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# SOME $q$-BETA AND MELLIN-BARNES INTEGRALS ON COMPACT LIE GROUPS AND LIE ALGEBRAS

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ABSTRACT. Multidimensional generalizations of beta type integrals of Barnes, Ramanujan, Askey-Wilson, and others are evaluated. These integrals are analogues of the summation theorems for multilateral hypergeometric series associated to the simple Lie algebras of classical type and type $G_2$. Many of these integrals can also be written as group integrals over a compact Lie group or conjugation invariant integrals over the corresponding Lie algebra.

## 1. INTRODUCTION

Ramanujan [42] evaluated the following integral:

$$\int_{-\infty}^{\infty} \frac{dt}{\Gamma(a+t)\Gamma(b+t)\Gamma(c-t)\Gamma(d-t)} = \frac{\Gamma(a+b+c+d-3)}{\Gamma(a+c-1)\Gamma(a+d-1)\Gamma(b+c-1)\Gamma(b+d-1)} \tag{1.1}$$

where $a, b, c, d \in \mathbb{C}$ and $\mathrm{Re}(a+b+c+d) > 3$. There is another integral due to Barnes [16] which seems related to (1.1),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)\,dt = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \tag{1.2}$$

where $\mathrm{Re}(a, b, c, d) > 0$. The integral (1.2) is called "Barnes' first lemma" and is an integral analogue of the Gauss summation theorem. It is also an extension of the classical beta integral (see [11]). The integrand in (1.2) is the weight function for a family of orthogonal polynomials called the continuous Hahn polynomials, which were found by Atakishiyev and Suslov [14] and generalized by Askey [8] (see also [13]).

Let $q$ be a real number, $0 < q < 1$. For any complex number $c$, define

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$$[c]_\infty = [c\,;\,q]_\infty = \prod_{k=0}^{\infty}(1 - cq^k)$$

and also $[c]_n = [c]_\infty/[cq^n]_\infty$, for any integer $n$. For any complex number $x$, we will also define $q^x = e^{x\log q}$. Similarly for $n \in \mathbb{Z}$ and $c \in \mathbb{C}$, the ordinary rising factorial is defined by $(c)_n = \Gamma(c+n)/\Gamma(c)$.

There is another pair of integrals with a similar relationship to one another as (1.1) has to (1.2),

$$\int_{-\infty}^{\infty} \frac{\prod_{j=1}^{4}\{[ia_je^u]_\infty[-ia_je^{-u}]_\infty\}}{[-qe^{2u}]_\infty[-qe^{-2u}]_\infty}\,du = \frac{\log(q^{-1})[q]_\infty \prod_{1\le j<k\le 4}[q^{-1}a_ja_k]_\infty}{[q^{-3}a_1a_2a_3a_4]_\infty} \tag{1.3}$$

where $|q^{-3}a_1a_2a_3a_4| < 1$, and

$$\frac{1}{2\pi}\int_0^{\pi} \frac{[e^{2i\theta}]_\infty[e^{-2i\theta}]_\infty}{\prod_{j=1}^{4}[a_je^{i\theta}]_\infty[a_je^{-i\theta}]_\infty}\,d\theta = \frac{[a_1a_2a_3a_4]_\infty}{[q]_\infty \prod_{1\le j<k\le 4}[a_ja_k]_\infty} \tag{1.4}$$

where $|a_j| < 1$ for $1 \le j \le 4$. The integral (1.4) was first evaluated by Askey and Wilson [13] and then also by others [7, 28, 29, 32, and 40] using different methods. The main reason the Askey-Wilson integral (1.4) has attracted so much attention is that the integrand is the weight function for a very general family of orthogonal polynomials in one variable which are called the Askey-Wilson polynomials [13]. The evaluation of the integral (1.4) together with the $q$-Pfaff-Saalschutz summation theorem were the keys to proving the orthogonality of the Askey-Wilson polynomials [13].

Just as the Askey-Wilson polynomials contain the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ as limiting cases, similarly the integral (1.4) has the classical beta integral as a limiting case (see [13]). Thus Askey calls the integral (1.4) a "$q$-beta integral". There are many one variable $q$-extensions of the beta integral, some of which are discussed in Askey [4–10], in Andrews and Askey [2], and in Askey and Roy [11].

From another point of view the integral (1.4) is closely tied to the theory of classical basic hypergeometric series in one variable. It is an integral analogue of the very-well-poised ${}_6\varphi_5$ summation theorem. This integral is also a $q$-analogue of a Mellin-Barnes integral (see [47]).

In the present paper multidimensional generalizations of integral (1.2) and (1.4) related to compact simple Lie groups and the associated Lie algebras will be given. The integrals (1.1) and (1.3) will also be generalized (see Theorem 1.6 below). We shall see that one way of roughly describing the relationship between (1.1) and (1.2) is that both are integral analogues of the ${}_2H_2$ summation theorem for bilateral hypergeometric series. Similarly the integrals (1.3) and (1.4) are both analogues of Bailey's ${}_6\psi_6$ summation theorem for bilateral basic hypergeometric series.

We will also mention one other integral which is an analogue of Ramanujan's ${}_1\psi_1$ summation theorem

$$(1.5)\qquad \int_0^\infty \frac{t^{x-1}[-a^t]_\infty[-bqt^{-1}]_\infty}{[-t]_\infty[-qt^{-1}]_\infty}\,dt = \frac{x[q^x]_\infty[q^{1-x}]_\infty[ab]_\infty}{\sin(\pi x)[aq^{-x}]_\infty[bq^x]_\infty[q]_\infty},$$

where $|a| < |q^x| < |b^{-1}|$. The integral (1.5) was evaluated independently by Askey and Roy [11], and Thiruvenkatacher and Venkatachaliengar [48] (see also Gasper [19]) and is an extension of and can be derived from an integral of Ramanujan [44, p. 182, 45, p. 195]. A multidimensional generalization of (1.5) will be given in Theorem 1.14 below.

The main object of this paper is to find integral analogues of the summation theorems [21–23] for very-well-poised hypergeometric series on Lie algebras. Generally speaking, for each summation theorem we give two distinct integral analogues. The first will be similar to (1.1), (1.3), or (1.5), and the second will be similar to (1.2) or (1.4). The first type of integral analogue turns out to be virtually a direct consequence of the corresponding summation theorem and is in fact equivalent to it. The proof of the second type of integral analogue is patterned after Barnes' proof of (1.2) and the Askey-Wilson proof of (1.4). Roughly speaking, one expands each of the second type of integral as a sum of series of residues. The series are of the form of a simple factor times an ordinary or basic very-well-poised hypergeometric series on the corresponding Lie algebra [22]. These series are summable by the analogue of Bailey's ${}_6\psi_6$ summation theorem (or Dougall's ${}_5H_5$ summation theorem) for the appropriate Lie algebra [22]. The resulting terms are then combined by means of either an elliptic function or a trigonometric function identity.

We will also show that the second type of integral analogue can be written as an integral over the corresponding compact simple Lie group or Lie algebra. In particular, it follows the results in §10 here that the Askey-Wilson integral (1.4) can be rewritten as an integral over the group $SU(2)$, the de Branges-Wilson integral [18, 50] as an integral over the Lie algebra $su(2)$ of $2\times 2$ skew-hermitian matrices with trace 0 and Barnes' first Lemma (1.2) as an integral over $u(1)$, the Lie algebra of $1 \times 1$ skew-hermitian matrices (i.e. the purely imaginary complex numbers). We will generalize the integral (1.4) to integrals over the compact classical groups $SU(n)$, $SO(2n+1, \mathbb{R})$, $SO(2n, \mathbb{R})$, $Sp(n)$, $U(n)$ and the compact real Lie group of type $G_2$. Similarly we will generalize the de Branges-Wilson integral to integrals over the corresponding simple Lie algebras. Finally we will generalize Barnes' first Lemma (1.2) to integrals over the Lie algebra $u(n)$ of $n \times n$ skew-hermitian matrices.

By means of the Weyl integration formula (identity (10.1)) and the corresponding formula for Lie algebras (identity (10.13)), these Lie group and Lie algebra integrals which generalize (1.2), (1.4) and the de Branges-Wilson integral are shown to be equivalent to integrals over a maximal torus or Cartan subalgebra respectively. Choosing coordinates for the maximal torus or a basis for the Cartan subalgebra, these integrals can be written as iterated contour integrals over the unit circle or imaginary axis respectively. It is in the form of iterated contour integrals that we actually prove these generalizations of the integrals (1.2), (1.4) and the de Branges-Wilson integral.

The first main result of this paper is the following generalization of integrals (1.1) and (1.3),

**Theorem 1.6.** *Let* $z_1, z_2, \ldots, z_n \in \mathbb{C}$ *with* $z_i - z_j$ *not a real number for all pairs* $1 \le i \le j \le n$. *Then* (1) *for* $n \ge 1$,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\prod\limits_{\substack{1\le i,j\le n \\ i\neq j}} \Gamma(1 + z_i + t_i - z_j - t_j)}{\prod\limits_{i=1}^{n+1}\prod\limits_{j=1}^{n} \Gamma(a_i + z_j + t_j)\Gamma(b_i - z_j - t_j)}\, dt_1 \cdots dt_n = \frac{\Gamma\left(-1 - 2n + \sum\limits_{i=1}^{n+1}(a_i + b_i)\right)}{\prod\limits_{i,j=1}^{n+1} \Gamma(a_i + b_j - 1)} \tag{1.7}$$

*where* $\mathrm{Re}(\sum_{i=1}^{n}(a_i + b_i)) > 2n + 1$;
(b) *for* $n \ge 2$ *let* $\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} t_i = 0$, *then*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\prod\limits_{i,j=1}^{n} \{[a_i q^{z_j+t_j}]_\infty [b_i q^{-z_j-t_j}]_\infty\}}{\prod\limits_{\substack{1\le i,j\le n \\ i\neq j}} [q^{1+z_i+t_i-z_j-t_j}]_\infty}\, dt_1\, dt_2 \cdots dt_{n-1} = \frac{[q]_\infty^{n-1}\left[q^{1-n}\prod\limits_{i=1}^{n} a_i\right]_\infty \left[q^{1-n}\prod\limits_{i=1}^{n} b_i\right]_\infty \prod\limits_{i,j=1}^{n}[q^{-1}a_i b_j]_\infty}{\left[q^{1-2n}\prod\limits_{i=1}^{n}(a_i b_i)\right]_\infty} \tag{1.8}$$

*where* $|q^{1-2n}\prod_{i=1}^{n}(a_i b_i)| < 1$;
(c) *for* $n \ge 2$ *and* $\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} t_i = 0$, *then*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\prod\limits_{\substack{1\le i,j\le n \\ i\neq j}} \Gamma(1 + z_i + t_i - z_j - t_j)}{\prod\limits_{i,j=1}^{n} \{\Gamma(a_i + z_j + t_j)\Gamma(b_i - z_j - t_j)\}}\, dt_1\, dt_2 \cdots dt_{n-1} = \frac{\Gamma\left(1 - 2n + \sum\limits_{i=1}^{n}(a_i + b_i)\right)}{\Gamma\left(1 - n + \sum\limits_{i=1}^{n} a_i\right)\Gamma\left(1 - n + \sum\limits_{i=1}^{n} b_i\right)\prod\limits_{i,j=1}^{n}\Gamma(a_i + b_j - 1)} \tag{1.9}$$

*where* $\mathrm{Re}(\sum_{i=1}^{n}(a_i + b_i)) > 2n - 1$;
(c) *for* $n \ge 1$ *let* $z_i + z_j$ *not be a real number for any pair* $1 \le i \le j \le n$,

*then*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{+\infty} \frac{\prod_{i=1}^{2n+2}\prod_{j=1}^{n}\{[a_i q^{z_j+t_j}]_\infty [a_i q^{-z_j-t_j}]_\infty\}}{\prod_{\substack{1\le i,j\le n\\ i\neq j}} [q^{1+z_i+t_i-z_j-t_j}]_\infty} \tag{1.10}$$
$$\cdot\left(\prod_{1\le i\le j\le n}[q^{1+z_i+t_i+z_j+t_j}]_\infty [q^{1-z_i-t_i-z_j-t_j}]_\infty\right)^{-1} dt_1\cdots dt_n$$
$$= \frac{[q]_\infty^n \prod_{1\le i<j\le 2n+2}[q^{-1}a_i a_j]_\infty}{\left[q^{-1-2n}\prod_{i=1}^{2n+2} a_i\right]_\infty}$$

*where* $|q^{-1-2n}\prod_{i=1}^{2n+2} a_i| < 1$ ;

(e) *for* $n \ge 1$ *and assumptions as in* (d), *then*

(1.11)

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\prod_{\substack{1\le i,j\le n\\ i\neq j}} \Gamma(1+z_i+t_i-z_j-t_j)}{\prod_{i=1}^{2n+2}\prod_{j=1}^{n}\{\Gamma(a_i+z_j+t_j)\Gamma(a_i-z_j-t_j)\}}$$
$$\cdot \prod_{1\le i\le j\le n}\{\Gamma(1+z_i+t_i+z_j+t_j)\Gamma(1-z_i-t_i-z_j-t_j)\}\, dt_1\cdots dt_n$$
$$= \frac{\Gamma\left(-1-2n+\sum_{i=1}^{2n+2} a_i\right)}{\prod_{1\le i<j\le 2n+2}\Gamma(a_i+a_j-1)}$$

*where* $\mathrm{Re}(\sum_{i=1}^{2n+2} a_i) > 2n+1$ ;

(f) *for* $n = 3$ *and* $\sum_{i=1}^{3} z_i = \sum_{i=1}^{3} t_i = 0$ *let* $z_i$ *not be a real number for all* $i$ , $1 \le i \le 3$ , *then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\prod_{i=1}^{4}\prod_{j=1}^{3}[a_i q^{z_j+t_j}]_\infty [q_i q^{-z_j-t_j}]_\infty \, dt_1\, dt_2}{\prod_{\substack{1\le i,j\le n\\ i\neq j}}[q^{1+z_i+t_i-z_j-t_j}]_\infty \prod_{j=1}^{3}\{[q^{1+z_j+t_j}]_\infty [q^{1-z_j-t_j}]_\infty\}} \tag{1.12}$$
$$= \frac{[q]_\infty^2 \left[q^{-3}\prod_{i=1}^{4} a_i\right]_\infty \prod_{1\le i\le j\le 4}[q^{-1}a_i a_j]_\infty \prod_{1\le i<j<k\le 4}[q^{-2}a_i a_j a_k]_\infty}{\left[q^{-7}\prod_{i=1}^{4} a_i^2\right]_\infty \prod_{i=1}^{4}[a_i]_\infty}$$

*where* $|q^{-7}\prod_{i=1}^{4} a_i^2| < 1$ ;

(g) *With assumptions of* (f)

(1.13)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\prod_{\substack{1\le i,j\le n\\ i\neq j}}\Gamma(1+z_i+t_i-z_j-t_j)\prod_{j=1}^{3}\Gamma(1+z_j+t_j)\Gamma(1-z_j-t_j)}{\prod_{i=1}^{4}\prod_{j=1}^{3}\{\Gamma(a_i+z_j+t_j)\Gamma(a_i-z_j-t_j)\}}1\,dt_1\,dt_2$$

$$=\frac{\Gamma\left(-7+2\sum_{i=1}^{4}a_i\right)\prod_{i=1}^{4}\Gamma(a_i)}{\Gamma\left(-3+\sum_{i=1}^{4}a_i\right)\prod_{1\le i\le j\le 4}\Gamma(a_i+a_j-1)\prod_{1\le i<j<k\le 4}\Gamma(a_i+a_j+a_k-2)}$$

*where* $2\,\mathrm{Re}(\sum_{i=1}^{4}a_i)>7$.

We will also prove the following generalization of integral (1.5):

**Theorem 1.14.** *For* $n\geq 1$, *let* $a_i, b_i, z_i\in\mathbb{C}$ *for* $1\le i\le n$ *with* $z_i-z_j$ *not a real number for all pairs* $1\le i<j\le n$. *Let* $x\in\mathbb{C}$ *and* $z=\sum_{i=1}^{n}z_i$. *Then*

(1.15)

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\left(\prod_{i=1}^{n}t_i\right)^{x-1}\prod_{i,j=1}^{n}\{[a_it_jq^{z_j}]_\infty[qb_it_j^{-1}q^{-z_j}]_\infty\}}{\left[-\prod_{i=1}^{n}t_i\right]_\infty\left[-q\prod_{i=1}^{n}t_i^{-1}\right]_\infty\prod_{\substack{1\le i,j\le n\\ i\neq j}}[t_it_j^{-1}q^{1+z_i-z_j}]_\infty}$$

$$=\frac{(\log(q^{-1}))^{n-1}\pi[q^x]_\infty[q^{1-x}]_\infty\prod_{i,j=1}^{n}[a_ib_j]_\infty}{[q]_\infty^{2-n}\sin(\pi x)\left[-q^{1-n+z-x}\prod_{i=1}^{n}a_i\right]_\infty\left[-q^{x-z}\prod_{i=1}^{n}b_i\right]_\infty}$$

*where*

$$\left|q^{1-n}\prod_{i=1}^{n}a_i\right|<|q^{x-z}|<\left|\prod_{i=1}^{n}b_i^{-1}\right|$$

*for convergence and we take a limit of the right-hand side when* $x$ *is an integer.*

The present paper is divided as follows. Section 2 of this paper is devoted to the proof of Theorem 1.6. Note that identity (1.1) is the $n=1$ case of identity (1.7), identity (1.3) is equivalent to the $n=2$ case of (1.8) and identity (1.5) is the $n=1$ case of (1.15). We prove in §3 Theorem 1.14 and in §4 an extension of Theorem 1.6. In §5 we prove a generalization of Barnes' first Lemma (1.2) associated to $u(n)$, the Lie algebra of $U(n)$ for $n\geq 1$. We also prove a partial ($q=0$) $q$-analogue of this result. In §6 we prove a generalization of the Askey-Wilson integral (1.4) for the group $SU(n)$, $n \geq 2$. In §7 a generalization of the Askey-Wilson integral for the compact symplectic group $Sp(n)$, $n \geq 1$, is proved. By specializing parameters we also prove the corresponding integrals for the special orthogonal groups $SO(2n+1, \mathbb{R})$ and $SO(2n, \mathbb{R})$, $n \geq 1$. In §8 an analogue of the Askey-Wilson integral is proved for the compact Lie group $G_2$. In §9 we discuss the Mellin-Barnes integrals associated to the corresponding Lie algebras. In §10 we give some Lie group and Lie algebra integral formulations of the previous results. Finally in §11 we give a family of related integral evaluations obtained by an application of the Weyl denominator formula (following Macdonald [35]).

As far as future applications of these results, the most obvious is to look for multivariate analogues of the Askey-Wilson orthogonal polynomials and the continuous Hahn polynomials, which should be orthogonal with respect to a weight function which is an integrand for some integral here. For the Lie algebra $su(n)$, multivariate Racah polynomials which are orthogonal with respect to a discrete measure have been found [20]. It is likely there should also be related polynomials orthogonal with respect to a continuous measure (or mixed discrete and continuous measure). There are probably $q$-analogues as well.

These integrals have also been used to evaluate a family of integrals [24] which are related to Selberg's beta integral [46]. In particular using the generalization of the Askey-Wilson integral for $Sp(n)$ one is able to prove a $q$-analogue of Selberg's integral [24]. By appropriate specializations of this $q$-Selberg integral one can prove the $q$-Macdonald-Morris root system conjectures [35, 39] for all affine root systems of classical type including $BC_n$ (see also Kadell [31]).

Finally, there is an important question left unanswered by this paper. We have provided many examples to show that classical Mellin-Barnes and beta type integral evaluations can be extended to multivariate integrals associated to various Lie groups and Lie algebras. However, we have failed to give a satisfactory group representation theoretic explanation for these integrals. They are natural extensions of important one variable integrals. Why are they natural objects from a group theory point of view? An answer to this question could yield significant applications of these integrals.

