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THE STRUCTURE OF HYPERFINITE BOREL EQUIVALENCE RELATIONS

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ABSTRACT. We study the structure of the equivalence relations induced by the orbits of a single Borel automorphism on a standard Borel space. We show that any two such equivalence relations which are not smooth, i.e., do not admit Borel selectors, are Borel embeddable into each other. (This utilizes among other things work of Effros and Weiss.) Using this and also results of Dye, Varadarajan, and recent work of Nadkarni, we show that the cardinality of the set of ergodic invariant measures is a complete invariant for Borel isomorphism of aperiodic nonsmooth such equivalence relations. In particular, since the only possible such cardinalities are the finite ones, countable infinity, and the cardinality of the continuum, there are exactly countably infinitely many isomorphism types. Canonical examples of each type are also discussed.

This paper is a contribution to the study of Borel equivalence relations on standard Borel spaces. We concentrate here on the study of the *hyperfinite* ones. These are by definition the increasing unions of sequences of Borel equivalence relations with finite equivalence classes but equivalently they can be also described as the ones induced by the orbits of a single Borel automorphism. They include therefore a great variety of examples, some of them discussed in §6. For instance, the equivalence relations: E_0 on $2^{\mathbb{N}}$ (where xE_0y iff x, y are eventually equal, i.e., $\exists n\forall m \geq n(x_m = y_m)$), E_t on $2^{\mathbb{N}}$ (where xE_ty iff x, y have equal tails, i.e., $\exists n\exists m\forall k(x_{n+k} = y_{m+k})$), $E(\mathbb{Z}, 2)$ on $2^{\mathbb{Z}}$ (where $xE(\mathbb{Z}, 2)y$ iff x is a shift of y), E_α on the unit circle $\mathbb{T}$ (where $\alpha \in \mathbb{T}$ and $xE_\alpha y$ iff x is the rotation of y by $n\alpha$, $n \in \mathbb{Z}$), $E(\mathbb{R}/\mathbb{Q})$ on $\mathbb{R}$ (the Vitali equivalence relation, i.e., $xE(\mathbb{R}/\mathbb{Q})y$ iff $x - y \in \mathbb{Q}$), are all hyperfinite.

Our main results in this paper provide a classification of hyperfinite Borel equivalence relations under two different notions of equivalence. The weaker one, which we call *bi-embeddability*, is the following: Given hyperfinite Borel equivalence relations E, F (on X, Y resp.) we say that E *embeds* into F, in symbols $E \sqsubseteq F$, if there is a Borel injection $f: X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$. Then E, F are *bi-embeddable*, in symbols $E \approx F$, if $E \sqsubseteq F$ and $F \sqsubseteq E$. As it turns out, except for the trivial class of *smooth* relations, i.e., those having Borel selectors, any two hyperfinite Borel equivalence relations are bi-embeddable; i.e., we have

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Theorem 1. *Let E, F be nonsmooth hyperfinite Borel equivalence relations. Then $E \approx F$.*

The strong notion of equivalence that we consider next is that of Borel isomorphism, where E, F (on X, Y resp.) are *Borel isomorphic*, in symbols $E \cong F$, if there is a Borel bijection $f\colon X \to Y$ with $xEy \Leftrightarrow f(x)Ff(y)$. By using Theorem 1 and recent work of Nadkarni [N2], as well as some classical results in ergodic theory, such as Dye's Theorem and the Ergodic Decomposition Theorem, we are able to classify completely hyperfinite Borel equivalence relations up to Borel isomorphism. For the nontrivial case of the *aperiodic* ones, i.e. those containing no finite equivalence classes, this works as follows:

Theorem 2. *Let E be an aperiodic, nonsmooth hyperfinite Borel equivalence relation. Then E is Borel isomorphic to exactly one of the following: E_t, $E_0 \times \Delta(n)$ (the product of E_0 with the equality relation on n elements) for $1 \le n \le \aleph_0$, $E^*(\mathbb{Z}, 2)$ (the restriction of $E(\mathbb{Z}, 2)$ to the aperiodic points of $2^{\mathbb{Z}}$).*

This theorem is equivalent to the following result providing a complete invariant for Borel isomorphism. Given a hyperfinite Borel equivalence relation E on X, induced by a Borel automorphism T, we say that a probability measure μ on X is *E-invariant* if it is T-invariant and *E-ergodic* if it is T-ergodic (i.e. every T-invariant set has measure 0 or 1). It is easy to check that this definition does not depend on T. Denote by $\mathscr{E}_0(E)$ the space of nonatomic (this is unnecessary if E is aperiodic), E-invariant, ergodic measures. Then we have

Theorem 2′. *The cardinal number $\operatorname{card}(\mathscr{E}_0(E))$ is a complete invariant for Borel isomorphism of aperiodic, nonsmooth hyperfinite Borel equivalence relations, i.e. for any two such E, F,*

$$E \cong F \Leftrightarrow \operatorname{card}(\mathscr{E}_0(E)) = \operatorname{card}(\mathscr{E}_0(F)).$$

This was conjectured by M. G. Nadkarni (see [CN2]), who proved first in [N3] the case when $\operatorname{card}(\mathscr{E}_0(E)) = \operatorname{card}(\mathscr{E}_0(F))$ is countable, by using his result in [N2] and Theorem 1.

This paper is organized as follows. In §1 we discuss in general *countable* (i.e., having countable equivalence classes) Borel equivalence relations, review a representation result of Feldman-Moore [FM], and discuss some of its consequences. In §2, we study the well-known notion of compressibility that plays an important role in the sequel. In §§3 and 4 we present some basic facts about invariant and quasi-invariant measures. The notion of hyperfinite Borel equivalence relation is discussed in §5, and in §6 various examples are presented. In §7, we prove Theorem 1 and some of its consequences. The hyperfiniteness of tail equivalence relations is established in §8. The classification Theorems 2 and 2′ are proved in §9, and §10 deals with an illustrative class of examples—the Lipschitz automorphisms of $2^{\mathbb{N}}$. Finally, §11 collects miscellaneous facts related to the results in this paper and other work in the literature.

There are several interesting open problems concerning hyperfiniteness. For example:

(1) Is the increasing union of a sequence of hyperfinite Borel equivalence relations hyperfinite?

(2) (Weiss) Is an equivalence relation induced by a Borel action of a countable amenable (or even abelian) group hyperfinite? (This is known to be true for the groups $\mathbb{Z}^n$—Weiss.)

(3) Is the notion of hyperfiniteness effective, i.e., if E is a Δ^1_1 hyperfinite equivalence relation on $2^{\mathbb{N}}$, is E induced by a Δ^1_1 automorphism of $2^{\mathbb{N}}$?

