

PREScribing SCALAR CURVATURES ON THE CONFORMAL CLASSES OF COMPLETE METRICS WITH NEGATIVE CURVATURE

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ABSTRACT. Let (M^n, g) be a complete noncompact Riemannian manifold with the curvature bounded between two negative constants. Given a function K on M^n , in terms of the behaviors of K at infinite, we give a fairly complete answer to when the K can be the scalar curvature function of a complete metric g_1 which is conformal to g .

1. INTRODUCTION

Let (M^n, g) be a n -dimensional Riemannian manifold with a metric g , $n \geq 3$. An interesting problem in Riemannian geometry is to study if one can conformally deform the metric g to a metric g_1 such that the scalar curvature of g_1 is a prescribed function K on M^n . In the case that M^n is compact and K is a constant, this is just the famous Yamabe problem. The Yamabe problem had drawn a lot of attention from mathematicians because the equivalent PDE problem stands for a critical case that the standard analytic machinery can apply. The solution of the Yamabe problem was first claimed by Yamabe in 1960 [26]. In 1968, Trüdinger [25] found a serious gap in Yamabe's proof and filled the gap for the case that the metric has nonpositive scalar curvature. Later Aubin [2] showed the solution for the case $n \geq 6$ and (M^n, g) is nonconformally flat. The remaining cases, which are the most difficult, were finally solved by Schoen [21]. Also see [15] for a unified proof in all cases. For recent progress of the general problem of prescribing scalar curvatures on a compact manifold, one may see [22] and [13].

In this paper, we consider the case that (M^n, g) is complete noncompact. Since in this case, the metric g is complete, we require the metric g_1 to also be complete. For the sake of clarity, we restate the problem.

Let (M^n, g) be a complete noncompact Riemannian manifold, $n \geq 3$. Given a function K on (M^n, g) , is there a complete metric g_1 , which is conformal to g and the scalar curvature of g_1 is K ?

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If we write $g_1 = u^{4/(n-2)}g$ for some $u > 0$ on M^n , then the solvability of the problem is equivalent to the solvability of the following PDE problem.

$$(P) \quad \begin{cases} -\gamma_n \Delta_g u + S_0 u = K u^{(n+2)/(n-2)} & \text{on } M^n, \\ u^{4/(n-2)}g & \text{is complete on } M^n, \\ u > 0 & \text{on } M^n, \end{cases}$$

where $\gamma_n = 4(n-1)/(n-2)$, Δ_g is the Laplace operator with respect to the metric g , and S_0 is the scalar curvature of g .

For problem (P), there have been various kinds of results. In 1982, Ni considered the case that (M^n, g) is the standard Euclidean space. By studying radial symmetric solutions and using the super- and subsolution method, Ni [17] was able to show some existence and nonexistence results for problem (P) in this case. In an attempt to understand the Yamabe problem for complete noncompact manifolds, Aviles and McOwen [6] showed that a complete metric, with scalar curvature bounded above by a negative constant outside a compact set, can always be conformally deformed to a complete metric with negative constant scalar curvature. While in [12], we demonstrated counterexamples to show that the Yamabe problem for complete noncompact manifolds is not always solvable. For the general problem of prescribing scalar curvatures on a complete manifold with negative curvature (the problem was raised as an open problem in [13]), Aviles and McOwen [5] showed the following existence result (Theorem 4 in [5]).

Theorem. *Suppose M^n is complete, noncompact and simply connected. For some positive constants A, B , sectional curvature $\text{Sec}(g)$ of g satisfies $-A^2 \leq \text{Sec}(g) \leq -B^2$. Let $r(x)$ be the distance of a point x to a fixed point with respect to the metric g . Then problem (P) is solvable if*

- (i) $A^2/B^2 \leq (n-1)^2/n(n-2)$;
- (ii) $-C_2 \leq K(x) \leq -C_1$ for $r(x) \geq r_0$, where C_2, C_1, r_0 are some positive constants;
- (iii) $\max_{M^n} K \leq C(r_0, A, B, n, C_1)(A^2/B^2 - (n-1)^2/n(n-2))$, where $C(r_0, A, B, n, C_1)$ is a constant.

Furthermore, if (ii) holds for $r_0 = 0$, then the conditions (i) and (iii) are not necessarily, and the solution is unique if $K \leq 0$ on M^n .

The goal of this paper is to continue to study the problem of prescribing scalar curvatures on complete noncompact manifolds with negative curvature. Our basic assumptions on the manifold (M^n, g) are

M^n is complete, noncompact and simply connected; the curvature of g satisfies $\text{Sec}(g) \leq -B^2$, $\text{Ric}(g) \geq -(n-1)A^2$ for some positive constants A, B .

Actually our results are true for weaker curvature assumptions. We will remark on this point at appropriate places. The results we get will give, in terms of the behaviors at infinity of the prescribed function K , a fairly complete answer to when K can be the scalar curvature of a complete metric g_1 which is conformal to g .

Here are our main results. We assume the basic assumptions for all results. Furthermore, we assume $n \geq 4$ in the first theorem and $n \geq 3$ in the rest. For the results corresponding to Theorem 1 when $n = 3$, see Theorem A and

Theorem B in §3. The distance of a point x to a fixed point with respect to the metric g will be denoted by $r(x)$

The first result says that if K is nonnegative near ∞ , then problem (P) is not solvable.

Theorem 1 ($n \geq 4$). *If there is $r_0 > 0$, such that $K(x) \geq 0$ for $r(x) \geq r_0$. Then problem (P) has no solutions.*

The second result says that if K tends to negative infinity as $r \rightarrow \infty$, then problem (P) is not solvable.

Theorem 2 ($n \geq 3$). *If there are positive constants C , r_0 , σ , such that $K(x) \leq -Ce^{\sigma r(x)}$ for $r(x) \geq r_0$, then problem (P) has no solutions.*

The third result says that problem (P) is solvable for the “rest cases”.

Theorem 3 ($n \geq 3$). *If $K \leq 0$ on M^n , and there are positive constants C_1 , C_2 , α , k , r_0 , such that $-C_2 \leq K(x) \leq -C_1 e^{-\alpha r(x)^k}$ for $r(x) \geq r_0$, then problem (P) has at least one solution.*

Using the inhomogeneity of the equation in problem (P), we see that Theorem 3 is a special case of the following more general result which we will prove.

Theorem 4 ($n \geq 3$). *Given positive constants A , B , r_0 , α , k , there is a constant $C(A, B, n, r_0, k, \alpha) > 0$, such that if $\max_{M^n} K \leq C(A, B, n, r_0, k, \alpha)$, and for some positive constant C_2 , $-C_2 \leq K(x) \leq -e^{-\alpha r(x)^k}$ for $r(x) \geq r_0$. Then problem (P) has at least one solution.*

It is clear that Theorem 4 generalizes Aviles and McOwen’s result.

From the analysis point of view, in the case that (M^n, g) has negative curvature, the difficulties for solving (P) mainly arise from the noncompactness of the manifold M^n and the generality of the metric. For nonexistence results, our idea is to look at the spherical average of the solution u on the geodesic spheres. By studying the ordinary differential inequalities for the spherical average induced from the equation in problem (P), we are able to prove that the metric $g_1 = u^{4/(n-2)}g$ is not complete for a solution u of problem (P), therefore showing a contradiction. Here one of the main technical difficulties is that the estimates of the volume element from the curvature assumptions only allow one control the first order derivatives of the volume element. This difficulty is overcome by an adoption of an idea used in the beautiful proofs of the unique continuation theorems appeared in [9] and [14] (see Remark 1 in §3). For existence results, the method is the super-sub-solution method; in this case, a careful computation and scaling of the metric are needed for us to prove the present result.

When $n = 2$, the problem of prescribing scalar curvatures has also been received extensive study. When (M^2, g) is of negative curvature, the problem is greatly simplified by the Ahlfors-Schwarz lemma. For further detailed results when $n = 2$, one might see [4, 5, 7, 16, 18, 19, 20].

The paper is organized as follows. In §2, we present some results that will be needed in the proofs. In §3, we prove the nonexistence results. The existence results will be proved in §4.

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2. PRELIMINARIES AND NOTATION

In this section, we fix notation and present some known results which will be needed in the proofs.

Let $\text{Sec}(g)$ be the sectional curvature of g , $\text{Ricc}(g)$ the Ricci curvature of g , and S_0 the scalar curvature of g .

Throughout the paper, we will assume that the manifold (M^n, g) satisfies (we refer to these assumptions as basic assumptions).

(M^n, g) is complete, noncompact and simply connected; for some positive constants A, B , $\text{Sec}(g) \leq -B^2$, $\text{Ricc}(g) \geq -(n-1)A^2$.

Fix a point, say o , in M^n , choose a normal coordinate center at o , denote the coordinate of points as x . By our assumptions on (M^n, g) , M^n is diffeomorphic to R^n under this normal coordinate. If we identify M^n with R^n in this way, we have two metrics g_0 and g on R^n (or on M^n), g is the original metric on M^n , g_0 is the Euclidean metric. Later on, when we refer to R^n with the Euclidean metric, we mean R^n (or M^n) with the metric g_0 .

Let $r(x)$ be the distance of x to o with respect to g . Then using the polar coordinates of x , we can write the metric g as

$$g = g_{ij} dx^i \otimes dx^j = dr^2 + r^2 g_{\theta_k \theta_l} d\theta^k \otimes d\theta^l.$$

Set $\sqrt{g} = \sqrt{\det(g_{ij})_{1 \leq i, j \leq n}}$, $\sqrt{g_\theta} = \sqrt{\det(g_{\theta_k \theta_l})_{1 \leq k, l \leq n-1}}$.

We will denote the volume form of g_0 by dx . Denote the volume form on the unit sphere S^{n-1} in the Euclidean space by $d\theta$. Then under the metric g , the volume form on the geodesic sphere $S_r = \{x | r(x) = r\}$ is $\sqrt{g}(r) d\theta$. Under the metric g_0 , the volume form on S_r is $r^{n-1} d\theta$.