## 2. Proof of Theorem 1.6

Let $f(z_1+t_1, z_2+t_2, \ldots, z_n+t_n)$ denote the integrand and $F$ denote the corresponding right-hand side for each of the integral identities (1.7)–(1.13). After some elementary computation we can rewrite several of the summation theorems for multilateral hypergeometric series associated to various Lie algebras

$$\sum_{y_1, \ldots, y_n=-\infty}^{\infty} f(z_1+y_1, z_2+y_2, \ldots, z_n+y_n) = F. \tag{2.1}$$

The summation theorems [21, Theorem 1.11, 22, Theorem 5.1 and 22, Theorem 8.2] can be rewritten as (2.1) where $f$ and $F$ are given in identities (1.7), (1.10) and (1.11) respectively. As an illustration of how the summation theorems can be rewritten as (2.1), consider [22, Theorem 5.1]. With a simple change of notation we have

$$\sum_{y_1,\dots,y_n=-\infty}^{\infty}\Bigg\{q^{-\sum_{i=1}^{n}(n+1-i)y_i}\prod_{i=1}^{n}\left(\frac{1-q^{2z_i+2y_i}}{1-q^{2z_i}}\right)$$
$$\cdot\prod_{1\le i<j\le n}\frac{(1-q^{z_i+y_i-z_j-y_j})(1-q^{z_i+y_i+z_j+y_j})}{(1-q^{z_i-z_j})(1-q^{z_i+z_j})}$$
$$\cdot\prod_{i=1}^{n+1}\prod_{k=1}^{n}\frac{[qa_i^{-1}q^{z_k}]_{y_k}[qa_i^{-1}a^{-z_k}]_{-y_k}}{[a_{n+1+i}q^{z_k}]_{y_k}[a_{n+1+i}q^{-z_k}]_{-y_k}}\Bigg\}$$

$$(2.2)\qquad =\frac{[q]_\infty^n\prod_{i,j=1}^{2n+1}[q^{-1}a_ia_j]_\infty}{\left[q^{-1-2n}\prod_{i=1}^{n}a_i\right]}$$
$$\cdot\frac{\prod_{1\le i<j\le n}\{[q^{1+z_i-z_j}]_\infty[q^{1+z_j-z_i}]_\infty[q^{1+z_i+z_j}]_\infty[q^{1-z_i-z_j}]_\infty\}}{\prod_{i=1}^{2n+2}\prod_{k=1}^{n}\{[a_iq^{z_k}]_\infty[a_iq^{-z_k}]_\infty\}}$$
$$\cdot\prod_{i=1}^{n}\{[q^{1+2z_i}]_\infty[q^{1-2z_i}]_\infty\}.$$

Now

$$q^{-\sum_{i=1}^{n}(n+1-i)y_i}\prod_{i=1}^{n}\left(\frac{1-q^{2z_i+2y_i}}{1-q^{2z_i}}\right)$$
$$\cdot\prod_{1\le i<j\le n}\frac{(1-q^{z_i+y_i-z_j-y_j})(1-q^{z_i+y_i+z_j+y_j})}{(1-q^{z_i-z_j})(1-q^{z_i+z_j})}$$
$$(2.3)\qquad =\prod_{i=1}^{n}\{[q^{1+2z_i}]_{2y_k}[q^{1-2z_i}]_{-2y_k}q^{2z_i^2-2(z_i+y_i)^2}\}$$
$$\cdot\prod_{1\le i<j\le n}\{[q^{1+z_i-z_j}]_{(y_i-y_j)}[q^{1-z_i+z_j}]_{(y_j-y_i)}$$
$$\cdot[q^{1+z_i+z_j}]_{(y_i+y_j)}[q^{1-z_i-z_j}]_{-(y_i+y_j)}$$
$$\cdot q^{\frac{1}{2}\{(z_i-z_j)^2+(z_i+z_j)^2-(z_i+y_j-z_j-y_j)^2-(z_i+y_i+z_j+y_j)^2\}}\}$$

and

$$(2.4)\qquad \prod_{i=1}^{n+1}\prod_{k=1}^{n}[qa_i^{-1}q^{z_k}]_{y_k}[qa_i^{-1}q^{-z_k}]_{-y_k}=\prod_{i=1}^{n+1}\prod_{k=1}^{n}\frac{q^{\{(z_k+y_k)^2-z_k^2\}}}{[a_iq^{-z_k}]_{-y_k}[a_iq^{z_k}]_{y_k}}.$$

Observe that for $(u_1,\dots,u_n)\in\mathbb{C}^n$ we have

$$\sum_{i=1}^{n}2u_i^2+\frac{1}{2}\sum_{1\le i<j\le n}\{(u_i-u_j)^2+(u_i+u_j)^2\}=(n+1)\sum_{i=1}^{n}u_i^2.$$

Setting $u_i=z_i$ or $z_i+y_i$, $1\le i\le n$, it follows that the $q^{z_i^2}$ and $q^{(z_i+y_i)^2}$ factors will cancel in the product of the expression on the right-hand side of

(2.3) with the right-hand side of (2.3) with the right-hand side of (2.4). Hence the left-hand of equation (2.2) equals

$$\sum_{y_1,\dots,y_n=-\infty}^{\infty}\left\{\frac{\prod_{i=1}^{n}[q^{1+2z_i}]_{2y_k}[q^{1-2z_i}]_{-2y_k}}{\prod_{i=1}^{2n+2}\prod_{k=1}^{n}[a_iq^{z_k}]_{y_k}[a_iq^{-z_k}]_{-y_k}}\cdot\prod_{1\le i<j\le n}[q^{1+z_i-z_j}]_{(y_i-y_j)}[q^{1-z_i+z_j}]_{(y_j-y_i)}\cdot[q^{1+z_i+z_j}]_{(y_i+y_j)}[q^{1-z_i-z_j}]_{-(y_i-y_j)}\right\}. \tag{2.5}$$

Substitute (2.3) into (2.2) and multiply both sides of equation (2.3) by

$$\frac{\prod_{i=1}^{2n+2}\prod_{k=1}^{n}\{[a_iq^{z_k}]_\infty[a_iq^{-z_k}]_\infty\}}{\prod_{1\le i<j\le n}\{[q^{1+z_i-z_j}]_\infty[q^{1+z_j-z_i}]_\infty[q^{1+z_i+z_j}]_\infty[q^{1-z_j-z_j}]_\infty\}}\cdot\prod_{i=1}^{n}\{[q^{1+2z_i}]_\infty[q^{1-2z_i}]_\infty\}^{-1}.$$

The result is

$$\sum_{y_1,\dots,y_n=-\infty}^{\infty}\left\{\frac{\prod_{i=1}^{2n+2}\prod_{k=1}^{n}([a_iq^{z_k+y_k}]_\infty[a_iq^{-z_k-y_k}]_\infty)}{\prod_{i=1}^{n}([q^{1+2z_i+2y_i}]_\infty[q^{1-2z_i-2y_i}]_\infty)}\cdot\prod_{1\le i<j\le n}([q^{1+z_i+y_i-z_j-y_j}]_\infty^{-1}[q^{1+z_j+y_j-z_i-y_i}]_\infty^{-1}\cdot[q^{1+z_i-y_i+z_j+y_j}]_\infty^{-1}[q^{1-z_i-y_i-z_j-y_j}]_\infty^{-1})\right\} \tag{2.6}$$

$$=\frac{[q]_\infty^n\prod_{i,j=1}^{2n+1}[q^{-1}a_ia_j]_\infty}{\left[q^{-1-2n}\prod_{i=1}^{n}a_i\right]_\infty}.$$

Identity (2.6) is the same as identity (2.1) where $f$ and $F$ are given in (1.10).

In general the summation formulas for multilateral basic hypergeometric series can be rewritten in a similar way where the power of $q$ factors can be shown to cancel by using identities (3.8), (3.9) and condition (3.5) from [22]. The summation formulas for the multilateral ordinary hypergeometric series can also be rewritten similarly and there is no problem with power of $q$ factors.

We will also consider the sum

$$\sum_{\substack{y_1, \dots, y_n=-\infty \\ y_1+\cdots+y_n=0}}^{\infty} f(z_1+y_1, z_2+y_2, \dots, z_n+y_n) = F. \tag{2.7}$$

Then the summation theorems [21, Theorem 1.15, 21, Theorem 1.13, 23, Theorem 1.1 and equation (2.28) and 23, Theorem 1.7 and equation (2.28)] can be rewritten as (2) where $f$ and $F$ are given in identities (1.8), (1.9), (1.12), and (1.13) respectively. The convergence conditions for each of these sums represented by equations (2.1) and (2.7) are identical to the conditions stated for the integral identities (1.7)–(1.13).

Now substitute $z_i + t_i$ in place of $z_i$ for $1 \le i \le n$ in equation (2.1). Then integrate both sides of (2.1) with respect to the variables $t_i$, $1 \le i \le n$:

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \sum_{y_1, \dots, y_n=-\infty}^{\infty} f(z_1+t_1+y_1, \dots, z_n+t_n+y_n)\, dt_1 \cdots dt_n \\ &\quad = \int_0^1 \cdots \int_0^1 F\, dt_1 \cdots dt_n = F. \end{aligned} \tag{2.8}$$

The left-hand side of (2.8) equals

$$\begin{aligned} &\sum_{y_1, \dots, y_n=-\infty} \int_{y_n}^{1+y_n} \int_{y_{n-1}}^{1+y_{n-1}} \cdots \int_{y_1}^{1+y_1} f(z_1+t_1, \dots, z_n+t_n)\, dt_1 \cdots d_n \\ &\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1+t_1, \dots, z_n+t_n)\, dt_1 \cdots dt_n. \end{aligned}$$

We then obtain the result

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1+t_1, \dots, z_n+t_n)\, dt_1 \cdots dt_n = F \tag{2.9}$$

where the convergence of the integral in (2.9) follows from the absolute convergence of the series on the left-hand side of (2.1). By a similar argument for $f$ and $F$ satisfying (2.7), we can integrate (2.7) to show that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1+t_1, \dots, z_n+t_n)\, dt_1 \cdots dt_{n-1} = F \tag{2.10}$$

where $\sum_{i=1}^n z_i = \sum_{i=1}^n t_i = 0$. This completes the proof of the identities (1.7)–(1.13) and Theorem 1.6.

## 3. Proof of Theorem 1.14

For $n = 1$ there are several proofs of identity (1.15) (see [11, 48 and 19]). We will assume that $n > 1$. With notation and assumptions as in Theorem 1.14 and $u \in \mathbb{C}$, define

$$\begin{aligned} &f(z_1, \dots, z_n; u) \\ &\quad = \frac{\prod_{i,j=1}^{n} \{[a_i q^{z_j}]_\infty [q b_i q^{-z_j}]_\infty\}}{\left[q^z u \prod_{i=1}^n b_i^{-1}\right]_\infty \left[q^{1-z} u^{-1} \prod_{i=1}^n b_i\right]_\infty \prod_{\substack{1\le i,j\le n \\ i\ne j}} [q^{1+z_i-z_j}]_\infty}. \end{aligned} \tag{3.1}$$

We can rewrite the $U(n)$ generalization of Ramanujan's ${}_1\psi_1$ summation theorem [21, Theorem 1.17] (see also [37]),

$$
\begin{aligned}
&\sum_{y_1,\ldots,y_n=-\infty}^{\infty} f(z_1+y_1,\, z_2+y_2,\, \ldots,\, z_n+y_n;\, u) \\
&= \frac{[q]_\infty^n \prod_{i,j=1}^{n} [a_i b_j]_\infty}{[u]_\infty \left[q^{1-n}u^{-1}\prod_{i=1}^{n}(a_i b_i)\right]_\infty} = F
\end{aligned}
\tag{3.2}
$$

where $|q^{1-n}\prod_{i=1}^{n}(a_i b_i)| < |u| < 1$ for convergence. If we replace $u$ by $uv$ with $v \in \mathbb{C}$, we may also write the $U(n)$ ${}_1\psi_1$ sum as

$$
\begin{aligned}
&\sum_{y_1,\ldots,y_n=-\infty}^{\infty} f(z_1+y_1,\, \ldots,\, z_n+y_n;\, u)v^y \\
&= \frac{[q]_\infty^n \prod_{i,j=1}^{n} [a_i b_j]_\infty \left[q^z uv\prod_{i=1}^{n} b_i^{-1}\right]_\infty \left[q^{1-z}(uv)^{-1}\prod_{i=1}^{n} b_i\right]_\infty}{[uv]_\infty \left[q^{1-n}(uv)^{-1}\prod_{i=1}^{n}(a_i b_i)\right]_\infty \left[q^z u\prod_{i=1}^{n} b_i^{-1}\right]_\infty \left[q^{1-z}u^{-1}\prod_{i=1}^{n} b_i\right]_\infty}
\end{aligned}
\tag{3.3}
$$

where $z = \sum_{i=1}^{n} z_i$, $y = \sum_{i=1}^{n} y_i$ and $|q^{1-n}\prod_{i=1}^{n}(a_i b_i)| < |uv| < 1$.

Now in (3.3) replace $z_i$ by $z_i + s_i$ for $1 \le i \le n$ and $v$ by $q^x$ for some $x \in \mathbb{C}$. Setting $s = \sum_{i=1}^{n} s_i$, multiply both sides by $q^{xs}$ and integrate with respect to $s_i$, $1 \le i \le n$. We obtain

(3.4)

$$
\begin{aligned}
&\int_0^1 \cdots \int_0^1 \sum_{y_1,\ldots,y_n=-\infty}^{\infty} f(z_1+s_1+y_1,\, \ldots,\, z_n+x_n+y_n;\, u)q^{x(y+s)}\, ds_1\cdots ds_n \\
&= \frac{[q]_\infty^n \prod_{i,j=1}^{n} [a_i b_j]_\infty}{[uq^x]_\infty \left[q^{1-n-x}u^{-1}\prod_{i=1}^{n}(a_i b_i)\right]_\infty} \\
&\quad \cdot \int_0^1 \cdots \int_0^1 \frac{\left[q^{z+s-x}u\prod_{i=1}^{n} b_i^{-1}\right]_\infty \left[q^{1-z-s-x}u^{-1}\prod_{i=1}^{n} b_i\right]_\infty}{\left[q^{z+s}u\prod_{i=1}^{n} b_i^{-1}\right]_\infty \left[q^{1-z-s}u^{-1}\prod_{i=1}^{n} b_i\right]_\infty}\, ds_1\cdots ds_n.
\end{aligned}
$$

Note that the integral on the right-hand side of (3.4) equals

$$
\int_0^1 \frac{\left[q^{z+s+x}u\prod_{i=1}^{n} b_i^{-1}\right]_\infty \left[q^{1-z-s-x}u^{-1}\prod_{i=1}^{n} b_i\right]_\infty}{\left[q^{z+s}\prod_{i=1}^{n} b_i^{-1}\right]_\infty \left[q^{1-z-s}u^{-1}\prod_{i=1}^{n} b_i\right]_\infty}\, ds.
$$

The left-hand side of (3.4) equals

$$\sum_{y_1,\dots,y_n=-\infty}^{\infty}\int_{y_n}^{1+y_n}\cdots\int_{y_1}^{1+y_1} f(z_1+s_1,\dots,z_n+s_n;u)q^{xs}\,ds_1\cdots ds_n \tag{3.5}$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(z_1+s_1,\dots,z_n+s_n;u)q^{xs}\,ds_1\cdots ds_n.$$

Set $u=-q^{-z}\prod_{i=1}^n b_i$ and $t_i=q^{s_i}$ for $1\le i\le n$. Then substituting (3.1) into the right-hand side of (3.5) we obtain

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\left(\prod_{i=1}^n t_i\right)^{x-1}\prod_{i,j=1}^n\{[a_it_iq^{z_j}]_\infty[qb_it_j^{-1}q^{-z_j}]_\infty\}\,dt_1\cdots dt_n}{\left[-\prod_{i=1}^n t_i\right]_\infty\left[-q\prod_{i=1}^n t_i^{-1}\right]_\infty\prod_{\substack{1\le i,j\le n\\ i\ne j}}[t_it_j^{-1}q^{1+z_i-z_j}]_\infty} \tag{3.6}$$
$$=\frac{(\log(q^{-1}))^n[q]_\infty^n\prod_{i,j=1}^n[a_ib_j]_\infty}{\left[-q^{-x-z}\prod_{i=1}^n b_i\right]_\infty\left[-q^{1-n+z-x}\prod_{i=1}^n a_i\right]_\infty}\cdot\int_0^1\frac{[-q^{s+x}]_\infty[-q^{1-s-x}]_\infty}{[q^{-s}]_\infty[-q^{1-s}]_\infty}\,ds.$$

From the identity corresponding to (3.6) for $n=1$ with $q^{z_1}=-1$ and applying (1.5), we find that

$$\int_0^1\frac{[-q^{s+x}]_\infty[-q^{1-s-x}]_\infty}{[-q^s]_\infty[-q^{1-s}]_\infty}\,ds=\frac{\pi[q^x]_\infty[q^{1-x}]_\infty}{\sin(\pi x)[q]_\infty^2\log(q^{-1})}. \tag{3.7}$$

Substituting (3.7) into (3.6) we obtain identity (1.15). The convergence condition $|q^{1-n}\prod_{i=1}^n a_i|<|q^{x-z}|<|\prod_{i=1}^n b_i^{-1}|$ now follows from the convergence condition for the sum (3.3). This completes the proof of Theorem 1.14.

## 4. Extension of Theorem 1.16

The identities (1.7)–(1.13) and also the $x=o$ case of identity (1.15) can be extended in a simple way (see [42]). In identities (1.7)–(1.13) let, as above, $f(z_1+t_1,\dots,z_n+t_n)$ denote the integrand on the left-hand side and $F$ the corresponding right-hand side of these identities. Similarly let $f(z_1,\dots,z_n;u)$ and $F$ be defined as in §3. Let $k_1,\dots,k_n$ be integers with at least one $k_j$ not equal to zero. With notation and assumptions as in identities (1.7), (1.10) and (1.11) then we have

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{2\pi i(\sum_{j=1}^n k_jt_j)}f(z_1+t_1,\dots,z_n+t_n)\,dt_1\cdots dt_n=0. \tag{4.1}$$

To prove (4.1) we rewrite the integral in (4.1) as

$$\int_0^1\cdots\int_0^1 e^{2\pi i(\sum_{j=1}^n k_jt_j)}\sum_{y_1,\dots,y_n=-\infty}^{\infty}f(z_1+t_1+y_1,\dots,z_n+t_n+y_n)\,dt_1\cdots dt_n$$
$$=F\int_0^1\cdots\int_0^1 e^{2\pi i(\sum_{j=1}^n k_jt_j)}\,dt_1\cdots dt_n=0$$

since some $k_j$ is nonzero. With notations and assumptions as in §3 we can similarly show that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{2\pi i(\sum_{j=1}^{n} k_j s_j)} f(z_1 + s_1, \ldots, z_n + s_n)\, ds_1 \cdots ds_n \tag{4.2}$$

where at least one $k_j$ is nonzero. With notations and assumptions as in (1.8), (1.9), (1.12), and (1.13) we can also show that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{2\pi i(\sum_{j=1}^{n-1} k_j t_j)} f(z_1 + t_1, \ldots, z_n + t_n)\, dt_1 \cdots dt_{n-1} = 0 \tag{4.3}$$

where at least one $k_j$ is not zero for $1 \le j \le n-1$.

## 5. A $u(n)$ generalization of Barnes' first lemma

In this section, we shall prove a generalization of Barnes' first lemma, which is an integral analogue of the Gauss summation theorem. The proof here is a generalization of Barnes' proof in one dimension given in Bailey [15]. The proof goes as follows: We expand the integral as a sum of several multiple series of residues of the poles. Each of these multiple series can be summed by means of Milne's $U(n)$ generalization of the Gauss summation theorem [38]. We are then left with a sum of several ratios of gamma functions. Finally, a trigonometric identity is used to combine this sum into a single ratio of gamma functions.

At the end of this section we prove a partial $q$-analogue (after setting $q = o$) of Theorem 5.1. It generalizes a result of Li and Soto-Andrade [33] (see also [10]).

The main result in this section is the following:

**Theorem 5.1.** *For $n \ge 1$, let $\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \ldots, \beta_{n+1} \in \mathbb{C}$ and assume that $\alpha_i + \beta_j$ is not a negative integer or zero for all $1 \le i, j \le n+1$. Then we have*

$$\begin{aligned} &\frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^{n+1} \prod_{j=1}^{n} \Gamma(\alpha_i - z_j)\Gamma(\beta_i + z_j)\, dz_1 \cdots dz_n}{\prod_{\substack{1 \le i, j \le n \\ i \ne j}} \Gamma(z_i - z_j)} \\ &= \frac{n! \prod_{k,j=1}^{n+1} \Gamma(\alpha_i + \beta_j)}{\Gamma\left(\sum_{i=1}^{n+1} (\alpha_i + \beta_i)\right)} \end{aligned} \tag{5.2}$$

*where the contours of integration are deformed so as to separate the sequences of poles going to the right $\{\alpha_i + k \mid 1 \le i \le n+1,\ k = 0, 1, 2, \ldots\}$ from the sequences of poles going to the left $\{-\beta_i - k \mid 1 \le i \le n+1,\ k = 0, 1, 2, \ldots\}$.*

*Proof.* Let us begin by assuming that $\mathrm{Re}(\alpha_i) > 0$ and $\mathrm{Re}(\beta_i) > 0$ for all $1 \le i \le n+1$, that $\mathrm{Re}(\sum_{i=1}^{n+1}(\alpha_i + \beta_i)) < 1$ and that $\alpha_i \ne \alpha_j + k$ for $1 \le i, j \le n+1$, $i \ne j$, and any integer $k$. We will remove these assumptions at the end of the proof.

Recall Stirling's formula

$$\Gamma(c+z) = \sqrt{2\pi}\, z^{c+z-\frac{1}{2}} e^{-z}(1 + O(1/z))$$

as $z \to \infty$ in the region $|\arg(z)| < \theta$, $0 < \theta < \pi$, where $c, z \in \mathbb{C}$. We also have the identity

$$\Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z)$$

and

$$|\sin(\pi z)| = O(e^{\pi |\operatorname{Im} z|}).$$

The integrand $F$ on the left-hand side of equation (5.2) can be rewritten as

$$F = \pi^{\frac{n(n+3)}{2}} \prod_{1 \le i < j \le n} (z_i - z_j) \tag{5.3}$$
$$\cdot \prod_{i=1}^{n+1} \prod_{j=1}^{n} \frac{\Gamma(\beta_i + z_j)}{\Gamma(1 - \alpha_i + z_j)} \frac{\prod_{1 \le i \le j \le n} \sin(\pi(z_j - z_i))}{\Gamma(1 - \alpha_i + z_j) \prod_{i=1}^{n+1} \prod_{j=1}^{n} \sin(\pi(\alpha_i - z_j))}.$$

Using Vandermonde determinant,

$$\prod_{1 \le i < j \le n} (z_i - z_j) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) z_{\pi(1)}^{n-1} z_{\pi(2)}^{n-2} \cdots z_{\pi(n)}^{o}$$

where $\operatorname{sgn}(\pi)$ is the sign of the permutation $\pi \in S_n$, and if we suppose that $|\alpha_i - z_j - k| > \varepsilon$ for all integers $k$ and $1 \le i \le n+1$, $1 \le j \le n$, and $\varepsilon > 0$, then we have that

$$|F| < A \left( \prod_{j=1}^{n} |z_j|^{\sum_{i=1}^{n} (\alpha_i + \beta_i) - 2} e^{-2x |\operatorname{Im} z_j|} \right), \tag{5.4}$$

for some constant $A > 0$ as $z_j \to \infty$ in the region $|\arg(z_j)| < \theta$, $0 < \theta < \pi$, for $j = 1, \ldots, n$. A consequence of inequality (5.4) is the integral on the left-hand side of equation (5.2) converges absolutely.

For any real number $r$, let $C_r$ be the line parallel to the imaginary axis going from $r - i\infty$ to $r + i\infty$. Assume that no poles of the integrand $F$ lie on the contour $C_r$ (for any variable $z_j$, $1 \le j \le n$). The integral on the left-hand side of equation (5.2) can be rewritten as

$$\frac{1}{(2xi)^n} \int_{(C_0)^n} F \, dz_1 \cdots dz_n \tag{5.5}$$
$$= \sum_{l=0}^{n} \frac{\binom{n}{l}}{(2\pi i)^n} \int_{(C_0 - C_r)^l} \int_{(C_r)^{n-l}} F \, dz_1 \cdots dz_n,$$

where we have used Fubini's theorem and the symmetry of the integrand $F$ with respect to permutation of the variables $z_j$, $1 \leq j \leq n$.

