

THE LIMITING BEHAVIOR OF THE KOBAYASHI-ROYDEN PSEUDOMETRIC

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ABSTRACT. We study the limit of the sequence of Kobayashi metrics of Riemann surfaces (when these Riemann surfaces form an analytic fibration in such a way that the total space of fibration becomes a complex surface), as the fibers approach the center fiber which is not in general smooth. We prove that if the total space is a Stein surface and the smooth part of the center fiber contains a component biholomorphic to a quotient of the disk by a Fuchsian group of first kind, then the Kobayashi metrics of the near-by fibers converge to the Kobayashi metric of this component as fibers tend to the center fiber.

INTRODUCTION

Let $\Phi: M \rightarrow \Delta$ be a holomorphic mapping from a complex surface M on the disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. Suppose that for each $c \neq 0$ $\Gamma_c = \Phi^{-1}(c)$ is a smooth noncompact Riemann surface and Γ_0^* is a smooth part of $\Gamma_0 = \Phi^{-1}(0)$. We shall investigate relations between the Kobayashi-Royden pseudometric $k_{\Gamma_0^*}$ on Γ_0^* and the limit of the Kobayashi-Royden pseudometric on nearby fibers. More precisely, we shall study the problem when the equality

$$(1) \quad \lim_{c \rightarrow 0} k_{\Gamma_c} = k_{\Gamma_0^*}$$

holds. In general, it is not so. In [PS, §2.2] there is an example of such mapping $\Phi: M \rightarrow \Delta$, where M is a holomorphically convex region in $\mathbb{C}^2$, every Γ_c is a disc, but $\lim_{c \rightarrow 0} k_{\Gamma_c} \neq k_{\Gamma_0^*}$. Zaidenberg found certain sufficient conditions, which imply (1) [Z]. But his result does not give the answer to the question whether (1) holds, when Φ is a polynomial of two complex variables and $M = \Phi^{-1}(\Delta)$. He supposed that the answer was positive. Let G be a Fuchsian group of the first kind. The Main Theorem of this paper says that, if M is a Stein surface and Γ_0^* contains a component R , which is biholomorphically equivalent to Δ/G , then $\lim_{c \rightarrow 0} k_{\Gamma_c} = k_R$. In particular, the Zaidenberg's conjecture is true. The last fact was announced in [Ka], where it was used to classify isotrivial polynomials on $\mathbb{C}^2$.

The paper is organized as follows. We present some terminology and formulate our main results in the first section. The second section contains a technical lemma about Fuchsian groups and its corollaries needed for the proof of the Main Theorem. This lemma asserts that two noncommutative nonelliptic elements of a Fuchsian group cannot move any point $z \in \Delta$ by a distance less than

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a certain $\varepsilon > 0$ at the same time. Next we handle the case of hyperbolic fibers $\{\Gamma_{b_j}\}$ with $b_j \rightarrow 0$. We consider universal holomorphic covering $f_j: \Delta \rightarrow \Gamma_{b_j}$ and find out when $\{f_j\}$ converge to an unramified mapping $f: D \rightarrow \Gamma_0^*$ on a certain maximal region $D \subset \Delta$. We also prove that $f(D)$ is a component of Γ_0^* and, if $D = \Delta$, then the Kobayashi-Royden pseudometric on $f(D)$ coincides with the limit of the Kobayashi-Royden pseudometric of nearby fibers. The result of the forth section says that D is simply connected in the case when M is a Stein surface. The last section contains the proof of the Main Theorem.

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1. FORMULATION OF THE MAIN THEOREM

First we fix terminology, notations and definitions that we shall use throughout the paper. Every manifold we are going to consider will be complex. If Y is a manifold, then TY is its holomorphic tangent bundle and $T_y Y$ is a tangent space at a point $y \in Y$. Put $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$, $\Delta = \Delta_1$, and $\Delta^* = \Delta - 0$. By a curve η in Y we mean a continuous mapping $\eta: [0, 1] \rightarrow Y$. A loop γ in Y is a curve with $\gamma(0) = \gamma(1)$. In other words, γ is a continuous mapping from $\partial\Delta$ to Y (as is frequently done, we use the symbol ∂ to denote boundaries). If $x \in \gamma(\partial\Delta)$, then we write $x \in \gamma$. Recall that a differential pseudometric on a complex manifold Y is a nonnegative homogeneous function on the tangent bundle TY , i.e., it is a function $p: TY \rightarrow \mathbb{R}$ such that $p(y, v) \geq 0$, $p(y, \lambda v) = |\lambda|p(y, v)$ for all $y \in Y$, $v \in T_y Y$, and $\lambda \in \mathbb{C}$. When p is continuous, we call the pseudometric continuous. If Y is connected and for each piecewise smooth curve η in Y there exists the integral $P(\eta) = \int_0^1 p(\eta(t), d\eta(t)) dt$, one can define the integral pseudometric $P(x, y) = \inf_{\eta} \{P(\eta) \mid \eta(0) = x, \eta(1) = y\}$. Of course, the integral pseudometric exists, when a proper differential pseudometric is continuous. The Kobayashi-Royden differential pseudometric is given by the formula

$$k_Y(y, v) = \inf_{\phi} \{1/r \mid \phi \in \text{Hol}(\Delta_r, Y), \phi(0) = y, d\phi(0) = v\}.$$

By Royden's theorem [R] it generates the integral pseudometric K_Y which coincides with the Kobayashi pseudometric on Y [Ko].

