mathfrak{B}$ of the integer $t - |x^k \cap x^t|$. It follows from Theorem 5.2 that the t-covering radius of $\mathfrak{B}$ is bounded from above by the t-degree of (the characteristic vector of) $\mathfrak{B}$. The reason is that the assumption $D_0^t(x^t) = \cdots = D_r^t(x^t) = 0$, for a given x^t, leads to a contradiction (when $\langle \boldsymbol{\beta}, \mathbf{1} \rangle \neq 0$).

It is interesting to see how the structure of the t-form space can be derived in a purely combinatorial framework, without explicit reference to the dual Hahn polynomials. Consider the $(t+1) \times (t+1)$ *tridiagonal matrix* $W^{k,t} = [w_{i,j}^{k,t}]_{i,j=0}^t$ constructed from the intersection numbers $w_{i,j}^{k,t}$ met in the proof of Theorem 3.2. By use of Definition 2.1 we obtain $W^{k,t}D^t = D^t C_1^{t,t}$. If $\mathbf{f}^t$ is a t-form for $\boldsymbol{\beta}$, i.e., if $\mathbf{f}^t D^t = c\mathbf{1}$, then this identity shows that $\mathbf{f}^t W^{k,t}$ also is a t-form for $\boldsymbol{\beta}$. Therefore, $W^{k,t}$ is an endomorphism of the t-form space of any design vector $\boldsymbol{\beta}$. In fact, we can state the following result, which is equivalent to Theorem 5.1. (The proof is omitted; see the comments below.)

Theorem 5.3. *Let* $W^{k,t} = [w_{i,j}^{k,t}]_{i,j=0}^t$. *The t-form space* $\mathscr{F}^t$ *is* $W^{k,t}$*-invariant. More precisely, if* $\mathbf{g}^t$ *is the generator t-form for* $\boldsymbol{\beta}$, *then the* $t+1-r$ *vectors* $\mathbf{g}^t(W^{k,t})^j$ *with* $j \in [0, t-r]$ *form a basis of* $\mathscr{F}^t$.

The connection between the settings of Theorems 5.1 and 5.3 can be described as follows. The recurrence relation for dual Hahn polynomials (Theorem 3.1) amounts to the matrix identity $W^{k,t}Q^{k,t} = Q^{k,t}\Lambda^t$, with $\Lambda^t = \operatorname{diag}(\lambda^t(0), \ldots, \lambda^t(t))$. Consider the $W^{k,t}$-image $\mathbf{f}^t W^{k,t}$ of a given vector $\mathbf{f}^t$ in $\mathbb{R}^{t+1}$. In view of the identity above, the dual Hahn transform of this image vector equals $\hat{\mathbf{f}}^t \Lambda^t$, and the associated polynomial is the product $\hat{f}^t(z)\lambda^t(z)$ reduced modulo $m^t(z)$.

Remark 5.2. It is possible to work out the theory of t-forms for any value of t with $0 \leq t \leq \lfloor v/2 \rfloor$. Let us briefly examine the case $t \geq k$ (instead of $t \leq k$ as above). In this situation, we have to replace t by k in the definition of the adjacency matrices, roughly speaking. Thus, the t-form space $\mathscr{F}^t$ of a design vector $\boldsymbol{\beta}$ in $\mathbb{R}\mathscr{P}^k$ now is a subspace of $\mathbb{R}^{k+1}$. It can be verified that the t-degree of $\boldsymbol{\beta}$ is independent of t, and counts the integers $s \in [1, k]$ that satisfy $\hat{b}^k(s) \neq 0$. The dual Hahn transform image of the t-form space of $\boldsymbol{\beta}$ is the same for all t (in the range $k \leq t \leq \lfloor v/2 \rfloor$). The appropriate transform is given by the polynomials $Q_i^{t,k}(z)$ with $i \in [0, k]$. This can be proved from the preceding results by use of the fact that the t-form space of $\boldsymbol{\beta}$ coincides with the k-form space of $\boldsymbol{\beta}\,(C_0^{t,k})^T$, a vector in $\mathbb{R}\mathscr{P}^t$.