There is an extensive literature on the subject of countable Borel equivalence relations in a measure theoretic framework, as it relates to both ergodic theory and the theory of operator algebras. The reader can consult the surveys C. C. Moore [Mo] and K. Schmidt [S2] about this. There is also some recent work in the Baire category framework; see Sullivan-Weiss-Wright [SWW]. In the descriptive set theoretic Borel context that we are interested in, relevant to us here is the work of Weiss [W2, W3], Chaube-Nadkarni [CN1, CN2], Nadkarni [N1, N2, N3], Wagh [Wa] as well as [K1, K2, K3], while further references can be found in the bibliography of these papers. Finally, for standard results in classical descriptive set theory that we use in this paper, see [Ku] or [Mos]. (Two particular such results that are used often below are: The image of a Borel set under a countable-to-1 Borel function is Borel [Mos, 4F.6]; a G_δ subset of a Polish space is Polish [Ku, §33, VI].)

In conclusion, we would like to thank both B. Weiss and M. G. Nadkarni, for helpful discussions and correspondence on the subject matter of this paper.

1. Countable Borel equivalence relations

Let X be a *standard Borel space*, i.e. a set equipped with a σ-algebra (its *Borel* sets) which is Borel isomorphic to the σ-algebra of the Borel sets in a Polish space. A *Borel equivalence relation* E on X is an equivalence relation which is Borel as a subset of X^2 (with the product Borel structure). For each Borel subset $Y \subseteq X$ we denote by $E \upharpoonright Y := E \cap Y^2$ the *restriction* of E to Y.

We want to consider some basic comparability relations among Borel equivalence relations.

Let (X, E), (X', E') be two Borel equivalence relations.

(i) We say that E is (*Borel*) *reducible* to E', in symbols $E \leq E'$, iff there is Borel $f\colon X \to X'$ such that $E = f^{-1}[E']$, i.e., $xEy \Leftrightarrow f(x)E'f(y)$. Any such reducing map f induces an injection $\tilde{f}\colon X/E \to X'/E'$ of the quotient spaces given by $\tilde{f}([x]_E) = [f(x)]_{E'}$ where $[x]_E$ denotes the E-equivalence class of x. We also use $E \approx^* E' :\Leftrightarrow E \leq E' \wedge E' \leq E$ for the *bi-reducibility* relation.

(ii) We say that E is (*Borel*) *embeddable* in E', in symbols $E \sqsubseteq E'$, if E is reducible to E' by an *injective* Borel map. We also use $E \approx E' :\Leftrightarrow E \sqsubseteq E' \wedge E' \sqsubseteq E$ for the *bi-embeddability* relation.

If we denote by $E \cong E'$ the relation of (*Borel*) *isomorphism* between E and E', i.e. the existence of a Borel bijection $f\colon X \to X'$ such that $xEy \Leftrightarrow f(x)E'f(y)$, then it is clear (since a Borel injective image of a Borel set is Borel) that

$$E \sqsubseteq E' \Leftrightarrow \exists \text{ Borel } Y \subseteq X'(E \cong E' \upharpoonright Y).$$

(iii) Finally, we say that E is (*Borel*) *invariantly embeddable* to E', in symbols $E \sqsubseteq^i E'$, if $E \cong E' \upharpoonright Y$, where Y is a Borel subset of X' *invariant* under E' (i.e., $y \in Y$, $zE'y \Rightarrow z \in Y$). By the usual Schroeder-Bernstein argument,

$$E \cong E' \Leftrightarrow E \sqsubseteq^i E' \wedge E' \sqsubseteq^i E.$$

Our primary goal here is to study *countable* Borel equivalence relations, i.e., Borel equivalence relations E for which every equivalence class $[x]_E$ is countable.

Let G be a countable group and X a standard Borel space. A *Borel action* is an action $(g, x) \mapsto g \cdot x$ of G on X (i.e. a map from $G \times X$ into X satisfying $1 \cdot x = x$, $gh \cdot x = g \cdot (h \cdot x)$) such that for each g, $g(x) := g \cdot x$ is Borel (thus a Borel automorphism of X). This is the same thing as saying that $(g, x) \mapsto g \cdot x$ is Borel from $G \times X$ into X, with G given the discrete Borel structure (and the product has as usual the product Borel structure). Given a Borel action of G on X, we denote by E_G the induced equivalence relation

$$xE_G y \Leftrightarrow \exists g \in G(y = g \cdot x).$$

This is clearly a countable Borel equivalence relation on X. Conversely we have

Theorem 1.1 (Feldman-Moore [FM]). *Let E be a countable Borel equivalence relation on a standard Borel space X. Then there are a countable group G and a Borel action of G on X such that $E = E_G$.*

We would like to mention first some consequences of this result.

Given any standard Borel space X and a countable group G, denote by X^G the set of maps from G into X with the usual product Borel structure. (If $\mathrm{card}(G) = n$, with $n \leq \aleph_0$, then X^G is Borel isomorphic to X^n.) There is a canonical Borel action of G on X^G given by $g \cdot p(h) = p(g^{-1}h)$ for $p \in X^G$, $g \in G$. We denote by $E(G, X)$ the corresponding equivalence relation. This equivalence relation, for $X = 2^{\mathbb{N}}$, is *invariantly universal* among all E_G in the following sense.

Proposition 1.2. *Let G be a countable group and E_G the equivalence relation induced by a Borel G-action on X. Then $E_G \sqsubseteq^i E(G, 2^{\mathbb{N}})$.*

Proof. Let $\{U_i\}_{i \in \mathbb{N}}$ be a sequence of Borel sets in X separating points. Define $f: X \to (2^{\mathbb{N}})^G$ by

$$f(x)(g)(i) = 1 \Leftrightarrow g^{-1} \cdot x \in U_i.$$

Then f is injective. Moreover $g \cdot f(x) = f(g \cdot x)$ so f maps X onto an invariant Borel subset Y of $(2^{\mathbb{N}})^G$ and shows, in particular, that $E_G \cong E(G, 2^{\mathbb{N}}) \upharpoonright Y$. □

By taking $G = F_\omega$, the free group with countably infinitely many generators, we obtain

Proposition 1.3. *Let E be a countable Borel equivalence relation. Then $E \sqsubseteq^i E(F_\omega, 2^{\mathbb{N}})$.*

Concerning the embeddability relation $\sqsubseteq$, one can obtain some tighter results. Let us note the following propositions.

Proposition 1.4. *Suppose G is a homomorphic image of H. Then $E(G, X) \sqsubseteq^i E(H, X)$.*

Proof. If $\pi: H \to G$ is an onto homomorphism, define $p \in X^G \mapsto p^* \in X^H$ by $p^*(h) = p(\pi h)$. Then $h \cdot p^* = (\pi h \cdot p)^*$, so this map shows that $E(G, X) \sqsubseteq^i E(H, X)$. □

Proposition 1.5. *If $G \subseteq H$ (i.e., G is a subgroup of H), then $E(G, X) \sqsubseteq E(H, X)$.*

Proof. Define $p \in X^G \mapsto p^* \in X^H$ by

$$p^*(h) = \begin{cases} p(h), & \text{if } h \in G, \\ x_0, & \text{if } h \notin G, \end{cases}$$

where x_0 is some fixed element of X. □

Proposition 1.6. $E(G, 2^{\mathbb{Z}-\{0\}}) \sqsubseteq E(G \times \mathbb{Z}, 3)$.