The following result estimates \sqrt{g} and $\sqrt{g_\theta}$.

Proposition 1. Let (M^n, g) be a complete Riemannian manifold. $\text{Ricc}(g) \geq -(n-1)A^2$, $\text{Sec}(g) \leq -B^2$. Then for $r > 0$,

$$(1) \quad -n(n-1)A^2 \leq S_0 \leq -n(n-1)B^2;$$

$$(2) \quad \left(\frac{\sinh Br}{Br} \right)^{n-1} \leq \sqrt{g_\theta}(r) \leq \left(\frac{\sinh Ar}{Ar} \right)^{n-1};$$

$$(3) \quad (n-1) \left(B \coth Br - \frac{1}{r} \right) \leq \frac{\partial \log \sqrt{g_\theta}}{\partial r} \leq (n-1) \left(A \coth Ar - \frac{1}{r} \right);$$

$$(4) \quad (n-1)B \coth Br \leq \frac{\partial \log \sqrt{g}}{\partial r} \leq (n-1)A \coth Ar;$$

Proof. (1) is trivial. (2), (3) are consequences of (4) if one notices that $r^{n-1} \sqrt{g_\theta} = \sqrt{g}$ and $\sqrt{g_\theta}(0) = 1$.

For (4), since $\Delta_g r = \partial \log \sqrt{g} / \partial r$, we only have to prove that for $r > 0$,

$$(n-1)B \coth Br \leq \Delta_g r \leq (n-1)A \coth Ar.$$

The left-hand inequality follows from the comparison theorem for the Hessian (for example, see Theorem A on p. 19 in [11].) The right-hand side inequality

follows from the argument presented in the proof of the Proposition 4.1 on p. 231 in [6]. \square

For notational convenience, we rewrite the equation in problem (P). Since the equation is inhomogeneous about the unknown function u , it is clear that (P) is equivalent to following problem. (We still denote it by (P).)

$$(P) \quad \begin{cases} -\Delta_g u + Su = Ku^{(n+2)/(n-2)} & \text{on } M^n, \\ u^{4/(n-2)}g & \text{is complete on } M^n; \\ u > 0 & \text{on } M^n. \end{cases}$$

where $S = \gamma_n^{-1}S_0 = ((n-2)/4(n-1))S_0$, and Δ_g is the Laplace operator with respect to the metric g .

Let $H^n(-1)$ be the hyperbolic space form, that is, $H^n(-1)$ is a complete, noncompact, simply connected manifold with constant sectional curvature -1 .

$$H_{loc}^2(M^n) = \{v|v, dv \in L_{loc}^2(M^n)\}.$$

For the solvability of (P), the following result will be needed in the proof of Theorem 4.

Proposition 2. *Let u^+ , u_- be two continuous functions in $H_{loc}^2(M^n)$, such that*

$$\begin{aligned} -\Delta_g u^+ + Su^+ &\geq K(u^+)^{(n+2)/(n-2)} && \text{on } M^n \text{ weakly;} \\ -\Delta_g u_- + Su_- &\leq K(u_-)^{(n+2)/(n-2)} && \text{on } M^n \text{ weakly;} \\ u_- &\leq u^+ && \text{on } M^n. \end{aligned}$$

Then there is a C^2 solution u to the equation in problem (P) that satisfies $u_- \leq u \leq u^+$.

Proof. See the proof of Proposition 2.1 on p. 229 in [6]. In fact, in the case that both u^+ and u_- are C^2 , the result is well known. The observation needed in the nonsmooth case is that the maximum principles are still available in this case. Then the proof of the smooth case can be adopted without much change. \square

3. NONEXISTENCE RESULTS

In this section, under the basic assumptions about the manifold, we prove nonexistence results for problem (P).

When $K \geq 0$ near ∞ , we deal with $n = 3$ and $n \geq 4$ separately. For $n \geq 4$, we can derive a better result at the present stage (Theorem 1). The case that $n = 3$ will be treated in Theorems A and B.

Theorem 1 ($n \geq 4$). *If there is $r_0 > 0$, such that $K(x) \geq 0$ for $r(x) \geq r_0$, then problem (P) has no solutions.*

Proof. We prove the result by contradiction. That is, if u is a solution of problem (P), we will show that $u^{4/(n-2)}g$ is not complete.

Without loss of generality (by multiplying g by a suitable constant), we may assume

$$A = 1, \quad B < \min\{1, (n-1)^2/16n(n-2)\}.$$

Then for $\nu = \{n(n-2)/4(n-1)^2\}B^2$, it is easy to verify

$$(5) \quad -\frac{1}{n-1}|S_1|\frac{1+\sqrt{1-4\nu}}{2\nu} + (n-1)(1-B) < 0$$

where $S_1 = -\{n(n-2)/4\}B^2 = -\gamma_n^{-1}n(n-1)B^2$. Therefore we can choose a constant $0 < \beta < 1$ such that for $b = \{\beta^2/(n-1)^2\}|S_1|$, we have

$$(6) \quad -\frac{\beta}{n-1}|S_1|\left(1 + \frac{1-2b+\sqrt{1-4b}}{2b}\right) + (n-1)(1-B) < 0.$$

Define

$$v(r) = \frac{1}{r^{n-1}} \int_{\partial B(r)} u \sqrt{g} \, d\theta,$$

and

$$F(r) = \frac{1}{r^{n-1}} \int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} \, d\theta \Big/ \frac{1}{r^{n-1}} \int_{\partial B(r)} u \sqrt{g} \, d\theta.$$

Then

$$(7) \quad \begin{aligned} v'(r) &= \left(\frac{1}{r^{n-1}} \int_{\partial B(r)} u \sqrt{g} \, d\theta \right)' = \left(\int_{S^{n-1}} u \sqrt{g_\theta} \, d\theta \right)' \\ &= \frac{1}{r^{n-1}} \int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} \, d\theta + \frac{1}{r^{n-1}} \int_{\partial B(r)} u \frac{\partial \log \sqrt{g_\theta}}{\partial r} \sqrt{g} \, d\theta. \end{aligned}$$

From the equation in (P), we have

$$\int_{B(r)} -\Delta_g u \, d\text{vol}_g + \int_{B(r)} Su \, d\text{vol}_g = \int_{B(r)} Ku^{(n+2)/(n-2)} \, d\text{vol}_g.$$

Integrating by parts, we get

$$\left(\int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} \, d\theta \right)' = \int_{\partial B(r)} (Su - Ku^{(n+2)/(n-2)}) \sqrt{g} \, d\theta.$$

A computation shows

$$\begin{aligned} F'(r) &= -\frac{n-1}{r}F(r) - F(r)^2 - \kappa(r)F(r) \\ &\quad + \frac{1}{r^{n-1}} \int_{\partial B(r)} (Su - Ku^{(n+2)/(n-2)}) \sqrt{g} \, d\theta \Big/ \frac{1}{r^{n-1}} \int_{\partial B(r)} u \sqrt{g} \, d\theta, \end{aligned}$$

where $\kappa(r)$ is defined as

$$\kappa(r) = \frac{1}{r^{n-1}} \int_{\partial B(r)} u \frac{\partial \log \sqrt{g_\theta}}{\partial r} \sqrt{g} \, d\theta \Big/ \frac{1}{r^{n-1}} \int_{\partial B(r)} u \sqrt{g} \, d\theta.$$

Since $-K \leq 0$ for $r \geq r_0$, and $S \leq -\gamma_n^{-1}n(n-1)B^2 = S_1$, we have

$$(8) \quad F'(r) \leq -[(n-1)/r]F(r) - F^2(r) - \kappa(r)F(r) + S_1 \quad \text{for } r \geq r_0.$$

We claim that there is $r_1 \geq r_0$, such that

$$(9) \quad F(r) < 0 \quad \text{for } r > r_1.$$

In fact, by (3), for $r > 0$,

$$(10) \quad (n-1)(B \coth Br - 1/r) \leq \kappa(r).$$

Hence there is r_2 , such that for $r > r_2$, $\kappa(r) > 0$. By (8), our claim follows if we can prove that there is a $r_1 > r_2$, such that $F(r_1) < 0$. If not, $F(r) > 0$ for $r > r_2$, then from (8), we have

$$F(r) \leq S_1(r - r_2) + F(r_2) \quad \text{for } r > r_2.$$

For r large this contradicts the assumption that $F(r) > 0$.

Now by (8),

$$(11) \quad F'(r) + ((n-1)/r + \kappa(r))F(r) \leq S_1.$$

Integrating (11), using (9), for $r_3 > r_1$, we get

$$(12) \quad F(r) \leq \frac{S_1}{r^{n-1}} \int_{r_3}^r t^{n-1} e^{-\int_t^r \kappa(\alpha) d\alpha} dt.$$

But from (3), it is easy to see that for $r > t$ (since $A = 1$ in our case)

$$(13) \quad e^{-\int_t^r \kappa(\alpha) d\alpha} \geq \left(\frac{\sinh t}{\sinh r} \right)^{n-1} \frac{r^{n-1}}{t^{n-1}}.$$

Equations (12) and (13) imply

$$F(r) \leq S_1 \int_{r_3}^r \left(\frac{\sinh t}{\sinh r} \right)^{n-1} dt.$$

Therefore

$$\begin{aligned} F(r) &\leq S_1 \int_{r_3}^r \frac{(1 - e^{-2r_3})^{n-1}}{e^{(n-1)r}} e^{(n-1)t} dt \\ &= \frac{S_1}{n-1} (1 - e^{-2r_3})^{n-1} (1 - e^{-(n-1)(r-r_3)}). \end{aligned}$$

Then for the $\beta < 1$ chosen at the beginning of the proof, there are r_3, r_4 , such that for $r \geq r_4 > r_3$,

$$(14) \quad F(r) \leq \frac{\beta}{(n-1)} S_1.$$

Since $S_1 < 0$, from (8), we have

$$F'(r) + ((n-1)/r + \kappa(r))F(r) \leq -(|S_1| + [\beta^2/(n-1)^2]S_1^2).$$

Now if we go through the proof of (14) once again, but replace S_1 by $-(|S_1| + \beta^2 S_1^2/(n-1)^2)$, we see that there is an r_5 such that for $r > r_5$,

$$(15) \quad F(r) \leq -\frac{\beta}{n-1} (|S_1| + \frac{\beta^2}{(n-1)^2} S_1^2).$$

In general, for the b defined at the beginning of the proof, if

$$F(r) \leq -\frac{\beta}{n-1} |S_1| (1 + bb_i) \quad \text{for some } b_i > 0 \text{ and } r > r_i,$$

we can conclude that there is r_{i+1} , such that for $r > r_{i+1}$,

$$(16) \quad F(r) \leq -\{\beta/(n-1)\} |S_1| (1 + b(1 + bb_i)^2).$$

Therefore we get a sequence $\{b_i\}_0^\infty$, which is defined by

$$b_{i+1} = (1 + bb_i)^2 \quad \text{for } i > 1, \quad b_0 = 1.$$

It is clear that the sequence $\{b_i\}$ is monotone increasing. Then there are two possibilities: (i) $\{b_i\}$ is bounded; (ii) $\{b_i\}$ is unbounded. In the following, we show that $u^{4/(n-2)}g$ is not complete in both cases.