Throughout the paper $\Phi: M \rightarrow \Delta$ is a holomorphic mapping from a smooth complex surface M on Δ such that for $c \neq 0$ $\Gamma_c = \Phi^{-1}(c)$ is a smooth Riemann surface. We shall say that $\Phi: M \rightarrow \Delta$ is a family of Riemann surfaces. The fiber $\Gamma_0 = \Phi^{-1}(0)$ can contain singular points. Denote the smooth part of Γ_0 by Γ_0^* . Let $\beta = \{b_j\} \subset \Delta^*$ be a sequence that tends to zero, let R be a component of Γ_0^* . We say that $\lim_{j \rightarrow \infty} k_{\Gamma_{b_j}} = k_R$ (or $\overline{\lim}_{j \rightarrow \infty} k_{\Gamma_{b_j}} \leq k_R$), if for each sequence $\{w_j \in T\Gamma_{b_j}\}$ that converges to $w \in TR$ in the topology of TM the equality $\lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}(w_j) = k_R(w)$ (or inequality $\overline{\lim}_{j \rightarrow \infty} k_{\Gamma_{b_j}}(w_j) \leq k_R(w)$) holds. If $\lim_{j \rightarrow \infty} k_{\Gamma_{b_j}} = k_R$ for each sequence β as above, then we say $\lim_{c \rightarrow 0} k_{\Gamma_c} = k_R$. In the same meaning $\overline{\lim}_{c \rightarrow \infty} k_{\Gamma_c} \leq k_R$. The following two results belong to Zaidenberg [Z].

Proposition 1.1. *For each component R of Γ_0^* the inequality $\overline{\lim}_{c \rightarrow \infty} k_{\Gamma_c} \leq k_R$ holds.*

Theorem 1.2. *Let $\overline{M}$ be a smooth compact surface and $\overline{\Gamma} \subset \overline{M}$ be an analytic curve in $\overline{M}$. Suppose that $M \subset \overline{M} - \overline{\Gamma}$, $\overline{\Gamma}_0 = \bigcap_{r>0} \overline{\Phi^{-1}(\Delta_r)}$, and $\Gamma_0 = \overline{\Gamma}_0 - \overline{\Gamma}$. If every component of Γ_0^* is hyperbolic, then $\lim_{c \rightarrow 0} k_{\Gamma_c} = k_R$.*

Zaidenberg conjectured that, if Φ is a polynomial on C^2 and $M = \Phi^{-1}(\Delta)$, then the assumption that all the components of Γ_0^* are hyperbolic can be omitted. We shall show that this hypothesis is correct. Recall that G is a Fuchsian group of the first kind, if the closure of the orbit $\{g(0) | g \in G\}$ in C contains $\partial\Delta$ [B]. In particular, in the polynomial case every hyperbolic component R of Γ_0^* has a representation $R \cong \Delta/G$, where G is a Fuchsian group of the first kind.

Main Theorem. *Let $\Phi: M \rightarrow \Delta$ be a family of Riemann surfaces. Suppose that M is a Stein manifold and Γ_0^* contains a component R that is biholomorphically equivalent to Δ/G , where G is a Fuchsian group of the first kind. Then $\lim_{c \rightarrow 0} k_{\Gamma_c} = k_R$.*

Note that, if R is nonhyperbolic, such a fact follows from Proposition 1.1. Hence we have

Corollary. *Let $\Phi: C^2 \rightarrow C$ be a polynomial. Then $\lim_{c \rightarrow 0} k_{\Gamma_c} = k_{\Gamma_0^*}$.*

We shall restrict ourselves to the case of connected fibers for $c \neq 0$ (in general case the proof is the same, but instead of Γ_c we have to use their components).

2. ONE PROPERTY OF FUCHSIAN GROUPS

We shall denote the Kobayashi metric on Δ by K_Δ .

Lemma 2.1. *For every $r > 0$ there exists $\varepsilon > 0$ such that for every Fuchsian group G , noncommutative elements $a', b' \in G$, and a point $z \in \Delta$ satisfying $0 < K_\Delta(z, a'(z)) < r$, either $K_\Delta(z, b'(z)) > \varepsilon$ or z is a fixed point of the mapping $b': \Delta \rightarrow \Delta$.*

Proof. Assume, to reach a contradiction, that for a certain $r > 0$ and each $\varepsilon > 0$ there exists a Fuchsian group G_ε , noncommutative elements $a'_\varepsilon, b'_\varepsilon \in G_\varepsilon$, and a point $z_\varepsilon \in \Delta$ such that $0 < K_\Delta(z_\varepsilon, a'_\varepsilon(z_\varepsilon)) < r$ and $0 < K_\Delta(z_\varepsilon, b'_\varepsilon(z_\varepsilon)) < \varepsilon$. We shall show that for a sufficiently small ε the group G_ε cannot be discontinuous. Without loss of generality, we set $z_\varepsilon = 0$. Let id be the identity element of G_ε . Since G_ε is a discontinuous group, one can find elements a_ε and b_ε satisfying

$$(2.1) \quad K_\Delta(0, b_\varepsilon(0)) = \min\{K_\Delta(0, g(0)) | g \in G_\varepsilon, g(0) \neq 0\},$$

$$(2.2) \quad K_\Delta(0, a_\varepsilon(0)) = \min\{K_\Delta(0, g(0)) | g \in G_\varepsilon, g(0) \neq 0, [b_\varepsilon, g] \neq \text{id}\}.$$

The mapping a_ε and b_ε can be represented in the form

$$\begin{aligned} a_\varepsilon(z) &= e^{i\theta_\varepsilon}(z + \alpha_\varepsilon)/(1 + \overline{\alpha}_\varepsilon z), & |\theta_\varepsilon| \in [0, \pi], \\ b_\varepsilon(z) &= e^{i\tau_\varepsilon}(z + \beta_\varepsilon)/(1 + \overline{\beta}_\varepsilon z), & |\tau_\varepsilon| \in [0, \pi]. \end{aligned}$$

We shall omit the index ε from now on, if it does not cause misunderstanding. Let us consider b as a function of two variables z and β . Expand b in power series of z , β , and $\overline{\beta}$. Then $b(z) = e^{i\tau}z + e^{i\tau}\beta$ up to the nonlinear terms. Hence for every natural m one can find a neighborhood of the origin

in $C^2 = \{(z, \beta)\}$ so that for all $n = 1, 2, \dots, m$,

$$b^n(z) = e^{in\tau} z + \sum_{l=1}^n e^{il\tau} \beta + O(|z|^2 + |\beta|^2)$$

in this neighborhood. Thus

$$b^n(0) = \beta \sum_{l=1}^n e^{il\tau} + O(|\beta|^2) = \beta e^{in\tau} (e^{in\tau} - 1) / (e^{i\tau} - 1) + O(|\beta|^2).$$