Proof. Define $p \in (2^{\mathbb{Z}-\{0\}})^G \mapsto p^* \in 3^{G\times\mathbb{Z}}$ by

$$p^*(g, n) = \begin{cases} p(g)(n), & \text{if } n \neq 0, \\ 2, & \text{if } n = 0. \end{cases}$$

Then $q = g \cdot p \Rightarrow q^* = (g, 0) \cdot p^*$. Conversely if $q^* = (g, n) \cdot p^*$ and $n = 0$ then $q = g \cdot p$. If $n \neq 0$, then $q^*(g_0, n_0) = p^*(g^{-1}g_0, n_0 - n)$, so $q(g_0)(n) = q^*(g_0, n) = p^*(g^{-1}g_0, 0) = 2$, a contradiction. □

Proposition 1.7. $E(G, 3) \sqsubseteq E(G \times \mathbb{Z}_2, 2)$.

Proof. View 0 as encoded by 00, 1 by 01, and 2 by 11. Define $p \in 3^G \mapsto p^* \in 2^{G\times\mathbb{Z}_2}$ by

$$\begin{aligned} p^*(g, i) &= 0, && \text{if } p(g) = 0, \\ &= 0, && \text{if } p(g) = 1,\ i = 0, \\ &= 1, && \text{if } p(g) = 1,\ i = 1, \\ &= 1, && \text{if } p(g) = 2. \end{aligned}$$

If $q = g\cdot p$, then $q^* = (g, 0)\cdot p^*$. Conversely, if $q^* = (g, i)\cdot p^*$ and $i = 0$, then $q = g\cdot p$. If $i = 1$ and $q(g_0) = 1$ for some g_0, then $q^*(g_0, 0) = 0$, $q^*(g_0, 1) = 1$ but $q^*(g_0, i) = p^*(g^{-1}g_0, 1 + i)$, thus $p^*(g^{-1}g_0, 0) = q^*(g_0, 1) = 1$, so $p(g^{-1}g_0) = 2$, while $p^*(g^{-1}g_0, 1) = q^*(g_0, 0) = 0$, a contradiction. So we must have that $q \in \{0, 2\}^G$ in which case it is easy to see that $q = g \cdot p$. □

Letting $F_n :=$ the free group with n generators, we have now from the preceding results that $E(F_2, 2)$ is *universal* among all countable Borel E in the following sense.

Proposition 1.8. *Let E be a countable Borel equivalence relation. Then $E \sqsubseteq E(F_2, 2)$.*

Proof. We have

$$\begin{aligned} E &\sqsubseteq E(F_\omega, 2^{\mathbb{N}}), && \text{by Proposition 1.3} \\ &\cong E(F_\omega, 2^{\mathbb{Z}-\{0\}}) \\ &\sqsubseteq E(F_2, 2^{\mathbb{Z}-\{0\}}), && \text{by Proposition 1.5} \\ & && \text{(as } F_\omega \text{ is embeddable in } F_2) \\ &\sqsubseteq E(F_2 \times \mathbb{Z}, 3), && \text{by Proposition 1.6} \\ &\sqsubseteq E(F_2 \times \mathbb{Z} \times \mathbb{Z}_2, 2), && \text{by Proposition 1.7} \\ &\sqsubseteq E(F_\omega, 2), && \text{by Proposition 1.4} \\ &\sqsubseteq E(F_2, 2), && \text{by Proposition 1.5. } \square \end{aligned}$$

We do not know whether every countable Borel E is of the form E_{F_2}, or, even more, whether $E \sqsubseteq^i E(F_2, 2)$ (see however Proposition 2.4 for an affirmative answer in a special case).

2. Compressibility

Given a countable Borel equivalence relation E on X, we denote by $[[E]]$ the set of Borel bijections $f: A \to B$ with A, B Borel subsets of X, which have the property that $\forall x \in A \ (f(x)Ex)$. Note that $f: A \to B$ belongs to $[[E]]$ iff, for any countable group G with $E = E_G$, there are Borel partitions $A = \bigcup_{i \in \mathbb{N}} A_i$, $B = \bigcup_{i \in \mathbb{N}} B_i$ and $g_i \in G$ with $g_i[A_i] = B_i$ and $g_i = f$ on A_i.

For Borel sets $A, B \subseteq X$ we let now $A \sim B :\Leftrightarrow \exists f \in [[E]](f: A \to B)$. This means that for any E-equivalence class C, if $A^C = A \cap C$, $B^C = B \cap C$, there is a 1-1 correspondence of A^C with B^C depending in a "uniform Borel" way on C. Let also

$$A \preceq B :\Leftrightarrow \exists \text{ Borel } B' \subseteq B(A \sim B').$$

Note that by the usual Schroeder-Bernstein argument

$$A \sim B \Leftrightarrow A \preceq B \wedge B \preceq A.$$

We discuss now the notion of compressibility of an equivalence relation arising in the measure theoretic aspects of this subject in the Hopf Theorem (see e.g. Weiss [W1] or Friedman [Fr]) and studied extensively in the Borel context in Nadkarni [N1, N2], Chaube-Nadkarni [CN1, CN2].

Let E be a countable Borel equivalence relation on X. We call E *compressible* if there is Borel $A \subseteq X$ with $X \sim A$ such that $X \backslash A$ is *full* (or a *complete section*), i.e., meets every E-equivalence class. This just means that each E-equivalence class can be mapped into a proper subset of itself in a uniform Borel way, i.e., every equivalence class is Dedekind infinite in a uniform Borel way. We have the following useful reformulations of compressibility.

Proposition 2.1. *Let E be a countable Borel equivalence relation on X. Then the following are equivalent*:

(1) *E is compressible.*

(2) *There is a pairwise disjoint sequence $\{A_n\}$ of full Borel sets with $A_n \sim A_m$ for all n, m.*

(3) *X is E-paradoxical, where a Borel subset $A \subseteq X$ is E-paradoxical if there are disjoint Borel $B, C \subseteq A$ with $A \sim B$ and $A \sim C$.*

Proof. (1) $\Rightarrow$ (2). Let $f: X \to B$, $f \in [[E]]$ with $X \backslash B = A$ full. Put $A_n = f^n[A]$.

(2) $\Rightarrow$ (3). Let $f_n: A_0 \to A_n$, $f_n \in [[E]]$ and let $E = E_G$, where $G = \{g_n\}$ is a countable group. Define $N: X \to \mathbb{N}$ by

$$N(x) = \text{least } n \text{ such that } g_n \cdot x \in A_0$$

(this exists since A_0 is full). Clearly N is Borel. Define now h_0, h_1 on X by

$$h_0(x) = f_{2N(x)}(g_{N(x)} \cdot x), \quad h_1(x) = f_{2N(x)+1}(g_{N(x)} \cdot x).$$

If $h_0[X] = B$, $h_1[X] = C$ then clearly $h_0: X \to B$, $h_1: X \to C$ are in $[[E]]$ and so $X \sim B$, $X \sim C$. But also $B \cap C = \varnothing$ and we are done.