In case (i), there exists a limit. Say $\lim_{i \rightarrow \infty} b_i = \lambda$.

Then $\lambda = (1 + b\lambda)^2$. Since $\lambda > 0$,

$$\lambda = (1 - 2b + \sqrt{1 - 4b})/2b^2.$$

λ is real only when $1 > 4b$. If λ is real, from (16) and (6), we get

$$-\frac{\beta}{n-1}|S_1|(1 + \frac{1-2b+\sqrt{1-4b}}{2b}) = -\frac{\beta}{n-1}|S_1|(1+b\lambda).$$

We see that there are $\eta < -(n-1)(1-B)$, and r_η , such that

$$(17) \quad F(r) \leq \eta \quad \text{for } r > r_\eta.$$

Then for $r > r_\eta$,

$$\frac{1}{r^{n-1}} \int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} \, d\theta \leq \eta \frac{1}{r^{n-1}} \int_{\partial B(r)} u \sqrt{g} \, d\theta.$$

By (7),

$$v'(r) \leq (\eta + \kappa(r))v(r) \quad \text{for } r > r_\eta.$$

Also from (3), we see that for any $\varepsilon > 0$, there is r_ε , such that

$$\kappa(r) \leq (n-1) + \varepsilon \quad \text{for } r > r_\varepsilon.$$

Then for $r_6 = r_\eta + r_\varepsilon$,

$$v'(r) \leq (\eta + (n-1) + \varepsilon)v(r) \quad \text{for } r > r_6.$$

Therefore for $r > r_6$, we get

$$(18) \quad v(r) \leq v(r_6)e^{(\eta+(n-1)+\varepsilon)(r-r_6)}.$$

Combining this with the definition of $v(r)$,

$$\int_{S^{n-1}} u(r, \theta) \sqrt{g_\theta} \, d\theta \leq C(r_6, \eta, u)e^{(\eta+(n-1)+\varepsilon)r}.$$

But (2) implies

$$\sqrt{g_\theta} \geq C(B, r_6)e^{(n-1)Br} \frac{1}{r^{n-1}} \quad \text{for } r > r_6.$$

Finally we get

$$(19) \quad \int_{S^{n-1}} u(r, \theta) \, d\theta \leq C(r_6, \eta, u, B)e^{(\eta+(n-1)(1-B)+\varepsilon)r}.$$

Since $\eta + (n-1)(1-B) < 0$, we can choose ε such that $\eta + (n-1)(1-B) + \varepsilon < 0$. Then since $2/(n-2) \leq 1$ for $n \geq 4$, we have

$$(20) \quad \begin{aligned} \int_{R^n} u^{2/(n-2)} dx &\leq C \int_0^\infty r^{n-1} \left(\int_{S^{n-1}} u(r, \theta) \, d\theta \right)^{2/(n-2)} \\ &\leq C \int_0^\infty r^{n-1} e^{2(\eta+(n-1)(1-B)+\varepsilon)r/(n-2)} < \infty. \end{aligned}$$

That is,

$$\int_{S^{n-1}} \left(\int_0^\infty r^{n-1} u^{2/(n-2)}(r, \theta) dr \right) d\theta < \infty.$$

Then for some $\theta_0 \in S^{n-1}$,

$$\int_0^\infty r^{n-1} u^{2/(n-2)}(r, \theta_0) dr < \infty.$$

Therefore

$$(21) \quad \int_0^\infty u^{2/(n-2)}(r, \theta_0) dr < \infty.$$

That is, for the metric $u^{4/(n-2)}g$, the curve $C = \{(r, \theta_0) | 0 < r < \infty\}$ has finite length. Thus the metric $u^{4/(n-2)}g$ is not complete.

For case (ii) where $\{b_i\}_0^\infty$ is unbounded, it is obvious that we can choose $\eta < -(n-1)(1-B)$ and r_η such that (17) is true. Then the same proof given above concludes that the metric $u^{4/(n-2)}g$ is also not complete. This completes the proof of Theorem 1. \square

Remark 1. In the proof, we did not take the second order derivative of $v(r)$ directly because the terms

$$\frac{1}{r^{n-1}} \int_{\partial B(r)} \frac{\partial u}{\partial r} \frac{\partial \log \sqrt{g_\theta}}{\partial r} \sqrt{g} d\theta; \quad \frac{1}{r^{n-1}} \int_{\partial B(r)} u \frac{\partial^2 \log \sqrt{g_\theta}}{\partial r^2} \sqrt{g} d\theta$$

cannot be estimated directly by the assumptions on the curvature and the equation in problem (P). This is the reason that we introduce the function $F(r)$. This idea has been used in [9] and [14] to study the unique continuation property for solutions of elliptic equations and systems. The same remark also applies to the proof of Theorem 2.

Remark 2. If one checks the proof of Theorem 1 carefully, it is easy to see that the proof gets through if one has the following estimates which are weaker than the estimates obtained in Proposition 1:

(a) There is $r_1 > 0$, such that

$$\frac{\partial \log \sqrt{g_\theta}}{\partial r} \geq 0 \quad \text{for } r > r_1;$$

(b) There is a function $\varepsilon(r)$, $\varepsilon(r) \rightarrow 0$ as $r \rightarrow \infty$, and $r_1 > 0$, such that

$$\frac{\partial \log \sqrt{g_\theta}}{\partial r} \leq (n-1)A + \varepsilon(r) \quad \text{for } r > r_1,$$

(c) There are positive constants C , r_1 and a function $\mu(r)$, $\mu(r) \rightarrow 0$ as $r \rightarrow \infty$, such that

$$\sqrt{g_\theta} \geq C e^{(n-1)Br + \mu(r)r} \quad \text{for } r > r_1.$$

Remark 3. There may exist a positive solution u to the equation in problem (P) for some K with $K \geq 0$ near ∞ . By Theorem 1, $u^{4/(n-2)}g$ is not complete.

Example 1. Let (M^n, g) be the space form. That is, $(M^n, g) = H^n(-1) = (D^n, g_2)$, where D^n is the unit ball in the Euclidean space. Since $g_2 = 4/(1-|x|^2)^2 dx^2$, $v = (2/(1-|x|^2))^{-(n-2)/2}$ satisfies the corresponding equation with $K = 0$. And $g_1 = v^{4/(n-2)} g_2 = dx^2$ is not complete on D^n .

Remark 4. From the proof, we see that the only place where we require $n > 3$ is to assure $2/(n-2) \leq 1$ in (20). But (19) is true for $n \geq 3$.

When $n = 3$, we can prove the following results

Theorem A ($n = 3$). *If there are positive constants C , r_0 , μ , such that*

$$(22) \quad K(x) \geq Cr(x)^{-\mu} \quad \text{for } r(x) > r_0,$$

then problem (P) has no solutions.

Proof. We will use some conclusions obtained in the proof of Theorem 1 to prove the result.

First of all, from the proof of (21), we only have to prove that for a solution u of problem (P)

$$(23) \quad \int_{R^3} u^2 dx < \infty.$$

From the equation in (P), we have

$$-\left(\int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} d\theta\right)' + \int_{\partial B(r)} Su\sqrt{g} d\theta = \int_{\partial B(r)} Ku^5\sqrt{g} d\theta.$$

Notice $S < 0$ and (2), for $r > 1$,

$$C(B)r^{-2}e^{2Br} \int_{\partial B(r)} Ku^5r^2 d\theta \leq -\left(\int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} d\theta\right)'.$$

Then for any $\xi > 0$,

$$C(B)r^\xi \int_{\partial B(r)} Ku^5r^2 d\theta \leq -r^{\xi+2}e^{-2Br} \left(\int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} d\theta\right)'.$$

By (9) and integration by parts, for $r > r_1 > r_0$,

$$\begin{aligned} C(B) \int_{r_1}^r t^\xi \int_{\partial B(t)} Ku^5t^2 d\theta dt &\leq - \int_{r_1}^r t^{\xi+2}e^{-2Bt} \left(\int_{\partial B(t)} \frac{\partial u}{\partial r} \sqrt{g} d\theta\right)' dt \\ &\leq -r^{\xi+2}e^{-2Br} \left(\int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} d\theta\right) - 2B \int_{r_1}^r t^{\xi+2}e^{-2Bt} \left(\int_{\partial B(t)} \frac{\partial u}{\partial r} \sqrt{g} d\theta\right) dt. \end{aligned}$$

Notice that (7) can be rewritten as

$$v'(r)r^2 = \int_{\partial B(r)} \frac{\partial u}{\partial r} \sqrt{g} d\theta + r^2\kappa(r)v(r).$$