It is easy to check that for each $\tau_0 \neq 2\pi k$ there is a neighborhood U of τ_0 and integer $n \geq 2$ so that for every $\tau \in U$,

$$(2.3) \quad |(e^{in\tau} - 1) / (e^{i\tau} - 1)| < 1.$$

Now one can see that $\tau_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Indeed, the preceding assumption implies

$$0 < |\alpha| < \tilde{r}, \quad 0 < |\beta| < \tilde{\varepsilon},$$

where $\tilde{\varepsilon} = (e^\varepsilon + 1)$ and $\tilde{r} = (e^r - 1) / (e^r + 1)$. Thus $\lim_{\varepsilon \rightarrow 0} |\beta_\varepsilon| = 0$. Assume $\overline{\lim}_{\varepsilon \rightarrow 0} |\tau_\varepsilon| > 1/m$. Then by (2.3) we can find $n \leq m$ with $|b^n(0)| < |b(0)|$. This contradicts (2.1). Thus $b_\varepsilon(z) \rightarrow z$ uniformly on compact subsets of Δ as $\varepsilon \rightarrow 0$. Let $\overline{\lim}_{\varepsilon \rightarrow 0} |\alpha_\varepsilon| = \alpha^0$. Since $|\alpha_\varepsilon| < \tilde{r}$, $a_\varepsilon \circ b_\varepsilon \circ a_\varepsilon^{-1}(z) \rightarrow z$ as $\varepsilon \rightarrow 0$. In particular, for any sufficiently small ε we have $|a_\varepsilon b_\varepsilon a_\varepsilon^{-1}(0)| < \alpha^0/2$. This implies either $\alpha^0 = 0$ or b and aba^{-1} are commutative. We shall prove that the last case does not hold. One can represent a and b as mappings of the upper half-plane. Then, if a is a hyperbolic element, we may put $a(z) = \lambda z$ with $\lambda > 0$ and if a is a parabolic element, we may put $a(z) = z + 1$ [A]. In both cases for any $b(z) = (pz + q)/(tz + s)$ with $p, q, t, s \in \mathbf{R}$ the direct computation shows that $[aba^{-1}, b] = \text{id}$, iff $[a, b] = \text{id}$. When a is an elliptic element, one may consider a as a mapping $a: \Delta \rightarrow \Delta$ given by the formula $a(z) = \lambda z$ with $\lambda^n = 1$ for a certain natural n . Again it is easy to show that $[aba^{-1}, b] = \text{id}$, iff $[a, b] = \text{id}$ for any Möbius transformation $b: \Delta \rightarrow \Delta$. But this contradicts (2.2). Therefore $\lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = 0$. Same arguments as above show that $\theta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence for any sufficiently small ε we have $|e^{i\theta_\varepsilon} - 1| + |e^{i\tau_\varepsilon} - 1| < 1/2$ and for an arbitrarily small α the following inequality holds

$$|b^{-1}a^{-1}ba(0)| \approx |e^{i\tau} - 1| |\alpha| + |e^{i\theta} - 1| |\beta| < |\alpha|/2 < |a(0)|$$

but $b^{-1}a^{-1}ba$ and b are not commutative, since $[a^{-1}, ba, b] \neq \text{id}$. This is a contradiction. $\square$

Corollary 2.2. *For every $r > 0$ there exists $\varepsilon > 0$ such that for every hyperbolic Riemann surface R , for every point $x \in R$, and for every couple of loops γ and μ that generate noncommutative elements of the fundamental group $\pi_1(R, x)$, the inequalities $K_R(\gamma) < \varepsilon$ and $K_R(\mu) < r$ do not hold simultaneously.*

The next three lemmas enable us to restate this corollary in a form which will be convenient for our following needs.

Lemma 2.3. *Let γ be a noncontractible loop on a Riemann surface R . Suppose that the corresponding element of the fundamental group $\pi_1(R)$ has a representation $[\gamma] = [\mu]^n$, where $[\mu] \in \pi_1(R)$ and the natural number $n \geq 2$. Then γ has points of self-intersection.*

Proof. Let H be the upper half-plane, and let $f: H \rightarrow R$ be a universal holomorphic covering. Then we can define the Möbius transformation $b: H \rightarrow H$ corresponding to $[\mu]$. If b is a hyperbolic transformation, one can choose f so that $b(z) = \lambda z$ with $\lambda > 0$ [A]. Let z_0 be a point in the inverse image of a point $x_0 \in \gamma$. Obviously, each curve in H that connects the points z_0 and $\lambda^n z_0$ contains points z' and z'' such that $z' = \lambda z''$. But this means that γ has the point of self-intersection $f(z')$. If b is parabolic, we may suppose that $b(z) = z + 1$. Again each curve that connects the points z_0 and $z_0 + n$ contains points z' and $z'' = z' + 1$. This implies the desired conclusion. $\square$

Lemma 2.4. *Let γ and μ be disjoint noncontractible loops in a Riemann surface R . Suppose that neither γ nor μ has points of self-intersection. Then γ and μ are homotopically equivalent, iff there is a region $U \subset R$ such that $\partial U = \gamma \cup \mu$ and U is topologically an annulus.*

