TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 341, Number 1, January 1994

AN EXPLICIT PLANCHEREL FORMULA FOR $U(2, 1)$

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ABSTRACT. The admissible duals of quasi-split unitary groups over nonarchimedean fields are determined. The set of irreducible unitarizable representations, and the Plancherel measure on the unitary dual, is given explicitly.

INTRODUCTION

Let $U(2,1)$ be the quasi-split unitary group associated to a separable quadratic extension E/F of nonarchimedean local fields. In this note, we give an explicit Plancherel formula for $G = U(2, 1)$. Moreover, the full unitary dual of G is explicitly determined.

In [8], the nonsupercuspidal part of the unitary dual of $SU(2, 1)$ is found, using an approach to the theory due to Harish-Chandra. The calculation of c-functions gives explicit formulas for the Plancherel factors μ_α. These factors give the Plancherel measure on the unitary principal series of $SU(2, 1)$ and $U(2, 1)$, and determine points of reducibility. The μ_α also allow one to classify all special representations (discrete series which are not supercuspidal) and all complementary series of the groups $SU(2, 1)$ and $U(2, 1)$. Further, the formal degrees of the special representations are found by calculating the residues at the corresponding poles of the functions μ_α. We note that the rank-1 group $SU(2, 1)$ has a countably infinite number of special representations, while the rank-1 group $SL(2, F)$ has only one special representation. Still, it is true for $SU(2, 1)$ that only finitely many special exponents occur, which answers a question of Clozel [2, 3], in this case. Also, the group $SU(2, 1)$ has infinitely many unitarizable nontempered elliptic representations.

An irreducible representation of G is called supercuspidal if its matrix coefficients are compactly supported modulo the center of G. Completely different techniques are required to classify the supercuspidal part of the dual. In [14], a method based on Hecke algebra isomorphisms developed by Howe and Moy is used to classify all discrete series of G in terms of minimal K-types, in the case that the residual characteristic p of F is odd, and the extension E/F is unramified. Further, the isomorphisms preserve Plancherel measure, so that the formal degrees of all supercuspidal, as well as all special representations, are determined by data transferred from certain other reductive groups G'. In [7], the supercuspidal representations of these groups are shown to be irreducibly induced from open compact subgroups, and their formal degrees are calculated.

Received by the editors September 26, 1990 and, in revised form, May 21, 1991.
1991 *Mathematics Subject Classification*. Primary 22E35; Secondary 22E46.

By constructing additional Hecke algebra isomorphisms, Jabon then transfers the supercuspidal data from the G' to G to explicitly determine the formal degrees of all supercuspidal representations of $U(2, 1)$ in this case.

In [4] Harish-Chandra proves the following Plancherel formula for reductive p-adic groups. Let S be a complete set of standard tori in G, no two of which are conjugate in G. For $A \in S$ and f in the Schwartz space $\mathscr{C}(G)$, define

$$f_A(x) = c^{-2}\gamma^{-1}|W|^{-1} \int_{\mathscr{E}^2(M)} \mu(\omega)d(\omega)(\Theta_\omega, r(x)f)\, d\omega,$$

where M is the centralizer of A in G, $W = N(A)/M$ is the Weyl group associated to A, and $\mathscr{E}^2(M)$ is the set of equivalence classes of irreducible discrete series representations of M. Here, $\omega \in \mathscr{E}^2(M)$ has formal degree $d(\omega)$, and $\mu(\omega)$ is Plancherel measure on the series of unitary induced representations $\mathrm{Ind}_P^G(\omega)$ of G, where $P = MN$ is a parabolic subgroup of G with Levi component M. Then the Plancherel formula is $f = \sum_{A \in S} f_A$.

The reader can consult [10], for example, for a detailed introduction and discussions concerning the Plancherel formula.

Since $G = U(2, 1)$ has rank one and is quasi-split, the only proper standard parabolic $P = MN$ is minimal, and M is abelian. The constant $c = c(G/P)$ equals 1. The Plancherel formula then becomes

$$\begin{aligned} f(1) &= \gamma^{-1}2^{-1} \int_{\mathscr{E}^2(M)} \mu(\omega)(\Theta_\omega, f)\, d\omega + \sum_{\text{d.s.}} d(\pi)(\Theta_\pi, f) \\ &= \gamma^{-1}2^{-1} \int_{\mathscr{E}^2(M)} \mu(\omega)(\Theta_\omega, f)\, d\omega + \sum_{\text{sp.}} d(\pi)(\Theta_\pi, f) + \sum_{\text{sc.}} d(\pi)(\theta_\pi, f) \end{aligned}$$

where the sum over the discrete series of G may be split into sums over the special representations and over the supercuspidal representations of G.

The set $\mathscr{E}^2(M)$ consists of a countable number of components, each of which is a complex variety. We will fix a base point λ in each component as in [18], and write $\omega = (\lambda, \nu)$ for a continuous parameter ν.

In this note we list all unitarizable representations of $G = U(2, 1)$, determine the support of the Plancherel measure, and give explicit formulas for the Plancherel measure $\mu(\omega) = \mu(\lambda, \nu)$ and the formal degrees $d(\pi)$, and for the constant $\gamma = \gamma(G/P)$.

The second-named author would like to thank the Department of Mathematics at Purdue University for its hospitality during his visit on leave during fall of 1987.

1. Preliminaries on $U(2, 1)$

Let E/F be a separable quadratic extension of nonarchimedean local fields, and let $x \mapsto \overline{x}$ be the Galois automorphism of E/F. Let $\mathbf{G}$ be the quasi-split algebraic group $\mathbf{U}(2, 1)$ and let $G = \mathbf{G}(F)$ be the group of F-rational points of $\mathbf{G}$.

We may realize G as the elements of $GL(3, E)$ which preserve the hermitian form

$$\langle (x_1, x_0, x_{-1}), (y_1, y_0, y_{-1}) \rangle = x_1\overline{y}_{-1} - x_0\overline{y}_0 + x_{-1}\overline{y}_1$$

on E^3. Thus the F-rational points of $\mathbf{G}$ are the fixed points in $GL(3, E)$ of the automorphism $g \mapsto Jg^{*-1}J^{-1}$ where $g^* = \overline{g}^t$ is the conjugate transpose of g and

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with respect to the standard basis $\{v_1, v_0, v_{-1}\}$ of E^3.