Example 5.1. Let $\boldsymbol{\beta}$ be the design vector for the Steiner system $S(5, 8, 24)$, that is, the unique 5-design $\mathfrak{B}$ of index 1 with $v = 24$, $k = 8$, and $|\mathfrak{B}| = 759$. Its inner distribution vector (Definition 3.4) is well known and easily calculated as $\mathbf{b}^8 = 759(1, 0, 0, 0, 280, 0, 448, 0, 30)$. From Definition 3.4, the components of the dual Hahn transform $\hat{\mathbf{b}}^8 = \mathbf{b}^8 Q^{8,8}$ of $\mathbf{b}^8$ are $\hat{\mathbf{b}}^8(0) = (3.11.23)^2$, $\hat{\mathbf{b}}^8(6) = (8.9.17.23)/13$, $\hat{\mathbf{b}}^8(8) = 5.8.23$, and $\hat{\mathbf{b}}^8(i) = 0$ for $i = 1, 2, 3, 4, 5, 7$. So from Theorem 4.1, $\boldsymbol{\beta}$ is a T-design vector for any $T \subseteq \{0, 1, 2, 3, 4, 5, 7\}$, and in particular, is a 5-design. From Definition 5.1, $A^8 = \{6, 8\}$, $A^7 = A^6 = \{6\}$, and $A^t = \varnothing$ for $0 \leq t \leq 5$. Hence from Definition 2.2 the respective t-degrees are $r^8 = 2$, $r^7 = r^6 = 1$, and $r^t = 0$ for $0 \leq t \leq 5$. For $t = 6, 7, 8$ the t-annihilator polynomials of type $(8, t)$ are, from Definitions 5.1, 5.2, and 3.3,

$$\hat{g}^6(z) = (6-z)(19-z) = 18Q_0^{8,6}(z) + 3Q_1^{8,6}(z) ,$$

$$\hat{g}^7(z) = (6-z)(19-z) = 2Q_0^{8,7}(z) + 2Q_1^{8,7}(z) ,$$

$$\hat{g}^8(z) = (6-z)(19-z)(8-z)(17-z) = 16Q_0^{8,8}(z) + 16Q_1^{8,8}(z) + 4Q_2^{8,8}(z) .$$

From Definition 2.2, the generator t-form identities directly deduced from these are:

$$6D_0^6(x^6) + D_1^6(x^6) = 6 ,$$

$$D_0^7(x^7) + D_1^7(x^7) = 1 ,$$

$$4D_0^8(x^8) + 4D_1^8(x^8) + D_2^8(x^8) = 4 ,$$

for every $x^t \in \mathscr{P}^t$. Note that the three equations above have 2, 2, and 3 nonnegative integer solutions respectively, namely (1, 0) or (0, 6), (1, 0) or (0, 1), and (1, 0, 0), (0, 1, 0) or (0, 0, 4). Theorem 5.2 shows how to calculate the whole t-distribution matrix D^t. Note that D^t is a $(t+1) \times \binom{24}{t}$ matrix, but by the calculations above, has only $r^t + 1$ distinct columns! (It is interesting to note here that any of the three equations above would imply that $\mathfrak{B}$ is a Steiner system.) As in Remark 5.2, it is also possible to calculate the t-form space for $9 \leq t \leq 12$, with t-degree $r^t = 2$. As an illustration,

$$20D_0^{12}(x^{12}) + 6D_1^{12}(x^{12}) + D_2^{12}(x^{12}) = 132 .$$

(Remember that $D_i^{12}(x^{12})$ counts the blocks of $\mathfrak{B}$ that have $8 - i$ points in common with a given 12-subset x^{12} of the 24-set V.)

Example 5.2. Let us briefly examine the simple situation characterized by $r^t = 1$, i.e., by $A^t = \{s_0\}$ for some s_0 with $1 \leq s_0 \leq t \leq k$. According to Theorem 5.1, this corresponds to T-design vectors $\boldsymbol{\beta}$ with $T = [1, t]\backslash\{s_0\}$. The t-annihilator polynomial is $\hat{g}^t(z) = (s_0 - z)(v + 1 - s_0 - z)$. From its dual Hahn expansion we obtain the following generator t-form identity:

$$[s_0(v+1-s_0) - t(v-k)]D_0^t(x^t) + (k+1-t)D_1^t(x^t)$$

$$= \binom{v}{t}^{-1}\binom{k}{t} s_0(v+1-s_0)\langle \boldsymbol{\beta} , \mathbf{1}\rangle .$$

The first two cases of Example 5.1 above illustrate this result, with $(t, s_0) = (6, 6)$ and $(7, 6)$, respectively.