Proof. Let $x_1 \in \gamma$ and $y_1 \in \mu$. Choose a curve $\nu_1: [0, 1] \rightarrow R$ so that $\nu_1(0) = x_1$, $\nu_1(1) = y_1$, ν_1 has no points of self-intersection and ν_1 intersects $\gamma \cup \mu$ at the points x_1 and y_1 only. Choose an analogous curve ν_2 so that ν_2 connects points $x_2 \in \gamma$ and $y_2 \in \mu$, and ν_2 is sufficiently close to, but disjoint from ν_1 . Then $\gamma - (x_1 \cup x_2)$ consists of two components γ_1 and γ_2 , and γ_1 is small enough. In the same way $\mu - (y_1 \cup y_2) = \mu_1 \cup \mu_2$, and μ_1 is small. Then there exists an open disc $D \subset R$ with $\partial D = \nu_1 \cup \nu_2 \cup \gamma_1 \cup \mu_1$. One can construct the loop $\eta = \nu_1 \cup \mu_2 \cup \nu_2 \cup \gamma_2$. Since γ and μ are homotopically equivalent, η must be contractible. By our construction η has no points of self-intersection. This implies the existence of the disc $U \subset R$ with $\partial U = \eta$. If $U \supset D$, then $U - \overline{D}$ contains the two components U_1 and U_2 . Each of them is a disc, $\partial U_1 = \gamma$ and $\partial U_2 = \mu$. This contradicts the assumption that γ and μ are noncontractible. Hence $U \cap D = \emptyset$. Obviously, $\overline{D} \cup \overline{U}$ is topologically a closed annulus and $\partial(\overline{U} \cup \overline{D}) = \gamma \cup \mu$. This completes the proof of the lemma.

Lemma 2.5. *Let γ and μ be noncontractible loops on a Riemann surface R , and neither γ nor μ has points of self-intersection. Suppose that $R - (\gamma \cup \mu)$ does not contain components that are topologically an annulus. Then for each $\varepsilon > 0$ there exists $r > 0$ such that, if $K_R(\gamma) < \varepsilon$ and $K_R(\mu) < \varepsilon$, then the distance between γ and μ in the Kobayashi metric is greater than r .*

Proof. Let $\nu: [0, 1] \rightarrow R$ be a curve that connects γ and μ so that $K_R(\nu)$ coincides with the distance between γ and μ . Let $\nu(0) = x_0 \in \gamma$. By Corollary 2.2 it is enough to verify that γ and $\gamma' = \nu^{-1} \circ \mu \circ \nu$ generate noncommutative elements $[\gamma]$ and $[\gamma']$ in $\pi_1(R, x_0)$. Since the group $\pi_1(R, x_0)$ is free, $[\gamma]$ and $[\gamma']$ are commutative, iff they belong to a cyclic subgroup. This implies that $[\gamma] = [\nu]^n$ and $[\gamma'] = [\nu]^l$ for a certain $[\nu] \in \pi_1(R, x_0)$. By Lemma 2.3 $k = l = 1$. Hence $[\gamma] = [\gamma']$. Therefore γ and μ must be homotopically equivalent. But this contradicts Lemma 2.4. $\square$

3. LIMITING BEHAVIOR OF HYPERBOLIC METRIC

From now on by R we denote a connected hyperbolic component of Γ_0^* .

Lemma 3.1. *Let α be a sequence of points in Δ^* that tends to zero. Suppose that for each $c \in \alpha$ the fiber Γ_c is a hyperbolic Riemann surface. Then for a certain infinite subsequence $\beta = \{b_j\} \subset \alpha$ there exists a differential pseudometric α_β*

on R such that $\alpha_\beta = \lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}$. Moreover, α_β is a continuous pseudometric and the equality $\alpha_\beta(v) = 0$ for a vector $v \in TR$ implies $\alpha_\beta \equiv 0$.

Proof. Let $\phi: \Delta \rightarrow R$ be a holomorphic embedding and $\phi(\Delta) = U$. It is easy to construct holomorphic embeddings $\phi_j: \Delta \rightarrow U_j \subset \Gamma_{b_j}$ so that $\phi_j(z) \rightarrow \phi(z)$ as $j \rightarrow \infty$ (e.g., see [Z]). Let ν_z denote the point $(z, d/dz) \in T\Delta$. We set $s_z^j = \phi_{j*}(\nu_z)$ and $s_z = \phi_*(\nu_z)$ (where ϕ_{j*} and ϕ_* are the induced mappings of the tangent bundles). Then $s_z^j \rightarrow s_z$ in topology of TM . Let $f_j: \Delta \rightarrow \Gamma_{b_j}$ be a universal holomorphic covering. Choose a connected component V_j of $f_j^{-1}(U_j)$ and a holomorphic mapping $g_j: \Delta \rightarrow \Delta$ so that the restriction of $g_j \circ \phi_j^{-1} \circ f_j$ to V_j is the identity mapping. One may suppose that $0 \in V_j$ and $g_j(0) = 0$. Let $\tilde{s}_z^j \in TV_j$ belong to the inverse image of the vector s_z^j under the mapping f_{j*} . Then $g_{j*}(\nu_z) = \tilde{s}_z^j$. On the other hand $g_{j*}(\nu_z) = g_j'(z)\nu_{g_j(z)}$. Hence, taking into consideration the equalities $k_\Delta(\tilde{s}_z^j) = k_{\Gamma_{b_j}}(s_z^j)$ and $k_\Delta(\nu_z) = 1/(1-|z|^2)$, we have $k_{\Gamma_{b_j}}(s_z^j) = |g_j'(z)|/(1-|g_j(z)|^2)$. Passing to a subsequence, if necessary, we suppose that $g_j(z) \rightarrow g(z)$ uniformly on compact subsets of Δ . By Hurwitz's theorem either $g'(z) \neq 0$ for every $z \in \Delta$ or $g'(z) \equiv 0$ (in the last case $g(z) \equiv 0$, since $g(0) = 0$). Therefore $\lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}(s_z^j) = |g'(z)|/(1-|g(z)|^2)$. Let $s_z^j = (x_j(z), t_j(z))$, where $x_j(z) \in U_j$ and $t_j(z) \in T_{x_j(z)}U_j$ (the notation, $s_z = (x(z), t(z))$ has the same meaning). A sequence $\{v_j | v_j = (x(z_j), \lambda_j t_j(z_j)); \lambda \in C\}$ converges to $v = (x(z), \lambda t(z))$ in the topology of TM , iff $z_j \rightarrow z$ and $\lambda_j \rightarrow \lambda$. Hence $\lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}(v_j) = |\lambda g'(z)|/(1-|g(z)|^2)$ and a proper limiting pseudometric exists on U . Let $\{U^j\}$ be a cover on R and each U^j be an open disc. We can repeat the above construction of the limiting pseudometric for each U^j instead of U . Application of the diagonal process completes the proof of the lemma. $\square$