Now let $N$ be a positive integer and $S_{n+1}$ be the set of permutations of the set $1, 2, \ldots, n+1$. Using Cauchy's theorem and identity (5.5), we rewrite the integral on the left-hand side of (5.2) as

(5.6)
$$\sum_{\sigma \in S_{n+1}} \left\{ \frac{\prod_{i=1}^{n+1} \prod_{j=1}^{n}{}' \Gamma(\beta_i + \alpha_{\sigma(j)}) \Gamma(\alpha_i - \alpha_{\sigma(j)})}{\prod_{1 \leq i < j \leq n} \Gamma(\alpha_{\sigma(i)} - \alpha_{\sigma(j)}) \Gamma(\alpha_{\sigma(j)} - \alpha_{\sigma(i)})} \right.$$
$$\left. \cdot \sum_{y_1, \ldots, y_n = 0}^{N-1} \prod_{1 \leq i < j \leq n} \frac{(\alpha_{\sigma(i)} + y_i) - (\alpha_{\sigma(j)} + y_j)}{(\alpha_{\sigma(i)} - \alpha_{\sigma(j)})} \prod_{i=1}^{n+1} \prod_{j=1}^{n} \frac{(\beta_i + \alpha_{\sigma(j)})_{y_j}}{(1 - \alpha_i + \alpha_{\sigma(j)})_{y_j}} \right\}$$
$$+ \sum_{l=0}^{n-1} \frac{\binom{n}{l}}{(2\pi i)^n} \int_{(C_0 - C_N)^l} \int_{(C_N)^{n-l}} F \, dz_1 \cdots dz_n$$

where $\prod'$ means that whenever the factor $\Gamma(0)$ occurs in the product it is replaced by 1. An application of inequality (5.4) shows that the limit as $N \to \infty$ of the expression (5.6) equals the expression

(5.7)
$$\sum_{\sigma \in S_{n+1}} \left\{ \frac{\prod_{i=1}^{n+1} \prod_{j=1}^{n}{}' \Gamma(\beta_i + \alpha_{\sigma(j)}) \Gamma(\alpha_i - \alpha_{\sigma(j)})}{\prod_{1 \leq i < j \leq n} \Gamma(\alpha_{\sigma(i)} - \alpha_{\sigma(j)}) \Gamma(\alpha_{\sigma(j)} - \alpha_{\sigma(i)})} \right.$$
$$\left. \cdot \sum_{y_1, \ldots, y_n = 0}^{\infty} \prod_{1 \leq i < j \leq n} \frac{(\alpha_{\sigma(i)} + y_i) - (\alpha_{\sigma(j)} + y_j)}{(\alpha_{\sigma(i)} - \alpha_{\sigma(j)})} \prod_{i=1}^{n+1} \prod_{j=1}^{n} \frac{(\beta_i + \alpha_{\sigma(j)})_{y_j}}{(1 - \alpha_j + \alpha_{\sigma(j)})_{y_j}} \right\},$$

where the infinite series in (5.7) converge absolutely since $1 \geq \mathrm{Re}(\sum_{i=1}^{n+1} (\alpha_i + \beta_i))$ (see [38] or [21]). It follows that the integral on the left-hand side of equation (5.2) equals expression (5.7).

The series in expression (5.7) can be evaluated by means of Milne's $U(n)$ generalization of the Gauss summation theorem [38]. We have

(5.8)
$$\sum_{y_1, \ldots, y_n = 0}^{\infty} \left\{ \prod_{1 \leq i < j \leq n} \frac{(\alpha_{\sigma(i)} + y_i) - (\alpha_{\sigma(j)} + y_j)}{(\alpha_{\sigma(i)} - \alpha_{\sigma(j)})} \prod_{i=1}^{n+1} \prod_{j=1}^{n} \cdot \frac{(\beta_i + \alpha_{\sigma(j)})_{y_j}}{(1 - \alpha_i + \alpha_{\sigma(1)})_{y_j}} \right\}$$
$$= \frac{\Gamma\left(1 - \sum_{i=1}^{n+1} (\alpha_i + \beta_i)\right) \prod_{i=1}^{n} \Gamma(1 + \alpha_{\sigma(i)} - \alpha_{\sigma(n+1)})}{\prod_{i=1}^{n+1} \Gamma(1 - \beta_i - \alpha_{\sigma(n+1)})}.$$

Substituting (5.8) into (5.7) and simplifying, we find that expression (5.7) equals

$$\begin{aligned}
&\Gamma\left(1-\sum_{i=1}^{n+1}(\alpha_i+\beta_i)\right)\prod_{i,j=1}^{n+1}\Gamma(\alpha_i+\beta_j)\\
&\quad\cdot\sum_{\sigma\in S_{n+1}}\frac{\prod_{i=1}^{n}\Gamma(1+\alpha_{\sigma(i)}-\alpha_{\sigma(n+1)})\Gamma(\alpha_{\sigma(n+1)}-\alpha_{\sigma(i)})}{\prod_{i=1}^{n+1}\Gamma(1-\beta_i-\alpha_{\sigma(n+1)})\Gamma(\beta_i+\alpha_{\sigma(n+1)})}\\
&=\Gamma\left(1-\sum_{i=1}^{n+1}(\alpha_i+\beta_i)\right)\prod_{i,j=1}^{n+1}\Gamma(\alpha_i+\beta_j)\\
&\quad\cdot\pi^{-1}\sum_{\sigma\in S_{n+1}}\frac{\prod_{i=1}^{n+1}\sin(\pi(\beta_i+\alpha_{\sigma(n+1)}))}{\prod_{i=1}^{n}\sin(\pi(\alpha_{\sigma(n+1)}-\alpha_{\sigma(i)}))}.
\end{aligned}\tag{5.9}$$

To complete the proof of Theorem 5.1, we will need the following:

**Lemma 5.10.** *With notation as above we have*

$$\sum_{\sigma\in S_{n+1}}\frac{\prod_{i=1}^{n+1}\sin(\pi(\beta_i+\alpha_{\sigma(n+1)}))}{\prod_{i=1}^{n}\sin(\pi(\alpha_{\sigma(n+1)}-\alpha_{\sigma(i)}))}=n!\sin\left(\pi\sum_{i=1}^{n+1}(\alpha_i+\beta_i)\right).\tag{5.11}$$

*Proof.* Let $H$ denote the sum on the left-hand side of equation (5.11). The value of the summand in $H$ is identical for all $\sigma\in S_{n+1}$ belonging to the coset of $S_n$ in $S_{n+1}$ such that $\sigma(n+1)=j$, for $j=1,\dots,n+1$. The order of each of these cosets in $n$. Hence we find that

$$H=n!\sum_{j=1}^{n+1}\frac{\prod_{i=1}^{n+1}\sin(\pi(\beta_i+\alpha_j))}{\prod_{\substack{i=1\\i\neq j}}^{n+1}\sin(\pi(\alpha_j-\alpha_i))}.\tag{5.12}$$

Setting $u_j=e^{\pi_i\alpha_j}$ and $v_j=e^{\pi_i\beta_j}$ for $j=1,\dots,n+1$ and where $i=\sqrt{-1}$, then we have

$$H=\frac{n!}{2i}\sum_{p=1}^{n+1}\frac{\prod_{j=1}^{n+1}(v_ju_p-v_j^{-1}u_p^{-1})}{\prod_{\substack{j=1\\j\neq p}}^{n+1}(u_pu_j^{-1}-u_p^{-1}u_j)}.\tag{5.13}$$

The sum on the right-hand side of equation (5.13) can be evaluated by means of the following

**Lemma 5.14** (Milne [36, Lemma 1.33]). *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be indeterminates with the $y_i$ distinct. We then have*

$$1 - x_1 x_2 \cdots x_n = \sum_{p=1}^{n} \left\{ (1 - x_p) \prod_{\substack{i=1 \\ i \neq p}}^{n} \frac{(y_p - x_i y_i)}{(y_p - y_i)} \right\}. \tag{5.15}$$

*By setting $x_j = u_j^{-2} v_j^{-2}$ and $y_j = u_j^2$ for $j = 1, \ldots, n+1$ in* (5.15), *we find that the right-hand side of equation* (5.13) *equals*

$$\begin{aligned} H &= \frac{n!}{2i} \prod_{j=1}^{n+1} (u_j v_j) \cdot \left( 1 - \prod_{j=1}^{n+1} (u_j v_j)^{-2} \right) \\ &= \frac{n!}{2i} \left( \prod_{j=1}^{n+1} (u_j v_j) - \prod_{j=1}^{n+1} (u_j v_j)^{-1} \right) \\ &= n! \sin \pi \left( \sum_{j=1}^{n+1} (\alpha_j + \beta_j) \right). \end{aligned} \tag{5.16}$$

This completes the proof of Lemma 5.10.

Now substitute identity (5.11) into the right-hand side of equation (5.9). It follows that the left-hand side of equation (5.2) equals

$$\begin{aligned} n!\Gamma & \left( 1 - \sum_{i=1}^{n+1} (\alpha_i + \beta_i) \right) \prod_{i,j=1}^{n+1} \Gamma(\alpha_i + \beta_j) \pi^{-1} \sin \pi \left( \sum_{i=1}^{n+1} (\alpha_i + \beta_i) \right) \\ &= n! \frac{\Gamma\left(1 - \sum_{i=1}^{n+1} (\alpha_i + \beta_i)\right) \prod_{i,j=1}^{n+1} \cdot \Gamma(\alpha_i + \beta_j)}{\Gamma\left(1 - \sum_{i=1}^{n+1} (\alpha_i + \beta_i)\right) \Gamma\left(\sum_{i=1}^{n+1} (\alpha_i + \beta_i)\right)} \\ &= \frac{n! \prod_{i,j=1}^{n+1} \Gamma(\alpha_i + \beta_j)}{\Gamma\left(\sum_{i=1}^{n+1} (\alpha_i + \beta_i)\right)}. \end{aligned}$$

This completes the proof of identity (5.2) under the assumptions stated at the beginning of the proof of Theorem 5.1 we can drop these assumptions by analytic continuation, thereby finishing the proof of Theorem 5.1.

We now state and prove a partial $q$-analogue of Theorem 5.1, generalizing a result of Li and Soto-Andrade [33] in the $n = 1$ case (see also [11]).

**Theorem 5.17.** *For $n \geq 1$, let $\max(|a_i|, |b_i|) < 1$ for $1 \leq i \leq n+1$ and $C^n$ be the $n$-fold product of the unit circle traversed in the positive direction. Then*

$$
\begin{aligned}
&\frac{1}{(2\pi i)^n} \int_{C^n} \frac{\prod\limits_{1\leq i<j\leq n} (1 - z_i z_j^{-1})(1 - z_j z_i^{-1})}{\prod\limits_{i=1}^{n+1} \prod\limits_{j=1}^{n} (1 - a_i z_j^{-1})(1 - b_i z_j)} \frac{dz_1\, dz_2 \cdots dz_n}{z_1 z_2 \cdots z_n} \\
&\quad = n! \cdot \frac{\left(1 - \prod\limits_{i=1}^{n+1} a_i b_i\right)}{\prod\limits_{i,j=1}^{n+1} (1 - a_i b_j)}.
\end{aligned}
\tag{5.18}
$$

*Proof*. Assume that the $a_i$ are nonzero and mutually distinct for $1 \leq i \leq n+1$. By expanding the integral in (5.18) in terms of residues, we find that the left-hand side of (5.18) equals

$$
\begin{aligned}
&n! \sum_{p=1}^{n+1} \prod_{\substack{i=1\\ i\neq p}}^{n+1} \left\{ (1 - a_p a_i^{-1}) \prod_{j=1}^{n+1} (1 - b_j a_i) \right\}^{-1} \\
&\quad = n! \prod_{i,j=1}^{n+1} (1 - a_i b_j)^{-1} \sum_{p=1}^{n+1} \frac{\prod\limits_{j=1}^{n+1} (1 - b_j a_p)}{\prod\limits_{\substack{i=1\\ i\neq p}}^{n+1} (1 - a_p a_i^{-1})}.
\end{aligned}
\tag{5.19}
$$

Setting $x_i = a_i b_i$ and $y_i = a_i^{-1}$, $1 \leq i \leq n+1$, in Lemma 5.14, we find that the r.h.s. of (5.19) equals

$$
\frac{n! \left(1 - \prod\limits_{i=1}^{n+1} a_i b_i\right)}{\prod\limits_{i,j=1}^{n+1} (1 - a_i b_j)}.
$$

By continuity we now drop the assumption that the $a_i$ are nonzero and mutually disjoint. This completes the proof of Theorem 5.17.

## 6. A MULTIDIMENSIONAL $q$-BETA INTEGRAL

In this section we will prove a multidimensional $SU(n)$ generalization of the Askey-Wilson of beta type integral [13, Theorem 2.1]. The proof here is largely modelled on that of Askey and Wilson in the one variable case. The Askey-Wilson proof has to be modified however due to the more complicated structure of the series of residues of the integral in several variables. For example, certain sets of terms in the series of residues are shown to cancel each other. Also a more powerful bilateral $SU(n)$ basic hypergeometric series summation theorem is needed to sum parts of the series of residues.

**Theorem 6.1.** *For $n \geq 2$, let $a_i, b_i \in \mathbb{C}$ and $|a_i|, |b_i| < 1$ for $1 \leq i \leq n$. Then*

$$\frac{1}{(2\pi i)^{n-1}} \int_{C^{n-1}} \frac{\prod\limits_{\substack{1\leq i,j\leq n \\ i\neq j}} [z_i z_j^{-1}]_\infty}{\prod\limits_{i,j=1}^{n} [a_i z_j^{-1}]_\infty [b_i z_j]_\infty} \frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_{n-1}}{z_{n-1}} \tag{6.2}$$
$$= \frac{n! \left[\prod\limits_{i=1}^{n}(a_i b_i)\right]_\infty}{[q]_\infty^{n-1} \left[\prod\limits_{i=1}^{n} a_i\right]_\infty \left[\prod\limits_{i=1}^{n} b_i\right]_\infty \prod\limits_{i,j=1}^{n} [a_i b_j]_\infty}$$

*where $\prod_{i=1}^{n} z_i = 1$ and $C^{n-1}$ is the $(n-1)$ fold direct product of the unit circle $C$ traversed in the positive direction.*

*Proof*. For the time being we will assume that $|q| < \prod_{i=1}^{n} |a_i b_i|$ and $a_i \neq a_j$, $b_i \neq b_j$ for $1 \leq i \neq j \leq n$. Let $A = \{a_1, \ldots, a_n\}$ and $B^{-1} = \{b_1^{-1}, \ldots, b_n^{-1}\}$. We will also assume that if $u = \prod_{i=1}^{l} v_i$ where $1 \leq l \leq n$ and $v_i \in A \cup B^{-1}$ (with possible repetitions) then $|u| \neq |q|^j$ for any $j \in \mathbb{Z}$. For $0 \leq k \leq n-1$ let $\Delta(k)$ be the set of all $\delta(k) = (\delta_1, \ldots, \delta_n)$ such that $\delta_i = (\gamma_i, \sigma_i)$ where

$$\begin{aligned}
&\gamma_i = A \cup B^{-1} \quad \text{for } 1 \leq i \leq k, \\
&\gamma_i = z_i \quad \text{for } k < i \leq n-1, \\
&\gamma_n = \prod_{i=1}^{n-1} \gamma_i^{-1}, \\
&\gamma_j \neq \gamma_l \quad \text{for } 1 \leq j \neq\leq k, \\
&\sigma_i = \begin{cases} 1 & \text{if } \gamma_i \in B^{-1} \text{ or } \gamma_1, \ldots, \gamma_{i-1} \subset A, \\ \text{either } -1 \text{ or } 1 & \text{otherwise for } 1 \leq i \leq k, \end{cases} \\
&\sigma_i = 1 \quad \text{for } k < i \leq n.
\end{aligned} \tag{6.3}$$

We will denote $\Delta(n-1)$ by $\Delta$ and $\delta(n-1)$ by $\delta$.

Let $C_\varepsilon$ be the circle of radius $\varepsilon$ centered at zero and traversed in the positive direction. We can choose the radius $\varepsilon$ so that the denominator of the integrand in (6.2) does not vanish for $|z_1| = \varepsilon$, $|z_i| = 1$, $2 \leq i \leq n-1$. Similarly, we will later choose $\varepsilon$ so that the circle $C_\varepsilon$ does not pass through the poles of other integrands we will consider.

For $0 \leq k \leq n-1$, $\delta(k) \in \Delta(k)$ and $\varepsilon \geq 0$, let $A_{\delta,\varepsilon}(k)$ be the set of all $(y_1, \ldots, y_n) \in Z^n$ such that

$$
\begin{aligned}
&y_i \geq 0 \quad \text{if } 1 \leq i < k\,,\ \gamma_i \in A \quad \text{and} \quad \sigma_i = 1\,;\\
&\sigma_i y_i \leq 0 \quad \text{and} \quad \left|\prod_{j=1}^{i} \gamma_j^{-1} q^{-y_j}\right| < 1 \quad \text{if } 1 \leq i < k \text{ and either}\\
&\qquad \gamma_i \in B^{-1} \quad \text{or} \quad \sigma_i = -1\,,\\
&y_k \geq 0 \quad \text{and} \quad \varepsilon < |\gamma_k q^{y_k}| \quad \text{if } \gamma_i \in A \quad \text{and} \quad \sigma_k = 1\,;\\
&\sigma_k y_k \quad \text{and} \quad \varepsilon < \left|\prod_{j=1}^{k} \gamma_j^{-1} q^{-y_j}\right| < 1 \text{ if either } \gamma_k \in B^{-1} \quad \text{or} \quad \sigma_k = -1\,;\\
&y_i = 0 \quad \text{if } k < i \leq n-1 \quad \text{and} \quad y_n = -\sum_{i=1}^{n-1} y_i.
\end{aligned}
\tag{6.4}
$$

Note that if $1 \leq i \leq k$ and $\gamma_1, \ldots, \gamma_i \subset B^{-1}$, then the condition $|\prod_{j=1}^{i} \gamma_j^{-1} q^{-y_j}| < 1$ is always satisfied when $y_i \leq 0$.

For $0 \leq k \leq n-1$, $\delta(k) \in \Delta(k)$ and $\varepsilon \geq 0$, we also define

$$
S_{\delta,\varepsilon}(k) = \sum_{(y_1,\ldots,y_n)\in A_{\delta,\varepsilon}(k)} \frac{\left(\prod_{i=1}^{n-1} \sigma_i\right) \prod_{1\leq i\neq j\leq n} [\gamma_i q^{y_i}/\gamma_j q^{y_j}]_\infty}{\prod_{i,j=1}^{n}{}' [a_i/\gamma_j q^{y_j}]_\infty [b_i \gamma_j q^{y_j}]_\infty}
\tag{6.5}
$$

where $\prod'$ means the usual product except that if $c = q^{-l}$ for some nonnegative integer $l$, then the factor $[c]_\infty$ in the product is replaced by $[q^{-l}]_l [q]_\infty$. The notation $1 \leq i \neq j \leq n$ is an abbreviation for $1 \leq i,\ j \leq n,\ i \neq j$. Simplifying, we obtain

$$
\begin{aligned}
S_{\delta,\varepsilon}(k) = {}& \frac{\left(\prod_{i=1}^{n-1} \sigma_i\right) \prod_{1\leq i\neq j\leq n} [\gamma_i/\gamma_j]_\infty}{\prod_{i,k=1}^{n}{}' [a_i/\gamma_j]_\infty [b_i\gamma_j]_\infty}\\
&\cdot \sum_{(y_1,\ldots,y_n)\in A_{\delta,\varepsilon}(k)} \prod_{1\leq i<j\leq n} \left(\frac{\gamma_i q^{y_i} - \gamma_j q^{y_j}}{\gamma_i - \gamma_j}\right) \prod_{i,j=1}^{n} \frac{[b_k \gamma_j]_{y_j}}{[q\gamma_j/a_i]_{y_j}}
\end{aligned}
\tag{6.6}
$$

where $(\gamma_i q^{y_i} - \gamma_j q^{y_j})/(\gamma_i - \gamma_j)$ is set equal to 1 if $\gamma_i = \gamma_j$ and $y_i = y_j = 0$ for $1 < i \neq j < n$. We will denote $\lim S_{\delta,\varepsilon}(k)$ by $S_\delta(k)$. Since $|q| < \prod_{i=1}^{n} |a_i b_i|$, the series $S_\delta(k)$ converges absolutely by Lemma 4.22 of [40].

We will now show by induction on $k$, for $0 \leq k \leq n-1$, that

$$
\begin{aligned}
&\frac{1}{(2\pi i)^{n-1}} \int_{C^{n-1}} \frac{\prod_{1\leq i\neq j\leq n} [z_i z_j^{-1}]_\infty}{\prod_{i,j=1}^{n} [a_i z_j^{-1}]_\infty [b_i z_j]_\infty} \frac{dz_1}{z_1} \cdots \frac{dz_{n-1}}{z_{n-1}}\\
&\quad = \frac{1}{(2\pi i)^{n-1-k}} \int_{C^{n-1-k}} \sum_{\delta(k)\in\Delta(k)} S_\delta(k) \frac{dz_{k+1}}{z_{k+1}} \cdots \frac{dz_{n-1}}{z_{n-1}}.
\end{aligned}
\tag{6.7}
$$

The identity (6.7) is trivially true for $k=0$. Suppose identity (6.7) is true for $0 \le k = l < n-1$. We will then show it is true for $k = l+1$.