(3) $\Rightarrow$ (1). If $f: X \to B$ is in $[[E]]$ and $g: X \to C$ is in $[[E]]$ with $B \cap C = \varnothing$, then clearly C and thus $X \backslash B$ is full, so E is compressible. $\square$

We call now a Borel subset $A \subseteq X$ *compressible* if $E \restriction A$ is compressible. A well-known basic fact (see [N1, 5.7]) about compressible sets is the following.

Proposition 2.2. *Let E be a countable Borel equivalence relation on X and $A \subseteq X$ a Borel set. If A is compressible, then $A \sim [A]_E$ where*

$$[A]_E = \{x \in X : \exists y[y \in A \wedge xEy]\}$$

is the E-saturation of A. In particular, $[A]_E$ is also compressible.

Proof. By Schroeder-Bernstein it is enough to show $[A]_E \preceq A$. By 2.1, let $\{A_n\}$ be pairwise disjoint Borel subsets of A full for $E \restriction A$ and thus for $E \restriction [A]_E$, with $A_n \sim A_m$. Then exactly as in the proof of $(2) \Rightarrow (3)$ of 2.1 (applied with $X = [A]_E$), we have that $[A]_E \preceq \bigcup_n A_n \subseteq A$. □

This has the following immediate corollary.

Proposition 2.3. *Let E, F be countable Borel equivalence relations. Then, if E is compressible and $E \sqsubseteq F$, we have $E \sqsubseteq^i F$. In particular, for compressible E, F: $E \approx F \Leftrightarrow E \cong F$.*

Proof. Let $E \sqsubseteq F$, so that $E \cong F \restriction A$, A a Borel set. Then A is compressible (for F). So $A \sim [A]_F$, thus $E \cong F \restriction A \cong F \restriction [A]_F$, i.e., $E \sqsubseteq^i F$. □

Another corollary is the following.

Proposition 2.4. *Let E be a countable Borel equivalence relation. If E is compressible, then $E \sqsubseteq^i E(F_2, 2)$ and so in particular E is induced by an action of F_2.*

We will provide now some further equivalents of compressibility. For each set Ω, let $I(\Omega) := \Omega \times \Omega$ be the largest equivalence relation on Ω. Put $I_\infty := I(\mathbb{N})$. For equivalence relations E on X and F on Y, $E \times F$ denotes the product (on $X \times Y$) where

$$(x, y)E \times F(x', y') :\Leftrightarrow xEx' \wedge yFy'.$$

Call an equivalence relation E *aperiodic* if every E-equivalence class is infinite. Finally, call a Borel equivalence relation E on X *smooth* if there is a Borel function $f: X \to Y$ (Y some standard Borel space) with $xEy \Leftrightarrow f(x) = f(y)$, i.e. $E \leq \Delta(Y)$, where $\Delta(\Omega) :=$ the equality on Ω. For countable Borel E, this is equivalent to the existence of a Borel *selector*, i.e., a Borel function $S: X \to X$ with $xEy \Rightarrow S(x) = S(y)$, $S(x)Ex$.

As usual, $E \subseteq F$ means that E is a *subequivalence relation* of F (i.e. $xEy \Rightarrow xFy$).

Proposition 2.5. *Let E be a countable Borel equivalence relation. Then the following are equivalent*:

(1) *E is compressible,*

(2) *$E \cong E \times I_\infty$,*

(3) *there is a smooth aperiodic Borel equivalence relation $F \subseteq E$.*

Proof. $(1) \Rightarrow (2)$. Clearly $E \cong (E \times I_\infty) \restriction (X \times \{0\})$ and so $E \cong (E \times I_\infty) \restriction [X \times \{0\}]_{E \times I_\infty} = E \times I_\infty$, by 2.3.

$(2) \Rightarrow (3)$. It is enough to show that (3) holds for $E \times I_\infty$. Define $F \subseteq E \times I_\infty$ by $(x, n)F(y, m) \Leftrightarrow x = y$.

(3) $\Rightarrow$ (1). Note that, since F admits a Borel selector and each F-equivalence class is infinite, F is compressible. But it is also straightforward to check that if F is compressible and $F \subseteq E$, then E is compressible. □

Note that in (3) $\Rightarrow$ (1), one only uses the following: $F \subseteq E$, F is smooth Borel and every E-equivalence class contains an infinite F-equivalence class.

For each set Ω, $E_0(\Omega)$ is the equivalence relation on $\Omega^{\mathbb{N}}$ given by

$$xE_0(\Omega)y :\Leftrightarrow \exists n \forall m \geq n(x_m = y_m).$$

We let $E_0 := E_0(2)$. We also denote by $E_t(\Omega)$ the equivalence relation on $\Omega^{\mathbb{N}}$ given by

$$xE_t(\Omega)y :\Leftrightarrow \exists n \exists m \forall k(x_{n+k} = y_{m+k})$$

and let again $E_t := E_t(2)$.

Consider then the following examples of compressible E:

(a) Let $X = \mathbb{N}^{\mathbb{N}}$ and $E = E_0(\mathbb{N})$. To see that E is compressible, note that $F \subseteq E$, where $xFy :\Leftrightarrow x' = y'$, with $x' = (x_1, x_2, \ldots)$ if $x = (x_0, x_1, x_2, \ldots)$.

(b) Let $X = 2^{\mathbb{N}}$ and $E = E_t$. Again E is compressible since $F \subseteq E$, where $xFy :\Leftrightarrow [x, y$ are eventually equal to $1] \vee [x = 1^n{}^\frown 0^\frown x' \wedge y = 1^m{}^\frown 0^\frown y'$ with $x' = y']$.

We can also use the present ideas to provide some alternative characterizations of the notion of bi-reducibility $E \approx^* F$ introduced in §1.

We call two Borel equivalence relations E, F *stably isomorphic*, in symbols $E \cong_s F$ if there are Borel sets A, B full for E, F resp., such that $E \restriction A \cong F \restriction B$.

We then have

Proposition 2.6. *Let E, F be countable Borel equivalence relations on X, Y resp. Then the following are equivalent*:

(1) $E \approx^* F$,

(2) $E \cong_s F$,

(3) $E \times I_\infty \cong F \times I_\infty$.

Proof. (3) $\Rightarrow$ (2). Let $f: X \times \mathbb{N} \to Y \times \mathbb{N}$ be a Borel isomorphism of $E \times I_\infty$, $F \times I_\infty$. Put $A = \{x \in X: \exists n \exists y(f(x, n) = (y, 0))\}$. For $x \in A$, let $N(x) =$ least n with $\exists y[f(x, n) = (y, 0)]$. Put $g(x) =$ the unique y with $f(x, N(x)) = (y, 0)$. Clearly g is Borel injective. Let $g[A] = B$. Clearly A, B are full and g shows that $E \restriction A \cong F \restriction B$.