Definition. Let $\beta = \{b_j\} \subset \Delta^*$ be a sequence that converges to zero, and let every fiber Γ_{b_j} hyperbolic. We shall say that β is an *admissible sequence* if there is a continuous differential pseudometric a_β on R such that $a_\beta = \lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}$ and the quality $a_\beta(v) = 0$ for a vector $v \in TR$ implies $a_\beta \equiv 0$. We will denote the corresponding integral pseudometric by A_β , and throughout the rest of the paper we will fix these notations β , a_β , and A_β for the above objects.

Lemma 3.2. Suppose that the a_β is a metric. Let $F = \{f_j\}$, where $f_j: \Delta \rightarrow \Gamma_{b_j}$ is a holomorphic universal covering with $f_j(0) \rightarrow x_0 \in R$ as $j \rightarrow \infty$. Then there is a nonempty open subset $D \subset \Delta$ that contains 0 and a subsequence $F_1 \subset F$ that converges to a mapping $f: D \rightarrow R$. Moreover

- (i) $f: D \rightarrow R$ is an unramified covering.
- (ii) F transforms the metric $k_\Delta|_D$ into the metric a_β .

Proof. First assume that f exists and prove (ii). Let $z \in D$, $x = f(z)$, and $x_j = f_j(z)$. Then $x_j \rightarrow x$ as $j \rightarrow \infty$. Choose a sequence $\{v_j | v_j \in T_{x_j}\Gamma_{b_j}\}$ that converges to a nonzero vector $v \in T_x R$ in the topology of TM . Let $\tilde{v}_j \in T_z \Delta$ belong to the inverse image of v_j under the mapping f_{j*} . Since β is admissible, $k_\Delta(\tilde{v}_j) = k_{\Gamma_{b_j}}(v_j) \rightarrow a_\beta(v)$. Thus for every j , $k_\Delta(\tilde{v}_j)$ is less than a certain common constant. Hence we may suppose that there is the limiting vector $\tilde{v}$ for the sequence $\{\tilde{v}_j\}$. Clearly, $k_\Delta(\tilde{v}) = a_\beta(v)$ and $f_*(\tilde{v}) = v$.

This implies (ii). Property (ii) means that, if f exists, then it must be locally homeomorphic. Let P be a sufficiently small neighborhood of x_0 such that P is biholomorphically equivalent to a ball, and all $U_j = P \cap \Gamma_{b_j}$ and $U = \Gamma_0 \cap P$ are discs. For every manifold N we will denote by $B(y, r, N) \subset N$ the ball of radius r in the metric K_N with the center at y . Let $B(x, r, A_\beta, R) \subset R$ be the analogous ball in the metric A_β with the center at x . Since a_β is a metric, there exists $r > 0$ such that $\overline{B(x_0, r, A_\beta, R)} \subset U$. Hence $\overline{B(f_j(0), r, \Gamma_{b_j})} \subset U_j$, when j is sufficiently large. The restriction of f_j to $\tilde{H}_0 = B(0, r, \Delta)$ is a homeomorphism between $\tilde{H}_0$ and $B(f_j(0), r, \Gamma_{b_j})$. The family $\{f_j|_{\tilde{H}_0} : \tilde{H}_0 \rightarrow P\}$ is normal. Pick out a converging subsequence $F_1 \subset F$ in this family. Let $f: \tilde{H}_0 \rightarrow B(x, r, A_\beta, R)$ be the limiting mapping. We have proved that $D \supset \tilde{H}_0$, i.e., D is not empty. Since f is locally homeomorphic and each $f_j|_{\tilde{H}_0}$ is a homeomorphism, one can easily check that the limiting mapping $f|_{\tilde{H}_0}$ is also homeomorphism. Set $H = B(x_0, r/3, A_\beta, R)$. Suppose there is a point $z \in D - \tilde{H}_0$ with $y = f(z) \in H$. Let $H_1 = B(y, 2r/3, A_\beta, R)$ and $\tilde{H}_1 = B(z, 2r/3, \Delta)$. Clearly $H_1 \subset U$. Repeating the above arguments we can choose a subsequence $F_2 \subset F_1$ so that the restriction F_2 to $\tilde{H}_1$ converges to a homeomorphism $g: \tilde{H}_1 \rightarrow H_1$. For every subsequence $F_3 \subset F_1 - F_2$ that converges to a mapping $h: \tilde{H}_1 \rightarrow H_1$ we have $g|_{\tilde{H}_1 \cap D} = h|_{\tilde{H}_1 \cap D} = f|_{\tilde{H}_1 \cap D}$. By the uniqueness theorem $h = g$. Thus one can take F_1 itself as F_2 and $D \supset \tilde{H}_1$. Since $H_1 \supset H$, $\tilde{H}_1$ contains a disc $\tilde{H}$ such that $f|_{\tilde{H}}: \tilde{H} \rightarrow H$ is a homeomorphism and $\tilde{H}_0 \cap \tilde{H} = \emptyset$ (indeed, $z \notin \tilde{H}_0$ and the restriction of f to $\tilde{H}_0$ is also a homeomorphism). We can consider H as a neighborhood of x_0 . Of course, analogous arguments enable us to find such a neighborhood for every point $x \in F(D)$. Hence $f: D \rightarrow F(D)$ is an unramified covering. In particular, $f(D)$ is an open set.