Define the special unitary group to be $SU(2, 1) = U(2, 1) \cap SL(3, E)$.

Equivalently, one could use the hermitian form

$$\langle (x_1, x_0, x_{-1}), (y_1, y_0, y_{-1}) \rangle = x_1\overline{y}_{-1} + x_0\overline{y}_0 + x_{-1}\overline{y}_1$$

on E^3.

Let A be a special torus of G and let M be the centralizer of A in G. Thus A is a split component of a parabolic subgroup of G with Levi factor $M = Z_G(A)$. Recall the complex structure on the set $\mathscr{E}(M)$ of smooth irreducible representations of M. Define $X(M)$ and $X(A)$ to be the groups of all rational characters of M and A, respectively, which are defined over F. Inclusion $A \to M$ defines an injective homomorphism $X(M) \to X(A)$. Let $a = \mathrm{Hom}(X(A), \mathbf{R}) = \mathrm{Hom}(X(M), \mathbf{R})$ be the real Lie algebra of A, and let $H: M \to \mathrm{Hom}(X(M), \mathbf{R})$ be the mapping determined by $|\chi(m)| = q^{\langle \chi, H(m) \rangle}$ for m in M. Set $a^* = X(A) \otimes \mathbf{R}$ and $a^*_{\mathbf{C}} = X(A) \otimes \mathbf{C}$.

Then $\nu \in a^*_{\mathbf{C}}$ defines a quasicharacter χ_ν of M by $\chi_\nu(m) = q^{i\langle \nu, H(m) \rangle}$. For λ in $\mathscr{E}^2(M)$, set $\lambda_\nu(m) = \lambda(m)\chi_\nu(m)$.

A maximal split torus in $G = U(2, 1)$ or $SU(2, 1)$ is isomorphic to the multiplicative group $F^\times$. We take the standard torus A to be the image of the coroot $x \mapsto \mathrm{diag}(x, 1, x^{-1})$, for x in $F^\times$. The centralizer M of A in $SU(2, 1)$ is isomorphic to the multiplicative group $E^\times$ via $x \mapsto \mathrm{diag}(x, \overline{x}/x, \overline{x}^{-1})$. The centralizer $\widetilde{M}$ of A in $U(2, 1)$ is the product of the group M and a compact group isomorphic to the group E^1 of norm 1 elements in $E^\times$, via $y \mapsto \mathrm{diag}(1, y, 1)$. Note that the Weyl element

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

fixes this compact group and sends $\mathrm{diag}(x, \overline{x}/x, \overline{x}^{-1})$ to $\mathrm{diag}(\overline{x}^{-1}, \overline{x}/x, x)$.

Thus the Weyl element w acts trivially on characters of the compact factor E^1, and acts on characters of $E^\times$ by sending a character $x \mapsto \lambda(x)$ to the character $x \mapsto \lambda(\overline{x}^{-1})$. Thus, if a character $\tilde{\lambda}$ of $\widetilde{M}$ restricts to a character λ of M, then $w\lambda = \lambda$ if and only if $w\tilde{\lambda} = \tilde{\lambda}$. Write $\tilde{\lambda} = (\sigma, \lambda)$ with σ a character of E^1 and λ a character of $E^\times$. The Plancherel measure on the unitary principal series of $U(2, 1)$ is determined by that for $SU(2, 1)$, and we write $\mu(\omega, \nu) = \mu(\sigma, \lambda, \nu) = \mu(\lambda, \nu)$. It follows that conditions for reducibility of unitary principal series, and the existence of complementary series and special representations for $U(2, 1)$, depend only on the restriction of $SU(2, 1)$.

Note that $w\lambda = \lambda$ if and only if $\lambda(x\overline{x}) = 1$ for all $x \in E^\times$.

Let $\overline{P} = M\overline{N}$ be the parabolic opposed to $P = MN$. Recall the definition of the constant $\gamma(G/P)$, e.g., in [4] or [17]. The product mapping from $\overline{N} \times P$ into G is a homeomorphism onto a dense open subset of G. The measure $d\overline{x}$ on the set of representatives $\overline{N}$ for the homogeneous space G/P is given by $\gamma^{-1}\delta_P(\overline{n})^{-1}\,d\overline{n}$, where δ_P is the modulus function of P defined by the formula $d_r x = \delta_F(x) d_l x$ relating left and right Haar measures on P. The value of $\gamma(G/P)$ is given by the integral

$$\gamma = \int_{\overline{N}} \delta_P(\overline{n})^{-1}\,d\overline{n},$$

where δ_P is extended to $G = KP$ by $\delta_P(kp) = \delta_P(p)$.

Let $q_F = q$ be the order of the residue field of F, and let q_E be the order of the residue field of E. Define $\mathrm{ord}_E(x)$ by $|x|_E = q_E^{-\mathrm{ord}_E(x)}$.

For G associated to an unramified extension E/F, $\gamma = 1 + q^{-3}$ if $\mathrm{ord}_E(2)$ is even, and $\gamma = 1 + q^{-1}$ if $\mathrm{ord}_E(2)$ is odd.

For G associated to a ramified extension E/F, $\gamma = 1 + q^{-1}$.

Haar measures are normalized so that $\int_K dk = 1$, $\int_{\overline{N}\cap K} d\overline{n} = 1$, and $\int_{K\cap P} d_l p = 1$. Realize $\overline{N}$ as

$$\overline{N} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & \overline{x} & 1 \end{pmatrix} | x, y \in E,\ x\overline{x} = y + \overline{y} \right\}.$$

Let $d_E x$ and $d_F z$ be Haar measures on E and F normalized so that the rings of integers $\mathscr{O}_E$ and $\mathscr{O}_F$ have volume equal to 1, and let $\{1, \tau\}$ be an $\mathscr{O}_F$-basis for $\mathscr{O}_E$.