Then (18), $\eta + 2(1 - B) + \varepsilon < 0$, (3), integration by parts, and $v(r) > 0$ imply that for $r_2 = r_1 + 1$,

$$\begin{aligned}
 (24) \quad & C(B) \int_{r_2}^r \int_{r_1}^s t^\xi \int_{\partial B(t)} K u^5 t^2 d\theta dt ds \\
 & \leq - \int_{r_2}^r s^{\xi+4} e^{-2Bs} (v'(s) - \kappa(s)v(s)) ds \\
 & \quad - 2B \int_{r_2}^r \int_{r_1}^s t^{\xi+4} e^{-2Bt} (v'(t) - \kappa(t)v(t)) dt ds \\
 & \leq C_1(A, B, r_1, \xi)r + C_2(A, B, r_1, \xi) \quad \text{for some constants } C_1, C_2.
 \end{aligned}$$

Now by Hölder inequality and (22), we have

$$C t^{\xi-\mu-3} \left(\int_{\partial B(t)} u^2 t^2 d\theta \right)^{5/2} \leq t^\xi \int_{\partial B(t)} K u^5 t^2 d\theta \quad \text{for } t > r_0$$

and

$$\begin{aligned}
 & \int_{r_2}^r \left(\int_{B(s) \setminus B(r_1)} u^2 dx \right)^{5/2} ds \\
 & \leq \int_{r_2}^r \left(\int_{r_1}^s t^{\xi-\mu-3} \left(\int_{\partial B(t)} u^2 t^2 d\theta \right)^{5/2} dt \right) \left(\int_{r_1}^s t^{-2(\xi-\mu-3)/3} dt \right)^{3/2} ds.
 \end{aligned}$$

If we choose $\xi = 6 + \mu$, then

$$\left(\int_{r_1}^s t^{-2(\xi-\mu-3)/3} dt \right)^{3/2} = \left(\int_{r_1}^s t^{-2} dt \right)^{3/2} < \infty.$$

Combining (24) with the above inequalities, we finally get that for $r > r_2$,

$$\int_{r_2}^r \left(\int_{B(s) \setminus B(r_1)} u^2 dx \right)^{5/2} ds \leq C_1(A, B, r_1, \mu)r + C_2(A, B, r_1, \mu).$$

Now it is clear that (23) is true. \square

We can do better for the hyperbolic space form. The argument in the following result actually works for all dimensions $n \geq 3$.

Theorem B. *If $(M^3, g) = H^3(-1)$ and there is a constant $r_0 > 0$ such that $K(x) \geq 0$ for $r(x) > r_0$, then problem (P) has no solutions.*

Proof. This result is basically a consequence of Theorem 5.1 in [23]. Let u be a solution of (P). By assumption, $(M^3, g) = H^3(-1) = (D^3, g_2)$ = hyperbolic space form, where D^3 is the 3-dimensional unit ball. Denote Euclidean coordinates on D^3 by x and $v = (2/(1 - |x|^2))^{1/2}$. Then $g_2 = v^4 dx^2$. Now $g_1 = u^4 g_2 = (uv)^4 dx^2$ is complete and the scalar curvature of g_1 is K . Therefore

$$\begin{aligned}
 & -\gamma_3 \Delta(uv) = K(uv)^5 \quad \text{on } D^3; \\
 & (uv)^4 dx^2 \quad \text{is complete on } D^3,
 \end{aligned}$$

where Δ is the Laplace operator of the standard Euclidean metric.

Let ϕ be the first eigenfunction of the following Dirichlet problem.

$$\begin{aligned} -\Delta\phi &= \lambda_1\phi & \text{on } D^3; \\ \phi &= 0 & \text{on } \partial D^3. \end{aligned}$$

Then we may assume $\phi > 0$ in D^3 . For a large constant C , look at the metric $g_3 = (uv + C\phi)^4 dx^2$ on D^3 . Since $\phi > 0$ and $(uv)^4 dx^2$ is complete on D^3 , g_3 is complete on D^3 . Furthermore, the scalar curvature of g_3 is

$$-\gamma_3(\Delta(uv + C\phi))(uv + C\phi)^{-5} = (K(uv)^5 + \lambda_1 C \gamma_3 \phi)(uv + C\phi)^{-5}.$$

Since $\lambda_1 > 0$ and $K(x) \geq 0$ for $|x| > 1 - \delta$ with some $0 < \delta < 1$, it is clear that we can choose a large C such that the scalar curvature of g_3 is nonnegative. Also since D^3 with the Euclidean metric dx^2 is conformal to a half-unit sphere S_+^3 with the standard metric g_4 , we have a metric g_3 which is complete, conformal to g_4 on the half-sphere, and has nonnegative scalar curvature. By Theorem 5.1 in [23], this kind of metric does not exist. That is, (P) has no solutions. \square

Now we consider the case that K tends to $-\infty$ as $r \rightarrow \infty$.

Theorem 2 ($n \geq 3$). *If there are $\sigma > 0$, $r_0 > 0$, $C > 0$ such that*

$$(25) \quad K(x) \leq -Ce^{\sigma r(x)} \quad \text{for } r(x) \geq r_0,$$

then problem (P) has no solutions.

Proof. As in proof of Theorem 1, we will prove that any positive solution u to the equation in problem (P) will not make $u^{4/(n-2)}g$ complete.

First of all, choose a constant q such that $q > n/(n-2)$ and for $p = 1 + 4/(n-2)q$

$$(26) \quad \sigma - 2(n-1)(p-1)(A-B) > 0.$$

From (26), we can choose a constant $\mu < 0$ such that

$$(27) \quad \sigma - (n-1)(p-1)(A-B) + (p-1)\mu > 0$$

and

$$(28) \quad \mu + (n-1)(A-B) < 0.$$

Now define

$$h(r) = \frac{1}{r^{n-1}} \int_{\partial B(r)} u^q \sqrt{g} d\theta$$

and

$$G(r) = \int_{\partial B(r)} u^{q-1} \frac{\partial u}{\partial r} \sqrt{g} d\theta.$$

Then it is easy to compute

$$(29) \quad h'(r) = q(1/r^{n-1})G(r) + \kappa_q(r)h(r)$$

where $\kappa_q(r)$ is defined as

$$\kappa_q(r) = \frac{1}{r^{n-1}} \int_{\partial B(r)} u^q \frac{\partial \log \sqrt{g}}{\partial r} \sqrt{g} d\theta \Big/ \frac{1}{r^{n-1}} \int_{\partial B(r)} u^q \sqrt{g} d\theta.$$

Multiply u^{q-1} on the both sides of the equation in the problem (P) and integrate by parts to get

$$\begin{aligned} & - \int_{\partial B(r)} u^{q-1} \frac{\partial u}{\partial r} \sqrt{g} d\theta + \int_{B(r)} (q-1) u^{q-2} |\nabla u|^2 d\text{vol}_g + \int_{B(r)} S u^q d\text{vol}_g \\ & = \int_{B(r)} K u^{qp} d\text{vol}_g. \end{aligned}$$

Therefore

$$(30) \quad G'(r) = \int_{\partial B(r)} (q-1) u^{q-2} |\nabla u|^2 \sqrt{g} d\theta + \int_{\partial B(r)} (S u^q - K u^{qp}) \sqrt{g} d\theta.$$

Set $S_2 = -\gamma_n^{-1} n(n-1)A^2 = -[n(n-2)/4]A^2$, and for $r \geq r_0$,

$$K_1(r) = \left\{ \frac{1}{r^{n-1}} \int_{\partial B(r)} |K|^{-1/(p-1)} \sqrt{g} d\theta \right\}^{-1}.$$

Then for $r \geq r_0$, by the Hölder inequality,

$$\left(\frac{1}{r^{n-1}} \int_{\partial B(r)} u^q \sqrt{g} d\theta \right)^p \leq \left(\frac{1}{r^{n-1}} \int_{\partial B(r)} |K| u^{qp} \sqrt{g} d\theta \right) K_1^{-(p-1)}.$$

Notice $K < 0$ for $r > r_0$ and $q > 1$, (30) implies

$$(31) \quad G'(r)/r^{n-1} \geq S_2 h(r) + K_1^{p-1} h(r)^p.$$

By (25) and (2), for $r \geq r_0$,

$$K_1 \geq C(r_0, A) r^{n-1} e^{-(n-1)Ar + \sigma r/(p-1)}.$$

Combing this with (31), for $r \geq r_0$, gives

$$(32) \quad G'(r) \geq r^{n-1} (S_2 h + C r^{(n-1)(p-1)} e^{\sigma r - (n-1)(p-1)Ar} h^p).$$

Rewrite (29) as

$$(e^{-\int_{r_0}^r \kappa_q(t) dt} h(r))' = \frac{q}{r^{n-1}} e^{-\int_{r_0}^r \kappa_q(t) dt} G(r).$$

Then from (32), we have

$$\begin{aligned} (33) \quad & (r^{n-1} e^{\int_{r_0}^r \kappa_q(t) dt} (e^{-\int_{r_0}^r \kappa_q(t) dt} h(r))')' \\ & \geq r^{n-1} (S_2 h + C r^{(n-1)(p-1)} e^{\sigma r - (n-1)(p-1)Ar} h^p). \end{aligned}$$

Define

$$(34) \quad w(r) = e^{-\int_{r_0}^r \kappa_q(t) dt} h(r).$$

Then (33) becomes

$$\begin{aligned} & (r^{n-1} e^{\int_{r_0}^r \kappa_q(t) dt} w'(r))' \\ & \geq q r^{n-1} (S_2 w + C r^{(n-1)(p-1)} e^{\sigma r - (n-1)(p-1)Ar + (p-1) \int_{r_0}^r \kappa_q(t) dt} w^p) e^{\int_{r_0}^r \kappa_q(t) dt}. \end{aligned}$$

That is ,

$$(35) \quad w'' + \left(\frac{n-1}{r} + \kappa_q(r) \right) w' \geq qS_2 w + C r^{(n-1)(p-1)} e^{\sigma r - (n-1)(p-1)Ar + (p-1) \int_{r_0}^r \kappa_q(t) dt} w^p.$$

Now (3) implies that for $r \geq r_0$,

$$e^{(p-1) \int_{r_0}^r \kappa_q(t) dt} \geq \frac{C(r_0, n, B)}{r^{(n-1)(p-1)}} e^{(n-1)(p-1)Br}.$$

Then

$$(36) \quad w'' + \left(\frac{n-1}{r} + \kappa_q(r) \right) w' \geq qS_2 w + C e^{\sigma r - (n-1)(p-1)(A-B)r} w^p.$$

We will prove that the function w approaches zero fast enough to conclude the metric $u^{4/(n-2)}g$ is not complete.