To check the equality $R = f(D)$ it is enough to prove that the set $f(D)$ is closed in R . Let x belong to the closure of $f(D)$ in R . Let P' be a sufficiently small neighborhood of x . Suppose that P' is biholomorphically equivalent to a ball, and $U' = P' \cap \Gamma_0$ and $\{U'_j = P' \cap \Gamma_{b_j}\}$ are discs. Choose $r > 0$ with $\overline{B(x, r, A_\beta, R)} \subset U'$. Then we can find a point $y \in f(D) \cap B(x, r/3, A_\beta, R)$. As we have seen, in this case $f(D) \supset B(y, 2r/3, A_\beta, R)$. Hence, $x \in f(D)$, which is the desired conclusion. $\square$

Corollary 3.3. *If the assumptions of Lemma 3.2 hold and $D = \Delta$, then $k_R = \lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}$.*

Proof. We shall use the notation of the proof of Lemma 3.2. If $D = \Delta$, then $f: D \rightarrow R$ is a universal holomorphic covering and

$$\lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}(v_j) = \lim_{j \rightarrow \infty} k_\Delta(\tilde{v}_j) = k_\Delta(\tilde{v}) = k_D(\tilde{v}) = k_R(v). \quad \square$$

4. STEIN CASE

From now on M is a Stein surface, and we will use the same notations R , $\beta = \{b_j\}$, a_β , $f_j: \Delta \rightarrow \Gamma_{b_j}$, $F = \{f_j\}$ and $f: D \rightarrow R$ as in the preceding section. Let a Riemann surface A be topologically an annulus. Denote the minimum of lengths of noncontractible loops in A by $l(A)$.

Proposition 4.1. *To each number $t > 0$ corresponds a positive number $r < 1$ so that the assumptions:*

- (i) L is a compact in Δ ;
- (ii) $0 \in L$;
- (iii) $\Delta - L$ is topologically an annulus;
- (iv) $l(\Delta - L) < t$

imply that $L \subset \Delta_r$.

Proof. Assume that the contrary. Then for a certain t and every $r < 1$ there is compact L_r that contains a point z_r with $|z_r| > r$ and satisfies (i)–(iv). Clearly $l(L - \Delta)$ is greater than $2K_\Delta(0, z_r)$. But $K_\Delta(0, z_r) \rightarrow \infty$ as $r \rightarrow 1$, and we have a contradiction with (iv). $\square$

According to [S] the Stein subvariety R has a tubular neighborhood $V \subset M$ that is biholomorphically equivalent to a neighborhood of the zero section in the normal bundle to R in M . Thus we have a holomorphic retraction $\tau: V \rightarrow R$. Let Q be a region in R with the compact closure. Then for a sufficiently small ε and every $c \in \Delta_\varepsilon$ the restriction τ to $\tau^{-1}(Q) \cap \Gamma_c$ is a holomorphic unramified covering, whose multiplicity over Q is equal to the multiplicity to zero of the function Φ on R .

Lemma 4.2. *Let γ be a loop in R without points of self-intersection. Let $\{\bar{\phi}_j | \bar{\phi}_j: \bar{\Delta} \rightarrow \Gamma_{b_j}\}$ be continuous embeddings that are holomorphic on Δ . Suppose that $\gamma_j = \bar{\phi}_j(\partial\Delta)$ belong to $\tau^{-1}(\gamma)$. Then γ is contractible.*

Proof. We shall consider the Stein manifold M as a closed analytic submanifold in C^n (e.g., see [GR]). Then each $\bar{\phi}_j$ has the following coordinate representation $\phi_j(z) = (\bar{\phi}_{j1}, \dots, \bar{\phi}_{jn})$. Denote the restriction $\bar{\phi}_j$ to Δ by ϕ_j , and let $\phi' = (\phi'_{j1}, \dots, \phi'_{jn})$ be the derivation of ϕ_j . As usual we shall use the symbol $\|\phi'_j(z)\|$ to denote the Euclidean length of the vector $\phi'_j(z)$. Suppose that the functions $\|\phi'_j\|$ converges to zero uniformly on compact subsets of Δ . Then there exists a sequence of points $\{z_j\} \subset \Delta$ with $|z_j| \rightarrow 1$ that satisfies $\|\phi'_j(z_j)\| \geq t/(1 - |z_j|^2)$ for a certain positive t . Indeed, otherwise it is easy to show that the maximal Euclidean distance between the points of γ_j , tends to zero as $j \rightarrow \infty$. But γ_j is close to $\tau(\gamma_j)$. This implies that γ must be a constant mapping, and we have a contradiction. Put $\bar{\psi}_j = \phi_j \circ \mu_j$, where $\mu_j(z) = (z + z_j)/(1 + \bar{z}_j z)$. Let $\psi_j = \bar{\psi}_j|_\Delta$. The loop γ belongs to a ball B in C^n . Hence for an arbitrary large j we have $\bar{\psi}_j(\partial\Delta) \subset B$. By the Maximum Principle $\bar{\psi}_j(\Delta) \subset B$. Therefore the family $\{\psi_j\}$ is normal. Passing to a subsequence, if necessary, we can suppose that $\{\psi_j\}$ converge to a mapping $\psi: \Delta \rightarrow \bar{R}$. Obviously, $\|\psi'_j(0)\| \geq t$, and, therefore, ψ is not constant. According to [Z, Lemma 2.2] $\psi(\Delta) \subset R$. Using a Möbius transformation again, if necessary, one may suppose that $\psi(0) \notin \gamma$. Choose an arbitrary small neighborhood N of γ in R so that N is topologically an annulus and $\psi(0) \notin N$. Then $N - \gamma$ consists of two components N_1 and N_2 , which are also annuli. Let μ_k be the component of the boundary of N_k other than γ . Obviously, $\psi_j(\Delta)$ must contain a component of either $\tau^{-1}(N_1) \cap \Gamma_{b_j}$ or $\tau^{-1}(N_2) \cap \Gamma_{b_j}$. Denote this component by L_j . Passing to a subsequence, we may suppose that $\tau(L_j) = N_1$ and $\tau|_{L_j}$ is a s -sheeted unramified covering, where s does not exceed the multiplicity of zero of the function Φ on R . Hence the Riemann surfaces $\{L_j\}$ are pairwise biholomorphically equivalent, and $l(L_j) = l(\psi_j^{-1}(L_j))$ does not

depend on j . Since $0 \notin \psi_j^{-1}(L_j)$, we see by Proposition 4.1 that there is a positive $r < 1$ such that $\Delta - \psi_j^{-1}(L_j) \subset \Delta_r$. Hence $\mu_1 \subset \psi(\Delta_r)$. This implies that μ_1 is contractible, and, therefore, γ is also contractible. $\square$