Example 5.3. We now give a "combinatorial construction" in the special case $s_0 = 1$. Suppose that $\mathfrak{B}_0$ is a t-design for $J(v-1, k)$ and $\mathfrak{B}_1$ is a t-design for $J(v-1, k-1)$, with the same point set V_0 (of cardinality $v - 1$). Consider the combinatorial structure in $J(v, k)$ with the point set $V = V_0 \cup \{\infty\}$ and with the block set $\mathfrak{B} = \mathfrak{B}_0 \cup \mathfrak{B}_1'$, where $\mathfrak{B}_1'$ consists of the k-sets $x \cup \{\infty\}$ with $x \in \mathfrak{B}_1$. We suppose that $\mathfrak{B}$ is **not** a 1-design. Then, the pair $(D_0^t(x^t), D_1^t(x^t))$ for $\mathfrak{B}$ assumes exactly two distinct values, one value in the case $\infty \in x^t$ and one (different) value in the other case. Hence there exists a nontrivial identity $g_0^t D_0^t(x^t) + g_1^t D_1^t(x^t) = c$ (for all $x^t \in \mathscr{P}^t$). The pair (g_0^t, g_1^t) is uniquely determined, within normalization, and g_1^t is not zero. Elementary "design computation" yields $g_0^t = v - t(v-k)$ and $g_1^t = k + 1 - t$. Therefore, our construction exemplifies the situation examined above, with $s_0 = 1$. As a conclusion, $\mathfrak{B}$ is a $[2, t]$-design for $J(v, k)$, although it is not a t-design.

Acknowledgment

The authors express their gratitude to the anonymous referee whose constructive remarks led to a more compact and systematic presentation of their results.

References

1. A. R. Calderbank and P. Delsarte, *On error-correcting codes and invariant linear forms*, SIAM J. Discrete Math. **6** (1993), 1–23.
2. A. R. Calderbank, P. Delsarte, and N. J. A. Sloane, *A strengthening of the Assmus-Mattson theorem*, IEEE Trans. Information Theory **IT-37** (1991), 1261–1268.
3. P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. **10** (1973).
4. ———, *Pairs of vectors in the space of an association scheme*, Philips Res. Rep. **32** (1977), 373–411.
5. ———, *Hahn polynomials, discrete harmonics, and t-designs*, SIAM J. Appl. Math. **34** (1978), 157–166.
6. C. F. Dunkl, *Spherical functions on compact groups and applications to special functions*, Sympos. Math. **22** (1977), 145–161.
7. ———, *An addition theorem for Hahn polynomials: the spherical functions*, SIAM J. Math. Anal. **9** (1978), 627–637.
8. R. L. Graham, S.-Y. R. Li, and W.-C. W. Li, *On the structure of t-designs*, SIAM J. Algebraic Discrete Methods **1** (1980), 8–14.
9. J. E. Graver and W. B. Jurkat, *The module structure of integral designs*, J. Combin. Theory Ser. A **15** (1973), 75–90.
10. S. Karlin and J. L. McGregor, *The Hahn polynomials, formulas and an application*, Scripta Math. **26** (1961), 33–46.
11. D. K. Ray-Chaudhuri and N. M. Singhi, *On existence of t-designs with large v and λ*, SIAM J. Discrete Math. **1** (1988), 98–104.
12. R. M. Wilson, *Inequalities for t-designs*, J. Combin. Theory Ser. A **34** (1983), 313–324.
13. ———, *On the theory of t-designs*, Enumeration and Designs (D. M. Jackson and S. A. Vanstone, eds.), Academic Press, New York, 1984, pp. 19–49.

Mathematical Sciences Research Center, AT&T Bell Laboratories, Murray Hill, New Jersey 07974

E-mail address: rc@research.att.com

Department of Computer Science, University of Louvain, Place Sainte Barbe 2, B1348 Louvain La Neuve, Belgium

E-mail address: delsarte@info.ucl.ac.be