Consider the poles of the terms of the series $S_\delta(l)$ with respect to the variable $z_{l+1}$. The term

$$\left(\prod_{i=1}^{n-1} \sigma_i\right) \prod_{1\le i\ne j\le n} [\gamma_i q^{y_i}/\gamma_i q^{y_j}]_\infty \Big/ \left(\prod_{i,j=1}^{n} [a_i/\gamma_j q^{y_j}]_\infty [b_i \gamma_j q^{y_j}]_\infty\right)$$

will have poles at

(1) $z_{l+1} = aq^p$ where $a \in A$, $a \ne \gamma_i$ for $1 \le i \le l$, and $p$ is a nonnegative integer;

(2) $z_{l+1} = bq^p \prod_{i=l+2}^{n-1} z_i^{-1} \prod_{i=1}^{l} \gamma_i^{-1} q^{-y_i}$ where $|z_{l+1}| < 1$, $b \in B$, $b^{-1} \ne \gamma_i$ for $1 \le i \le l$, and $p$ is a nonnegative integer;

(3) $z_{l+1} = a^{-1} q^{-p} \prod_{i=l+2}^{n-1} z_i^{-1} \prod_{i=1}^{l} \gamma_i^{-1} q^{-y_i}$ where $|z_{l+1}| < 1$, $a \in A$, $a \ne \gamma_i$ for $1 \le i \le l$, and $p$ is a nonnegative integer.

Poles of type 1 correspond to $\sigma_{l+1} = 1$ and poles of type 3 to $\sigma_{l+1} = -1$. Note that a pole of type 3 cannot occur if $\{\gamma_1, \dots, \gamma_l\} \subset A$.

Applying Cauchy's theorem, it follows that

$$\begin{aligned}
&\frac{1}{(2\pi i)^{n-1-l}} \int_{C^{n-1-l}} \sum_{\delta(l)\in\Delta(l)} S_\delta(l) \frac{dz_{l+1}}{z_{l+1}} \cdots \frac{dz_{n-1}}{z_{n-1}} \\
&\quad = \frac{1}{(2\pi i)^{n-1}} \int_{C^{n-2-l}} \int_{C_\varepsilon} \sum_{\delta(l)\in\Delta(l)} S_\delta(l) \frac{dz_{l+1}}{z_{l+1}} \cdots \frac{dz_{n-1}}{z_{n-1}} \\
&\qquad + \frac{1}{(2\pi i)^{n-2-l}} \int_{C^{n-2-l}} \sum_{\delta(l+1)\in\Delta(l+1)} S_{\delta,\varepsilon}(l+1) \frac{dz_{l+2}}{z_{l+2}} \cdots \frac{dz_{n-1}}{z_{n-1}}.
\end{aligned}$$

Now

$$\int_{C_\varepsilon} \sum_{\delta(l)\in\Delta(l)} S_\delta(l) \frac{dz_{l+1}}{z_{l+1}} \to 0 \tag{6.8}$$

as $\varepsilon \to 0$. This is because for $\delta(l) \in \Delta(l)$, $|z_{l+1}| = \varepsilon$, $|z_i| = 1$, $l+1 < i < n$, the series $S_\delta(l)$ is bounded in absolute value by a series of the form

$$D \sum_{\substack{y_1+\cdots+y_n=0 \\ |u_{l+1} q^{y_{l+1}}| = \varepsilon}} \sum_{\pi\in S_n} \left| \prod_{i=1}^{n} (u_{\pi(i)} q^{y_{\pi(i)}})^{n-i} \prod_{j,h=1}^{n} \frac{[b_j u_h]_{y_h}}{[q u_h/a_j]_{y_h}} \right| \tag{6.9}$$

where $D > 0$ is a constant, $u_i = \gamma_i$ for $1 \le i \le l$, $|q| < |u_{l+1}| < 1$, $u_{l+1} q^{y_{l+1}} = z_{l+1}$, $u_i = z_i$ for $l+1 \le i < n$, $\prod_{i=1}^{n} u_i = 1$, and $S_n$ is the symmetric group on $n$ letters. By Lemma 4.22 of [21] this series converges absolutely and uniformly in $z_i$ for $l+1 \le i < n$. Letting $\varepsilon = \varepsilon_0 |q|^{y_{l+1}}$ for some $|q| < \varepsilon_0 \le 1$, it follows that $|S_\delta(l)| \to 0$ as $y_{l+1} \to \infty$. Hence

$$\begin{aligned}
&\frac{1}{(2\pi i)^{n-1-l}} \int_{C^{n-1-l}} \sum_{\delta(l)\in\Delta(l)} S_\delta(l) \frac{dz_{l+1}}{z_{l+1}} \cdots \frac{dz_{n-1}}{z_{n-1}} \\
&\quad = \frac{1}{(2\pi k)^{n-2-l}} \int_{C^{n-2-l}} \sum_{\delta(l+1)\in\Delta(l+1)} S_\delta(l+1) \frac{dz_{l+2}}{z_{l+2}} \cdots \frac{dz_{n-1}}{z_{n-1}}.
\end{aligned}$$

Identity (6.7) now follows by induction.

Denoting $S_\delta(n-1)$ by $S_\delta$, the case $k=n-1$ of identity (6.7) is simply

$$\frac{1}{(2\pi i)^{n-1}}\int_{C^{n-1}}\frac{\prod\limits_{1\le i\ne j\le n}[z_iz_j^{-1}]_\infty}{\prod\limits_{i,j=1}^{n}[a_iz_j^{-1}]_\infty[b_iz_j]_\infty}\frac{dz_1}{z_1}\cdots\frac{dz_{n-1}}{z_{n-1}}=\sum_{\delta\in\Delta}S_\delta. \tag{6.10}$$

We shall show that some of the $S_\delta$ on the right-hand side of equation (6.10) cancel.

Let $1\le j+1\le l+1<n$ and $\delta=(\delta_1,\dots,\delta_n)\in\Delta$ satisfy

$$\begin{aligned}&\{\gamma_1,\dots,\gamma_j\}\subset B^{-1}\\&\{\gamma_{j+1},\dots,\gamma_l\}\subset A\quad\text{with } \sigma_i=1 \text{ for } j+1\le i\le l,\ \text{ and } \gamma_{l+1}\in B^{-1}.\end{aligned}\tag{6.11}$$

Let $\delta'=(\delta_1',\dots,\delta_n')\in\Delta$ with $\delta_i'=(\gamma_i',\sigma_i')$ for $1\le i\le n$ satisfy

$$\begin{gathered}\gamma_i'=\gamma_i,\ \sigma_i'=\sigma_i\quad\text{for } 1\le i\le n,\ \ i\ne j+1,\ \ l+1;\\ \gamma_{j+1}'=\gamma_{l+1},\ \sigma_{j+1}'=1\quad\text{and}\quad\gamma_{l+1}'=\gamma_{j+1},\ \sigma_{l+1}'=-1.\end{gathered}$$

Note that $\delta'\in\Delta$ is of the form

$$\begin{aligned}&\{\gamma_1',\dots,\gamma_{j+1}'\}\subset B^{-1},\\&\{\gamma_{j+2}',\dots,\gamma_l'\}\subset A\quad\text{with } \sigma_i'=1 \text{ for } j+2\le i\le l,\end{aligned}\tag{6.12}$$

and

$$\gamma_{l+1}'\in A\quad\text{with } \sigma_{l+1}'=-1.$$

There is clearly a one-to-one correspondence between such $\delta$ and $\delta'$ satisfying conditions (6.11) and (6.12) respectively.

Examining the conditions (6.4), we see that the set

$$A_{\delta,0}(n-1)=\{(y_1,y_2,\dots,y_n)\mid(y_{\tau(1)},y_{\tau(1)},\dots,y_{\tau(n)})\in A_{\delta',0}(n-1)\}$$

where $\tau$ is the transposition $(j+1,\ l+1)$ and $\delta',\delta'$ are as above. Since $\prod_{i=1}^{n-1}\sigma_i=-\prod_{i=1}^{n-1}\sigma_i'$, it follows from equation (6.6) that

$$S_\delta=-S_{\delta'}.\tag{6.13}$$

Therefore in the sum $\sum_{\delta\in\Delta}S_\delta$, the only terms $S_\delta$ not cancelling are for $\delta\in\Delta$ which for some $j$, $0\le j\le n-1$, satisfy

$$\{\gamma_1,\dots,\gamma_j\}\subset B^{-1},\qquad\{\gamma_{j+1},\dots,\gamma_{n-1}\}\subset A,\tag{6.14}$$

and $\sigma_i=1$ for all $i$, $1\le i\le n$.

To evaluate $S_\delta$ for $\delta\in\Delta$ satisfying conditions (6.14), we will need a $SU(n)$ generalization of the ${}_6\psi_6$ summation theorem which is proved in [21, Theorem 1.15]

**Theorem 6.15.** *With* $|q| < 1$ *and* $|q^{1-n}\prod_{i=1}^n (d_i/c_i)| < 1$, *we have*

$$\sum_{\substack{y_1,\dots,y_n=-\infty\\ y_1+\dots+y_n=0}}^{\infty} \prod_{1\le i<j\le n}\left(\frac{u_iq^{y_i}-u_jq^{y_j}}{u_i-u_j}\right)\prod_{i,k=1}\frac{[c_iu_k/u_i]_{y_k}}{[d_iu_k/u_i]_{y_k}}$$

$$(6.16)\qquad = \frac{[q]_\infty^{n-1}\prod_{i,k=1}^n[(d_iu_k)/(c_ku_i)]_\infty \prod_{1\le i\neq j\le n}[qu_i/u_j]_\infty}{\left[q^{1-n}\prod_{i=1}^n(d_i/c_i)\right]_\infty \prod_{i,k=1}^n[(qu_i)/(c_iu_k)]_\infty[d_iu_k/u_i]_\infty} \cdot \left[q\Big/\prod_{i=1}^n c_i\right]_\infty \left[q^{1-n}\prod_{i=1}^n d_i\right]_\infty$$

*where* $u_i\neq u_j$ *for* $1\le i<j\le n$, *and* $c_iu_j/u_i\neq q^l$ *and* $q^{-1}d_iu_j/u_i\neq q^{-l}$ *for a positive integer* $l$ *and* $1\le i,\ j\le n$.

Setting $c_i = b_i\gamma_i$, $d_i = q\gamma_i/a_i$ and $u_i=\gamma_i$ for $1\le i\le n$ and using formula (6.6), we obtain

$$(6.17)\qquad S_\delta = \frac{[q]_\infty^{n-1}\prod_{1\le i\neq j\le n}[\gamma_i/\gamma_j]_\infty[q\gamma_j/\gamma_i]_\infty}{\prod_{i,j=1}^n{}'[a_i/\gamma_j]_\infty[b_i\gamma_j]_\infty} \cdot \frac{\left[q/\prod_{i=1}^n b_i\right]_\infty\left[q/\prod_{i=1}^n a_i\right]_\infty\prod_{i,j=1}^n[q/(a_ib_j)]_\infty}{\left[q\prod_{i=1}^n(a_ib_i)^{-1}\right]_\infty\prod_{i,j=1}^n[q/(b_i\gamma_j)]_\infty[q\gamma_j/a_i)]_\infty}$$

where $\delta\in\Delta$ satisfies the conditions (6.14) and $\prod'$ is defined as in (6.5). For $c\in\mathbb{C}$ we define $\sigma(c) = [c]_\infty[q/c]_\infty$. With $\delta\in\Delta$ satisfying (6.14), we then have

$$(6.18)\qquad S_\delta = \frac{\left[\prod_{i=1}^n(a_ib_i)\right]_\infty}{\left[\prod_{i=1}^n a_i\right]_\infty\left[\prod_{i=1}^n b_i\right]_\infty\prod_{i,j=1}^n[a_ib_j]_\infty}\cdot T_\delta$$

where

$$(6.18b)\qquad T_\delta = \frac{[q]_\infty^{n-1}\sigma\left(\prod_{i=1}^n a_i\right)\sigma\left(\prod_{i=1}^n b_i\right)\prod_{i,j=1}^n\sigma(a_ib_j)\prod_{1\le i\neq j\le n}\sigma(\gamma_i/\gamma_j)}{\sigma\left(\prod_{i=1}^n(a_ib_i)\right)\prod_{i,j=1}^n{}'\sigma(a_i/\gamma_j)\sigma(b_i\gamma_j)}$$

and the product $\prod'$ means that if $c=1$, then the factor $\sigma(c)$ is replaced by $[q]_\infty^2$.

As a consequence of (6.10), (6.13), (6.14) and (6.18a, b), the proof of identity (6.2) is reduced to providing the following identity:

$$(6.19)\qquad T_\delta = n![q]_\infty^{1-n}$$

when the sum is over all $\delta\in\Delta$ satisfying (6.14).

Set $U(a_1, \dots, a_n, b_1, \dots, b_n) = \sum T_\delta$ with the sum over $\delta \in \Delta$ satisfying (6.14). Choose some $k$, $1 \le k \le n$, and fix the variables $a_i$, $i \ne k$, and all $b_i$, $1 \le i \le n$. Setting $a_k = e^{\alpha_k}$, then $U(e^{\alpha_k}) = U(a_1, \dots, a_n, b_1, \dots, b_n)$ is an elliptic function of the variable $\alpha_k \in \mathbb{C}$ with periods $\log q$ and $2\pi i$. This can be verified for each term $T_\delta$ and hence for the sum. We claim that $U(e^{\alpha_k})$ has at most one $a$ simple pole in each period parallelogram and hence is independent of the variable $a_k = e^{\alpha_k}$.

To show this, consider the possible poles of $U(e^{\alpha_k})$. They are simple and there are translates of the following types in each period parallelogram:

(i) $\prod_{i=1}^n (a_i b_i) = 1$,

(ii) $a_i a_k^{-1} = 1$ for $i \ne k$,

(iii) $a_k \prod_{i=1}^l b_{m(i)}^{-1} \prod_{j=1}^{n-l-1} a_{p(j)} = 1$

where $l \ge 1$, $1 \le m(1) < m(2) < \cdots < m(l) \le n$, $1 \le p(1) < p(2) < \cdots < p(n-l-1) \le n$ and $p(j) \ne k$ for all $j$, $1 \le j \le n-l-1$. For the poles of types (ii) and (iii) we will define an involution $\delta \to \delta'$ on the set of $\delta \in \Delta$ satisfying (6.14) such that the residues with respect to the variable $T_\delta$ and $T_{\delta'}$ will cancel. $\delta = (\delta_1, \dots, \delta_n)$ and $\delta' = (\delta'_1, \dots, \delta'_n)$ with $\delta_i = (\gamma_i, \sigma_i)$ and $\delta'_i = (\gamma'_i, \sigma'_i)$ for $i = 1, 2, \dots, n$, then we will have $\sigma'_1 = \sigma'_2 = \cdots = \sigma'_n = 1$ and $\{\gamma'_1, \gamma'_2, \dots, \gamma'_n\} = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ at the pole $a_k = c$. If $(1 - a_k^{-1}c)^{-1}$ will be a factor of $T_{\delta'}$, then $(1 - c^{-1}a_k)^{-1}$ will be a factor of $T_{\delta'}$ and vice versa. It will then follow that the residues of $T_\delta$ and $T_{\delta'}$ at $a_k = c$ will cancel.

Suppose $a_i a_k^{-1} = 1$ with $i \ne k$. Then we define the involution $\delta \to \delta'$ as follows: If both $a_i, a_k \in \{\gamma_1, \dots, \gamma_{n-1}\}$ or both $a_i, a_k \notin \{\gamma_1, \dots, \gamma_{n-1}\}$, then set $\delta' = \delta$. In these cases the residue of $T_\delta$ at $a_k = a_i$ (with respect to the variable $a_k$) is zero. If only one of the pair $a_i, a_k \in \{\gamma_1, \dots, \gamma_{n-1}\}$, then simply interchange $a_i$ and $a_k$, leaving $\sigma'_1 = \sigma'_2 = \cdots = \sigma'_n = 1$. For example, if $\gamma_l = a_k$ and $a_i \notin \gamma_1, \dots, \gamma_{n-1}$, then define $\gamma'_j = \gamma_j$ for $j \ne l$, $1 \le j \le n-1$, and set $\gamma'_l = a_i$. We also have $\gamma'_n = \prod_{i=1}^{n-1} (\gamma'_i)^{-1}$. Clearly there is a one-to-one correspondence between such $\delta$ and $\delta'$. The residues of $T_\delta$ and $T_{\delta'}$ at $a_k = a_i$ with respect to the variable $a_k$ cancel. It follows that $U(e^{\alpha_k}) = \sum T_\delta$ has no poles when $a_k = a_i$.

Suppose $a_k \prod_{i=1}^l b_{m(i)}^{-1} \prod_{j=1}^{n-l-1} a_{p(j)} = 1$ with notation as above. We define the involution $\delta \to \delta'$ as follows: If $\{\gamma_1, \dots, \gamma_{n-1}\} \not\subset \{a_k, b_{m(1)}^{-1}, \dots, b_{m(l)}^{-1}, a_{p(1)}, \dots, a_{p(n-l-1)}\}$, then set $\delta' = \delta$. In this case the residue of $T_\delta$ at $a_k = \prod_{i=1}^l b_{m(i)} \prod_{j=1}^{n-l-1} a_{p(j)}^{-1}$ with respect to $a_k$ is zero. Now suppose that $\{\gamma_1, \dots, \gamma_{n-1}\} \subset \{a_k, b_{m(1)}^{-1}, \dots, b_{m(l)}^{-1}, a_{p(1)}, \dots, a_{p(n-l-1)}\}$. If $\{a_j\} = \{a_k, b_{m(1)}^{-1}, \dots, b_{m(l)}^{-1}, a_{p(1)}, \dots, a_{p(n-l-1)}\} - \{\gamma_1, \dots, \gamma_{n-1}\}$, then suppose $\gamma_l = b_s^{-1}$. Define $\delta'$ by setting $\gamma'_i = \gamma_i$ for $i \ne l$, $1 \le i \le n-1$, and $\gamma'_l = a_j$ with $\sigma'_1 = \cdots = \sigma'_n = 1$. We also set $\gamma'_n = \prod_{i=1}^{n-1} (\gamma'_i)^{-1}$. If $\{b_s^{-1}\} = \{a_k, b_{m(1)}^{-1}, \dots, b_{m(l)}^{-1}, a_{p(1)}, \dots, a_{p(n-l-1)}\} - \{\gamma_1, \dots, \gamma_{n-1}\}$, then suppose $\gamma_l = a_j$. Define $\delta'$ by setting $\gamma'_i = \gamma_i$ for $i \ne l$, $1 \le j \le n-1$, and $\gamma'_l = b_s^{-1}$ with $\sigma'_1 = \cdots = \sigma'_n = 1$. Again we set $\gamma'_n = \prod_{i=1}^{n-1} (\gamma'_i)^{-1}$. As before, one verifies that there is a one-to-one correspondence between such $\delta$ and $\delta'$ and that the residues of $T_\delta$ and $T_{\delta'}$ at $a_k = \prod_{i=1}^l b_{m(i)} \prod_{j=1}^{n-l-1} a_{p(j)}^{-1}$ with respect to $a_k$ cancel. It follows that $U(e^{\alpha_k})$ has no poles of type (iii).

We have just shown that the only possible poles of $U(e^{\alpha_k})$ in a period parallelogram is the single simple pole of type (i). Since $U(e^{\alpha_k})$ is an elliptic function, then it must be a constant. Hence, $U(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is independent of all the variables $a_i$, $1 \le i \le n$. In a similar way one shows that $U(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is also independent of all the variables $b_i$, $1 \le i \le n$, and is therefore a constant.

To evaluate the constant $U(a_1, \ldots, b_n)$ set $a_i b_i = 1$ for $i = 1, \ldots, n-1$. One then verifies the only $T_\delta$ which do not vanish are those for which one and only one member of each pair $\{a_i, b_i^{-1}\}$, $1 \le i < n-1$, belongs to the set $\{\gamma_1, \ldots, \gamma_{n-1}\}$. For those $\delta$ satisfying this condition and also the conditions (6.14), $T_\delta$ equals the constant $[q]_\infty^{1-n}$. The number of such $\delta$ is

$$\sum_{k=0}^{n-1} \binom{n-1}{k} k!(n-1-k)! = n!.$$

This finishes the proofs of identities (6.19) and (6.2).

By analytic continuation, we can remove the assumptions made at the beginning of the proof of Theorem 6.1. This completes the proof of Theorem 6.1.

## 7. A GENERALIZATION OF THE ASKEY-WILSON INTEGRAL FOR THE SYMPLECTIC GROUPS

In this section we will prove a generalization of the Askey-Wilson integral for the symplectic group $Sp(n)$. The method of proof is similar to that of Theorem 6.1 and is modelled on the Askey and Wilson proof in one variable. We first expand the integral as a sum of several multiple series of residues. These multiple series are, up to a factor, basic hypergeometric series very-well-posed on the Lie algebra $Sp(n)$. They are evaluated by means of a generalization of the ${}_6\psi_6$ summation theorem [22]. Using an elliptic function identity, the resulting sum is then combined into a single quotient of infinite products.

The main result in this section is the following.

**Theorem 7.1.** *For $n \ge 1$, let $a_i \in \mathbb{C}$ for $1 \le i \le 2n+2$. Assume that the pairwise products of $\{a_1, a_2, \ldots, a_{2n+2}\}$ as a multiset (i.e. both $a_i^2$ and $a_i a_j$ are considered among the products) do not belong to the set $\{q^j,\ j = 0, -1, -2, \ldots\}$. Then*

$$\begin{aligned}\frac{1}{(2\pi i)^n}\int_{C^n} &\frac{\prod\limits_{1\le i<j<n}[z_iz_j]_\infty[z_i^{-1}z_j^{-1}]_\infty[z_iz_j^{-1}]_\infty[z_i^{-1}z_j]_\infty}{\prod\limits_{i=1}^{2n+2}\prod\limits_{j=1}^{n}[a_iz_j]_\infty[a_iz_j^{-1}]_\infty} \\ &\cdot\prod_{j=1}^{n}[z_j^2]_\infty[z_j^{-2}]_\infty\frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_n}{z_n} \qquad (7.2)\\ &= \frac{n!2^n\left[\prod\limits_{i=1}^{2n+2}a_i\right]_\infty}{[q]_\infty^n\prod\limits_{1\le i<j\le 2n+2}[a_ia_j]_\infty},\end{aligned}$$

*where the contour $C$ is the unit circle traversed in the positive direction, but with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to infinity such a contour exists by the assumptions above.*

*Proof*. Let us begin by assuming that $|a_i| < 1$ for all $1 \le i \le 2n+2$, that $|q| < |\prod_{i=1}^{2n+2} a_i| < 1$ and that $a_i a_j^{-1} \neq q^k$ for $1 \le i \neq j \le 2n+2$ and any integer $k$. We will remove these assumptions at the end of the proof.