Set

$$(37) \quad y(r) = \xi e^{\mu r}$$

where the constant ξ will be specified in a moment and μ is determined by (27) and (28).

Then

$$\begin{aligned} y'' + \left(\frac{n-1}{r} + \kappa_q(r) \right) y' - qS_2 y - C e^{\sigma r - (n-1)(p-1)(A-B)r} y^p \\ = y(r) \left\{ \mu^2 + \left(\frac{n-1}{r} + \kappa_q(r) \right) \mu - qS_2 - C e^{\sigma r - (n-1)(p-1)(A-B)r + (p-1)\mu r} \xi^{p-1} \right\}. \end{aligned}$$

Therefore by (27) and (3), there is $r_1 > 1$, such that for arbitrary $\xi > 1$, when $r > r_1$,

$$(38) \quad y'' + \left(\frac{n-1}{r} + \kappa_q(r) \right) y' \leq qS_2 y + C e^{\sigma r - (n-1)(p-1)(A-B)r} y^p.$$

For the constant C in (38), choose ξ large enough such that

$$(39) \quad w(r_1) - \xi e^{\mu r_1} < 0$$

and

$$(40) \quad qS_2 + pC\xi^{p-1} > 0.$$

Set $z(r) = w(r) - y(r)$. From (36) and (38), we have that for $r > r_1$,

$$(41) \quad z'' + \left(\frac{n-1}{r} + \kappa_q(r) \right) z' \geq (qS_2 + C e^{\sigma r - (n-1)(p-1)(A-B)r} p \rho(r)^{p-1}) z(r),$$

where either $w(r) \leq \rho(r) \leq y(r)$ or $y(r) \leq \rho(r) \leq w(r)$.

Now we claim that $z(r)$ cannot achieve a local positive maximum on (r_1, ∞) . In fact if $z(r)$ does achieve a local positive maximum at $r_3 > r_1$, then there are $r_4 < r_5 \leq \infty$ such that $r_4 < r_3 < r_5$ and $z(r) \geq 0$ on (r_4, r_5) . But $z(r) \geq 0$ implies $w(r) \geq y(r) = \xi e^{\mu r}$ and then $\rho(r) \geq \xi e^{\mu r}$. Therefore (40), (41), and $r_1 > 1$ imply

$$(42) \quad \begin{aligned} z'' + \left(\frac{n-1}{r} + \kappa_q(r) \right) z' &\geq (qS_2 + C e^{\sigma r - (n-1)(p-1)(A-B)r} p \rho(r)^{p-1}) z(r) \\ &\geq (qS_2 + Cp\xi^{p-1} e^{\sigma r - (n-1)(p-1)(A-B)r + (p-1)\mu r}) z(r) \geq 0 \quad \text{on } (r_4, r_5). \end{aligned}$$

But at r_3 , $z(r_3) > 0$, $z'(r_3) = 0$, $z''(r_3) \leq 0$. Thus we have a contradiction from (42). That is, our claim is true.

Observing (39), we see that there are only two possibilities:

- (i) $z(r) \leq 0$ on (r_1, ∞) ;
- (ii) for some $r_6 > r_1$, $z(r) > 0$, $z'(r) \geq 0$ on (r_6, ∞) .

We will prove that $u^{4/(n-2)}g$ is not complete if (i) is true and (ii) will not happen. If (i) holds,

$$(43) \quad w(r) \leq \xi e^{\mu r} \quad \text{for } r \geq r_1.$$

Combining this with (34) and the definition of $h(r)$, we have

$$\int_{S^{n-1}} u^q(r, \theta) \sqrt{g_\theta} d\theta \leq \xi e^{\mu r + \int_{r_0}^r \kappa_q(t) dt} \quad \text{for } r \geq r_1.$$

Then by (2),

$$\int_{S^{n-1}} u^q(r, \theta) d\theta \leq \xi e^{\mu r + \int_{r_0}^r \kappa_q(t) dt} \frac{(Br)^{n-1}}{(\sinh Br)^{n-1}} \quad \text{for } r \geq r_1.$$

Using (3), we get

$$\int_{S^{n-1}} u^q(r, \theta) d\theta \leq C(A, B, r_1, n) e^{\mu r + (n-1)(A-B)r} \quad \text{for } r \geq r_1.$$

From (28) and noting that $q > 2/(n-2)$, we can proceed as in the proof of (21) to conclude that the metric $u^{4/(n-2)}g$ is not complete.

If (ii) holds, pick up $r_7 > r_6$. Then $z_1(r) = z(r)/z(r_7)$ satisfies $z_1(r) > 1$, $z_1'(r) \geq 0$ for $r > r_7$, and (41) implies

$$\begin{aligned} z_1'' + ((n-1)/r + \kappa_q(r))z_1' \\ \geq (qS_2 + C(A, B, \xi, n, r_7))e^{\sigma r - (n-1)(p-1)(A-B)r + (p-1)\mu r} z_1(r) \quad \text{for } r > r_7. \end{aligned}$$

Here we used the fact that $w(r) \geq y(r)$. Then $\rho(r) \geq y(r)$ for $r > r_1$.

Denote $\gamma = \frac{1}{2}(\sigma - (n-1)(p-1)(A-B) + (p-1)\mu)$. Then there is $r_8 > r_7$ such that for $r_8 > r_7$,

$$z_1'' + ((n-1)/r + \kappa_q(r))z_1' \geq C e^{\gamma r} z_1(r).$$

Since $z_1'(r_8) \geq 0$, we get

$$(44) \quad z_1'(r) \geq C \int_{r_8}^r \frac{s^{n-1}}{r^{n-1}} e^{-\int_s^r \kappa_q(t) dt} e^{\gamma s} z_1(s) ds.$$

Combining (44) with (3), we have

$$(45) \quad z_1'(r) \geq \frac{C}{e^{(n-1)Ar}} \int_{r_8}^r e^{(n-1)As + \gamma s} z_1(s) ds \quad \text{for } r > r_8.$$

Since $z_1(r) \geq 1$ for $r > r_7$, there is $r_9 > r_8$ such that

$$z_1'(r) \geq \frac{C}{e^{(n-1)Ar}} \frac{e^{(n-1)Ar + \gamma r}}{(n-1)A + \gamma} (1 - e^{((n-1)A + \gamma)(r_8 - r)}) \geq C e^{\gamma r} \quad \text{for } r > r_9 > r_8.$$

That is $w'(r) - y'(r) \geq C e^{\gamma r}$ for $r > r_9$. Since $\mu < 0$, $\gamma > 0$, there is r_{10} such that

$$(46) \quad w'(r) \geq C e^{\gamma r} \quad \text{for } r \geq r_{10}.$$

Then for some $r_{11} > r_{10}$,

$$(47) \quad w(r) > Ce^{\gamma r} \quad \text{for } r > r_{11}.$$

Denote $\zeta = \frac{1}{2}(\sigma - (n-1)(p-1)(A-B))$. From (36) and (47), we have

$$(48) \quad w'' + ((n-1)/r + \kappa_q(r))w' \geq Ce^{\zeta r} w^p \quad \text{for } r > r_{12} > r_{11}.$$

Notice that $\zeta > 0$, (46), (47) and (48) imply (similar to the proof of (45))

$$\begin{aligned} w'(r) &\geq C \int_{r_{12}}^r \frac{1}{e^{(n-1)Ar}} e^{(n-1)As + \zeta s} w^p ds \\ &\geq C \int_{r_{12}}^r \frac{1}{e^{(n-1)Ar}} e^{(n-1)As + \zeta s + \gamma ps} ds \\ &\geq Ce^{\gamma pr} \quad \text{for } r > r_{13} > r_{12}. \end{aligned}$$

Thus for some $r_{14} > r_{13}$,

$$(49) \quad w(r) \geq Ce^{\gamma pr} \quad \text{for } r > r_{14}.$$

Using (46), (48), and (49), go through the proof of (49) once again, we see that there is $r_{15} > r_{14}$ such that

$$(50) \quad w(r) \geq Ce^{\gamma p^2 r} \quad \text{for } r > r_{15}.$$

Repeating the above procedure, in general, we can prove that for any integer k , there is r_k such that

$$(51) \quad w(r) \geq Ce^{\gamma p^k r} \quad \text{for } r > r_k.$$

Denote $\nu = (2(n-1)A + \zeta)/\gamma p^k$. Since $p > 1$, we can choose k large enough such that

$$(52) \quad 2p/(p+1+\nu) > 1.$$

Now fixing one of these k and combining (48) with (3), we have

$$(53) \quad e^{(n-1)Ar} w'(r) \geq C \int_{r_k}^r e^{(n-1)As + \zeta s} w^p ds.$$

Define

$$\psi(r) = \int_{r_k}^r e^{(n-1)As + \zeta s} w^p ds.$$

Then

$$(54) \quad \psi'(r) = e^{(n-1)Ar + \zeta r} w^p.$$

Using $\psi(r)$, (53) can be rewritten as

$$\begin{aligned} e^{2(n-1)Ar + \zeta r} w^p w'(r) &\geq Ce^{(n-1)Ar + \zeta r} w^p \int_{r_k}^r e^{(n-1)As + \zeta s} w^p ds \\ &= C\psi(r)\psi'(r) \quad \text{for } r > r_k. \end{aligned}$$