(2) $\Rightarrow$ (1). Let A, B be Borel full with $E \restriction A \cong F \restriction B$ via f. Let $E = E_G$ with $G = \{g_n\}$ a countable group. Define for each $x \in X$, $N(x) =$ least n with $g_n \cdot x \in A$ and $g: X \to Y$ by $g(x) = f(g_{N(x)} \cdot x)$. Then g is Borel and reduces E to F, thus $E \leq F$. Similarly $F \leq E$, so $E \approx^* F$.

(1) $\Rightarrow$ (3). Suppose $f: X \to Y$ reduces E to F and $g: Y \to X$ reduces F to E. Let $\tilde{f}: X/E \to Y/F$ be the induced injection given by $\tilde{f}([x]_E) = [f(x)]_F$ and similarly for $\tilde{g}: Y/F \to X/E$. By Schroeder-Bernstein applied to $\tilde{f}$, $\tilde{g}$, we can partition X into E-invariant Borel sets A, B and Y into F-invariant Borel sets C, D such that $\tilde{f}$ maps A/E onto C/F and $\tilde{g}$ maps D/F onto B/E. Therefore $f[A]$ is full in C and $g[D]$ is full in B. Note that since f, g are countable-to-1, $f[A]$, $g[D]$ are Borel sets.

We will show that $(E \times I_\infty) \restriction (B \times \mathbb{N}) \cong (F \times I_\infty) \restriction D'$, where D' is Borel full in $D \times \mathbb{N}$, and $(F \times I_\infty) \restriction (C \times \mathbb{N}) \cong (E \times I_\infty) \restriction A'$, where A' is Borel full in $A \times \mathbb{N}$. Since $E \times I_\infty$, $F \times I_\infty$ are compressible, this will imply that $(E \times I_\infty) \restriction (B \times \mathbb{N}) \cong (F \times I_\infty) \restriction (D \times \mathbb{N})$ and $(F \times I_\infty) \restriction (C \times \mathbb{N}) \cong (E \times I_\infty) \restriction (A \times \mathbb{N})$, so $E \times I_\infty \cong F \times I_\infty$.

Let us work with $(F \times I_\infty) \restriction (C \times \mathbb{N})$, the other case being similar. Let $(F \times I_\infty) = E_H$, with $H = \{h_n\}$ a countable group. Given $z \in C \times \mathbb{N}$, let $N(z)$ be the least n with $h_n \cdot z = (y, 0)$ and $y \in f[A]$. Put $h_{N(z)} \cdot z = (p(z), 0)$, p Borel. Let $f^*: f[A] \to A$ be Borel with $f(f^*(y)) = y$, which exists since f is countable-to-1. Finally put $q(z) = (f^*(p(z)), N(z))$. Then q is a Borel injection, $A' = q[C \times \mathbb{N}]$ is full in $A \times \mathbb{N}$, and q shows that $(F \times I_\infty) \restriction (C \times \mathbb{N}) \cong (E \times I_\infty) \restriction A'$. □

We close this section with the following question:

Let E, F be aperiodic countable Borel equivalence relations. If $E \approx^* F$, is it true that $E \approx F$?

Note that this is equivalent to asking whether $E \times I_\infty \sqsubseteq E$ for all aperiodic E. A counterexample could be provided, for example, by finding a property $\mathscr{P}(E)$ of aperiodic countable Borel equivalence relations, which is preserved under restriction (i.e. $\mathscr{P}(E) \Rightarrow \mathscr{P}(E \restriction A)$) but not extension from a full subset (i.e., $\mathscr{P}(E \restriction A)$ does not necessarily imply $\mathscr{P}(E)$, whenever A is full for E).

3. Invariant and quasi-invariant measures

Let X be a standard Borel space. By a *measure* on X we will always mean a *σ-finite Borel measure* on X. If μ is a measure on X and $\mu(X) < \infty$ we call μ *finite*; if $\mu(X) = 1$, μ is a *probability* measure. For any Borel $f: X \to Y$, we define $f\mu$ to be the measure $f\mu(A) = \mu(f^{-1}[A])$.

For any measures μ, ν on X, $\mu << \nu$ means that μ is *absolutely continuous* to ν (i.e., $\nu(A) = 0 \Rightarrow \mu(A) = 0$), and we let

$$\mu \sim \nu \;:\Leftrightarrow\; \mu << \nu \wedge \nu << \mu.$$

This is an equivalence relation on measures, whose equivalence classes $[\mu]_\sim$ (or just $[\mu]$) are called *measure classes.* Note that for any μ there is a probability measure ν with $\nu \in [\mu]$. A measure μ is *orthogonal* to ν, in symbols $\mu \perp \nu$, if there is a Borel partition $X = A \cup B$ with $\mu(A) = \nu(B) = 0$.

Now let G be a countable group acting in a Borel way on X. A measure μ is called *G-invariant* if $g \cdot \mu = \mu$ (here $g \cdot \mu$ is the measure $g\mu$ for $g(x) = g \cdot x$ so that $g \cdot \mu(A) = \mu(g^{-1} \cdot A)$) and *$G$-quasi-invariant* if $g \cdot \mu \sim \mu$.

Suppose now E is a countable Borel equivalence relation on X and μ a measure on X. We call μ *E-invariant* if μ is G-invariant for any G acting in a Borel way on X with $E = E_G$. It is easy to see that this property is independent of G and is equivalent to either of the properties below:

(1) If $f \in [[E]]$, $f: A \to B$, then $\mu(A) = \mu(B)$.

(2) If $f \in [[E]]$, $f: X \to X$, then $f\mu = \mu$.

With the preceding notation, we call μ *E-quasi-invariant,* if for any G with $E = E_G$, μ is G-quasi-invariant. Again it is easy to see that this is independent of G and equivalent to the following properties:

(1) For any Borel $A \subseteq X$, if $\mu(A) = 0$ then $\mu([A]_E) = 0$.

(2) If $f \in [[E]]$, $f: A \to B$ and $\mu(A) = 0$, then $\mu(B) = 0$.

A measure μ on X is caled *E-ergodic* if, for any Borel invariant A, $\mu(A) = 0$ or $\mu(X \backslash A) = 0$. It is called *E-nonatomic* if $\mu([x]_E) = 0$ for each E-equivalence class $[x]_E$, i.e., $\mu(\{x\}) = 0$ for each point x.