For any nonnegative integer $N$ let $C_N$ be the circle of radius $|q|^N$ centered at zero and traversed in direction. Let

$$F = \frac{\prod_{1\le i<j\le n} [z_i z_j]_\infty [z_i^{-1} z_j^{-1}]_\infty [z_i z_j^{-1}]_\infty [z_i^{-1} z_j]_\infty \prod_{j=1}^{n} [z_j^2]_\infty [z_j^{-2}]_\infty}{\prod_{i=1}^{2n+2} \prod_{j=1}^{n} [a_i z_j]_\infty [a_i z_j^{-1}]_\infty}.$$

Then the integral on the left-hand side of (7.2) can be rewritten as

$$\begin{aligned} &\frac{1}{(2\pi i)^n} \int_{(C_0)^n} F \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \\ (7.3) \qquad &= \sum_{l=0}^{n} \frac{\binom{n}{l}}{(2\pi i)^n} \int_{(C_0 - C_N)^l} \int_{(C_N)^{n-l}} F \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \end{aligned}$$

where we have used Fubini's theorem and the symmetry of the integrand $F$ with respect to permutation of the variables $z_j$, $1 < j < n$.

Now let $S$ be the set of all injective mappings $\pi \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 2n+2\}$. For each $\pi \in S$ choose a subset $J(\pi) \subset \{1, \ldots, 2n+2\}$ where the order of $J(\pi)$ equals $n+1$ and $J(\pi)$ contains the image of $\pi$. Let $J'(\pi) = \{1, \ldots, 2n+2\} = J(\pi)$. Using Cauchy's theorem and identity (7.3), we rewrite the l.h.s. of (7.2) as

$$\begin{aligned} &\sum_{\pi \in S} \left\{ \frac{\prod_{1\le i<j\le n} [a_{\pi(i)} a_{\pi(j)}]_\infty [a_{\pi(i)}^{-1} a_{\pi(j)}^{-1}]_\infty [a_{\pi(i)} a_{\pi(j)}^{-1}]_\infty [a_{\pi(i)}^{-1} a_{\pi(j)}]_\infty}{\prod_{i=1}^{2n+2} \prod_{j=1}^{n}{}' [a_i a_{\pi(j)}]_\infty [a_i a_{\pi(j)}^{-1}]_\infty} \right. \\ &\quad \cdot \prod_{j=1}^{n} [a_{\pi(j)}^2]_\infty [a_{\pi(j)}^{-2}]_\infty \sum_{y_1, \ldots, y_n = 0}^{N-1} \\ (7.4) \qquad &\quad \cdot \left\{ q^{-\sum_{i=1}^{n} (n+1-i) y_i} \prod_{1 \le i < j \le n} \frac{(1 - a_{\pi(i)} a_{\pi(j)} q^{y_i + y_j})(1 - a_{\pi(i)} a_{\pi(j)}^{-1} q^{y_i - y_j})}{(1 - a_{\pi(i)} a_{\pi(j)})(1 - a_{\pi(i)} a_{\pi(j)}^{-1})} \right. \\ &\quad \left. \left. \cdot \prod_{j=1}^{n} \frac{(1 - a_{\pi(j)}^2 q^{2y_i})}{(1 - a_{\pi(j)}^2)} \prod_{j=1}^{n} \frac{\prod_{i \in J'(\pi)} [a_i a_{\pi(j)}]_{y_j} [a_i a_{\pi(j)}^{-1}]_{y_j}}{\prod_{i \in J(\pi} [q a_i^{-1} a_{\pi(j)}]_{y_j} [q a_i^{-1} a_{\pi(j)}^{-1}]_{y_j}} \right\} \right\} \\ &+ \sum_{l=0}^{n-1} \frac{\binom{n}{l}}{(2\pi i)^n} \int_{(C_0 - C_N)^l} \int_{(C_N)^{n-l}} F \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \end{aligned}$$

where $\prod'$ means that whenever the factor $[1]_\infty$ occurs in the product it is replaced by $[q]_\infty$.

Now suppose $z_i \in C_{k_i}$ where the $k_i$ are nonnegative integers for $1 \le i \le n$, then we have $z_i = u_i q^{k_i}$ for $u_i \in C_0$, $1 \le i \le n$. We find that

$$F(z_1, \dots, z_n) = \frac{\prod\limits_{1\le i<j\le n} [u_iu_j]_\infty [u_i^{-1}u_j^{-1}]_\infty [u_iu_j^{-1}]_\infty [u_i^{-1}u_j]_\infty}{\prod\limits_{i=1}^{2n+2} \prod\limits_{j=1}^{n} [a_iu_j]_\infty [a_iu_j^{-1}]_\infty} \cdot \prod_{j=1}^{n} [u_j^2]_\infty [u_j^{-2}]_\infty \cdot B(k_1, \dots, k_n),$$

where

$$B(k_1, \dots, k_n) = q^{-\sum_{i=1}^n (n+1-i)k_i} \left[\prod_{j=1}^{n} \left(\frac{1-u_j^2 q^{k_j}}{1-u_j^2}\right)\right] \cdot \prod_{1\le i<j\le n} \frac{(1-u_iu_jq^{k_i+k_j})(1-u_iu_j^{-1}q^{k_i-k_j})}{(1-u_iu_j)(1-u_iu_j^{-1})} \cdot \prod_{j=1}^{n} \left\{ \frac{\prod\limits_{i=1}^{n+1} [a_iu_j]_{y_j} [a_iu_j^{-1}]_{y_j}}{\prod\limits_{i=n+2}^{2n+2} [qa_i^{-1}u_j]_{y_j} [qa_i^{-1}u_j^{-1}]_{y_j}} \right\}.$$

From [22, §2] it follows that the series

$$\sum_{k_1, \dots, k_n=-\infty}^{\infty} B(k_1, \dots, k_n)$$

is absolutely and uniformly convergent for $u_i \in C_0$, $1 \le i \le n$. Consequently we have for $0 \le l \le n-1$, that

$$\lim_{N\to\infty} \sup_{\substack{z_i \in C_0 \cup C_N,\, 1\le i\le l \\ z_j \in C_N,\, l+1\le j\le n}} |F(z_1, \dots, z_n)| = 0,$$

and hence

$$\lim_{N\to\infty} \sum_{l=0}^{n-1} \frac{\binom{n}{l}}{(2\pi i)^n} \int_{(C_0-C_N)^l} \int_{(C_N)^{n-l}} F \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = 0. \tag{7.5}$$

Taking the limit of expression (7.4) as $N \to \infty$ and applying identity (7.5), we find that the l.h.s. of equation (7.2) equals

(7.6)

$$\sum_{\pi\in S}\left\{\frac{\prod_{1\le i<j\le n}[a_{\pi(i)}a_{\pi(j)}]_\infty[a^{-1}_{\pi(i)}a^{-1}_{\pi(j)}]_\infty[a_{\pi(i)}a^{-1}_{\pi(j)}]_\infty[a^{-1}_{\pi(i)}a_{\pi(j)}]_\infty}{\prod_{i=1}^{2n+2}\prod_{j=1}^{n}{}'[a_i a_{\pi(j)}]_\infty[a_i a^{-1}_{\pi(j)}]_\infty}\right.$$

$$\cdot\prod_{j=1}^{n}[a^2_{\pi(j)}]_\infty[a^{-2}_{\pi(j)}]_\infty\sum_{y_1,\ldots,y_n=0}^{\infty}$$

$$\cdot\left\{q^{-\sum_{i=1}^{n}(n+1-i)y_i}\prod_{1\le i<j\le n}\frac{(1-a_{\pi(i)}a_{\pi(j)}q^{y_i+y_j})(1-a_{\pi(i)}a^{-1}_{\pi(j)}q^{y_i-y_j})}{(1-a_{\pi(i)}a_{\pi(j)})(1-a_{\pi(i)}a^{-1}_{\pi(j)})}\right.$$

$$\left.\left.\cdot\prod_{j=1}^{n}\frac{(1-a^2_{\pi(j)}q^{2y_j})}{(1-a^2_{\pi(j)})}\prod_{j=1}^{n}\frac{\prod_{i\in J'(\pi)}[a_i a_{\pi(j)}]_{y_j}[a_i a^{-1}_{\pi(j)}]_{-y_j}}{\prod_{i\in J(\pi)}[qa_i^{-1}a_{\pi(j)}]_{y_j}[qa_i^{-1}a^{-1}_{\pi(j)}]_{-y_j}}\right\}\right\}$$

with notation as in (7.4) and where the infinite series converge absolutely since

$$|q|<\left|\prod_{i=1}^{2n+2}a_k\right|\quad\text{(see [22]).}$$

The infinite series in expression (7.6) can be summed by a generalization of Bailey's ${}_6\psi_6$ summation theorem for basic hypergeometric series very-well-posed on $Sp(n)$:

**Theorem 7.7** [22, Theorem 5.1]. *Let* $c_i,\ b_i\in\mathbb{C}$ *for* $i=1,\ldots,n+1$ *and* $(z_1,\ldots,z_n)\in\mathbb{C}^n$. *Suppose* $|q^{-n}\cdot\prod_{i=1}^{n+1}(b_i/c_i)|<1$. *Then*

$$\sum_{y_1,\ldots,y_n=-\infty}^{\infty}\left\{q^{-\sum_{i=1}^{n}(n+1-i)y_i}\prod_{j=1}^{n}\frac{(1-q^{2(z_j+y_j)})}{(1-q^{2z_j})}\right.$$

$$\cdot\prod_{1\le i<j\le n}\frac{(1-q^{z_i+y_i-z_j-y_j})(1-q^{z_i+y_i+z_j+y_j})}{(1-q^{z_i-z_j})(1-q^{z_i+z_j})}$$

$$\left.\cdot\prod_{i=1}^{n+1}\prod_{j=1}^{n}\frac{[c_iq^{z_j}]_{y_j}[c_iq^{-z_j}]_{-y_j}}{[b_iq^{z_j}]_{y_j}[b_iq^{-z_j}]_{-y_j}}\right\}$$

$$(7.8)\qquad =\frac{[q]^n_\infty\prod_{i,j=1}^{n+1}[b_i/c_j]_\infty\prod_{1\le i<j\le n+1}[q/(c_ic_j)_\infty[q^{-1}b_ib_j]_\infty}{\left[q^{-n}\prod_{i=1}^{n+1}(b_i/c_i)\right]_\infty}$$

$$\cdot\frac{\prod_{1\le i<j\le n}[q^{1+z_i+z_j}]_\infty[q^{1-z_i-z_j}]_\infty[q^{1+z_i-z_j}]_\infty[q^{1+z_j-z_i}]_\infty}{\prod_{i=1}^{n+1}\prod_{j=1}^{n}[b_iq^{z_j}]_\infty[b_iq^{-z_j}]_\infty[qc_i^{-1}q^{z_j}]_\infty[qc_i^{-1}q^{-z_j}]_\infty}$$

$$\cdot\prod_{j=1}^{n}[q^{1+2z}]_\infty[q^{1-2z_j}]_\infty.$$

With notation as above, set

$$\{c_1, \dots, c_{n+1}\} = \{a_i \mid i \in J'(\pi)\}, \{b_1, \dots, b_{n+1}\} = \{qa_i^{-1} \mid i \in J(\pi)\}$$

and $q^{z_j} = a_{\pi(j)}$ for $j = 1, \dots, n$. With these substitutions, the $(y_1, \dots, y_n)$ term in the series on the l.h.s. of (3.8) vanishes whenever $y_j < 0$ for some $j$, $1 \le j \le n$. For $c \in \mathbb{C}$, $c \ne 0$, define

$$\sigma(c) = [c]_\infty [q/c]_\infty. \tag{7.9}$$

Not substituting identity (7.8) into expression (7.6), we find that the l.h.s. of equation (7.2) equals

$$\frac{\left[\prod_{i=1}^{2n+1} a_i\right]_\infty}{\prod_{1\le i<j\le 2n+2} [a_i a_j]_\infty} \sum_{\pi\in S} T_\pi, \tag{7.10}$$

where

$$\begin{aligned} T_\pi = {} & \frac{[q]_\infty^n \prod_{1\le i<j\le 2n+2} \sigma(a_i a_j)}{\sigma\left(\prod_{i=1}^{2n+2} a_i\right)} \prod_{j=1}^n \sigma(a_{\pi(j)}^2)\sigma(a_{\pi(j)}^{-2}) \\ & \cdot \frac{\prod_{1\le i<j\le n} \sigma(a_{\pi(i)}a_{\pi(j)})\sigma(a_{\pi(i)}^{-1}a_{\pi(j)}^{-1})\sigma(a_{\pi(i)}a_{\pi(j)}^{-1})\sigma(a_{\pi(i)}^{-1}a_{\pi(j)})}{\prod_{i=1}^{2n+2} \prod_{j=1}^{n}{}' \sigma(a_i a_{\pi(j)})\sigma(a_i a_{\pi(j)}^{-1})} \end{aligned} \tag{7.11}$$

and the product $\prod'$ means that if $c = 1$, then the factor $\sigma(c)$ is replaced by $[q]_\infty^2$.

As a consequence (7.10) and (7.11), the proof of identity (3.2) is reduced to proving the following identity:

$$\sum_{\pi\in S} T_\pi = n!2^n [q]_\infty^{-n}. \tag{7.12}$$

The proof of identity (7.12) now proceeds similarly to that of identity (6.19) in §6. Set

$$U(a_1, a_2, \dots, a_{2n+2}) = \sum_{\pi\in S} T_\pi.$$

Choose some $k$, $1 \le k \le 2n+2$, and fix the variables $a_i$, $i \ne k$. Setting $a_k = e^{\alpha_k}$, then $U(e^{\alpha_k}) = U(a_1, \dots, a_{2n+2})$ is an elliptic function of the variable $\alpha_k \in \mathbb{C}$ with periods $\log q$ and $2\pi i$. This can be verified for each term $T_\pi$ and hence for the sum. We claim that $U(e^{\alpha_k})$ has at most one pole, a simple pole in each period parallelogram and hence is constant in the variable $\alpha_k$ and in $a_k = e^{\alpha_k}$.

The possible poles of $U(e^{\alpha_k})$ with respect to the variable $\alpha_k$ are the following two types:

(1) $e^{\alpha_k} = a_k = q^m \prod_{i=1\,;\,i\neq k}^{2n+2} a_i^{-1}$ for some integer $m$;
(2) $e^{\alpha_k} = a_k = q^m a_l$ for some integer $m$ and $1 \le l \le 2n+2$, $k \neq l$.

Consider the poles of type (2). Since the function $U(e^{\alpha_k})$ is elliptic, it is sufficient to consider only the poles $a_k = a_l$, $k \neq l$. For each such pole $a_k = a_l$, we will define an involution $\pi \to \pi'$ on the set $S$. Let $\pi \in S$. If both $k, l \in$ image of $\pi$, then define $\pi' = \pi$. Similarly if both $k, l \notin$ image of $\pi$, then define $\pi' = \pi$. If $k = \pi(j)$ for some $j$, $1 \le j \le n$, and $l \notin$ image of $\pi$, then define the map $\pi' \in S$ by

$$\pi'(i) = \begin{cases} \pi(i) & \text{for } 1 \le i \le n,\ i \neq j; \\ l & \text{for } i \neq j. \end{cases}$$

Finally, if $k \notin$ image of $\pi$ and $l = \pi(j)$ for some $j$, $1 \le j \le n$, then define the map $\pi' \in S$ by

$$\pi'(i) = \begin{cases} \pi(i) & \text{for } 1 \le i \le n,\ i \neq j; \\ k & \text{for } i = j. \end{cases}$$

The map $\pi \to \pi'$ is clearly an involution of $S$. If $T_\pi$ has a pole at $a_k = a_l$, then $T_{\pi'}$ also has a pole at $a_k = a_l$ and the residues of $T_\pi$ and $T_{\pi'}$ with respect to the variable $a_k$ cancel at $a_k = a_l$. It follows that $U(e^{\alpha_k})$ has no poles of type (2).

We have shown that the elliptic function $U(e^{\alpha_k})$ has at most one simple pole in each period parallelogram. Hence $U(a_1, \ldots, a_{2n+2})$ must be constant in the variable $a_k$. Since $k$ is arbitrary, $1 \le k \le 2n+2$, we have that $U(a_1, \ldots, a_{2n+2})$ is a constant in $a_1, \ldots, a_{2n+2}$.

To evaluate the constant $U(a_1, \ldots, a_{2n+2})$, set $a_i a_{n+i} = 1$ for $i = 1, \ldots, n$. Let $S'$ be the subset of all $\pi \in S$ such that one and only one member of each pair $\{i, n+i\}$, $1 \le i \le n$, belongs to the image of $\pi$. The order of $S'$ is $2^n n!$. One verifies that $T_\pi = 0$ if $\pi \notin S'$ and $T_\pi = [q]_\infty^{-n}$ if $\pi \in S$. Therefore we find that $U(a_1, \ldots, a_{2n+2}) = 2^n n! [q]^{-n}$.

By analytic continuation we may drop the assumptions made at the beginning of the proof of Theorem 7.1. This completes the proof of Theorem 7.1.

*Remark* 7.13. By specialization of the parameters in identity (7.2), one can evaluate the corresponding integrals associated to the compact groups $SO(2n+1, \mathbb{R})$ and $SO(2n, \mathbb{R})$. Recall that for $c \in \mathbb{C}$, we have the identity

$$[c]_\infty[-c]_\infty[cq^{\frac{1}{2}}]_\infty[-cq^{\frac{1}{2}}]_\infty = [c^2]_\infty.$$

Hence, if we set $a_{2n} = -1$, $a_{2n+1} = q^{\frac{1}{2}}$ and $a_{n2+2} = -q^{\frac{1}{2}}$ in identity (7.2) we obtain the following

**Theorem 7.14** ($SO(2n+1)$). *For $n \ge 1$, let $a_i \in \mathbb{C}$ for $1 \le i \le 2n-1$. Assume that the pairwise products of $\{a_1, a_2, \ldots, a_{2n-1}\}$ as a multiset do not belong to the set $\{q^j,\ j = 0, -1, -2, \ldots\}$. Then*

$$(7.15)\quad \begin{aligned} &\frac{1}{(2\pi i)^n}\int_{C^n} \frac{\prod\limits_{1\le i<j\le n} [z_iz_j]_\infty[z_i^{-1}z_j^{-1}]_\infty[z_iz_j^{-1}]_\infty[z_i^{-1}z_j]_\infty}{\prod\limits_{i=1}^{2n-1}\prod\limits_{j=1}^{n}[a_iz_j]_\infty[a_iz_j^{-1}]_\infty} \\ &\qquad\cdot \prod_{j=1}^{n}[z_j]_\infty[z_j^{-1}]_\infty \frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_n}{z_n} \\ &= \frac{n!2^n\left[q\prod\limits_{i=1}^{2n-1}a_i\right]_\infty}{[q]_\infty^n \prod\limits_{1\le i<j\le 2n-1}[a_ia_j]_\infty \prod\limits_{i=1}^{2n-1}[-a_i]_\infty[q^{\frac12}a_i]_\infty[-q^{\frac12}a_i]_\infty} \\ &= \frac{n!2^n\left[q\prod\limits_{i=1}^{2n-1}a_i\right]_\infty \prod\limits_{i=1}^{2n-1}[a_i]_\infty}{[q]_\infty^n \prod\limits_{1\le i<j\le 2n-1}[a_ia_j]_\infty \prod\limits_{i=1}^{2n-1}[a_i^2]_\infty}, \end{aligned}$$

*where the contour $C$ is the unit circle traversed in the positive direction, but with suitable deformations to separate the sequence of poles converging to zero from the sequences of poles diverging to infinity. Note that we have used the identity $[-q^{\frac12}]_\infty[q^{\frac12}]_\infty[-q]_\infty = [q\,;q^2]_\infty[-q\,;q]_\infty = 1$.*

If we set $a_{2n-1} = 1$ in identity (7.15), we obtain the following

**Theorem 7.16** ($SO(2n)$). *With notation and assumptions similar to the above and $n \ge 1$, then we have*

(7.17)

$$\begin{aligned} &\frac{1}{(2\pi i)^n}\int_{C^n} \frac{\prod\limits_{1\le i<j\le n} [z_iz_j]_\infty[z_i^{-1}z_j^{-1}]_\infty[z_iz_j^{-1}]_\infty[z_i^{-1}z_j]_\infty}{\prod\limits_{i=1}^{2n-2}\prod\limits_{j=1}^{n}[a_iz_j]_\infty[a_iz_j^{-1}]_\infty} \frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_n}{z_n} \\ &= \frac{n!2^{n-1}\left[q\prod\limits_{i=1}^{2n-2}a_i\right]}{[q]_\infty^n \prod\limits_{1\le i<j\le 2n-2}[a_ia_j]_\infty \prod\limits_{i=1}^{2n-2}[a_i^2]_\infty}. \end{aligned}$$

*Remark* 7.18. With notation and assumptions as in Theorem 7.1, if we set $q = 0$ in identity (7.2), then we obtain the identity

$$(7.19)\quad \begin{aligned} &\frac{1}{(2\pi i)^n}\int_{C^n} \frac{\prod\limits_{1\le i<j\le n} (1-z_iz_j)(1-z_i^{-1}z_j^{-1})(1-z_iz_j^{-1})(1-z_i^{-1}z_j)}{\prod\limits_{i=1}^{2n+2}\prod\limits_{j=1}^{n}(1-a_iz_j)(1-a_iz_j^{-1})} \\ &\qquad\cdot \prod_{j=1}^{n}(1-z_j^2)(1-z_j^{-2})\frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_n}{z_n} = \frac{n!2^n\left(1-\prod\limits_{i=1}^{2n+2}a_i\right)}{\prod\limits_{1\le i<j\le 2n+2}(1-a_ia_j)}. \end{aligned}$$

Setting $a_{2n+2} = -1$ in identity (7.19), we obtain the corresponding identity for $SO(2n+1)$:

$$\frac{1}{(2\pi i)^n}\int_{C^n} \frac{\prod_{1\le i<j\le n}(1-z_iz_j)(1-z_i^{-1}z_j^{-1})(1-z_iz_j^{-1})(1-z_i^{-1}z_j)}{\prod_{i=1}^{2n+2}\prod_{j=1}^{n}(1-a_iz_j)(1-a_iz_j^{-1})} \cdot\prod_{j=1}^{n}(1-z_j)(1-z_j^{-1})\frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_n}{z_n} \tag{7.20}$$
$$= \frac{n!2^n\left(1+\prod_{i=1}^{2n+1}a_i\right)}{\prod_{1\le i<j\le 2n+1}(1-a_ia_j)\prod_{i=1}^{2n+1}(1+a_i)}.$$

The setting $a_{2n+1} = 1$ in identity (7.20), we obtain the corresponding identity for $SO(2n)$;

(7.21)
$$\frac{1}{(2\pi i)^n}\int_{C^n} \frac{\prod_{1\le i<j\le n}(1-z_iz_j)(1-z_i^{-1}z_j^{-1})(1-z_iz_j^{-1})(1-z_i^{-1}z_j)}{\prod_{i=1}^{2n}\prod_{j=1}^{n}(1-a_iz_j)(1-a_iz_j^{-1})}\frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dx_n}{z_n}$$
$$= \frac{n!2^{n-1}\left(1+\prod_{i=1}^{2n}a_i\right)}{\prod_{1\le i<j\le 2n}(1-a_ia_j)\prod_{i=1}^{2n}(1-a_i^2)}.$$

One should compare these identities (7.19)–(7.21) with those found in Littlewood [34].