Write the entry y in $\overline{N}$ satisfying $y + \overline{y} = x\overline{x}$ as $y = x\overline{x}/2 + z\tau$, with $x \in E$, $z \in F$, assuming $\mathrm{char}(F)$ not equal to 2. Then normalized Haar measure on $\overline{N}$ is given by $|2|_F^{-1}\, d_E x\, d_F z$ if $\mathrm{ord}_E 2$ is even, and by $q_E^{1/2}|2|_F^{-1}\, d_E x\, d_F z$ if $\mathrm{ord}_E 2$ is odd (which can happen only if E/F is unramified).

If $\mathrm{char}(F)$ equals 2, write $y = z + x\overline{x}\tau(\tau + \overline{\tau})^{-1}$ with x in E and z in F. Then normalized Haar measure on $\overline{N}$ is given by $|\tau(\tau+\overline{\tau})^{-1}|_E^{1/2}\, d_E x\, d_F z$ if $\mathrm{ord}_E\, \tau(\tau+\overline{\tau})^{-1}$ is even, and by $q_E^{1/2}|\tau(\tau+\overline{\tau})^{-1}|_E^{1/2}\, d_E x\, d_F z$ if $\mathrm{ord}_E\, \tau(\tau+\overline{\tau})^{-1}$ is odd.

2. Unitary principal series

Let $P = MN$ be a minimal parabolic subgroup of G. Recall the complex structure on the set $\mathscr{E}^2(M)$; $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ defines a quasicharacter χ_ν of M by $\chi_\nu(m) = q^{i\langle \nu, H(m)\rangle}$. For λ in $\mathscr{E}^2(M)$, set $\lambda_\nu(m) = \lambda(m)\chi_\nu(m)$.

In our case, A is one-dimensional and λ is a character of $M = E^\times$. If we choose the complex coordinate s of $\mathfrak{a}_{\mathbf{C}}^*$ by setting $\lambda_s(x) = \lambda(x)|x|_E^s$, for $x \in E^\times$ and $s \in \mathbf{C}$, then $q_E^{-s} = q_F^{-\nu}$. We set $z = q_F^{-\nu}$, so that $|x|_E^s = z^n$ for the integer $n = \mathrm{ord}_E(x)$.

Define the (smooth) principal series representation $\mathrm{Ind}(P, G; \lambda_\nu)$ to be left translation in the space of locally constant functions on G which transform on the right by

$$f(gmn) = \delta_P(m)^{-1/2}\lambda_\nu(m)^{-1} f(g)$$

for all g in G, m in M, and n in N.

Let K be the standard maximal compact subgroup of G, so that $G = KP = KMN$. One could also define the representation of G on the Hilbert space of functions transforming by the above rule, whose restrictions to K are in $L^2(K)$.

An analysis of the standard intertwining operator on certain K-types [8] allows the computation of Harish-Chandra's c-functions, and hence the Plancherel factors μ_α. Note that such analysis [9] then allows one to verify a conjecture of Langlands [11] that a certain product of root numbers and quotients of Artin L-functions provides a valid normalization of the intertwining operators in the case of the minimal principal series representations of quasi-split reductive groups. The Plancherel factors may thus be given by a uniform expression [9]. Cancellation of zeros and poles may occur in this expression. Since we need the location of the zeros and poles, and the residues at the poles, we give the explicit formulas below for the Plancherel factors $\mu(\omega) = \mu(\sigma, \lambda, \nu)$ from the calculation of the c-functions in [8].

Shahidi [16] proves the conjecture of Langlands on normalization of intertwining operators, assuming the inducing representation is generic, for series of representations induced from arbitrary parabolics of quasi-split groups. Assuming two natural conjectures in harmonic analysis of p-adic groups, he then proves the conjecture in general.

Note that work of Macdonald [12] decomposes $L^2(G/\ /K)$, and one may compare the Plancherel formula for K-bi-invariant functions with our formulas to check the constants associated to choice of Haar measure on G.

One should also note that the μ-functions are products of Euler factors, as shown in general in [18], and that Silberger's result does in fact agree with our calculation of c-functions and Plancherel measure for $U(2, 1)$.

A unitary principal series representation $\mathrm{Ind}(P, G; \sigma, \lambda_\nu)$ of $U(2, 1)$ is reducible if and only if the character λ_ν of $E^\times$ is nontrivial, but has trivial restriction to $F^\times$, by the theorem in §7 of [8].

Plancherel measure $\mu(\sigma, \lambda, \nu) = \mu(\lambda, \nu)$ on the unitary principal series representations of $U(2, 1)$ is given by the following formulas. Set $z = q_E^{-s} = q^{-\nu}$, with ν purely imaginary.

I. Suppose that the extension E/F is unramified.

a. Let $\lambda_0 = 1$, so λ_ν is the orbit of unramified characters. The Plancherel factor is then given by

$$\mu(\lambda, \nu) = \gamma^2 \frac{(1-z)(1+z)(1-z^{-1})(1+z^{-1})}{(1-q^{-2}z)(1+q^{-1}z)(1-q^{-2}z^{-1})(1+q^{-1}z^{-1})}$$

and the corresponding unitary principal series are all irreducible.

b. Suppose $\lambda_0 = \lambda$ is ramified of degree $h \geq 1$ and has trivial restriction to $F^\times$. Then

$$\mu(\lambda, \nu) = \gamma^2 q^{2h} \frac{(1+z)(1+z^{-1})}{(1+q^{-1}z)(1+q^{-1}z^{-1})}.$$

The reducible unitary principal series in these cases correspond to the value $z = 1$, i.e., to $\nu = 0$.

c. Otherwise, suppose λ is ramified of degree $h \geq 1$ and the character of $E^\times$ given by $x \mapsto \lambda(x\overline{x})$ is ramified of degree $h' \geq 1$. The restriction of λ to $F^\times$ is then also ramified of degree $h' \geq 1$. Then w fixes no character in the corresponding orbit and the Plancherel factor is constant, given by

$$\mu(\lambda, \nu) = \gamma^2 q^{2h+h'}.$$

The corresponding unitary principal series for these orbits are all irreducible.