Then for $r > r_k$, by (51),

$$C\psi(r)\psi'(r) \leq w'(r)w^p(e^{\gamma p^k r})^\nu \leq Cw'(r)w^p w^\nu = Cw'(r)w^{p+\nu}.$$

Integrating the above inequality from r_k to r , we have

$$w^{p+\nu+1}(r) \geq C\psi^2(r) + w^{p+\nu+1}(r_k) - C\psi^2(r_k) \quad \text{for } r > r_k.$$

Combining this with (51) and the definition of $\psi(r)$, we can choose $R_k > r_k$ such that

$$w^{p+\nu+1}(r) \geq C\psi^2(r) \quad \text{for } r > R_k,$$

i.e.,

$$w^p(r) \geq C\psi^{2p/(p+\nu+1)}(r) \quad \text{for } r > R_k.$$

Therefore, by (54), we have

$$\psi'(r) \geq Ce^{(n-1)Ar+\zeta r} \psi^{2p/(p+\nu+1)}(r) \quad \text{for } r > R_k.$$

Finally, integrating the above inequality from R_k to r , we get

$$(55) \quad \frac{1}{1-2p/(p+\nu+1)} \left(\frac{1}{\psi^{2p/(p+\nu+1)-1}(r)} - \frac{1}{\psi^{2p/(p+\nu+1)-1}(R_k)} \right) \\ \geq C \int_{R_k}^r e^{(n-1)As+\zeta s} ds.$$

Since $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$, from (52) we get a contradiction to (55) as $r \rightarrow \infty$. That is, the case (ii) is impossible. \square

Remark 5. The actual estimates about the volume element that we needed in the proof of Theorem 2 are (b) in Remark 2 and

(d) there is a function $\chi(r)$, $\chi(r) \rightarrow 0$ as $r \rightarrow \infty$, and $r_1 > 0$ such that

$$\frac{\partial \log \sqrt{g_\theta}}{\partial r} \geq (n-1)B + \chi(r) \quad \text{for } r > r_1;$$

(e) there are positive constants C , r_1 and a function $\eta(r)$, $\eta(r) \rightarrow 0$ as $r \rightarrow \infty$, such that

$$\sqrt{g_\theta} \leq Ce^{(n-1)Ar+\eta(r)r} \quad \text{for } r > r_1.$$

Remark 6. All arguments in the proofs of Theorems 1, A, and 2 are near ∞ . Therefore if near ∞ on (M^n, g) we have the estimates in Remarks 2 and 5, the conclusions of these theorems hold. For example, we can apply our results to the manifold (M_1^n, g_2) , where $(M_1^n \setminus E_1, g_2) = (M^n \setminus E_0, g)$, E_0 is a compact set in M^n , E_1 is a subset in M_1^n , and (M^n, g) is a manifold satisfying our basic assumptions.

Remark 7. One also observes that there might exist positive solutions to the equation in problem (P) satisfying (25), as shown in the following example.

Example 2. Let (M^n, g) be a manifold satisfying our basic assumptions. Set $v = \phi(r)e^{wr}$, where ϕ is a cutoff function which vanishes near 0 and equals to 1 for $r > 1$. Because $\partial \log \sqrt{g}/\partial r$ is bounded for $r > 1$, we can choose $w < 0$, $|w|$ large, such that

$$w^2 + \frac{\partial \log \sqrt{g}}{\partial r} w + |S| > 0 \quad \text{for } r \text{ large.}$$

Then for

$$K(x) = (-\Delta_g v + Sv)v^{-(n+2)/(n-2)}$$

v satisfies the corresponding equation, and K satisfies (25). But $v^{4/(n-2)}g$ is not complete because $w < 0$.

4. EXISTENCE RESULTS

In this section, under the basic assumptions on the manifold, we prove Theorem 4. Notice that there is a constant $r(k, \alpha)$ such that for $r > r(k, \alpha)$, $-e^{-\alpha r^k} \leq -r^{2k}e^{-\alpha r^{k+1}}$. Then in order to prove Theorem 4, we only have to prove the following result.

Theorem 5 ($n \geq 3$). *Given constants $A, B, r_0, k, \alpha > 0$, there is a constant $C(A, B, n, r_0, k, \alpha) > 0$, such that if*

$$(56) \quad \max_{M^n} K \leq C(A, B, n, r_0, k, \alpha),$$

and for some positive constant C_2 ,

$$-C_2 \leq K(x) \leq -r(x)^{2(k-1)}e^{-\alpha r(x)^k} \quad \text{for } r(x) \geq r_0.$$

Then problem (P) has at least one solution.

Using the inhomogeneity of the equation in problem (P), we see that Theorem 5 is equivalent to

Theorem 5* ($n \geq 3$). *Given constants $A, B, r_0, k, \alpha > 0$, there is a constant $C(A, B, n, r_0, k, \alpha) > 0$ such that if for some constants C_1, C_2 $\max_{M^n} K \leq C_1 C(A, B, n, r_0, k, \alpha)$, and*

$$-C_2 \leq K(x) \leq -C_1 r(x)^{2(k-1)}e^{-\alpha r(x)^k} \quad \text{for } r(x) \geq r_0.$$

Then problem (P) has at least one solution.

Remark 8. Condition (56) is automatically satisfied if $\max_{M^n} K \leq 0$. Therefore we have

Theorem 3 ($n \geq 3$). *If $K \leq 0$ on M^n , and there are positive constants C_1, C_2, α, k, r_0 such that $-C_2 \leq K(x) \leq -C_1 r(x)^{2(k-1)}e^{-\alpha r(x)^k}$ for $r(x) \geq r_0$, then problem (P) has at least one solution.*

Remark 9. Compare Theorem 5 with Theorem 4 in [5]; we removed the restriction on the ratio of A^2/B^2 . In Theorem 4 of [5], the authors assumed

$$A^2/B^2 \leq (n-1)^2/n(n-2).$$

Remark 10. Also, one should notice that Theorem 5 applies to the case that K might approach 0 as $r \rightarrow \infty$. This case is not covered by Theorem 4 in [5].

Proof of Theorem 5. Choose a constant ϱ such that

$$(57) \quad \varrho = 10^7 n(n-2) \frac{A^2}{r_0} + \frac{1}{r_0} + 1.$$

We multiply the metric g by the constant ϱ^2 . That is consider $g_5 = \varrho^2 g$. Since g_5 is conformal to g by a constant multiple, we only have to solve problem (P) for the metric g_5 . For convenience we denote g_5 by g . Then

$$(58) \quad \text{Ricc}(g) \geq -(n-1) \frac{A^2}{\varrho^2}; \quad \text{Sec}(g) \leq -\frac{B^2}{\varrho^2}.$$

The condition about K becomes

$$(59) \quad -C_2 \leq K(x) \leq -\left(\frac{r(x)}{\varrho}\right)^{2(k-1)} e^{-\alpha r(x)^k/\varrho^k} \quad \text{for } r(x) \geq \varrho r_0,$$

since the distance is scaled by the constant ϱ .

Without loss of generality, we may assume $k > 2$. Let

$$(60) \quad m_1 = \max_{M^n} K.$$

We first consider $m_1 > 0$. By multiplying the solution by a constant, using the inhomogeneity of the equation, we only have to solve the following PDE problem:

$$(P^*) \quad \begin{aligned} & -\Delta_g u + Su = (K/m_1)u^{(n+2)/(n-2)} \quad \text{on } M^n; \\ & u^{4/(n-2)}g \quad \text{is complete on } M^n; \\ & u > 0 \quad \text{on } M^n. \end{aligned}$$

We will solve problem (P^*) by constructing a pair of super- and sub-solutions.

Choose a smooth function $\phi(r)$ on $(0, \infty)$ as follows:

- (1) $\phi(r) \geq 0$ on $(0, \infty)$;
- (2) $\phi'(r) \geq 0$ on $(0, \infty)$;
- (3) $\phi(r) = 0$ on $(0, \varrho r_0)$;
- (4) $\phi(r) = 1$ on $(3\varrho r_0, \infty)$;
- (5) $\phi(2\varrho r_0) = \varepsilon_0$;
- (6) $|\phi'(r)| \leq C(\varrho r_0)^{-1}\varepsilon_0$, $|\phi''(r)| \leq C(\varrho r_0)^{-2}\varepsilon_0$ on $(\varrho r_0, 2\varrho r_0)$.

The constant $0 < \varepsilon_0 < 1$ will be specified later.

- (7) $|\phi'(r)| \leq C(\varrho r_0)^{-1}$, $|\phi''(r)| \leq C(\varrho r_0)^{-2}$ on $(2\varrho r_0, 3\varrho r_0)$.

The super solution we will look for is of the form

$$v = v_1\phi + (1 - \phi)v_2$$

where $v_1 = e^{w_1 r^k}$, $v_2 = z_2 e^{w_2 r} f(r)$. The constants z_2, w_1, w_2 , and the function $f(r)$ will be specified later.

Let

$$(61) \quad Lv = (-\Delta_g v + Sv)v^{-(n+2)/(n-2)}.$$

We want to compute Lv on four different intervals.

- (1) $0 \leq r \leq \varrho r_0$,
- (2) $r \geq 3\varrho r_0$,
- (3) $\varrho r_0 \leq r \leq 2\varrho r_0$,
- (4) $2\varrho r_0 \leq r \leq 3\varrho r_0$.

(1) On $(0, \varrho r_0)$. Notice that if $v = v(r)$, a function only depending on r , we have

$$\Delta_g v = \frac{\partial^2 v}{\partial r^2} + \frac{\partial \log \sqrt{g}}{\partial r} \frac{\partial v}{\partial r}.$$

Then it is easy to verify

$$(62) \quad \begin{aligned} Lv_2 = z_2^{-\frac{4}{(n-2)}} e^{-\frac{4}{(n-2)} w_2 r} \Bigg\{ & - (w_2^2 f(r) + 2w_2 f'(r) + f''(r)) \\ & - \frac{\partial \log \sqrt{g}}{\partial r} (w_2 f(r) + f'(r)) + S f(r) \Bigg\} f^{-\frac{(n+2)}{(n-2)}}. \end{aligned}$$

Set

$$f(r) = \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right)^\xi$$

where μ, ξ will be specified later.