*Remark* 7.22. We finally remark that an integral analog of the summation theorem for the multilateral hypergeometric series very-well-poised on the affine root system $B_n$ [22, Theorem 6.1] can be obtained by setting $a_{2n+1} = q^{\frac{1}{2}}$ and $a_{2n+2} = -q^{\frac{1}{2}}$ in identity (7.2). Also by specializing identities (1.10) and (1.11) one can obtain identities corresponding to the Lie algebras of types $B_n$ and $D_n$ and the affine root system of type $B_n^v$. For example, to obtain from (1.10) an identity corresponding to $B_n$ set $a_{2n+2} = q^{\frac{1}{2}}$, $a_{2n+1} = -q^{\frac{1}{2}}$ and $a_{2n} = -q$ in (1.10).

## 8. An analogue of the Askey-Wilson integral for $G_2$

In this section we will prove an analogue of the Askey-Wilson integral for the compact Lie group of type $G_2$. The method of proof is similar to that of Theorem 6.1. The only real change is that there is a slightly different proof of the elliptic function identity used at the end.

The main result in this section is the following

**Theorem 8.1.** *Let* $a_1, a_2, a_3, a_4 \in \mathbb{C}$ *and* $|a_i| < 1$ *for* $1 \le i \le 4$. *Then*

$$\text{(4.2)}\quad \frac{1}{(2\pi i)^2}\int_{C^2} \frac{\prod\limits_{\substack{1\le i,j\le 3\\ i\ne j}} [z_i z_j^{-1}]_\infty \prod\limits_{j=1}^{3}[z_j]_\infty[z_j^{-1}]_\infty}{\prod\limits_{i=1}^{4}\prod\limits_{j=1}^{3}[a_i z_j]_\infty[a_i z_j^{-1}]_\infty}\frac{dz_1}{z_1}\frac{dz_2}{z_2} = \frac{12\left[\prod\limits_{i=1}^{4} a_i^2\right]_\infty \prod\limits_{i=1}^{4}[a_i]_\infty}{[q]_\infty^2\left[\prod\limits_{i=1}^{4} a_i\right]_\infty \prod\limits_{1\le i\le j\le 4}[a_i a_j]_\infty \prod\limits_{1\le i<j<k\le 4}[a_i a_j a_k]_\infty}$$

*where* $\prod_{i=1}^{3} z_i = 1$ *and* $C^2$ *is two-fold direct product of the unit circle* $C$ *traversed in the positive direction.*

*Proof.* We will assume temporarily that $|q| < \prod_{i=1}^{4}|a_i^2|$, $a_i \ne a_j$ for $1 \le i \ne j \le 4$, and $a_i \ne a_j a_k$ for $1 \le i, j, k \le 4$ with $i, j, k$ not necessarily distinct. Let

$$A = \{a_1, a_2, a_3, a_4\} \quad \text{and} \quad A^{-1} = \{a_1^{-1}, \dots, a_4^{-1}\}.$$

For $0 \le k \le 2$, let $\Delta(k)$ be the set of all $\delta(k) = (\delta_1, \delta_2, \delta_3)$ such that $\delta_i = (\gamma_i, \sigma_i)$, where

$$\text{(8.3)}\quad \begin{aligned} &\gamma_i \in A \cup A^{-1} \quad \text{for } 1 \le i \le k,\\ &\gamma_i = z_i \quad \text{for } k < i \le 2,\\ &\gamma_3 = \gamma_1^{-1}\gamma_2^{-1},\\ &\gamma_1 \ne \gamma_2 \quad \text{and} \quad \gamma_1 \ne \gamma_2^{-1},\\ &\sigma_1, \sigma_3 = 1 \quad \text{and} \quad \sigma_2 = \begin{cases} 1 \text{ if } \gamma_2 \in A^{-1} \text{ or } \gamma_1 \in A,\\ \text{either } 1 \text{ or } -1 \text{ otherwise.}\end{cases}\end{aligned}$$

We will denote $\Delta(2)$ by $\Delta$ and $\delta(2)$ by $\delta$. With this notation, the proof of Theorem 8.1 begins completely analogously to that of Theorem 6.1. For $\varepsilon > 0$, we define $A_{\delta,\varepsilon}(k)$ exactly as in (6.4). Similarly to (6.5), for $\delta \in \Delta$ we define

$$\text{(8.4)}\quad \begin{aligned} S_\delta &= S_{\delta,0}(2)\\ &= \sum_{(y_1,y_2,y_3)\in A_{\delta,0}(2)} \left\{ \left(\prod_{i=1}^{2}\sigma_i\right) \prod_{1\le i\ne j\le 3}[\gamma_i q^{y_i}/\gamma_j q^{y_j}]_\infty \right.\\ &\qquad \left. \frac{\prod\limits_{j=1}^{3}[\gamma_j q^{y_j}]_\infty[\gamma_j^{-1} q^{-y_j}]_\infty}{\prod\limits_{i=1}^{4}\prod\limits_{j=1}^{3}{}'[a_i\gamma_j q^{y_j}]_\infty[q_i\gamma_j^{-1}q^{-y_j}]_\infty} \right\}\end{aligned}$$

where $\prod'$ means that whenever the factor $[q^{-l}]_\infty$ occurs in the product for some nonnegative integer $l$, then $[q^{-l}]$ is replaced by $[q^{-l}]_l[q]_\infty$. Similarly to

(6.6), after simplication we obtain

(8.5)
$$S_\delta = \frac{\left(\prod_{i=1}^{2}\sigma_i\right)\prod_{1\le i\neq j\le 3}[\gamma_i/\gamma_j]_\infty\prod_{j=1}^{3}[\gamma_j]_\infty[\gamma_j^{-1}]_\infty}{\prod_{i=1}^{4}\prod_{j=1}^{3}{}'[a_i\gamma_j]_\infty[a_i\gamma_j^{-1}]_\infty]}$$
$$\cdot \sum_{(y_1,y_2,y_3)\in A_{\delta,0}(2)}\left\{q^{y_1+3y_2}\left(\frac{1-\gamma_3\gamma_1^{-1}q^{y_3-y_1}}{1-\gamma_3\gamma_1^{-1}}\right)\left(\frac{1-\gamma_1\gamma_2^{-1}q^{y_1-y_2}}{1-\gamma_1\gamma_2^{-1}}\right)\right.$$
$$\cdot\left(\frac{1-\gamma_3\gamma_2^{-1}q^{y_3-y_2}}{1-\gamma_3\gamma_2^{-1}}\right)\left(\frac{1-\gamma_1 q^{y_1}}{1-\gamma_1}\right)\left(\frac{1-\gamma_3 q^{y_3}}{1-\gamma_3}\right)$$
$$\left.\cdot\left(\frac{1-\gamma_2^{-1}q^{-y_2}}{1-\gamma_2^{-1}}\right)\prod_{i=1}^{4}\prod_{j=1}^{3}\frac{[a_i\gamma_j]_{y_j}}{[qa_i^{-1}\gamma_j]_{y_j}}\right\}.$$

Since $|q| < \prod_{i=1}^4 |a_i|^2$, the series $S_\delta$ converges absolutely (see [23, Theorem 1.1 and expression (2.28)]).

Completely analogously to the proof in §6, we obtain the following identity similar to equation (6.10):

(8.6)
$$\frac{1}{(2\pi i)^2}\int_{C^2}\frac{\prod_{1\le i\neq j\le 3}[z_iz_j^{-1}]_\infty\prod_{j=1}^{3}[z_j]_\infty[z_j^{-1}]_\infty}{\prod_{i=1}^{4}\prod_{j=1}^{3}[a_iz_j][a_iz_j^{-1}]_\infty}\frac{dz_1\,dz_2}{z_1z_2} = \sum_{\delta\in\Delta}S_\delta.$$

As in §6, we shall show that some of the $S_\delta$ cancel in the sum $\sum_{\delta\in\Delta}S_\delta$.

Let $\delta = (\delta_1, \delta_2, \delta_3) \in \Delta$ satisfy

(8.7)
$$\gamma_1\in A,\quad \gamma_2\in A^{-1}\quad\text{and}\quad \sigma_1=\sigma_2=1.$$

Let $\delta' = (\delta_1', \delta_2', \delta_3') \in \Delta$ with $\delta_1' = (\gamma_1', \sigma_i')$ for $1\le i\le 3$ satisfy $\gamma_1' = \gamma_2$, $\gamma_2' = \gamma_1$, $\sigma_1 = 1$ and $\sigma_2 = -1$. Note that $\delta'\in\Delta$ satisfies

(8.8)
$$\gamma_1'\in A^{-1},\quad \gamma_2'\in A,\ \sigma_1=1\quad\text{and}\quad \sigma_2=-1.$$

There is clearly a one-to-one correspondence between such $\delta$ and $\delta'$ satisfying conditions (8.7) and (8.8) respectively.

As in §6, we see that
$$A_{\delta,0}(2) = \{(y_1, y_2, y_3) \mid (y_2, y_1, y_3)\in A_{\delta',0}(2)\}$$
with $\delta$, $\delta'$ as above. Since $\sigma_1\sigma_2 = -\sigma_1'\sigma_2'$, it follows from equation (8.5) that

(8.9)
$$S_\delta = -S_{\delta'}.$$

Therefore in the sum $\sum_{\delta\in\Delta}S_\delta$, the only terms $S_\delta$ not cancelling are for $\delta\in\Delta$ which satisfy

(8.10)
$$\sigma_i = 1 \text{ for } 1\le i\le 3 \text{ and if } \gamma_1\in A,\text{ then } \gamma_2\in A.$$

To evaluate $S_\delta$ for $\delta\in\Delta$ satisfying conditions (8.10), we will need a $G_2$ analogue of the ${}_6\psi_6$ summation theorem which is proved in [23, Theorem 1.1 and expression (2.28)]

**Theorem 8.11.** *With* $|q| < 1$ *and* $|q| < \prod_{i=1}^4 |c_i|^2$, *we have*

$$
\begin{aligned}
&\sum_{\substack{y_1, y_2, y_3=-\infty \\ y_1+y_2+y_3=0}}^{\infty} \left\{ q^{y_1+3y_2} \left( \frac{1-q^{u_3+y_3-u_1-y_1}}{1-q^{u_3-u_1}} \right) \right. \\
&\qquad \cdot \left( \frac{1-q^{u_1+y_1-u_2-y_2}}{1-q^{u_1-u_2}} \right) \left( \frac{1-q^{u_3+y_3-u_2-y_2}}{1-q^{u_3-u_2}} \right) \left( \frac{1-q^{u_1+y_1}}{1-q^{u_1}} \right) \\
&\qquad \left. \cdot \left( \frac{1-q^{u_3+y_3}}{1-q^{u_3}} \right) \left( \frac{1-q^{-u_2-y_2}}{1-q^{-u_2}} \right) \prod_{i=1}^4 \prod_{j=1}^3 \frac{[c_i q^{u_j}]_{y_j}}{[qc_i^{-1} q^{u_j}]_{y_j}} \right\} \\
&= \frac{[q]_\infty^2 \left[ q \prod_{i=1}^4 c_i^{-1} \right]_\infty \prod_{1 \le i \le j \le 4} [q(c_i c_j)^{-1}]_\infty \prod_{1 \le i < j < k \le 4} [q(c_i c_j c_k)^{-1}]_\infty}{\left[ q \prod_{i=1}^4 c_i^{-2} \right]_\infty \prod_{i=1}^4 [qc_i^{-1}]_\infty \prod_{i=1}^4 \prod_{j=1}^3 [c_i^{-1} q^{1+u_j}]_\infty [c_i^{-1} q^{1-u_j}]_\infty} \\
&\qquad \cdot \prod_{j=1}^3 [q^{1+u_j}]_\infty [q^{1-u_j}]_\infty \prod_{\substack{1 \le j, k \le 3 \\ j \ne k}} [q^{1+u_j-u_k}]_\infty ,
\end{aligned} \tag{8.12}
$$

*where* $u_1 + u_2 + u_3 = 0$.

For $\delta \in \Delta$ satisfying condition (8.10), let $c_i = a_i$ and $q^{u_i} = \gamma_i$ for $1 \le i \le 3$ in formula (8.12). We then obtain

$$
S_\delta = \frac{\left[ \prod_{i=1}^4 a_i^2 \right]_\infty \prod_{i=1}^4 [a_i]_\infty}{\left[ \prod_{i=1}^4 a_i \right]_\infty \prod_{1 \le i \le j \le 4} [a_i a_j]_\infty \prod_{1 \le i \le j \le k \le 4} [a_i a_j a_k]_\infty} T_\delta , \tag{8.13}
$$

where

$$
\begin{aligned}
T_\delta = {} & \frac{[q]_\infty^2 \sigma\left( \prod_{I=1}^4 a_i \right) \prod_{1 \le i \le j \le 4} \sigma(a_i a_j) \prod_{1 \le i < j < k \le 4} \sigma(a_i a_j a_k)}{\sigma\left( \prod_{i=1}^4 a_i^2 \right) \prod_{i=1}^4 \sigma(a_i)} \\
& \cdot \frac{\prod_{\substack{1 \le i, j \le 3 \\ i \ne j}} \sigma(\gamma_i/\gamma_j) \prod_{j=1}^3 \sigma(\gamma_j) \sigma(\gamma_j^{-1})}{\prod_{i=1}^4 \prod_{j=1}^3{}' \sigma(a_i \gamma_j) \sigma(a_i \gamma_j^{-1})}
\end{aligned} \tag{8.14}
$$

and where $\sigma(c)$ and $\prod'$ are defined as in (6.18), (7.9), and (7.11).

As a consequence of equations (8.6), (8.9), (8.13), and (8.14), the proof of identity (8.2) is reduced proving the following identity:

$$
\sum T_\delta = 12[q]_\infty^{-2} , \tag{8.15}
$$

where the sum is over all $\delta \in \Delta$ satisfying (8.10).

As in §§6 and 7, we set $U(a_1, a_2, a_3, a_4) = \sum T_\delta$ with the sum over $\delta \in \Delta$ satisfying (8.10). As before, choose some $k$, $1 \le k \le 4$, and fix the variables

$a_i$ for $i \neq k$. Setting $a_k = e^{\alpha_k}$, then $U(e^{\alpha_k}) = U(a_1, \ldots, a_4)$ is an elliptic function of the variable $\alpha_k \in \mathbb{C}$ with periods $\log q$ and $2\pi i$. Again this can be verified for each $T_\delta$ in the sum $\sum T_\delta$. We will show that $U(e^{\alpha_k})$ has at most three possible poles, which are simple, in each period parallelogram. A closer examination of the residue at each of these three possible poles shows that, up to a factor the residues are elliptic functions involving the other variables $a_i$, $i \neq k$. We will show that these elliptic functions have no poles in a period parallelogram. Hence these elliptic functions are constant. A simple evaluation will then show that they are identically zero. Consequently, since the residues of the possible poles of the original function $U(e^{\alpha_k})$ vanish, then $U(e^{\alpha_k})$ must also be constant. It follows that $U(a_1, \ldots, a_4)$ is constant in all variables $a_1, \ldots, a_4$. This constant will then be found by a special evaluation.

Let us begin by considering the possible poles of $U(e^{\alpha_k})$. They are simple and there are translates of the following types in each period parallelogram:

(1) $\prod_{i=1}^{4} a_i = \pm q^{\frac{1}{2}}$ or $-1$,
(2) $a_i a_k^{-1} = 1$ for $i \neq k$,
(3) $a_i a_j a_k^{-1} = 1$ for $i, j$ and $k$ mutually distinct,
(4) $a_i a_j^{-1} a_k = 1$ for $i, j$ and $k$ mutually distinct.

The poles of type (1) will be examined later. For the poles of types (2), (3) and (4), we will define an involution $\delta \to \delta'$ on the set of $\delta \in \Delta$ satisfying (8.10) such that the residues with respect to the variable $e^{\alpha_k} = a_k$ of $T_\delta$ and $T_{\delta'}$ will cancel. If $\delta = (\delta_1, \delta_2, \delta_3)$ and $\delta' = (\delta_1', \delta_2', \delta_3')$ with $\delta_i = (\gamma_i, \sigma_i)$ and $\delta_i' = (\gamma_i', \sigma_i')$ for $i = 1, 2, 3$, then will have $\sigma_1' = \sigma_2' = \sigma_3' = 1$ and $\{\gamma_1', \gamma_2', \gamma_3'\} = \{\gamma_1, \gamma_2, \gamma_3\}$ at the pole $a_k = c$. If $(1 - (c/a_k))^{-1}$ is a factor of $T_\delta$, then $(1 - (a_k/c))^{-1}$ will be a factor of $T_{\delta'}$ and vice versa. It will then follow that the residues of $T_\delta$ and $T_{\delta'}$ at $a_i = c$ will cancel.

Suppose $a_i a_k^{-1} = 1$ with $i \neq k$. Then we define the involution $\delta \to \delta'$ as follows: If both $a_i, a_k \in \{\gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1}\}$ or both $a_i, a_k \notin \{\gamma_1, \gamma_2, \gamma_2^{-1}, \gamma_2^{-1}\}$, then set $\delta' = \delta$. In these cases the residue of $T_\delta$ at $a_k = a_i$ (with respect to the variable $e^{\alpha_k} = a_k$) is zero. If only one of the pair $a_i, a_k \in \{\gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1}\}$, then simply interchange $a_i$ and $a_k$, leaving $\sigma_1' = \sigma_2' = 1$. For example, if $\gamma_1 = a_k$ and $\gamma_2 = a_j^{-1}$ with $j \neq i$, then define $\gamma_1' = a_i$ and $\gamma_2' = a_j^{-1}$ for $\delta' = (\delta_1', \delta_2', \delta_3')$ with $\delta' = (\gamma_l', \sigma_l')$, $1 \leq l \leq 3$. Clearly there is a one-to-one correspondence between such $\delta$ and $\delta'$. The residue of $T_\delta$ and $T_{\delta'}$ at $a_k = a_i$ with respect to the variable $a_k$ cancel. It follows that $U(e^{\alpha_k}) = \sum T_\delta$ has no poles when $a_k = a_i$.

Suppose $a_i a_j a_k^{-1} = 1$ for $i, j$, and $k$ mutually distinct. Then we define an involution $\delta \to \delta'$ as follows: Let $D = \{a_i a_j, (a_i a_j)^{-1}, a_i a_k^{-1}, a_j a_k^{-1}, a_i^{-1} a_k, a_j^{-1} a_k\}$. Suppose $\gamma_3 \notin D$, then define $\delta = \delta'$. In this case the residue of $T_\delta$ at $a_k = a_i a_j$ (with respect to the variable $e^{\alpha_k} = a_k$) is zero. If $\gamma_3 \in D$, then we will define $\delta' = (\delta_1', \delta_2', \delta_3')$ with $\delta_l' = (\gamma_l', \sigma_l)$, $1 \leq l \leq 3$, in a case-by-case fashion. In each case set $\sigma_1' = \sigma_2' = \sigma_3' = 1$ and $\gamma_3' = (\gamma_1' \gamma_2')^{-1}$. If $\gamma_3 = a_i a_j$, then define $\gamma_1' = \gamma_1$ and $\gamma_2' = a_k$. If $\gamma_3 = (a_i a_j)^{-1}$, then define $\gamma_1' = a_k^{-1}$ and $\gamma_2' = \gamma_2$. If $\gamma_3 = a_i a_k^{-1}$ then define $\gamma_1' = \gamma_1 = a_i^{-1}$ and $\gamma_2' = a_j^{-1}$. Similarly if $\gamma_3 = a_j a_k^{-1}$, then define $\gamma_1' = \gamma_1 = a_j^{-1}$ and $\gamma_2' = a_i^{-1}$. If $\gamma_3 = a_i^{-1} a_k$, then define $\gamma_1' = a_j$ and $\gamma_2' = \gamma_2 = a_i$. Similarly if $\gamma_3 = a_j^{-1} a_k$, then define $\gamma_1' = a_i$

and $\gamma_2' = \gamma_2 = a_j$. One now verifies that there is one-to-one correspondence between such $\delta$ and $\delta'$ and that the residues of $T_\delta$ and $T_{\delta'}$ at $a_k = a_i a_j$ with respect to the variable $a_k$ cancel. It follows that $U(e^{\alpha_k}) = \sum T_\delta$ has no poles when $a_k = a_i a_j$.