Let us notice first a few simple facts:

Proposition 3.1. *Let E be a countable Borel equivalence relation and μ a measure. Then there is an E-quasi-invariant measure μ^* such that:*

(1) *If A is Borel E-invariant, then $\mu(A) = \mu^*(A)$.*

(2) *If μ is E-ergodic, so is μ^*.*

(3) *μ^* is a least, in the sense of $<<$, measure such that μ^* is E-quasi-invariant and $\mu << \mu^*$.*

Proof. Let $E = E_G$ with $G = \{g_1, g_2, \dots\}$. First, we claim that we can write X (the space on which E lives) as $X = \bigcup_n A_n$, with A_n Borel, such that for all i, $\mu(g_i \cdot A_n) < \infty$. To prove this, call a Borel set $A \subseteq X$ *nice* if $\mu([A]_E) > 0$ and $\mu(g_i \cdot A) < \infty$, for all i. We claim that nice sets exist: Indeed, fix a Borel set B with $\infty > \mu(B) > 0$. Then write $g_i \cdot B = \bigcup_n B_{i,n}$, where the sets $B_{i,n}$ are Borel, increasing with n and $\mu(B_{i,n}) < \infty$. For each i, fix $n = n(i)$ such that $\mu(g_i^{-1} \cdot B_{i,n}) > \mu(B) - \mu(B)/2^{i+2}$. Put $A = \bigcap_i g_i^{-1} \cdot B_{i,n(i)}$. Then $\mu(A) > 0$, so $\mu([A]_E) > 0$, and $g_i \cdot A \subseteq B_{i,n(i)}$, so $\mu(g_i \cdot A) < \infty$. By a simple exhaustion argument now, we can find a sequence C_i of nice Borel sets with $\bigcup_i [C_i] = X$. Let $\{A_n\} = \{g_j \cdot C_i\}$.

Fix now such a sequence $\{A_n\}$ and choose a sequence $\{\alpha_i\}$ of positive reals such that $\alpha = \sum \alpha_i < \infty$ and $\sum \alpha_i \mu(g_i \cdot A_n) < \infty$, for all n. Define then μ^* by

$$\mu^*(A) = \alpha^{-1} \sum \alpha_i \mu(g_i \cdot A).$$

Clearly $\mu << \mu^*$ and (1), (2) hold. For (3), assume $\mu << \nu$ and ν is E-quasi-invariant. We will show that $\mu^* << \nu$: For any Borel $A \subseteq X$, $\nu(A) = 0 \Rightarrow \forall g \in G(\nu(g \cdot A) = 0) \Rightarrow \forall g \in G(\mu(g \cdot A) = 0) \Rightarrow \mu^*(A) = 0$. □

Proposition 3.2. *Let E be a countable Borel equivalence relation and $A \subseteq X$ a full Borel set. Let μ be a measure on A which is $E \restriction A$-invariant. Then there is a unique E-invariant measure ν on X with $\nu(B) = \mu(B)$ for all Borel sets $B \subseteq A$. If μ is nonatomic or ergodic (for $E \restriction A$), so is ν (for E).*

(In general, ν will not be finite even if μ is.)

Proof. Let $E = E_G$, $G = \{g_0, g_1, \dots\}$ a countable group, with $g_0 = 1$. Put $A_i = g_i \cdot A$, $B_i = A_i \setminus \bigcup_{j<i} A_j$, so that $\{B_i\}$ is a Borel partition of X. Put

$$\nu(B) = \sum_{i=0}^{\infty} \mu(g_i^{-1} \cdot (B \cap B_i)).$$

Clearly $\nu(B) = \mu(B)$, for $B \subseteq A$ $(= A_0 = B_0)$.

To see that ν is E-invariant, note that, if for Borel $B \subseteq X$, $g \in G$ we define

$$B_{i,k} = \{x \in B \cap B_i \colon g \cdot x \in B_k\},$$

then $\{B_{i,k}\}$ is a Borel partition of B. So to show $\nu(B) = \nu(g \cdot B)$, it is enough to show that $\nu(B_{i,k}) = \nu(g \cdot B_{i,k})$. Equivalently, assume that $B \subseteq B_i$ and $g \cdot B \subseteq B_k$. Now $\nu(B) = \mu(g_i^{-1} \cdot B)$ and $\nu(g \cdot B) = \mu(g_k^{-1} g \cdot B)$. Define $h \colon g_i^{-1} \cdot B \to g_k^{-1} g \cdot B$ by $h = g_k^{-1} g g_i \restriction g_i^{-1} \cdot B$, so that $h \in [[E \restriction A]]$. Thus by the $E \restriction A$-invariance of μ we have $\mu(g_i^{-1} \cdot B) = \mu(g_k^{-1} g \cdot B)$ and we are done.

Assume now μ is nonatomic. Then clearly so is ν. If μ is ergodic, let $B \subseteq X$ be E-invariant. Then $A \cap B$ is $E \restriction A$-invariant. Say $\mu(A \cap B) = 0$. Then $\nu(A \cap B) = 0$, so $\nu([A \cap B]_E) = \nu(B) = 0$. Similarly if $\mu(A \backslash B) = 0$. □

Proposition 3.3. *Let E be a countable Borel equivalence relation and $A \subseteq X$ a full Borel set. Let μ be a probability measure on A which is $E \restriction A$-quasi-invariant. Then there is a probability measure ν on X such that ν is E-quasi-invariant and $\nu(B) = \nu(A) \cdot \mu(B)$ for all Borel sets $B \subseteq A$. If μ is nonatomic or ergodic (for $E \restriction A$), so is ν (for E).*

Proof. Let $E = E_G$, $G = \{g_i\}$, $g_0 = 1$. Define A_i, B_i as in 3.2 and put

$$\nu(B) = c_\mu \sum_{i=0}^{\infty} 2^{-i-1} \mu(g_i^{-1} \cdot (B \cap B_i))$$

for $B \subseteq X$ Borel, where c_μ is a positive constant such that $\nu(X) = 1$. It is easy to check that $\nu(B) = \nu(A) \cdot \mu(B)$, for $B \subseteq A$.

We will check next that ν is E-quasi-invariant. Let $B \subseteq X$ be Borel with $\nu(B) = 0$; we will show that $\nu([B]_E) = 0$. We can assume that $B \subseteq B_i$ for some i. Since $[B]_E = [g_i^{-1} \cdot B]_E$, it suffices to show that

$$C \subseteq A \text{ Borel} \wedge \mu(C) = 0 \Rightarrow \nu([C]_E) = 0.$$

For that it is enough again to show that $\mu(g_i^{-1} \cdot ([C]_E \cap B_j)) = 0$ for each j. But $g_j^{-1} \cdot ([C]_E \cap B_j) \subseteq [C]_{E \restriction A}$, and $\mu([C]_{E \restriction A}) = 0$ by the $E \restriction A$-quasi-invariance of μ.

If μ is nonatomic, clearly so is ν. If μ is $E \restriction A$-ergodic, then ν is E-ergodic as in 3.2. □

In particular (from 3.2), if $E \sqsubseteq F$ and E has an invariant (nonatomic, ergodic) measure, then so does F.

We now have the following result which is a special case of the theorem in Harrington-Kechris-Louveau [HKL], but was already proved earlier in Effros [E1, E2] and Weiss [W2].