II. Suppose that E/F is a ramified extension.

d. Let λ_ν be the orbit of unramified characters. Then

$$\mu(\lambda, \nu) = \gamma^2 \frac{(1-z)(1-z^{-1})}{(1-q^{-1}z)(1-q^{-1}z^{-1})}.$$

The reducible unitary principal series in this case corresponds to the value $z = -1$, i.e., to $\nu = \pi i/\ln(q)$.

e. Let λ_0 be ramified of degree $h \geq 1$ with trivial restriction to $F^\times$. The Plancherel measure is constant on these series, given by

$$\mu(\lambda, \nu) = \gamma^2 q^h.$$

Reducible unitary principal series occur for the two values $z = 1$ and $z = -1$.

f. Let λ_0 be ramified of degree $h \geq 1$ with restriction to $F^\times$ equal the character of order 2 associated to the extension E/F by class field theory. Then

$$\mu(\lambda, \nu) = \gamma^2 q^{h-1} \frac{(1-z)(1+z)(1-z^{-1})(1+z^{-1})}{(1-q^{-1/2}z)(1+q^{-1/2}z)(1-q^{-1/2}z^{-1})(1+q^{-1/2}z^{-1})}.$$

The corresponding unitary principal series are all irreducible for these orbits.

g. Otherwise, w fixes no character in the orbit of λ. Then λ must be a ramified character of degree $h \geq 1$ such that $x \mapsto \lambda(x\overline{x})$ is ramified on $E^\times$ of degree $h' \geq 1$, and the restriction of λ to $F^\times$ is ramified of degree $h'' \geq 1$. Then the unitary principal series representations for such orbits are all irreducible, and the Plancherel factor is constant

$$\mu(\lambda, \nu) = \gamma^2 q^{h+h'-h''}.$$

We note that if d is the differential exponent of the extension E/F and $e = 2$ is the degree of ramification, then $eh''-e < h'+d \leq eh''$, at least if $h''-1 > 2d$.

If, in case I, $\mathrm{ord}_E(2)$ is odd, the formulas should be divided by the constant $q_E = q^2$, which comes from the normalization of Haar measure on $\overline{N}$.

3. Special representations

Special representations are discrete series representations which are not supercuspidal, hence are contained in a representation induced from a proper parabolic. Since our G have rank one, any special must be contained in an $\mathrm{Ind}(P, G; \lambda_\nu)$. Replacing λ_ν by $w\lambda_\nu$ if necessary, we may assume that ν lies in the right half-plane $\mathrm{Re}\,\nu > 0$. Recall that λ is unitary.

By criteria of Casselman, since $\dim(A/Z) = 1$, representation $\mathrm{Ind}(P, G; \lambda_\nu)$ contains a special representation as a composition factor iff $\mathrm{Ind}(P, G; \lambda_\nu)$ is reducible and $\nu \neq 0$. See 5.4.5.2 and 5.4.5.3 of [17]. Further, by 5.4.2.4 of [17], if $\nu_0 \neq 0$, then $\mathrm{Ind}(P, G; \lambda_{\nu_0})$ is reducible if, and only if, $\mu(\lambda, \nu)$ has a pole at $\nu = \nu_0$. This occurs if, and only if, the intertwining operator $A(w, \lambda_{\nu_0})$ has a nontrivial kernel. In this case, the composition series of $\mathrm{Ind}(P, G; \lambda_{\nu_0})$ has length 2. Such points of reducibility λ_{ν_0} will be called special points.

Since $\dim(A/Z) = 1$, the formal degree of the discrete series component π_{λ, ν_0} of an induced representation $\mathrm{Ind}(P, G; \lambda_{\nu_0})$ corresponding to a special point is given as follows by Theorem 5.5.4.3 of [17]. The formal degree of a special representation is expressed in terms of the residue at the corresponding pole of Harish-Chandra's μ-function by the formula

$$d(\pi_{\lambda, \nu_0}) = (-1)[L_A^* : L^*(\lambda)]\gamma^{-1} \operatorname{Res}_{\nu=\nu_0} \mu(\lambda : \nu),$$

assuming that Haar measure is normalized so that $\int_K dk = 1$.

Here, $L_A^* = \{\nu \in a^* | \langle \nu, H(A)\rangle \subseteq 2\pi\mathbf{Z}/\log q\}$, $L^*(\lambda) = \{\nu \in a^* | \lambda \otimes \nu \simeq \lambda\}$, and $L_M^* = \{\nu \in a^* | \langle \nu, H(M)\rangle \subseteq 2\pi\mathbf{Z}/\log q\}$. In general, there is an inclusion of lattices $L_M^* \subseteq L^*(\lambda) \subseteq L_A^*$.

In our case, $M = E^\times$, $A = F^\times$, $L^*(\lambda) = L_M^*$, and by definition of the ramification index e, the index $[L_A^* : L_M^*]$ equals 1 if E/F is unramified and equals 2 if E/F is ramified.

The residue is defined by

$$\mathrm{Res}_{\nu=\nu_0}\, \mu(\lambda : \nu) = \left(1 - \frac{z(\nu_0)}{z}\right) \mu(\lambda : \nu(z))|_{z=z(\nu_0)}.$$

We illustrate the calculation of the formal degree in the case E/F is unramified and λ is ramified of degree $h \geq 1$, with trivial restriction to $F^\times$. Set $z = q_E^{-s} = q^{-2s} = q^{-\nu}$. Then from (Ib) above,

$$\mu(\sigma, \lambda, \nu) = \mu(\lambda, \nu) = \gamma^2 q^{2h} \frac{(1+z^{-1})(1+z)}{(1+q^{-1}z^{-1})(1+q^{-1}z)}.$$