Now suppose $a_i a_j^{-1} a_k = 1$ for $i, j$ and $k$ mutually distinct. We define the involution $\delta \to \delta'$ as follows: Let $E = \{a_i a_j^{-1}, a_i^{-1} a_j, a_i a_k, (a_i a_k)^{-1}, a_j^{-1} a_k, a_j a_k^{-1}\}$. Suppose $\gamma_3 \notin E$, then define $\delta' = \delta$. In this case the residue of $T_\delta$ and $a_k = a_i^{-1} a_j$ with respect to the variable $a_k$ is zero. If $\gamma_3 \in E$, then we will define $\delta'$ in a case-by-case fashion. In each case set $\sigma_1' = \sigma_2' = \sigma_3' = 1$ and $\gamma_3' = (\gamma_1' \gamma_1')^{-1}$. If $\gamma_3 = a_i a_j^{-1}$, then define $\gamma_1' = a_i^{-1}$ and $\gamma_2' = \gamma_2 = a_k^{-1}$. If $\gamma_3 = a_i^{-1} a_j$, then define $\gamma_1' = a_k$ and $\gamma_2' = \gamma_2 = a_i$. If $\gamma_3 = a_i a_k$, then define $\gamma_1' = \gamma_1$ and $\gamma_2' = a_j$. If $\gamma_3 = a_i^{-1} a_k^{-1}$, then define $\gamma_1' = a_j^{-1}$ and $\gamma_2' = \gamma_2$. If $\gamma_3 = a_j^{-1} a_k$, then define $\gamma_1' = \gamma_1 = a_i^{-1}$. If $\gamma_3 = a_j a_k^{-1}$, then define $\gamma_1' = a_i$ and $\gamma_2' = \gamma_2 = a_k$. As above, one verifies that there is a one-to-one correspondence between such $\delta$ and $\delta'$ and that the residues of $T_\delta$ and $T_{\delta'}$ at $a_k = a_i^{-1} a_j$ with respect to the variable $a_k$ cancel. It follows that $U(e^{\alpha_k}) = \sum T_\delta$ has no poles when $a_k = a_i^{-1} a_j$. Therefore the elliptic function $U(e^{\alpha_k})$ has no poles of types (2), (3) and (4).

Finally, we consider the possible of type (1) where $\prod_{i=1}^4 a_i = \pm q^{\frac{1}{2}}$ or $-1$. For example, let us suppose that $a_k = q^{\frac{1}{2}} \prod_{i=1\,;\, i \neq k}^4 a_i^{-1}$. For each $\delta \in \Delta$ satisfying condition (5.10), let $\operatorname{Res}(T_\delta)$ be the residue of $T_\delta$ with respect to the variable $a_k$ at the pole $a_k = q^{\frac{1}{2}} \prod_{i=1\,;\, i \neq k}^4 a_i^{-1}$. Now choose any variable $a_l$, $l \neq k$, and fix the other variables $a_i$ for $i \neq l, k$. Set $a_l = e^{\alpha_l}$ with $a_l \in \mathbb{C}$. Then the function $R_\delta(e^{\alpha_l}) = R_\delta(a_l) = (\prod_{i=1\,;\, i \neq k}^4 a_i) \cdot \operatorname{Res}(T_\delta)$ is an elliptic function in $a_l$ with periods $\log q$ and $2\pi i$.

Consider the function $V(e^{\alpha_l}) = \sum R_\delta(e^{\alpha_l})$, where the sum is over all $\delta \in \Delta$ satisfying condition (8.10). The possible poles of $V(e^{\alpha_l}) = V(a_l)$ in the variable $\alpha_l$ are essentially the same as those of types (2), (3), and (4) of the function $U(a_l) = U(e^{\alpha_l})$. In other words, in each period parallelogram, the possible poles of $V(e^{\alpha_l})$ are translates of the following types:

(i) $a_i a_l^{-1} = 1$ for $i \neq l$,
(ii) $a_i a_j a_l^{-1} = 1$ for $i, j$, and $l$ mutually distinct,
(iii) $a_i a_j^{-1} a_l = 1$ for $i, j$, and $l$ mutually distinct,

where we also permit $k = i$ or $j$. Note that after the substitution $a_k = q^{\frac{1}{2}} \prod_{i=1\,:\, i \neq k}^4 a_i^{-1}$ the possible poles of types (i), (ii), and (iii) are all distinct and simple. The proof that the sum of the residues of $\sum T_\delta = U(a_k)$ vanish at the poles of types (2), (3), and (4) goes over with essentially no change (except with the variable $a_l$ in place of $a_k$) to show that the sum of the residues of $\sum R_\delta = V(a_l)$ vanish at the poles of types (i), (ii), and (iii).

To evaluate the constant $V$ (as a function of all the variables $a_i$, $i \neq k$), set $\prod_{i=1\,;\, i \neq k i}^4 a_i = 1$ where we assume that the set $\{a_i, a_j, q\}$ is algebraically independent over the rational numbers for some pair $i, j$ with $i, j \neq k$ and $i \neq j$. One now checks that in each $R_\delta$ only the factors $\sigma(a_k^2)$ and $\sigma(\prod_{i=1\,;\, i \neq k}^4 a_i)$ vanish. Hence each $R_\delta$ has two zeroes in the numerator and at most open zero in the denominator. It follows that $\sum R_\delta = V$ is zero. Therefore the function $U(a_k)$ has no pole at $a_k = q^{\frac{1}{2}} \prod_{i=1\,;\,i\neq k}^{4} a^{-1}$.

An entirely similar argument also shows that $U(a_k)$ has no poles at $a_k = -q^{\frac{1}{2}} \prod_{i=1\,;\,i\neq k}^{4} a_i^{-1}$ and $a_k = -\prod_{i=1\,;\,i\neq k}^{4} a_i^{-1}$. It follows that $U(e^{\alpha_k})$ has no poles of type (1) and therefore no poles at all. Since $U(e^{\alpha_k})$ is an elliptic function, it must be constant in the variable $\alpha_k$ (or $a_k$). Hence $U(a_1, a_2, a_3, a_4)$ is constant in all the variables $a_1, \dots, a_4$.

To evaluate the constant $U(a_1, \dots, a_4)$, we set $a_1 = -1$ and $a_2 = q^{\frac{1}{2}}$ with $a_3$ and $a_4$ independent). One then checks that in the sum $\sum T_\delta = U(a_1, \dots, a_4)$ all terms $T_\delta$ vanish except where

$$\gamma_3 \in \{a_1, a_2, (a_1a_2)^{-1}, a_1^{-1}a_2, a_1a_2^{-1}\}, \tag{8.16}$$

with notation as above. For the $\delta \in \Delta$ satisfying conditions (8.10) and (18.6), one checks that the values of $T_\delta$ are then identical. There are six such $\delta \in \Delta$ satisfying (8.10) and (8.16). After an elementary computation making use of the identity $[c^2; q]_\infty = [c; q]_\infty[-c; q]_\infty[cq^{\frac{1}{2}}; q]_\infty[-cq^{\frac{1}{q}}]_\infty$ for $c \in \mathbb{C}$, we find that

$$T_\delta = \frac{2[q; q^2]_\infty^2[-q; q]_\infty^2}{[q; q]_\infty^2} \tag{8.17}$$

for all $\delta \in \Delta$ satisfying (8.10) and (8.16). Since $[q; q^2]_\infty[-q; q]_\infty = 1$, then identity (8.17) reduces to

$$T_\delta = \frac{2}{[q; q]_\infty^2}. \tag{8.18}$$

Hence $U(a_1, \dots, a_4) = \sum T_\delta = 12[q]_\infty^2$. This completes the proof of identity (8.15). By analytic continuation we may drop the assumptions made at the beginning of the proof of Theorem 8.1. This completes the proof of Theorem 8.1.

## 9. Some multidimensional Mellin-Barnes integrals

We will compute the limiting cases as $q \to 1$ in Theorems 6.1, 7.1, and 8.1.

**Theorem 9.1.** *For $n \geq 2$, let $a_i, b_i \in \mathbb{C}$ with $\mathrm{Re}(a_i)$, $\mathrm{Re}(b_i) > 0$ for $1 \leq i \leq n$. Then*

$$\begin{aligned} &\frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \frac{\prod_{i,j=1}^{n} \Gamma(a_i - z_j)\Gamma(b_i + z_j)\, dz_1\, dz_2 \cdots dz_{n-1}}{\prod_{\substack{1\leq i,j\leq n\\ i\neq j}} \Gamma(z_i - z_j)} \\ &= \frac{n!\Gamma\left(\sum_{i=1}^{n} a_i\right)\Gamma\left(\sum_{i=1}^{n} b_i\right)\prod_{i,j=1}^{n}\Gamma(a_i + b_j)}{\Gamma\left(\sum_{i=1}^{n}(a_i + b_i)\right)} \end{aligned} \tag{9.2}$$

*where $\sum_{i=1}^{n} z_i = 0$ and the contours of integration are the imaginary axis.*

**Theorem 9.3.** *For $n \geq 1$, let $a_i \in \mathbb{C}$ for $1 \leq i \leq 2n+2$. Assume that the pairwise sums of $\{a_1, a_2, \dots, a_{2n+2}\}$ as a multiset (i.e. both $2a_i$ and $a_i + a_j$*

*are considered among the sums) do not belong to the set* $\{0, -1, -2, \dots\}$. *Then*

$$\begin{aligned}(9.4)\qquad &\frac{1}{(2\pi i)^n}\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}\frac{\prod_{i=1}^{2n+2}\prod_{j=1}^{n}\Gamma(a_i+z_j)\Gamma(a_i-z_j)}{\prod_{1\le i<j\le n}\Gamma(z_i+z_j)\Gamma(-z_i-z_j)\Gamma(z_i-z_j)\Gamma(z_i-z_j)}\\ &\cdot\frac{1}{\prod_{j=1}^{n}\Gamma(2z_j)\Gamma(-2z_j)}\,dz_1\,dz_2\cdots dz_n\\ &=\frac{n!2^n\prod_{1\le i<j\le 2n+2}\Gamma(a_i+a_j)}{\Gamma\left(\sum_{i=1}^{2n+2}a_i\right)},\end{aligned}$$

*where the contours of integration are deformed so as to separate the sequence of poles going to the right* $\{a_i+k \mid 1\le i\le 2n+2\},\ k=0,1,2,\dots\}$ *from the sequences of poles going to the left* $\{-a_i-k \mid 1\le i\le 2n+2,\ k=0,1,2,\dots\}$.

**Theorem 9.5.** *Let* $a_i\in\mathbb{C}$ *and* $\mathrm{Re}(a_i)>0$ *for* $1\le i\le 4$. *Then*

$$\begin{aligned}(9.6)\qquad &\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{i=1}^{4}\prod_{j=1}^{3}\Gamma(a_i+z_j)\Gamma(a_i-z_j)}{\prod_{\substack{1\le i,j\le 3\\ i\ne j}}\Gamma(z_i-z_j)\prod_{j=1}^{3}\Gamma(z_j)\Gamma(-z_j)}\,dz_1\,dz_2\\ &=\frac{12\Gamma\left(\sum_{i=1}^{4}a_i\right)\prod_{1\le i\le j\le 4}\Gamma(a_i+a_j)\prod_{1\le i<j<k\le 4}\Gamma(a_i+a_j+a_k)}{\Gamma\left(2\sum_{i=1}^{4}a_i\right)\prod_{i=1}^{4}\Gamma(a_i)}\end{aligned}$$

*where* $z_1+z_2+z_3=0$ *and the contours of integration are the imaginary axis.*

*Proofs of Theorems* 9.1, 9.3, *and* 9.5. The proof of these limiting cases is similar to the proof of Theorem 5.1 in §5. One begins by making assumptions on the parameters analogous to those made in the proof of Theorem 6.1, 7.1, and 8.1. Then one obtains ordinary hypergeometric series (very-well-posed on the various Lie algebras [22]) expansions of the Mellin-Barnes integrals in place of the corresponding basic hypergeometric series expansion used in the proofs of Theorem 6.1, 7.1, and 8.1. One makes estimates for the integrands similar to (5.4), which can be used to prove the convergence of the integrals and to show the integrals equal the corresponding series expansion. For Theorems 9.1 and 9.5, to show the integrals equal the corresponding sum of hypergeometric series, one needs to prove an analogue of the limit (6.8) and the estimate (6.9). This can be done by applying Stirling's formula (as in (5.3) and (5.4)) and using the convergence estimates for the ordinary $SU(n)$ or $G_2$ hypergeometric series (see [21 and 23]). For Theorems 9.1 and 9.5, the cancellation between the ordinary series corresponding to the $S_\delta$ (see (6.5) and (6.6)) occurs in exactly the same way as in the proofs of Theorem 6.1 and 8.1. The remaining series in all the Theorems 9.1, 9.3, and 9.5 are summable by the corresponding ordinary (Lie algebra) hypergeometric series summation formulas (see [22]). Finally, the limiting cases ($q \to 1$) of the elliptic function identities (6.19), (7.12), and (8.15) give trigonometric identities, which are used to combine the terms into a single quotient of products of gamma functions.

To take the limits of the elliptic functions (or theta functions) one can use the $q$-analogue of the gamma function defined by F. H. Jackson

$$\Gamma_q(x) = \frac{[q\,;\,q]_\infty (1-q)^{1-x}}{[q^x : q]_\infty}$$

for $0 < q < 1$. We have the following limit (see Askey [4] and Gosper [1, Appendix A]):

$$\lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x).$$

This completes the proofs of identities (9.2), (9.4), and (9.6) under the assumptions made at the beginning. These assumptions are then dropped by analytic continuation, completing the proofs of Theorems 9.1, 9.3, and 9.5.

*Remark* 9.7. Recall the Legendre duplication formula

$$\Gamma(2x) = \pi^{-\frac{1}{2}} 2^{2x-1} \Gamma(x)\Gamma(x+\tfrac{1}{2}).$$

Using this formula and setting $a_{2n+2} = \frac{1}{2}$ and then $a_{2n+1} = 0$ in identity (9.4), we obtain the following identities associated to $SO(2n+1)$ and $SO(2n)$ respectively:

$$\begin{aligned}
&\frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^{2n+1} \prod_{j=1}^{n} \Gamma(a_i + z_j)\Gamma(a_i - z_j)}{\prod_{1 \le i < j \le n} \Gamma(z_i + z_j)\Gamma(-z_i - z_j)\Gamma(z_i - z_j)\Gamma(z_j - z_i)} \\
(9.8) \qquad &\cdot \frac{1}{\prod_{j=1}^{n} \Gamma(z_j)\Gamma(-z_j)} \, dz_1 \cdots dz_n \\
&= \frac{n! 2^n}{(4\pi)^n} \frac{\prod_{1 \le i < j \le 2n+1} \Gamma(a_i + a_j) \prod_{i=1}^{2n+1} \Gamma(\frac{1}{2} + a_i)}{\Gamma\left(\frac{1}{2} + \sum_{i=1}^{2n+1} (a_i)\right)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^{2n} \prod_{j=1}^{n} \Gamma(a_i + z_j)\Gamma(a_i - z_j)\, dz_1 \cdots dz_n}{\prod_{1 \le i < j \le n} \Gamma(z_i + z_j)\Gamma(-z_i - z_j)\Gamma(z_i - z_j)\Gamma(z_j - z_i)} \\
(9.9) \qquad &= \frac{n! 2^{n-1} \Gamma\left(\sum_{i=1}^{2n} a_i\right) \prod_{1 \le i < j \le 2n} \Gamma(a_i + a_j) \prod_{i=1}^{2n} \Gamma(2a_i)}{\Gamma\left(2 \sum_{i=1}^{2n} a_i\right)},
\end{aligned}$$

where notation and assumptions in (9.8) and (9.9) are similar to those in identity (9.4).

## 10. The corresponding group integrals

The integrals in Theorems 5.17, 6.1, 7.1, 7.14, 7.16, 8.1, and identities (7.20) and (7.21) can all be written as integrals over the corresponding compact Lie groups with respect to the normalized Haar measure. Similarly, the integrals in §9 and Theorem 5.1 can all be written as integrals over the corresponding Lie algebra with respect to conjugation invariant measures. We will follow the exposition given in Macdonald [35].

Let $G$ be a compact connected Lie group, $T$ a maximal torus of $G$, such that $R$ is the root system of $(G, T)$. Let $B = \{\alpha_1, \ldots, \alpha_l\}$ be a basis of $R$, which defines a system of positive roots. Let $W$ be the Weyl group of $R$ and $|W|$ the order of $W$. For $\alpha \in R$, let $e^\alpha$ be the corresponding character of $T$.

For $t \in T$, define $|\Delta(t)|^2 = \prod_{\alpha\in R}(1-e^\alpha(t))$. Then Weyl's integration formula [47] states

$$\int_G f(x)\,dx = \frac{1}{|W|}\int_T |\Delta(t)|^2 f(t)\,dt \tag{10.1}$$

for any continuous class function $f$ and $G$, where $dx$ and $dt$ are normalized Haar measures ($\int_G dx = \int_T dt = 1$).

Let $\pi_\lambda$ be any representation of $G$ acting on the space $V(\lambda)$ over $\mathbb{C}$ with highest weight $\lambda$. Let $\pi(\lambda)$ denote the set of weights occurring in $V(\lambda)$ (corresponding to characters $e^\mu$ of $T$, $\mu \in \pi(\lambda)$). Let $M_\lambda(\mu)$ denote the multiplicity of the weight $\mu$ in $V(\lambda)$. Let $\pi^*$ be the representation contragredient to $\pi$ and Ad be the adjoint representation of $G$.

Let $r$, $S$ be nonnegative integers and $a_i, b_j \in \mathbb{C}$ for $1 \le i \le r$ and $1 \le j \le s$. Assume that $|a_i|, |b_j| < 1$ for all $i$ and $j$. Consider the following integral:

$$\int_G \frac{\prod_{k=1}^{\infty} \det(1 - q^k \operatorname{Ad} x)}{\prod_{k=0}^{\infty} \det\left(\prod_{i=1}^{r}(1 - q^k a_i \pi_\lambda(x)) \prod_{j=1}^{s}(1 - q^k b_j \pi_\lambda^*(x))\right)}\, dt, \tag{10.2}$$

where 1 represents the identity map and $dx$ is the normalized Haar measure on $G$. Using Weyl's integration formula (10.1), we can rewrite the integral (10.2) as

$$\frac{[q]_\infty^l}{|W|}\int_T \frac{\prod_{\alpha\in R}[e^\alpha(t)]_\infty}{\prod_{\mu\in\pi(\lambda)}\left\{\prod_{i=1}^{r}[a_i e^\mu(t)]_\infty \prod_{j=1}^{s}[b_j e^\mu(t^{-1})]_\infty\right\}^{m_\lambda(u)}}\, dt \tag{10.3}$$

where $dt$ is the normalized Haar measure on $T$.

We now list the group integrals corresponding to the previous integral theorems.

**Theorem 10.4.** *Let notation be as above.*

(a) *For $n \geq 2$, let $\sigma = \pi_\lambda$ be the natural $n$-dimensional representation of $SU(n)$, where $\lambda = \lambda_1$ is the first fundamental weight* [17, *Appendix*]. *Then we have* (*cf. Theorem* 6.1)

$$\int_{SU(n)} \frac{\prod_{k=1}^{\infty} \det(1 - q^k \operatorname{Ad} x)}{\prod_{k=0}^{\infty} \prod_{i=1}^{n} [\det[(1 - q^k a_i \sigma(x))(1 - q^k b_i \sigma^*(x))]} dx = \frac{\left[\prod_{i=1}^{n} (a_i b_i)\right]_\infty}{\left[\prod_{i=1}^{n} a_i\right]_\infty \left[\prod_{i=1}^{n} b_i\right]_\infty \prod_{i,j=1}^{n} [a_i b_j]_\infty} \tag{10.5}$$

*where $\sigma^*$ is the contragredient of $\sigma$.*

(b) *For $n \geq 1$, let $\sigma = \pi_\lambda$ be the natural $2n$-dimensional representation of $Sp(n)$, where $\lambda = \lambda_1$ is the first fundamental weight* [17, *Appendix*]. *Then we have* (*cf. Theorem* 7.1)

$$\int_{Sp(n)} \frac{\prod_{k=1} \det(1 - q^k \operatorname{Ad} x)}{\prod_{k=0}^{\infty} \prod_{i=1}^{2n+2} \det(1 - q^k a_i \sigma(x))} dx = \frac{\left[\prod_{i=1}^{2n+2} a_i\right]_\infty}{\prod_{1 \leq i < j \leq 2n+2} [a_i a_j]_\infty}. \tag{10.6}$$

(c) *For $n \geq 1$, let $\sigma = \pi_\lambda$ be the natural $(2n+1)$-dimensional representation of $SO(2n+1)$, where $\lambda = \lambda_1$ is the first fundamental weight* [17, *Appendix*]. *Then we have* (*cf. Theorem* 7.14)

$$\int_{SO(2n+1)} \frac{\prod_{k=1}^{\infty} \det(1 - q^k \operatorname{Ad} x)}{\prod_{k=0}^{\infty} \prod_{i=1}^{2n-1} \det(1 - q^k a_i \sigma(x))} dx = \frac{\left[q \prod_{i=1}^{2n-1} a_i\right]_\infty}{\prod_{1 \leq i < j \leq 2n-1} [a_i a_j]_\infty \prod_{i=1}^{2n-1} [a_i^2]_\infty}. \tag{10.7}$$

*With assumptions and notation as in* (10.7), *we also have* (*cf. identity* (7.20)).

$$\int_{SO(2n+1)} \frac{dx}{\prod_{i=1}^{2n+1} \det(1 - a_i \sigma(x))} = \frac{\left(1 + \prod_{i=1}^{2n+1} a_i\right)}{\prod_{1 \leq i < j \leq 2n+1} (1 - a_i a_j) \prod_{i=1}^{2n+1} (1 - a_i^2)}. \tag{10.8}$$

(d) *For $n \geq 1$, let $\sigma = \pi_\lambda$ be the natural $2n$-dimensional representation of $SO(2n)$, where $\lambda = \lambda_1$ is the first fundamental weight. Then we have* (*cf. Theorem* 7.16)

$$\int_{SO(2n)} \frac{\prod_{k=1}^{\infty} \det(1 - q^k \operatorname{Ad} x)}{\prod_{k=0}^{\infty} \prod_{i=1}^{2n-2} \det(1 - q^k a_i \sigma(x))} dx = \frac{\left[q \prod_{i=1}^{2n-2} a_i\right]_\infty}{\prod_{1 \leq i < j \leq 2n-2} [a_i a_j]_\infty \prod_{i=1}^{2n-2} [a_i^2]_\infty}. \tag{10.9}$$

*With assumptions and notation as in* (10.9), *we also have* (*cf. identity* (7.21)):

$$\int_{SO(2n)} \frac{dx}{\prod_{i=1}^{2n} \det(1 - a_i\sigma(x))} = \frac{\left(1 + \prod_{i=1}^{2n} a_i\right)}{\prod_{1\le i<j\le 2n} (1 - a_i a_j) \prod_{i=1}^{2n} (1 - a_i^2)}. \tag{10.10}$$

(e) *Let* $G_2$ *be a compact connected real form of* $G_2$ *and let* $\sigma = \pi_\lambda$ *be the irreducible 7-dimensional representation of* $G_2$, *where* $\lambda = \lambda_1$ *is the first fundamental weight. Then we have* (*cf. Theorem* 8.1)

$$\begin{aligned} &\int_{G_2} \frac{\prod_{k=1}^{\infty} \det(1 - q^k \operatorname{Ad} x)}{\prod_{k=0}^{\infty} \prod_{i=1}^{r} \det(1 - q^k a_i \sigma(x))}\, dx \\ &\quad = \frac{\left[\prod_{i=1}^{r} a_i^2\right]_\infty}{\left[\prod_{i=1}^{4} a_i\right]_\infty \prod_{1\le i\le j\le 4} [a_i a_j]_\infty \prod_{1\le i<j<k\le 4} [a_i a_j a_k]_\infty}. \end{aligned} \tag{10.11}$$

(f) *For* $n \ge 1$, *let* $\sigma = \pi_\lambda$ *be the natural n-dimensional representation of* $U(n)$. *Then we have* (*cf. Theorem* 5.17)

$$\int_{U(n)} \frac{dx}{\prod_{i=1}^{n+1} \det[(1 - a_i\sigma(x))(1 - b_i\sigma(x))]} = \frac{\left(1 - \prod_{i=1}^{n+1} a_i b_i\right)}{\prod_{i,j=1}^{n+1} (1 - a_i b_j)}. \tag{10.12}$$

The integrals in §9 and Theorem 5.12 may also be written as integrals over the corresponding Lie algebras. We first need to define a matrix argument generalization of the classical gamma function. This definition is probably not new (e.g. see [26]). Let $X$ be any $n \times n$ complex matrix and let $I$ represent the $n \times n$ identity matrix. Then define

$$\Gamma(I + X) = \prod_{k=1}^{\infty} \left[\left(\frac{k+1}{k}\right)^{\operatorname{tr}(X)} \det\left(I + \frac{1}{k}X\right)^{-1}\right] \tag{10.13}$$

where $\operatorname{tr}(X)$ is the trace of $X$. If $X$ is a complex number (i.e. a $1 \times 1$ matrix), then definition (10.13) reduces to the classical definition. Also if $X$ is conjugate (i.e. $X = AYA^{-1}$) to a diagonal matrix $Y$ with diagonal entries $(y_1, y_2, \dots, y_n)$, then $\Gamma(I + X) = \prod_{i=1}^{n} \Gamma(1 + y_i)$.