A computation shows

$$\begin{aligned} & w_2^2 f(r) + 2w_2 f'(r) + f''(r) + \frac{\partial \log \sqrt{g}}{\partial r} (w_2 f(r) + f'(r)) + |S| f(r) \\ &= \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right)^{\xi-2} \left\{ w_2^2 \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right)^2 + 2\mu \xi w_2 \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right) \frac{1}{(\cosh \mu r)^2} \right. \\ &\quad + \mu^2 \xi (\xi - 1) \frac{1}{(\cosh \mu r)^4} - 2\mu^2 \xi \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right) \frac{\sinh \mu r}{(\cosh \mu r)^3} \\ &\quad + \frac{\partial \log \sqrt{g}}{\partial r} w_2 \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right)^2 \\ &\quad + \mu \xi \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right) \frac{1}{(\cosh \mu r)^2} \frac{\partial \log \sqrt{g}}{\partial r} \\ &\quad \left. + |S| \left(1 + \frac{\sinh \mu r}{\cosh \mu r}\right)^2 \right\}. \end{aligned}$$

Define

$$t = \frac{\sinh \mu r}{\cosh \mu r} \quad \text{for } r \geq 0.$$

Using $(\cosh \mu r)^2 - (\sinh \mu r)^2 = 1$, we have
(63)

$$\begin{aligned} & w_2^2 f(r) + 2w_2 f'(r) + f''(r) + \frac{\partial \log \sqrt{g}}{\partial r} (w_2 f(r) + f'(r)) + |S| f(r) \\ &= (1+t)^{\xi-2} \left\{ \left(w_2^2 + \frac{\partial \log \sqrt{g}}{\partial r} w_2 + |S| \right) (1+t)^2 + 2\mu \xi w_2 (1+t)(1-t^2) \right. \\ &\quad + \mu^2 \xi (\xi - 1) (1-t^2)^2 - 2\mu^2 \xi (1+t)(1-t^2)t \\ &\quad \left. + \mu \xi (1+t)(1-t^2) \frac{\partial \log \sqrt{g}}{\partial r} \right\} \\ &= (1+t)^\xi \left\{ w_2^2 + \frac{\partial \log \sqrt{g}}{\partial r} w_2 + |S| + 2\mu \xi w_2 (1-t) + \mu^2 \xi (\xi - 1) (1-t)^2 \right. \\ &\quad \left. - 2\mu^2 \xi (1-t)t + \mu \xi (1-t) \frac{\partial \log \sqrt{g}}{\partial r} \right\}. \end{aligned}$$

By (58),

$$(64) \quad \frac{\partial \log \sqrt{g}}{\partial r} > 0 \quad \text{for } r > 0.$$

Now in (63), we choose $\mu < 0$, $\xi = \frac{1}{2}$, and $w_2 = -\mu^2$ (the value of μ will be specified in a moment). We have

$$\begin{aligned} (65) \quad & w_2^2 f(r) + 2w_2 f'(r) + f''(r) + \frac{\partial \log \sqrt{g}}{\partial r} (w_2 f(r) + f'(r)) + |S| f(r) \\ &\leq (1+t)^{1/2} \{ \mu^4 + |S| - \mu^3(1-t) - \frac{1}{4} \mu^2 (1-t)^2 - \mu^2 (1-t)t \}. \end{aligned}$$

Here we used the fact that for $\mu < 0$, $r \geq 0$, $-1 < t < 0$. Furthermore, for $0 < r < 3\varrho r_0$,

$$|t| = \left| \frac{\sinh \mu r}{\cosh \mu r} \right| = \frac{1 - e^{2\mu r}}{1 + e^{2\mu r}} \leq 4\mu^2 r \leq 12\mu^2 \varrho r_0.$$

Then (65) implies (also use the fact that $-1 < t \leq 0$.)

$$\begin{aligned}
 (66) \quad & w_2^2 f(r) + 2w_2 f'(r) + f''(r) + \frac{\partial \log \sqrt{g}}{\partial r} (w_2 f(r) + f'(r)) + |S| f(r) \\
 & \leq (1+t)^{1/2} \{ \mu^4 + |S| + 2|\mu|^3 - \frac{1}{4}\mu^2 + 2\mu^2 |t| \} \\
 & = (1+t)^{1/2} \{ |S| + \mu^2(-\frac{1}{4} + 2|t| + \mu^2 + 2|\mu|) \} \\
 & \leq (1+t)^{1/2} \{ |S| + \mu^2(-\frac{1}{4} + 24\mu^2 \varrho r_0 + \mu^2 + 2|\mu|) \}.
 \end{aligned}$$

By (58),

$$(67) \quad |S| \leq \frac{n(n-2)}{4} \frac{A^2}{(\varrho r_0)^2}.$$

Now in (66), choose $\mu = -1/\sqrt{768\varrho r_0}$. Then from (66) and (57), we have

$$\begin{aligned}
 (68) \quad & w_2^2 f(r) + 2w_2 f'(r) + f''(r) + \frac{\partial \log \sqrt{g}}{\partial r} (w_2 f(r) + f'(r)) + |S| f(r) \\
 & \leq (1+t)^{1/2} \left\{ \frac{1}{768\varrho r_0} \left(-\frac{1}{4} + \frac{24}{768} + \frac{1}{768\varrho r_0} + \frac{2}{\sqrt{768\varrho r_0}} \right) + \frac{n(n-2)}{4} \frac{A^2}{(\varrho r_0)^2} \right\} \\
 & \leq (1+t)^{1/2} \left\{ \frac{1}{768\varrho r_0} \left(-\frac{1}{4} + \frac{24}{768} + \frac{1}{768} + \frac{2}{\sqrt{768}} \right) + \frac{n(n-2)}{4} \frac{A^2}{(\varrho r_0)^2} \right\} \\
 & \leq (1+t)^{1/2} \left\{ -\frac{1}{6144\varrho r_0} + \frac{n(n-2)}{4} \frac{A^2}{(\varrho r_0)^2} \right\} < 0.
 \end{aligned}$$

Here we also used a consequence of (57) that $\varrho r_0 > 1$.

From (62) and (68), we can choose $z_2 = z_2(A, n, r_0) > 0$ small enough such that

$$(69) \quad Lv_2 \geq 1 \quad \text{on } (0, 3\varrho r_0).$$

Therefore

$$(70) \quad Lv_2 \geq K/m_1 \quad \text{on } (0, \varrho r_0).$$

(2) On $(3\varrho r_0, \infty)$. We compute

$$(71) \quad Lv_1 = -e^{-4w_1 r^k/(n-2)} \left(k^2 w_1^2 r^{2(k-1)} + k(k-1)w_1 r^{k-2} + \frac{\partial \log \sqrt{g}}{\partial r} k w_1 r^{k-1} + |S| \right).$$

Choose $w_1 = \frac{1}{4}(n-2)\alpha/\varrho^k$. By (4),

$$\begin{aligned}
 (72) \quad & \frac{\partial \log \sqrt{g}}{\partial r} \leq (n-1) \left(\frac{A}{\varrho} \right) \frac{\cosh(Ar/\varrho)}{\sinh(Ar/\varrho)} \\
 & \leq (n-1) \left(\frac{A}{\varrho} \right) \frac{1 + e^{-6Ar_0}}{1 - e^{-6Ar_0}} \quad \text{for } r \geq 3\varrho r_0.
 \end{aligned}$$

Then

$$\begin{aligned}
 & k^2 w_1^2 r^{2(k-1)} + k(k-1)w_1 r^{k-2} + \frac{\partial \log \sqrt{g}}{\partial r} k w_1 r^{k-1} + |S| \\
 & \leq r^{2(k-1)} C_3(n, A, r_0, k, \alpha) \quad \text{for } r \geq 3\varrho r_0.
 \end{aligned}$$

Assume

$$(73) \quad 1/m_1 \geq C_3(n, A, r_0, k, \alpha) \varrho^{2(k-1)}$$

(later we will choose the constant $C(A, B, n, r_0, k, \alpha)$ in the assumption of Theorem 5 such that (73) is true). Then we have

$$(74) \quad Lv_1 \geq -\frac{1}{m_1} \left(\frac{r}{\varrho}\right)^{2(k-1)} e^{-\alpha r^k/\varrho^k} \geq \frac{K}{m_1} \quad \text{for } r \geq 3\varrho r_0.$$

(3) On $(\varrho r_0, 2\varrho r_0)$. By a computation, we have

$$(75) \quad Lv = (v)^{-\frac{n+2}{n-2}} \left\{ \phi v_1^{\frac{n+2}{n-2}} Lv_1 + (1-\phi) v_2^{\frac{n+2}{n-2}} Lv_2 - \frac{\partial \log \sqrt{g}}{\partial r} (v_1 - v_2) \phi'(r) - (v_1 - v_2) \phi''(r) - 2\phi'(v'_1 - v'_2) \right\}.$$

From properties (1)–(6) of ϕ , Proposition 1, and the construction of v_1, v_2 , we see that there is a constant $C^*(n, A, B, r_0, k, \alpha)$ such that for $\varrho r_0 \leq r \leq 2\varrho r_0$,

$$Lv \geq (v)^{-(n+2)/(n-2)} \{-\varepsilon_0 C^*(n, A, B, r_0, k, \alpha) + (1-\varepsilon_0) v_2^{(n+2)/(n-2)} Lv_2\}.$$

Using (69) gives

$$(76) \quad Lv \geq (v)^{-(n+2)/(n-2)} \{-\varepsilon_0 C^*(n, A, B, r_0, k, \alpha) + (1-\varepsilon_0) v_2^{(n+2)/(n-2)}\}.$$