Theorem 3.4 (Effros [E1, E2], Weiss [W2]). *Let E be a countable Borel equivalence relation. Then the following are equivalent:*

(1) *E is not smooth;*
(2) $E_0 \sqsubseteq E$;
(3) *E admits a nonatomic, ergodic, (quasi-)invariant measure.*

Since it is easy to check that every smooth countable Borel E on an uncountable space X admits a nonatomic invariant measure, it follows that *every* countable Borel equivalence relation E on an uncountable space X admits a nonatomic invariant measure.

Not every countable Borel equivalence relation admits a *finite* invariant measure. It is clear that if E is compressible it cannot admit such a measure. The following basic result of Nadkarni shows that this is the only obstruction and provides a fundamental relation between compressibility and existence of finite invariant measures.

Theorem 3.5 (Nadkarni [N2]). *Let E be a countable Borel equivalence relation. Then the following are equivalent:*

(1) *E is not compressible,*
(2) *E admits an invariant probability measure.*

(In [N2], this is stated and proved only for hyperfinite E (see §5), but the proof can be easily generalized to arbitrary countable E.)

§4. The spaces of invariant and quasi-invariant measures

Let X be a standard Borel space. Denote by $\mathscr{M}(X)$ the space of probability measures on X equipped with the Borel structure generated by the maps $\mu \mapsto \mu(A)$, where $A \subset X$ is Borel. This is a standard Borel space (see, e.g., [Va]). If now E is a countable Borel equivalence relation on X, put

$$\mathscr{I}_0(E) := \{\mu \in \mathscr{M}(X) : \mu \text{ is } E\text{-invariant and nonatomic}\},$$
$$\mathscr{E}_0(E) := \{\mu \in \mathscr{I}_0(E) : \mu \text{ is } E\text{-ergodic}\}.$$

Then $\mathscr{I}_0(E)$, $\mathscr{E}_0(E)$ are Borel subsets of $\mathscr{M}(X)$ (see again [Va]). Let also

$$\mathscr{Q}\mathscr{I}_0(E) := \{\mu \in \mathscr{M}(X) : \mu \text{ is } E\text{-quasi-invariant and nonatomic}\},$$
$$\mathscr{Q}\mathscr{E}_0(E) := \{\mu \in \mathscr{Q}\mathscr{I}_0(E) : \mu \text{ is } E\text{-ergodic}\}.$$

Again $\mathscr{Q}\mathscr{I}_0(E)$ is Borel, a fact which can be seen as follows: Since any two uncountable Borel spaces are Borel isomorphic, we can assume that $X = 2^{\mathbb{N}}$. Let $\{C_n\}$ be an enumeration of the clopen subsets of $2^{\mathbb{N}}$. Then if $E = E_G$, with $G = \{g_0, g_1, \ldots\}$, we have that a probability measure μ on $2^{\mathbb{N}}$ is in $\mathscr{Q}\mathscr{I}_0(E)$ iff (1) $\forall i \forall n \exists m \forall j[\mu(C_j) < 1/(m+1) \Rightarrow \mu(g_i \cdot C_j) < 1/(n+1)]$ and (2) $\forall n \exists j_1 \cdots j_k[\{C_{j_1}, \ldots, C_{j_k}\}$ is a partition of $2^{\mathbb{N}}$ and $\mu(C_{j_p}) < 1/(n+1)$, $1 \le p \le k]$. Although we do not need it, it can be shown that $\mathscr{Q}\mathscr{E}_0(E)$ is also Borel. (This has been proved by A. Ditzen, using the result in [KP] and the method of [Va, Theorem 4.1].)

We will discuss $\mathscr{I}_0$, $\mathscr{E}_0$ (for hyperfinite E) in §9. Here we want to say a few things about $\mathscr{Q}\mathscr{I}_0$, $\mathscr{Q}\mathscr{E}_0$. Clearly both of these are $\sim$-invariant (where $\sim$ is equivalence of measures). For nonsmooth E, $\sim$ on $\mathscr{Q}\mathscr{E}_0(E)$ is quite complicated. The fact below strengthens results in Krieger [Kr2] and Katznelson-Weiss [KW].

Proposition 4.1. *Let E be a nonsmooth countable Borel equivalence relation. Then there is a Borel injection $F: 2^{\mathbb{N}} \to \mathscr{M}(X)$ with* range$(F) \subseteq \mathscr{Q}\mathscr{E}_0(E)$ *and $xE_0y \Leftrightarrow F(x) \sim F(y)$.*

Proof. Since E is not smooth, we have by 3.4 that there is a Borel injection $f: 2^{\mathbb{N}} \to X$ with $xE_0y \Leftrightarrow f(x)Ef(y)$.

For each $\mu \in \mathscr{M}(2^{\mathbb{N}})$, let $f\mu \in \mathscr{M}(X)$ be its image under f. So $f\mu$ is a measure on $f[2^{\mathbb{N}}] = A$. Applying 3.3 to $f\mu$ (with $X = [A]_E$ there), we can define an injective Borel map $g: \mathscr{Q}\mathscr{I}_0(E_0) \to \mathscr{Q}\mathscr{I}_0(E)$ such that $g[\mathscr{Q}\mathscr{E}_0(E_0)] \subseteq \mathscr{Q}\mathscr{E}_0(E)$ and $\mu \sim \nu \Leftrightarrow g(\mu) \sim g(\nu)$. So it is enough to show that there is a Borel injection $h: 2^{\mathbb{N}} \to \mathscr{M}(2^{\mathbb{N}})$ with range$(h) \subseteq \mathscr{Q}\mathscr{E}_0(E_0)$ and $xE_0y \Leftrightarrow h(x) \sim h(y)$. This follows easily by a standard argument applying a result of Kakutani (see, e.g., [HS, 22.38]). For each $x \in 2^{\mathbb{N}}$, let μ_x be the product measure on $2^{\mathbb{N}}$, where for $x(n) = 0$ the nth coordinate is given the $(1/2, 1/2)$ measure, while for $x(n) = 1$ it is given the $(3/4, 1/4)$ measure. Then $xE_0y \Leftrightarrow \mu_x \sim \mu_y$. It is easy to check that μ_x is E_0-quasi-invariant and is well known that it is E_0-ergodic (see, e.g., [GM, 6.4.6]). Put $h(x) = \mu_x$. $\square$

Finally we verify that the structure of $\mathcal{QI}_0(E)$, $\mathcal{QE}_0(E)$ modulo $\sim$ depends only on the stable isomorphism type of E (recall $\cong_s$ and 2.6 here).