We have a Lie algebra analogue of Weyl's integration formula (see [35 and 22]). Let $g$ be the Lie algebra associated to the compact Lie group $G$. Let $t$ be a Cartan subalgebra of $g$ and $f$ a $G$-invariant function on $g$. Let $d\xi$ and $d\tau$ be suitably normalized Lebesgue measures. If $|p(\tau)|^2 = \prod_{\alpha\in R} |\alpha(\tau)|$ for $\tau \in t$, then we have

$$\int_g f(\xi)\, d\xi = \frac{1}{|W|} \int_t f(\tau)|p(\tau)|^2\, d\tau. \tag{10.14}$$

We use the same notation as above, except that $\pi_\lambda$ is a finite dimensional representation of the Lie algebra $g$ and $\pi_\lambda^*$ is the contragredient representation.

The representation ad is the adjoint representation of $g$. Similarly to the above, let $r$ and $s$ be nonnegative integers and $a_i, b_j \in \mathbb{C}$ for $1 \le i \le r$ and $1 \le j \le s$. Assume that $\mathrm{Re}(a_i)$ and $\mathrm{Re}(b_j) > 0$ for all $i$ and $j$, and consider the following integral:

$$\int_g \frac{\prod_{i=1}^{r} \Gamma(a_i I + \pi_\lambda(\xi)) \prod_{j=1}^{s} \Gamma(b_j I + \pi_\lambda^*(\xi))\, d\xi}{\Gamma(I + \mathrm{ad}\,\xi)} \tag{10.15}$$

where $I$ is the identity map and $d\xi$ is a suitably normalized Lebesgue measure on $g$. Using the integration formula (8.14), we can rewrite the integral (8.15) as

$$\frac{1}{|W|} \int_t \frac{\prod_{\mu \in \pi(\lambda)} \left\{ \prod_{i=1}^{r} \Gamma(a_i + \mu(\tau)) \prod_{j=1}^{s} \Gamma(b_j - \mu(\tau)) \right\}^{m_\lambda(\mu)}}{\prod_{\alpha \in R} \Gamma(\alpha(\tau))}\, d\tau$$

where $d\tau$ is a suitable normalized measure on the Cartan subalgebra $t$.

We now list the Lie algebra integrals corresponding to some of the previous theorems.

**Theorem 8.16.** *Let notation be as above.*

(a) *For $n \ge 1$, let $\sigma$ be the natural $n$-dimensional representation of $u(n)$, the Lie algebra of skew-hermitian matrices. Then we have* (*cf.* *Theorem* 5.1)

$$\int_{u(n)} \frac{\prod_{i=1}^{n+1} \{\Gamma(a_i I + \sigma(\xi)) \Gamma(b_i I + \sigma^*(\xi))\}}{\Gamma(I + \mathrm{ad}\,\xi)}\, d\xi = \frac{\prod_{i,j=1}^{n+1} \Gamma(a_i + b_j)}{\Gamma\left(\sum_{i=1}^{n+1} (a_i + b_i)\right)}. \tag{10.17}$$

(b) *If $n \ge 2$, let $\sigma$ be the natural $n$-dimensional representation of $su(n)$, the Lie algebra of skew-hermitian matrices with trace* 0. *Then we have* (*cf.* *Theorem* 9.1)

$$\begin{aligned} &\int_{su(n)} \frac{\prod_{i=1}^{n} \{\Gamma(a_i I + \sigma(\xi)) \Gamma(b_i I + \sigma^*(\xi))\}}{\Gamma(I + \mathrm{ad}\,\xi)}\, d\xi \\ &\quad = \frac{\Gamma\left(\sum_{i=1}^{n} a_i\right) \Gamma\left(\sum_{i=1}^{n} b_i\right) \prod_{i,j=1}^{n} \Gamma(a_i + b_j)}{\Gamma\left(\sum_{i=1}^{n} (a_i + b_i)\right)}. \end{aligned} \tag{10.18}$$

(c) *For $n \ge 1$, let $\sigma$ be the natural $2n$-dimensional representation of $sp(n)$, the Lie algebra of the compact group $Sp(n)$. Then* (*cf.* *Theorem* 9.3)

$$\int_{sp(n)} \frac{\prod_{i=1}^{2n+2} \Gamma(a_i I + \sigma(\xi))}{\Gamma(I + \mathrm{ad}\,\xi)}\, d\xi = \frac{\prod_{1 \le i < j \le 2n+2} \Gamma(a_i + a_j)}{\Gamma\left(\sum_{i=1}^{2n+2} a_i\right)}. \tag{10.19}$$

(d) *For $n \geq 2$, let $\sigma$ be the natural $n$-dimensional representation of the Lie algebra of skew-symmetric real matrices, where $n$ can be even or odd. Then* (*cf. identities* (9.8) *and* (9.9))

$$\int_{so(n)} \frac{\prod_{i=1}^{n} \Gamma(a_i I + \sigma(\xi))}{\Gamma(I + \operatorname{ad}\xi)}\, d\xi = \frac{\Gamma\left(\sum_{i=1}^{2n} a_i\right) \prod_{1 \leq i < j \leq n} \Gamma(a_i + a_j) \prod_{I=1}^{n} \Gamma(2a_i)}{\Gamma\left(2\sum_{i=1}^{n} a_i\right)}. \tag{10.20}$$

(e) *Let $\sigma$ be the 7-dimensional irreducible representation of the Lie algebra $g_2$ associated to the compact Lie group $G_2$. Then* (*cf. Theorem* 9.5)

$$\int_{g_2} \frac{\prod_{i=1}^{4} \Gamma(a_i I + \sigma(\xi))}{\Gamma(I + \operatorname{ad}(\xi))}\, d\xi = \frac{\Gamma\left(\sum_{i=1}^{4} a_i\right) \prod_{1 \leq i \leq j \leq n} \Gamma(a_i + a_j) \prod_{1 \leq i < j < k \leq 4} \Gamma(a_i + a_j + a_k)}{\Gamma\left(2\sum_{i=1}^{4} a_i\right)}. \tag{10.21}$$

*Remark* 10.22. We can tie together Theorems 10.4 and 10.16 in the following way: First we define a generalization of the $q$-gamma function (see [5]). For $q$ real, $0 < q < 1$, $X$ an $n \times n$ complex matrix, let $q^X = e^{(\log q^X)} = e^{X \log q}$ and define

$$\Gamma_q(X) = \frac{[q]_\infty^n (1-q)^{n - \operatorname{tr}(X)}}{\prod_{k=0}^{\infty} \det(I - q^k q^X)}$$

W. Gosper's proof [1] in one dimensional can be generalized to show that

$$\lim_{q \to 1^-} \Gamma_q(I + X) = \Gamma(I + X).$$

Also the map $X \to q^X$ gives a (noninjective) map between the Lie algebra $g$ and the Lie group $G$ (as a group of matrices). We can rewrite the identities (10.5), (10.6), and (10.11) of Theorem 10.4 in terms of $q$-gamma functions. For example if we replace $a_i$, $b_j$ by $q^{a_i}$ and $q^{b_j}$, then identity (10.5) can be rewritten as

$$\frac{1}{(1-q)^{n^2-1}} \int_{SU(n)} \frac{\prod_{i=1}^{n} \Gamma_q(a_i I + \sigma(X)) \Gamma_q(b_i I + \sigma^*(X))\, du}{\Gamma_q(I + \operatorname{ad} X)} = \frac{\Gamma_q\left(\sum_{i=1}^{n} a_i\right) \Gamma_q\left(\sum_{i=1}^{n} b_i\right) \prod_{i,j=1}^{n} \Gamma_q(a_i + b_j)}{\Gamma_q\left(\sum_{i=1}^{n} (a_i + b_i)\right)} \tag{10.23}$$

with notation as in Theorem 6.16 where $u = q^X$ with $X \in su(n)$ and where $du$ is the normalized Haar measure on $Su(n)$. At least formally, the limit as $q \to 1^-$ of identity (10.23) is identity (10.18).

For the Lie groups $G = Sp(n)$ and $G_2$ there are, identities similar to (10.23) which correspond to the identities (10.6) and (10.11). The factor in front of the integral is $(1-q)^{-\dim g}$ where $\dim g$ is the (real) dimension of the Lie algebra $g$ (or the Lie group $G$). Again, the formal limit of these identities as $q \to 1^-$ will be the identities (10.19) and (10.21) respectively.

## 11. Some related integrals

We will use the Weyl denominator formula [27, p. 138] to prove some integral identities which are equivalent to the identities (5.18), (6.2), (7.2), (7.15), (7.17), (7.20), (7.21), and (8.2). The main argument here follows Macdonald [35].

With notation as in §10, consider the following form of the Weyl denominator formula:

$$\sum_{w\in W}\prod_{\alpha>0}\frac{1}{(1-e^{-w\alpha})} = \sum_{w\in W}\frac{e^{w\rho}}{\prod_{\alpha>0}(e^{\frac{w\alpha}{2}} - e^{\frac{-w\alpha}{2}})} = 1 \tag{11.1}$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. Using identity (11.1), we multiply the integrands on the left-hand side of identities (5.18), (6.2), (7.2), (7.15), (7.17), (7.20), (7.21), and (8.2) by the sum

$$\sum_{w\in W}\prod_{\alpha>0}(1-e^{-w\alpha}(t))^{-1}$$

for the appropriate root system $R$ and the corresponding Weyl group $W$, and when $t \in T$ is written in terms of the variables $z_1, \dots, z_n$.

For example, consider the identity (2.2). We find

$$\frac{1}{(2\pi i)^{n-1}}\sum_{w\in S_n}\int_{C_{n-1}}\frac{\prod_{1\le i<j\le n}[z_{w(i)}z_{w(j)}^{-1}]_\infty[qz_{w(j)}z_{w(i)}^{-1}]_\infty}{\prod_{i,j=1}^{n}[a_iz_j^{-1}]_\infty[b_iz_j]_\infty}\frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_{n-1}}{z_{n-1}}$$

$$= \frac{n!\left[\prod_{I=1}^{n}(a_ib_i)\right]_\infty}{[q]^{n-1}\left[\prod_{i=1}^{n}a_i\right]_\infty\left[\prod_{i=1}^{n}b_i\right]_\infty\prod_{i,j=1}^{n}[a_ib_j]_\infty}$$

where $z_1\cdots z_n = 1$ and $S_n$ is the symmetric group on $n$ letters. Observe that the Haar measure $(2\pi i)^{1-n}(z_1z_2\cdots z_{n-1})^{-1}dz_1\cdots dz_{n-1}$ on $T$ is invariant under permutations of the variables $z_1, \dots, z_n$. Similarly the product $\prod_{i,j=1}^{n}[a_iz_j^{-1}]_\infty[b_iz_j]_\infty$ is invariant under permutations of the variables $z_j$. Hence for $w \in S_n$, we find that

$$\int_{C^{n-1}} \frac{\prod_{1\le i<j\le n} [z_{w(i)}z_{w(j)}^{-1}]_\infty [q z_{w(j)} z_{w(i)}^{-1}]_\infty}{\prod_{i,j=1}^{n} [a_i z_j^{-1}]_\infty [b_i z_j]_\infty} \frac{dz_1}{z_1}\cdots\frac{dz_{n-1}}{z_{n-1}}$$

$$= \int_{C^{n-1}} \frac{\prod_{1\le i<j\le n} [z_{w(i)}z_{w(j)}^{-1}]_\infty [q z_{w(j)} z_{w(i)}]_\infty}{\prod_{i,j=1}^{n} [a_i z_{w(j)}^{-1}]_\infty [b_i z_{w(j)}]_\infty} \frac{dz_{w(1)}}{z_{w(1)}}\cdots\frac{dz_{w(n-1)}}{z_{w(n-1)}}$$

$$= \int_{C^{n-1}} \frac{\prod_{1\le i<j\le n} [z_i z_j^{-1}]_\infty [q z_j z_i^{-1}]_\infty}{\prod_{i,j=1}^{n} [a_i z_j^{-1}]_\infty [b_i z_j]_\infty} \frac{dz_1}{z_1}\cdots\frac{dz_{n-1}}{z_{n-1}}.$$

We therefore obtain the following

**Theorem 11.2.** *With notations and assumptions as in Theorem* 6.1, *we have*

$$\frac{1}{(2\pi i)^{n-1}} \int_{C^{n-1}} \frac{\prod_{1\le i<j\le n} [z_i z_j^{-1}]_\infty [q z_j z_i^{-1}]_\infty}{\prod_{i,j=1}^{n} [q_i z_j^{-1}]_\infty [b_i z_j]_\infty} \frac{dz_1}{z_1}\cdots\frac{dz_{n-1}}{z_{n-1}}$$

$$(11.3) \qquad = \frac{\left[\prod_{i=1}^{n}(a_i b_i)\right]_\infty}{[q]_\infty^{n-1}\left[\prod_{i=1}^{n} a_i\right]_\infty \left[\prod_{i=1}^{n} b_i\right]_\infty \prod_{i,j=1}^{n}[a_i b_j]_\infty}$$

*where* $z_1\cdots z_n = 1$.

By a similar argument we also obtain

**Theorem 11.4.** *With assumptions and notation as in the corresponding previous identities, we have*

(a) (*cf. identity* (7.2))

$$\frac{1}{(2\pi i)^n} \int_{C^n} \frac{\prod_{1\le i<j\le n} [z_i z_j]_\infty [q z_i^{-1} z_j^{-1}]_\infty [z_i z_j^{-1}]_\infty [q z_i^{-1} z_j]_\infty}{\prod_{i=1}^{2n+2}\prod_{j=1}^{n} [a_i z_j]_\infty [a_i z_j^{-1}]_\infty}$$

$$(11.5) \qquad \cdot \prod_{j=1}^{n} [z_j^2]_\infty [q z_j^{-2}]_\infty \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$= \frac{\left[\prod_{i=1}^{2n+2} a_i\right]_\infty}{[q]_\infty^n \prod_{1\le i<j\le 2n+2} [a_i a_j]_\infty};$$

(b) (*cf. identity* (7.15))

$$\frac{1}{(2\pi i)^n}\int_{C^n}\frac{\prod_{1\le i<j\le n}[z_iz_j]_\infty[qz_i^{-1}z_j^{-1}]_\infty[z_iz_j^{-1}]_\infty[qz_i^{-1}z_j]_\infty}{\prod_{i=1}^{2n-1}\prod_{j=1}^{n}[a_iz_j]_\infty[q_iz_j^{-1}]_\infty}$$

$$(11.6)\qquad \cdot\prod_{j=1}^{n}[z_j]_\infty[qz_j^{-1}]_\infty\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$=\frac{\left[q\prod_{i=1}^{2n-1}a_i\right]_\infty\prod_{i=1}^{2n-1}[a_i]_\infty}{[q]_\infty^n\prod_{1\le i<j\le 2n-1}[a_ia_j]_\infty\prod_{i=1}^{2n-1}[a_i^2]_\infty};$$

(c) (*cf. identity* (7.17))

$$(11.7)\qquad \frac{1}{(2\pi i)^n}\int_{C^n}\frac{\prod_{1\le i<j\le n}[z_iz_j]_\infty[qz_i^{-1}z_j^{-1}]_\infty[z_iz_j^{-1}]_\infty[qz_i^{-1}z_j]_\infty}{\prod_{i=1}^{2n-2}\prod_{j=1}^{n}[a_iz_j]_\infty[a_iz_j^{-1}]_\infty}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$=\frac{\left[q\prod_{i=1}^{2n-2}a_i\right]_\infty}{[q]_\infty^n\prod_{1\le i<j\le 2n-2}[a_ia_j]_\infty\prod_{i=1}^{2n-2}[a_i^2]_\infty};$$

(d) (*cf. identity* (7.20))

$$(11.8)\qquad \frac{1}{(2\pi i)^2}\int_{C^n}\frac{\prod_{1\le i<j\le n}(1-z_iz_j)(1-z_iz_j^{-1})\prod_{j=1}^{n}(1-z_j)}{\prod_{i=1}^{2n+1}\prod_{j=1}^{n}(1-a_iz_j)(1-a_iz_j^{-1})}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$=\frac{\left(1+\prod_{i=1}^{2n+1}a_i\right)}{\prod_{1\le i<j\le 2n+1}(1-a_ia_j)\prod_{i=1}^{2n+1}(1+a_i)};$$

(e) (*cf. identity* (7.21))

$$(11.9)\qquad \frac{1}{(2\pi i)^n}\int_{C^n}\frac{\prod_{1\le i<j\le n}(1-z_iz_j)(1-z_iz_j^{-1})}{\prod_{i=1}^{2n}(1-a_iz_j)(1-a_iz_j^{-1})}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$=\frac{\left(1+\prod_{i=1}^{2n}a_i\right)}{\prod_{1\le i<j\le 2n}(1-a_ia_j)\prod_{i=1}^{2n}(1-a_i^2)};$$

(f) (*cf. identity* (8.2))

$$\begin{aligned}
(11.10)\qquad &\frac{1}{(2\pi i)^2}\int_{C^2}\frac{[z_3z_1^{-1}]_\infty[qz_1z_3^{-1}]_\infty[z_1z_2^{-1}]_\infty[qz_2z_1^{-1}]_\infty}{\prod_{i=1}^{4}\prod_{j=1}^{3}[a_iz_j]_\infty[a_iz_j^{-1}]_\infty}\\
&\cdot[z_3z_2^{-1}]_\infty[qz_2z_3^{-1}]_\infty[z_1]_\infty[qz_1^{-1}]_\infty[z_3]_\infty[qz_3^{-1}]_\infty\\
&\cdot[z_2^{-1}]_\infty[qz_2]_\infty\frac{dz_1}{z_1}\frac{dz_2}{z_2}\\
&=\frac{\left[\prod_{i=1}^{4}a_i^2\right]_\infty\prod_{i=1}^{4}[a_i]_\infty}{[q]_\infty^2\left[\prod_{i=1}^{2}a_i\right]_\infty\prod_{1\le i\le j\le 4}[a_ia_j]_\infty\prod_{1\le i<j<k\le 4}[a_ia_ja_k]_\infty}
\end{aligned}$$

*where* $z_1z_2z_3=1$.

(g) (*cf. identity* (5.18))

$$\begin{aligned}
(11.11)\qquad &\frac{1}{(2\pi i)^n}\int_{C^n}\frac{\prod_{1\le i<j\le n}(1-z_iz_j^{-1})}{\prod_{i=1}^{n+1}\prod_{j=1}^{n}(1-a_iz_j^{-1})(1-b_iz_j)}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\\
&=\frac{\left(1-\prod_{i=1}^{n+1}a_ib_j\right)}{\prod_{i,j=1}^{n+1}(1-a_ib_j)}.
\end{aligned}$$

*Remark* 11.12. The right-hand sides of the limiting cases ($q\to 1^-$) of identities (11.3), (11.5)–(11.7), and (11.10) all vanish by a simple symmetry argument. Also, by the method above we can find identities related to (1.8), (1.10), (1.12), (1.15), and the specialization of (1.10) corresponding to the Lie algebras of types $B_n$ and $D_n$.

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