We may assume that $\mathrm{Re}\,\nu > 0$. Then a pole occurs for $\nu_0 = 1 + \pi i/(\log q)$, so that a special point occurs if and only if $z = z(\nu_0) = -q^{-1}$. Then

$$\begin{aligned}\mathrm{Res}_{\nu=\nu_0}\, \mu(\lambda, \nu) &= \gamma^2 q^{2h} \left(1 - \frac{-q^{-1}}{z}\right) \left.\frac{(1+z^{-1})(1+z)}{(1+q^{-1}z^{-1})(1+q^{-1}z)}\right|_{z=-q^{-1}} \\ &= \gamma^2 q^{2h} \left.\frac{(1+z^{-1})(1+z)}{1+q^{-1}z}\right|_{z=-q^{-1}} \\ &= \gamma^2 q^{2h} \frac{(1-q)(1-q^{-1})}{1-q^{-2}} = -\gamma^2 q^{2h} \frac{q-1}{1+q^{-1}}.\end{aligned}$$

Thus the formal degree of the corresponding special representation in such an orbit is given by

$$\begin{aligned}d(\pi_{\lambda,\nu_0}) &= (-1)\gamma^{-1}(-\gamma^2) q^{2h} \frac{q-1}{1+q^{-1}} = \gamma q^{2h} \frac{q-1}{1+q^{-1}} \\ &= (1+q^{-3}) q^{2h} \frac{q-1}{1+q^{-1}} = (q^2 - q + 1) q^{2h-2}(q-1).\end{aligned}$$

For the parabolic rank 1 case, special points in an orbit $\mathrm{Ind}(P, G; \omega_\nu)$ are associated to those unitary representations ω fixed by the nontrivial Weyl group element for which the associated unitary representation $\mathrm{Ind}(P, G; \omega)$ is irreducible. This correspondence is one-to-one [18]. In each connected component of $\mathscr{E}^2(M)$ for which such ω exist, we always choose such an ω as base point.

For $U(2,1)$, special points (σ, λ, ν) are associated in [8] to those characters λ of $E^\times$ whose restriction to $A = F^\times$ is the character of order 2 of $F^\times$ associated to the extension E/F by class field theory, and to the trivial character of $E^\times$. Only two special exponents occur. Thus the conjecture of Clozel [2] on the finiteness of the number of special exponents is true in this case. See also the Introduction to [3].

Special representations of $U(2,1)$ and their formal degrees are given as follows. Let I denote the Iwahori subgroup.

I. Suppose that the extension E/F is unramified.

a. Let $\lambda_0 = 1$, so λ_ν is the orbit of unramified characters. Two special representations occur in this orbit. The special occurring for $z = q^{-2}$ (the Steinberg representation) has formal degree

$$\gamma\frac{(q-1)(q^2+1)}{1+q^{-3}} = (q-1)(q^2+1) = \frac{(q-1)(q^2+1)}{(q^3+1)\,\mathrm{vol}(I)}$$

while the special occurring for $-q^{-1}$ has formal degree

$$\gamma\frac{q(q-1)}{1+q^{-3}} = q(q-1) = \frac{q(q-1)}{(q^3+1)\,\mathrm{vol}(I)}.$$

b. Suppose λ_0 is ramified of degree $h \geq 1$ and has trivial restriction to $F^\times$.

Let $\lambda_\nu(x) = \lambda_0(x)|x|_E^s$. One special representation occurs in each such orbit, for the value $z = -q^{-1}$, which has formal degree

$$\gamma q^{2h+1}\frac{q-1}{q+1} = (q^2-q+1)q^{2h-2}(q-1) = \frac{q^{2h-2}(q-1)}{(q+1)\,\mathrm{vol}(I)}.$$

c. Otherwise, the Weyl element w fixes no point in the orbit. Then λ is ramified of degree $h \geq 1$ and the restriction to $F^\times$ is ramified of degree $h' \geq 1$. No special points occur in these orbits.

II. Suppose that E/F is a ramified extension.

d. Let $\lambda_\nu(x) = |x|_E^\nu$ be the orbit of unramified characters. A special representation (the Steinberg representation) occurs for the value $z = q^{-1}$, which has formal degree

$$2\gamma\frac{q(q-1)}{q+1} = 2(q-1) = \frac{2(q-1)}{(q+1)\,\mathrm{vol}(I)}.$$

e. Let λ_0 be ramified of degree $h \geq 1$ with trivial restriction to $F^\times$. No special points occur in these orbits.

f. Let λ_0 be ramified of degree $h \geq 1$ with restriction to $F^\times$ equal the character of order 2 associated to the extension E/F by class field theory. Two special representations occur in each such orbit, for the values $z = q^{-1/2}$ and $-q^{-1/2}$. They have common formal degree

$$2\gamma\frac{q^h(q-1)}{2(q+1)} = q^{h-1}(q-1) = \frac{q^{h-1}(q-1)}{(q+1)\,\mathrm{vol}(I)}.$$

g. Otherwise, w fixes no point in the orbit. Then λ is a ramified character of degree $h \geq 1$ such that $x \mapsto \lambda(x\overline{x})$ is a ramified character of $E^\times$ of degree $h' \geq 1$. No special points occur in these orbits.

Note that the formal degrees are all integers if Haar measure on G is normalized so that the volume of K is 1. However, in the case that E/F is unramified and $\mathrm{ord}_E(2)$ is odd, the expressions above for the formal degrees should be divided by q^2.

4. Complementary series and nontempered Langlands quotients

In this section we give the unitarizable representations of $U(2,1)$ which do not support Plancherel measure. These are included so that we list the full unitary dual of G.

We may assume that the parameter ν is in the right half-plane.

Recall that the Weyl element w satisfies $w^2 = 1$. Suppose that λ is a unitary character of $E^\times$ for which $\mu(\lambda, 0) = 0$, i.e., for which $W_\lambda = W'_\lambda$ in the theory of R-groups, i.e., such that the normalized operator $\mathscr{A}(w, \lambda, 0)$ is scalar. Then $w\lambda = \lambda$, and these are exactly the characters whose orbits contain a special point. These are also exactly the characters fixed by w for which $\operatorname{Ind}(P, G; \lambda)$ is irreducible.