Lemma 4.3. *The pseudometric a_β generated by an admissible sequence β is a metric on R in the case when R is different from Δ , Δ^* , or an annulus.*

Proof. Let $\gamma_1, \dots, \gamma_k$ be disjoint noncontractible loops in R without points of self-intersection such that they are not pairwise homotopically equivalent, for each i the set $R - \gamma_i$ is not connected, and every γ_i is a component of the boundary of a compact $L \subset R$. Let L_j be a component of $\tau^{-1}(L) \cap \Gamma_{b_j}$. One may suppose that $\tau|_{L_j}: L_j \rightarrow L$ is a s -sheeted unramified covering for all j . Let $\{\gamma_{ij}^l | l = 1, \dots, l_{ij} \leq s\}$ be the components of $\tau^{-1}(\gamma_i) \cap L_j$. If R has a positive genus, we can suppose that L contains a loop μ without points of self-intersection so that $L - \mu$ is connected and $\mu \cap \bigcup_{j=1}^k \gamma_j = \emptyset$. In this case we denote one of the components of $\gamma^{-1}(\mu) \cap \Gamma_{b_j}$ by μ_j . Assume, to reach a contradiction, that $a_\beta \equiv 0$. Then $K_{\Gamma_{b_j}}(\gamma_{ij}^l), K_{\Gamma_{b_j}}(\mu_j) \rightarrow 0$ as $j \rightarrow \infty$ and the distance between each pair of these loops in the Kobayashi metric on Γ_{b_j} also tends to zero. By Lemma 2.5 all of these loops must be homotopically equivalent. Since $\tau|_{L_j}: L_j \rightarrow L$ is an unramified covering, $L_j - \mu_j$ is connected. Hence by Lemma 2.4 μ_j cannot be homotopically equivalent to any component of the boundary of L_j , or in other words, to any γ_{ij}^l . Therefore it remains to consider the case when R is biholomorphically equivalent to a region in C . Then under the assumptions of the lemma one may suppose that $k \geq 3$. Thus we have, at least, three loops γ_1, γ_2 , and γ_3 . By Lemma 2.5 there is a region $U_j \subset \Gamma_{b_j}$ such that $\partial U_j = \gamma_{1j}^1 \cup \gamma_{2j}^1$ and U_j is topologically an annulus. Note that U_j does not belong to L_j (otherwise, using Lemmas 2.3 and 2.4 it is easy to show that γ_1 and γ_2 are homotopically equivalent). Moreover, since the component of $\Gamma_{b_j} - L_j$ whose boundary contains γ_{3j}^1 is different from a disc according to Lemma 4.2, U_j does not contain L_j . Hence U_j is a component of $\Gamma_{b_j} - L_j$. Taking γ_{3j}^1 instead of γ_{2j}^1 we can construct a component V_j of $\Gamma_{b_j} - L_j$ so that $\partial V_j = \gamma_{1j}^1 \cup \gamma_{3j}^1$ and V_j is topologically an annulus. Since $\partial V_j \cap \partial U_j = \gamma_{1j}^1$, $V_j = U_j$. Then $\partial U_j = \partial V_j$, and this leads to a contradiction. Therefore a_β is not trivial. By Lemma 3.1 $a_\beta(v) \neq 0$ for each $v \in TR$. This completes the proof of the lemma. $\square$

Lemma 4.4. *Let M be a Stein surface and let D be the same as in Lemma 3.2. Then D is simply connected.*

Proof. Assume that D is not simply connected. Then there is a couple of discs d and d' such that $\bar{d} \subset \Delta$, $d' \subset d$, d does not belong to D , and $\bar{d} - d' \subset D$. We again consider M as a submanifold in C^n . The set $f(\bar{d} - d')$ belongs to a certain ball in C^n . Same arguments as in Lemma 4.2 show that the family $\{f_j|_d\}$ is normal. Let $\tilde{f}: d \rightarrow \bar{R}$ be a limiting mapping. This mapping is unique, since it coincides with f on $d - d'$. In particular, it is nonconstant. The set $f(d)$ does not contain singular points of Γ_0 , because otherwise $f_j(d)$ must intersect Γ_0 for an arbitrary large j [Z]. Hence $\tilde{f}(d) \subset R$, i.e., $d \subset D$. But this contradicts our assumption. $\square$

Corollary 4.5. *Lemma 4.2 holds without the condition that γ has no point of self-intersection.*

5. PROOF OF THE MAIN THEOREM

We keep the same notation R , β , a_β , $F = \{f_j\}$, $f: D \rightarrow R$ as in the preceding section. By Lemmas 3.2, 4.3, and 4.4 we suppose that the family F converges to the mapping $f: D \rightarrow R$ on a nonempty simply connected region $D \subset \Delta$ with $0 \in D$, and f is an unramified covering, which transforms the metric $k_\Delta|_D$ into the metric a_β . Let G_j be the Fuchsian group such that $f_j(z) = f_j(z')$ iff $z' = g(z)$ for a certain $g \in G_j$. We say that a Möbius transformation h is *limiting* for $\{G_j\}$, if there is a sequence $\{g_j | g_j \in G_j\}$ that converges to h uniformly on compact subsets of Δ . Let G be the group of holomorphic one-to-one mappings D to D such that $f(z) = f(z')$, if $z' = g(z)$ for a certain $g \in G$.