Now we see that there is a small $\varepsilon_0 = \varepsilon_0(n, A, B, r_0, k, \alpha) > 0$ such that if ϕ is defined by this ε_0 , we have

$$Lv \geq 0 \quad \text{on } (\varrho r_0, 2\varrho r_0).$$

Since $K \leq 0$ on $(\varrho r_0, 2\varrho r_0)$,

$$(77) \quad Lv \geq K/m_1 \quad \text{on } (\varrho r_0, 2\varrho r_0).$$

(4) On $(2\varrho r_0, 3\varrho r_0)$. By (69) and (75), we have that on $(2\varrho r_0, 3\varrho r_0)$,

$$(78) \quad Lv \geq (v)^{-(n+2)/(n-2)} \left\{ \phi v_1^{(n+2)/(n-2)} Lv_1 - \frac{\partial \log \sqrt{g}}{\partial r} (v_1 - v_2) \phi'(r) - (v_1 - v_2) \phi''(r) - 2\phi'(v'_1 - v'_2) \right\}.$$

Now if at r_1 with $2\varrho r_0 \leq r_1 \leq 3\varrho r_0$,

$$\begin{aligned} & \phi v_1^{(n+2)/(n-2)} Lv_1(r_1) - \frac{\partial \log \sqrt{g}}{\partial r} (v_1 - v_2) \phi'(r_1) \\ & - (v_1 - v_2) \phi''(r_1) - 2\phi'(r_1)(v'_1(r_1) - v'_2(r_1)) \geq 0, \end{aligned}$$

we certainly have

$$(79) \quad Lv(r_1) \geq 0.$$

If at r_1 with $2\varrho r_0 \leq r_1 \leq 3\varrho r_0$,

$$\begin{aligned} & \phi v_1^{(n+2)/(n-2)} Lv_1(r_1) - \frac{\partial \log \sqrt{g}}{\partial r} (v_1 - v_2) \phi'(r_1) \\ & - (v_1 - v_2) \phi''(r_1) - 2\phi'(r_1)(v'_1(r_1) - v'_2(r_1)) \leq 0, \end{aligned}$$

and noting that on $(2\rho r_0, 3\rho r_0)$, $\phi \geq \varepsilon_0$, then $v \geq \varepsilon_0 v_1$ on $(2\rho r_0, 3\rho r_0)$, we have

$$(80) \quad \begin{aligned} Lv(r_1) \geq (\varepsilon_0 v_1)^{-(n+2)/(n-2)} \Big\{ & \phi v_1^{(n+2)/(n-2)} Lv_1(r_1) \\ & - \frac{\partial \log \sqrt{g}}{\partial r} (v_1(r_1) - v_2(r_1)) \phi'(r_1) \\ & - (v_1 - v_2) \phi''(r_1) - 2\phi'(r_1)(v_1'(r_1) - v_2'(r_1)) \Big\}. \end{aligned}$$

Hence in all cases there is a constant $C_4(n, A, B, r_0, k, \alpha)$ such that

$$(81) \quad Lv \geq -C_4(n, A, B, r_0, k, \alpha) \quad \text{on } (2\rho r_0, 3\rho r_0).$$

Then in order to have

$$(82) \quad Lv \geq K/m_1 \quad \text{on } (2\rho r_0, 3\rho r_0),$$

we only need that on $(2\rho r_0, 3\rho r_0)$,

$$-\frac{1}{m_1} \left(\frac{r}{\rho} \right)^{2(k-1)} e^{-\alpha r^k/\rho^k} \leq -C_4(n, A, B, r_0, k, \alpha).$$

It is enough to have

$$(83) \quad \frac{1}{m_1} \geq \frac{1}{(2r_0)^{2(k-1)}} e^{\alpha(3r_0)^k} C_4(n, A, B, r_0, k, \alpha).$$

Now it is clear that if we choose the constant $C(A, B, n, r_0, k, \alpha)$ in the assumption of Theorem 5 in such a way that

$$(84) \quad \begin{aligned} & C^{-1}(A, B, n, r_0, k, \alpha) \\ & = \max \left\{ \frac{1}{(2r_0)^{2(k-1)}} e^{\alpha(3r_0)^k} C_4(n, A, B, r_0, k, \alpha), \right. \\ & \quad \left. C_3(n, A, r_0, k, \alpha) \rho^{2(k-1)} \right\}. \end{aligned}$$

Then (73) and (83) are true. Therefore (82) is true. Now (70), (74), (77), and (82) imply that v is a supersolution to the equation in problem (P^*) .

On the other hand, since $w_1 > 0$, we can choose a small constant $z_3 > 0$ such that $z_3 \leq v$. But when z_3 is small, we have

$$(85) \quad Lz_3 = z_3^{-4/(n-2)} S \leq K/m_1 \quad \text{on } M^n$$

by the assumption on K and the fact that $S \leq C(n, A, r_0) < 0$. Therefore for z_3 small, z_3 is a subsolution to the equation in problem (P^*) .

Now Proposition 2 yields a solution u to the equation in problem (P^*) satisfying $z_3 \leq u \leq v$. It is clear that this solution is positive and also makes $u^{4/(n-2)}g$ complete because g is complete and $u \geq z_3$.

In the case that $\max K \leq 0$, we choose a positive constant m_1 such that $1/m_1 > C^{-1}(n, A, B, r_0, k, \alpha)$. Then we can proceed as in the previous case to conclude the existence of a solution of problem (P^*) . Therefore there is no restriction for $\max_{M^n} K$ when $K \leq 0$. \square

Remark 11. From the proof of Theorem 5, one can check that the estimates needed for the volume element are (64), (72), and

$$\frac{\partial \log \sqrt{g}}{\partial r} < C(r_1, r_2) \quad \text{for } 0 < r_1 < r < r_2.$$

Thus Theorem 5 actually holds for the following type of complete, noncompact, simply connected manifolds (M^n, g) :

- (1) $S_0 < -C < 0$;
- (2) $\text{Ric}(g) \geq -(n-1)A^2$, $\text{Sec}(g) \leq 0$.

Remark 12. Condition (56) is not a best possible condition in the following sense.

Claim. Let (M^n, g) be a manifold satisfying our basic assumptions. Given positive constants $A, B, r_0, k, \alpha, C_2$; for any $m > 0$, there is a function K_m on (M^n, g) such that

$$-C_2 \leq K_m(x) \leq -r(x)^{2(k-1)}e^{-\alpha r(x)^k} \quad \text{for } r(x) \geq r_0.$$

But $\max_{M^n} K_m \geq m$, and problem (P) is solvable for this K_m .

Sketch of the construction. Choose a function K on M^n such that

$$-C_2 \leq K(x) \leq -r(x)^{2(k-1)}e^{-\alpha r(x)^k} \quad \text{for } r(x) \geq r_0.$$

By (4),

$$\frac{\partial \log \sqrt{g}}{\partial r} \geq C(B, n) \frac{1}{r} \quad \text{for } 0 < r < 1.$$

Also notice that (65) actually holds as

$$\begin{aligned} & w_2^2 f(r) + 2w_2 f'(r) + f''(r) + \frac{\partial \log \sqrt{g}}{\partial r} (w_2 f(r) + f'(r)) + |S|f(r) \\ & \leq (1+t)^{1/2} \{ \mu^4 + |S| - \mu^3(1-t) - \frac{1}{4}\mu^2(1-t)^2 - \mu^2(1-t)t \} \\ & \quad - \mu^2 \frac{\partial \log \sqrt{g}}{\partial r} (1+t)^{1/2}. \end{aligned}$$

Now replace m_1 in the definition of (60) by

$$m_1 = \max_{r(x) \geq 1} K(x).$$

Then for the same choice of μ, z_2, w_2, ξ , we see that we can modify K on $\{x | r(x) \leq 1\}$ to get a K_m satisfying the requirement in the Claim and problem (P) is solvable for this K_m by the same proof given in the proof of Theorem 5. \square

Remark 13. From Remark 12, one might guess that probably no restriction on $\max K$ is necessary. In the following, we give an example to show that the situation might be complicated.

Example 3. Consider the hyperbolic space form $H^n(-1)$. Let

$$v_1 = \left(\frac{1}{4(\cosh r/2)^2} \right)^{(n-2)/2},$$

Then it is easy to see that $-\Delta_g v_1 + S v_1 = 0$. Also set $v_2 = e^{wr}$ with $w < -(n+2)/2$ and

$$w^2 + w \frac{\partial \log \sqrt{g}}{\partial r} + |S| \geq C > 0 \quad \text{for } r \geq r_1$$

where r_1 is some number. Such w exists because $\partial \log \sqrt{g} / \partial r$ is bounded for $r > 1$.

Now consider the metric $g_6 = (v_1 + v_2)^{4/(n-2)} g$. It is clear that g_6 is not complete because $w < 0$. Furthermore the scalar curvature of g_6 is

$$\begin{aligned} K &= \gamma_n (v_1 + v_2)^{-(n+2)/(n-2)} (-\Delta_g (v_1 + v_2) + S(v_1 + v_2)) \\ &= -\gamma_n (v_1 + v_2)^{-(n+2)/(n-2)} \left(w^2 + w \frac{\partial \log \sqrt{g}}{\partial r} + |S| \right) e^{wr} \\ &\approx -C e^{(w+(n+2)/2)r} \quad \text{as } r \rightarrow \infty. \end{aligned}$$

Therefore for some C_1, C_2, r_0, K satisfies

$$-C_2 \leq K(x)/C_1 \leq -e^{(w+(n+2)/2)r(x)} \quad \text{for } r(x) > r_0.$$

That is, we demonstrated a function K/C_1 which satisfies the assumptions of Theorem 5 except (56), but the corresponding equation in problem (P) has a positive solution $u = v_1 + v_2$ which does not make $u^{4/(n-2)} g$ complete.

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