The adjoint of the normalized intertwining operator $\mathscr{A}(w, \lambda, \nu)$ is $\mathscr{A}(w, \lambda, -w\bar{\nu})$. Suppose next that $w\nu = -\bar{\nu}$, i.e., that ν is real, after perhaps replacing λ by the product of λ and the unramified character of $E^\times$ of order 2.

Then $\mathscr{A}(w, \lambda, \nu)$ is selfadjoint. One defines an invariant hermitian form for the representation $\operatorname{Ind}(P, G; \lambda_\nu)$ by

$$\langle f, g \rangle = (f, \mathscr{A} g),$$

where $(\ ,\ )$ is the pairing between $\operatorname{Ind}(P, G; \lambda_\nu)$ and its contragredient given by integration over K.

Then one gets the Stein complementary series as usual, letting ν range from 0 out to the first point of reducibility, i.e., out to the special point corresponding to such a λ. One uses the fact that $\operatorname{Ind}(P, G; \lambda_0)$ is irreducible and a continuity argument to show that the form is definite. See the discussion in Chapter XVI of the book [10] by Knapp, and also Silberger's results for p-adic groups in [18] and [19].

There are also unitarizable, nontempered elliptic representations which occur as Langlands quotients (see, e.g., Chapter XI of [1]) at the points of reducibility at the ends of the complementary series. The composition series at a point of reducibility (i.e., at a special point) has length 2, and these representations are the quotients of $\operatorname{Ind}(P, G; \lambda_\nu)$ by the associated special representations. Using the fact that the special representation is exactly the kernel of the intertwining operator $A(w, \lambda_\nu)$, one then easily checks that the pairing $\langle f_1, f_2 \rangle = (f_1, A(w, \lambda_\nu) f_2)$ induces a unitary inner product on the quotients at the ends of complementary series.

Note that these quotients are usually infinite-dimensional. Consider, for example, the restrictions to $SU(2, 1)$. Recall first that for p-adic groups, the composition series of $\operatorname{Ind}(\lambda_1)$ and $\operatorname{Ind}(\lambda_2)$ are disjoint unless $\lambda_1 = w\lambda_2$ for some element of the Weyl group. See, e.g., 5.4.4.1 of [17]. One obtains a countably infinite number of quotients which are pairwise inequivalent. The kernel of a finite-dimensional representation of a p-adic group must be an open normal subgroup. Thus the trivial representation is the only irreducible finite-dimensional representation of the group $SU(2, 1)$. The trivial representation occurs in the representation $\operatorname{Ind}(\delta_P^{1/2})$ of $SU(2, 1)$. Hence all other Langlands quotients are infinite-dimensional. Finally, note that the only finite-dimensional representations of $U(2, 1)$ which occur are the one-dimensional characters which factor through the determinant, i.e., the representations which restrict to the trivial representation of $SU(2, 1)$.

5. Supercuspidal representations

For this section only, we assume that the residual characteristic p is not equal to 2. Then Howe-Moy theory will apply to the groups $U(2, 1)$ associated to any separable quadratic extension E/F. We also take the extension E/F to be unramified in this section. Thus $E = F(\sqrt{\varepsilon})$ where ε is a nonsquare unit.

The irreducible representations of $G = (2, 1)$ are classified in this case in [14] in terms of Hecke algebra isomorphisms and nondegenerate representations. In particular, all supercuspidal representations are classified by this method, and in [7] their formal degrees are calculated.

Each square-integrable representation occurs discretely in the unitary dual, and the Plancherel measure is the formal degree.

First we must describe Moy's nondegenerate representations for G. These are specific irreducible representations of specific compact open subgroups of G. Let ϖ be a prime element in F and let $\mathcal{P}_E = \varpi\mathcal{O}_E$. There are three conjugacy classes of parahoric subgroups in G represented by

$$K = \mathbf{G}(\mathcal{O}_F), \quad L = \left\{ g \in G \,\middle|\, g \in \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_e & \mathcal{P}_E^{-1} \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \end{pmatrix} \right\},$$

and the Iwahori

$$I = \left\{ g \in G \,\middle|\, g \in \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix} \right\}.$$

The groups K, L, and I have natural well-known filtrations [15] which we will now describe. Set $K_0 = K$ and $K_i = \{x \in K | x \equiv I \bmod \mathcal{P}_E\}$ for $i \geq 1$. For purposes of duality, let

$$l_0 = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{P}_E^{-1} \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}, \qquad l_1 = \begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix},$$

and $l_{2i+j} = \varpi^i l_j$ for i any integer and $j \in \{0, 1\}$. Then define the filtration of L by $L_0 = L$ and $L_i = \{x \in L | x \in 1 + l_i\}$ for $i \geq 1$. Finally for I, let

$$i_0 = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}, \quad i_1 = \begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix}, \quad i_2 = \begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix},$$

and $i_{3i+j} = \varpi^i i_j$ for i any integer and $j \in \{0, 1, 2\}$. Then $I_0 = I$, and $I_i = \{x \in I | x \in 1 + i_i\}$ for $i \geq 1$. For the purposes of representation theory, we need another filtration of I, of length 4. Let

$$i_0^\flat = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix}, \qquad i_1^\flat = \begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix},$$

$$i_2^\flat = \begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix}, \qquad i_3^\flat = \begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix},$$

and $i_{4i+j}^\flat = \varpi^i i_j^\flat$ for i any integer and $j \in \{0, 1, 2, 3\}$. The nonstandard filtration of I is then $I_0^\flat = I$, $I_i^\flat = \{x \in I | x \in 1 + i_i^\flat\}$. Morris has only recently put this unusual filtration into a general context [13].