Lemma 5.1. *The set H of limiting Möbius transformations is a subgroup of G of finite index.*

Proof. By construction, H is a group and for each pair $z, z' \in D$ the equality $h(z) = z'$ for an element $h \in H$ implies $f(z) = f(z')$. Hence $H \subset G$. As in the preceding section $\tau: V \rightarrow R$ is a holomorphic retraction of a Stein neighborhood V of R . Consider all the loops $\{\gamma: \partial\Delta \rightarrow R\}$ such that $\gamma(1) = f(0)$ and for an arbitrary large j there is a loop γ_j in Γ_{b_j} with $\gamma_j(1) = f_j(0)$ and $\gamma = \tau \circ \gamma_j$. These loops generate a subgroup H_1 of finite index in $\pi_1(R, f(0))$. This index does not exceed the multiplicity of zero of the function Φ on R . Since $\pi_1 \cong G$, one can consider H_1 as a subgroup in G as well. Let γ be a loop in R with $[\gamma] \in H_1$ and $\{\gamma_j \in \Gamma_{b_j}\}$ be the corresponding loops, which converge to γ uniformly. Consider the mappings $\nu_j: \mathbf{R} \rightarrow \Delta$ and $\nu: \mathbf{R} \rightarrow D$ such that $f_j \circ \nu_j(t) = \gamma_j(e^{2\pi i t})$, $f \circ \nu(t) = \gamma(e^{2\pi i t})$, and $\nu(0) = \nu_j(0) = 0$. Since $\gamma_j \rightarrow \gamma$ and $f_j \rightarrow f$, one can see that $\nu_j \rightarrow \nu$ uniformly. By $\tilde{\gamma}_j$ and $\tilde{\gamma}$ we will denote the elements of the Fuchsian groups G_j and G that correspond $[\gamma_j]$ and $[\gamma]$ respectively. Clearly, $\tilde{\gamma}_j^k(t) = \nu_j(t+k)$ and $\tilde{\gamma}^k(t) = \nu(t+k)$ for each integer k . This means $\tilde{\gamma}_j \rightarrow \tilde{\gamma}$ as $j \rightarrow \infty$. Hence $H_1 \subset H$ and H is a subgroup of G of finite index. $\square$

Let $\tilde{R} \rightarrow R$ be an unramified covering that corresponds to the subgroup $H \subset \pi_1(R)$. Then, since D is simply connected, the mapping $\tilde{f}: D \rightarrow D/H \cong \tilde{R}$ is a universal holomorphic covering. Recall that by the hypotheses of Main Theorem G is isomorphic to a Fuchsian group of the first kind G' , acting on Δ . More precisely, there is a biholomorphic mapping $\varphi: \Delta \rightarrow D$ such that φ generates isomorphism between G and G' . Therefore H is isomorphic to a subgroup H' of finite order in G' . Hence H' is a Fuchsian group of the first kind as well. According to [G, §3, Lemma 3] it is easy to check now that, since the closure of the orbits $\{h'(0) | h' \in H'\}$ coincides with $\partial\Delta$, the closure of orbits $\{h(0) | h \in H\}$ must coincide with ∂D . Assume that z is a point of $\partial D \cap \Delta$. Choose an arbitrary small neighborhood U of z and element $\tilde{v}$, $\tilde{\eta} \in H$ so that $\tilde{v}(0)$ and $\tilde{\eta} \circ \tilde{v}(0) \in U \cap D$. Let $\tilde{\mu}$, $\tilde{\gamma} \in H$ be noncommutative elements. Then $\tilde{\eta}$, $\tilde{v}^{-1}$, $\tilde{\gamma} \circ \tilde{v}^{-1}$, $\tilde{\mu} \circ \tilde{v}^{-1}$ cannot belong to a cyclic subgroup of H . Hence one of the pairs $\tilde{\eta}$ and $\tilde{v}^{-1}$, $\tilde{\eta}$ and $\tilde{\gamma} \circ \tilde{v}^{-1}$ or $\tilde{\eta}$ and $\tilde{\mu} \circ \tilde{v}^{-1}$ are not commutative. Consider the corresponding noncommutative pair of elements in G_j for a sufficiently large

j . Put $z' = \tilde{v}(0)$. Application of Lemma 2.1 to the above pair and the point z' leads to a contradiction. Thus $D = \Delta$ and by Corollary 3.3 $k_R = \lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}$. This implies immediately that for every sequence $\{b_j\} \subset \Delta^*$ with hyperbolic fibers $\{\Gamma_{b_j}\}$ and $b_j \rightarrow 0$ $k_R = \lim_{j \rightarrow \infty} k_{\Gamma_{b_j}}$. The last thing we need to confirm is that if there exists a sequence $\{b_j\} \rightarrow 0$ with nonhyperbolic fibers $\{\Gamma_{b_j}\}$ then R cannot be hyperbolic. Assume that such a sequence exists. Then Γ_{b_j} is biholomorphically equivalent to C or C^* . Hence R has no handle, for if it had, then all of the fibers Γ_{b_j} would have handles as well for sufficiently large j . Since a Fuchsian group of the first kind corresponds to the Riemann surface R , R is different from Δ , Δ^* or an annulus. Thus $\pi_1(R)$ has, at least, two generators $[\gamma_1]$ and $[\gamma_2]$. One may suppose that the loops γ_1 and γ_2 have no points of self-intersection. Note that the proof of Lemma 4.2 does not use the assumption that $\{\Gamma_{b_j}\}$ are hyperbolic, i.e., it remains true without this assumption. Thus, since Γ_{b_j} is biholomorphically equivalent to C or C^* either γ_1^k or γ_2^k must be approximated by contractible loops in $\{\Gamma_{b_j}\}$ for a certain integer k . This contradicts Lemma 4.2. Hence there is no sequence $\{b_j\} \rightarrow 0$ with nonhyperbolic fibers $\{\Gamma_{b_j}\}$. The main theorem is proved.

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