If Q represents one of the above parahoric subgroups and $\{Q_i\}_{i=0}^\infty$ represents one of the above filtrations of Q, then Q_i/Q_j is an abelian group for $1 \leq$

$i \leq j \leq 2i$; we must explicitly describe these abelian filtration quotients. To accomplish this, we introduce a bilinear form on g, the Lie algebra of G, by

$$\langle x, y\rangle = \mathrm{tr}_{E/F}(\mathrm{tr}(xy)).$$

It is nondegenerate. If h is a subalgebra of g, let $h^{\perp}$ be the orthogonal complement of h in g. Express Q_i as $\{x \in Q | x \in 1 + \tilde{q}_i\}$ where $\tilde{q}_i$ is some lattice in $GL_3(E)$, and set $q_i = \tilde{q}_i \cap g$ for $i \geq 1$. Then the Cayley transform $c(x) = (1-x)(1+x)^{-1}$ maps p_i bijectively onto P_i and induces a group isomorphism $Q_i/Q_j \cong q_i/q_j$ in the range $1 \leq i \leq j \leq 2i$. We obtain the characters of Q_i/Q_j as follows. Direct calculation reveals that $q_i^{\perp} = q_{-i-d+1}$ where d is the "period" of the filtration $\{Q_i\}$. Thus $d = 1$ for K, $d = 2$ for L, and $d = 3$ or 4 for I depending upon which filtration one is using. Then by duality,

$$(Q_i/Q_j)^{\wedge} \cong q_{-j-d+1}/q_{-i-d+1}$$

for $1 \leq i \leq j \leq 2i$. To make this more explicit, let ψ be a character of the additive group of F with conductor $\mathscr{O}_F$. Then we send the coset $\mathbf{a}$ of $a \in q_{-j-d+1}$ to the character $\Omega_{\mathbf{a}}$ where

$$\Omega_{\mathbf{a}}(c(x)) = \psi\left(\frac{\langle x, -\mathbf{a}\rangle}{2}\right) \quad \text{for } x \in q_i. \tag{5.1}$$

An important special case is when $Q = K$ and $j = i+1$. Then $q_i = \varpi^i g(\mathscr{O}_F)$ where $g(\mathscr{O}_F) = g \cap M_3(\mathscr{O}_E)$ and q_{-i-1}/q_{-i} is naturally isomorphic with $g(\overline{F})$ (where $\overline{F}$ denotes the residue field of F) via multiplication by ϖ^{i+1}. In short, the characters of K_i/K_{i+1} are in one-to-one correspondence with the $\overline{F}$-points of the Lie algebra g; we will make use of this identification below.

We now define *nondegenerate representations* for G. To define *nondegenerate representations of level* 1, observe that a parahoric modulo its first standard filtration subgroup is isomorphic to the rational points of a reductive group defined over $\overline{F}$. in fact,

$$K/K_1 \cong U(2,1)(\overline{F}), \quad L/L_1 \cong U(1,1) \times U(1)(\overline{F}), \quad I/I_1 \cong A(\overline{F}).$$

A nondegenerate representation of level 1 is a pair (Q, Ω) where Q is a standard parahoric (K, L, or I) and Ω is a representation of Q obtained by composing an irreducible cuspidal representation of Q/Q_1 with the natural map $Q \to Q/Q_1$.

A *nondegenerate representation of unramified type* is a pair $(K_i, \Omega_{\mathbf{a}})$ with $i \geq 1$ and $\Omega_{\mathbf{a}} \in (K_i/K_{i+1})^{\wedge}$, where $\varpi^{i+1}\mathbf{a}$ is represented by a nonscalar semisimple element of $g(\overline{F})$.

Finally we define *nondegenerate representations of ramified type.* Consider the following sets $\tilde{\mathbf{a}}$. In (5.2a, b, c, d) a is in $\mathscr{O}_F^{\times}$, in (5.2e) a and c are in $\mathscr{O}_F$ and b is in $(\mathscr{O}_F^{\times})^2$.

TABLE 1. Ramified filtration subgroups

$\mathbf{a}$	Q
(5.2a)	I_{3i-1}
(5.2b)	L_{2i-1}
(5.2c)	$I^{\flat}_{4i-2}$
(5.2d)	I_{3i-2}
(5.2e)	L_{2i-2}

TABLE 2. Transferring subgroups for supercuspidal representations

Nondegenerate representation	Transferring group
(K, Ω), Ω cuspidal	1
(L, Ω), Ω cuspidal	1
$(K_i, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ elliptic	unramified anisotropic torus
$(K_i, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ semisimple $\mathbf{a}$ not elliptic, not scalar	$U(1, 1) \times U(1)(F)$
(Q, ρ), ramified type (5.1a, b, c, d)	ramified anisotropic torus
(Q, ρ), ramified type (5.1e′)	unramified anisotropic torus
(Q, ρ), ramified type (5.1e″)	$U(2) \times U(1)(F)$

(5.2a) $$\varpi^{-i-1}\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varpi a\sqrt{\varepsilon} & 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathscr{P}_E & \mathscr{P}_E & \mathscr{O}_E \\ \mathscr{P}_E & \mathscr{P}_E & \mathscr{P}_E \\ \mathscr{P}_E^2 & \mathscr{P}_E & \mathscr{P}_E \end{pmatrix}\right\},$$

(5.2b) $$\varpi^{-i-1}\left\{\begin{pmatrix} 0 & 1 & 0 \\ \varpi a & 0 & 1 \\ 0 & \varpi a & 0 \end{pmatrix} + \begin{pmatrix} \mathscr{P}_E & \mathscr{P}_E & \mathscr{O}_E \\ \mathscr{P}_E^2 & \mathscr{P}_E & \mathscr{P}_E \\ \mathscr{P}_E^2 & \mathscr{P}_E^2 & \mathscr{P}_E \end{pmatrix}\right\},$$

(5.2c) $$\varpi^{-i-1}\left\{\begin{pmatrix} 0 & 0 & \sqrt{\varepsilon} \\ 0 & 0 & 0 \\ \varpi a\sqrt{\varepsilon} & 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathscr{P}_E & \mathscr{P}_E & \mathscr{P}_E \\ \mathscr{P}_E & \mathscr{P}_E & \mathscr{P}_E \\ \mathscr{P}_E^2 & \mathscr{P}_E & \mathscr{P}_E \end{pmatrix}\right\},$$

(5.2d) $$\varpi^{-i-1}\left\{\begin{pmatrix} 0 & 0 & a\sqrt{\varepsilon} \\ \varpi & 0 & 0 \\ 0 & \varpi a & 0 \end{pmatrix} + \begin{pmatrix} \mathscr{P}_E & \mathscr{P}_E & \mathscr{P}_E \\ \mathscr{P}_E^2 & \mathscr{P}_E & \mathscr{P}_E \\ \mathscr{P}_E^2 & \mathscr{P}_E^2 & \mathscr{P}_E \end{pmatrix}\right\},$$

(5.2e)
$$\varpi^{-i-1}\left\{\begin{pmatrix} \varpi a\sqrt{\varepsilon} & 0 & \sqrt{\varepsilon} \\ 0 & \varpi c\sqrt{\varepsilon} & 0 \\ \varpi^2 b\sqrt{\varepsilon} & 0 & \varpi a\sqrt{\varepsilon} \end{pmatrix} + \begin{pmatrix} \mathscr{P}_E^2 & \mathscr{P}_E & \mathscr{P}_E \\ \mathscr{P}_E^2 & \mathscr{P}_E^2 & \mathscr{P}_E \\ \mathscr{P}_E^3 & \mathscr{P}_E^2 & \mathscr{P}_E^2 \end{pmatrix}\right\} \qquad (i \geq 2).$$

We will distinguish two subcases in (5.2e) according to whether or not $(a-c)^2$ is congruent to b modulo $\mathscr{P}_F$. Let

(5.2e′) $\tilde{\mathbf{a}}$ as in (5.2e) with $(a-c)^2 \neq b \bmod \mathscr{P}_F$,

(5.2e″) $\tilde{\mathbf{a}}$ as in (5.2e) with $(a-c)^2 = b \bmod \mathscr{P}_F$.

Let $\mathbf{a}$ denote $\tilde{\mathbf{a}} \cap g$. Each set $\mathbf{a}$ defines a character $\Omega_{\mathbf{a}}$ on a certain standard or nonstandard parahoric filtration subgroup Q as indicated in Table 1.

TABLE 3. $U(2, 1)(F)$ supercuspidal formal degrees

Nondegenerate representation	Formal degrees $(\times[\mathrm{vol}\, K]^{-1})$
(K, Ω), Ω cuspidal	$(q-1)(q+1)^2$ $(q-1)(q^2-q+1)$ $q(q-1)$ (unique unipotent cuspidal of $U(2, 1)(\overline{F})$)
(L, Ω), Ω cuspidal	$(q-1)(q^2-q+1)$
$(K_i, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ elliptic	$(q-1)(q+1)^2 q^{3i}$ (cubic torus) $(q-1)(q^2-q+1)q^{3i}$ (quadratic torus)
$(K_i, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ semisimple $\mathbf{a}$ not elliptic, not scalar	$(q-1)(q^2-q+1)q^{2i}$ (level 1 $U(1, 1) \times U(1)(F)$ rep) $(q-1)(q^2-q+1)q^{2i+j}$ (unramified $U(1, 1) \times U(1)(F)$ rep) $(q-1)(q+1)(q^2-q+1)q^{2i+j-1}$ (ramified $U(1, 1) \times U(1)(F)$ rep)
$(I_{3i-1}, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ type (5.2a)	$(q-1)(q+1)^2(q^2-q+1)q^{3i-2}$ (cubic torus)
$(L_{2i-1}, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ type (5.2b)	$(q-1)(q+1)(q^2-q+1)q^{3i-2}$ (quadratic torus)
$(I^{\flat}_{4i-2}, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ type (5.2c)	$(q-1)(q+1)(q^2-q+1)q^{3i+2}$ (quadratic torus)
$(I_{3i-2}, \Omega_{\mathbf{a}})$, $i \geq 1$, $\mathbf{a}$ type (5.2d)	$(q-1)(q+1)^2(q^2-q+1)q^{3i-3}$ (cubic torus)
$(L_{2i-2}, \Omega_{\mathbf{a}})$, $i \geq 2$, $\mathbf{a}$ type (5.2e′)	$(q-1)(q+1)(q^2-q+1)q^{3i-3}$ (quadratic torus)
$(L_{2i-2}, \Omega_{\mathbf{a}})$, $i \geq 2$, $\mathbf{a}$ type (5.2e″)	$(q-1)(q+1)(q^2-q+1)q^{2i-3}$ (level 1 $U(2) \times U(1)(F)$ rep) $(q-1)(q+1)(q^2-q+1)q^{2i+j-3}$ (unramified $U(2) \times U(1)(F)$ rep) $(q-1)(q+1)^2(q^2-q+1)q^{2i+j-3}$(ramified $U(2) \times U(1)(F)$ rep)

Define two nondegenerate representations (Q, Ω) and (Q', Ω') to be associate if either (1) Q and $\mathbf{Q}'$ are parahoric subgroups, $Q/Q_1 \cong Q'/Q_1'$ and $\Omega \cong \Omega'$, or (2) $\Omega = \Omega_{\mathbf{a}}$, $\Omega' = \Omega_{\mathbf{a}'}$ with some element of $\mathbf{a}$ conjugate to some element of $\mathbf{a}'$.

Moy [14] showed that for every irreducible admissible representation π of G, there exists a one-dimensional character χ such that $\pi \otimes \chi$ contains a nondegenerate representation (unique up to associates). Furthermore he classified the irreducible representations which contain each nondegenerate representation (Q, ρ) by classifying the representations of the Hecke algebra $H(G//Q, \tilde{\rho})$. He accomplished the latter by constructing a Plancherel measure preserving isomorphism of $H(G//Q, \tilde{\rho})$ with $H(G'//Q', \tilde{\rho}')$ where G' is a different reductive group defined over F. Jabon [7] expressed each of the supercuspidal representations of the various G''s as induced representations from open compact subgroups, calculated their formal degrees, and then transferred the supercuspidal data over to G. This required constructing additional Hecke algebra isomorphisms for the different kinds of supercuspidal representations of the G''s. We refer the reader to [7] for more details. Tables 2 and 3 list the transferring groups as well as the formal degrees of the supercuspidals.

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