Proposition 4.2. *Let E, F be countable Borel equivalence relations on the spaces X, Y resp. If $E \cong_s F$, then there are Borel maps $\Phi: \mathcal{QI}_0(E) \to \mathcal{QI}_0(F)$ and $\Psi: \mathcal{QI}_0(F) \to \mathcal{QI}_0(E)$ such that $\mu << \nu \Leftrightarrow \Phi(\mu) << \Phi(\nu)$, $\mu \perp \nu \Leftrightarrow \Phi(\mu) \perp \Phi(\nu)$ and similarly for Ψ, and $\Psi(\Phi(\mu)) \sim \mu$, $\Phi(\Psi(\mu)) \sim \mu$. Thus Φ induces a bijection $\tilde{\Phi}: \mathcal{QI}_0(E)/\sim \to \mathcal{QI}_0(F)/\sim$ with inverse $\tilde{\Psi}$. Moreover, $\tilde{\Phi}$ maps $\mathcal{QE}_0(E)/\sim$ onto $\mathcal{QE}_0(F)/\sim$.*

Proof. Fix full Borel sets A, B for E, F resp. and a Borel isomorphism g of $E \restriction A$ with $F \restriction B$. It suffices clearly to show that there is a Borel map $g: \mathcal{QI}_0(E) \to \mathcal{QI}_0(E \restriction A)$ such that $\mu << \nu \Leftrightarrow g(\mu) << g(\nu)$, $\mu \perp \nu \Leftrightarrow g(\mu) \perp g(\nu)$ and similarly a Borel map $h: \mathcal{QI}_0(E \restriction A) \to \mathcal{QI}_0(E)$ with the same properties, such that $gh(\mu) \sim \mu$, $hg(\mu) \sim \mu$, with the further property that g, h map ergodic measures to ergodic measures.

To define g, let $\mu \in \mathcal{QI}_0(E)$. Then $\mu(A) > 0$, as A is full, so define a probability measure $g(\mu)$ on A by $g(\mu)(B) = \mu(B)/\mu(A)$ for Borel $B \subseteq A$. It is clear that $g: \mathcal{QI}_0(E) \to \mathcal{QI}_0(E \restriction A)$. To define h, let $\mu \in \mathcal{QI}_0(E \restriction A)$ and let $h(\mu) = \nu$ be defined as in the proof of 3.3. Note then that $g(h(\mu)) = \mu$. We check next that $hg(\mu) \sim \mu$. If $\mu(B) = 0$, then $\mu([B]_E) = 0$, from which it follows that $hg(\mu)(B) = 0$. If $hg(\mu)(B) = 0$, then $hg(\mu)([B]_E) = 0$, so $g(\mu)([B]_E \cap A) = 0$, thus $\mu([B]_E \cap A) = 0$, and, as $[[B]_E \cap A]_E = [B]_E$, $\mu([B]_E) = 0$.

If $\mu << \nu$, clearly $g(\mu) << g(\nu)$. If $\mu << \nu$, we verify that $h(\mu) << h(\nu)$. If $h(\nu)(B) = 0$, then (in the notation of the proof of 3.3) $\nu(g_i^{-1} \cdot (B \cap B_i)) = 0$, so $\nu(A \cap [B \cap B_i]_E) = 0$ (since $[B \cap B_i]_E = [g_i^{-1} \cdot (B \cap B_i)]_E$), thus $\nu(A \cap [B]_E) = 0$, $\mu(A \cap [B]_E) = 0$, therefore $h(\mu)(B) = 0$.

Preservation of orthogonality is easy to check, as is the fact that both g, h preserve ergodicity. □

5. Hyperfinite equivalence relations

A countable Borel equivalence E is called *hyperfinite* if there is an increasing sequence $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ of finite Borel equivalence relations with $E = \bigcup_n E_n$ (i.e., $xEy \Leftrightarrow \exists n(xE_ny)$). A *finite* equivalence relation is one for which *all* equivalence classes are finite.

Remark. The condition that the sequence be increasing is crucial. Every countable Borel equivalence relation E can be written as a union $E = \bigcup_n E_n$, where each E_n is a Borel equivalence relation all of whose equivalence classes have cardinality at most 2. (This can be seen from the proof of Theorem 1.1 in [FM]: It is shown there that there is a sequence of Borel automorphisms $\{f_n\}$ which are of order 2, i.e., $f_n^2 =$ identity, such that $xEy \Leftrightarrow \exists n(f_n(x) = y)$. Put now $E_n = \{(x, y): x = y \vee f_n(x) = y\}$.)

The next result gives a series of equivalent formulations of the notion of hyperfiniteness. The equivalence of (1) and (2) is due to Weiss [W2], the direction (4) $\Rightarrow$ (1) to Weiss [W2], Slaman-Steel [SS] (see also Krieger [Kr1, 4.1.1]), and (1) $\Rightarrow$ (4) to Slaman-Steel [SS].

Theorem 5.1. *Let E be a countable Borel equivalence relation on X. Then the following are equivalent:*

(1) *E is hyperfinite*;

(2) *$E = \bigcup_{n=1}^{\infty} E_n$, where E_n are finite Borel equivalence relations, $E_n \subseteq E_{n+1}$, and each E_n-equivalence class has cardinality at most n*;

(3) *$E = \bigcup_n E_n$, where E_n are smooth Borel equivalence relations, $E_n \subseteq E_{n+1}$*;

(4) *$E = E_{\mathbb{Z}}$, i.e. there is a Borel automorphism T of $\mathbb{X}$ with $xEy \Leftrightarrow \exists n \in \mathbb{Z}$ $(T^n(x) = y)$*;

(5) *There is a Borel assignment $C \mapsto <_C$ giving for each E-equivalence class C a linear order $<_C$ of C of order type finite or $\mathbb{Z}$. (That $C \mapsto <_C$ is Borel means that the relation*

$$R(x, y, z) :\Leftrightarrow x <_{[z]_E} y$$

is Borel.)

Proof. First note that (4), (5) are easily equivalent. Indeed, if (4) holds and assuming without loss of generality that $X = \mathbb{R}$ and letting $<$ be the usual ordering of $\mathbb{R}$, we define for each E-equivalence class C,

$$\begin{aligned} x <_C y :\Leftrightarrow x, y \in C \wedge [(&C \text{ is finite } \wedge x < y) \\ &\vee (C \text{ is infinite } \wedge \exists n > 0\, [T^n(x) = y])]. \end{aligned}$$

Conversely, given $C \mapsto <_C$, we define T by

$$\begin{aligned} T(x) :\Leftrightarrow (&y \text{ is the successor of } x \text{ in } <_{[x]_E}) \vee (x \text{ is the last} \\ &\text{element of } <_{[x]_E} \text{ and } y \text{ is the first element}). \end{aligned}$$

Using Theorem 1.1, it is easy to verify that T is Borel. We prove now the other equivalences.

(1) $\Rightarrow$ (2). Let $E = \bigcup_{n=1}^{\infty} R_n$, $R_n \subseteq R_{n+1}$, $R_1 = \Delta(X)$, R_n finite Borel equivalence relations. Define E_n as follows: $E_1 = R_1$. For any $n \geq 2$, let $X_n = \{x \in X\colon \mathrm{card}([x]_{R_n}) \leq n\}$, $X_{n-1} = \{x \notin X_n\colon \mathrm{card}([x]_{R_{n-1}}) \leq n\}$, $X_{n-2} = \{x \notin X_n \cup X_{n-1}\colon \mathrm{card}([x]_{R_{n-2}}) \leq n\}, \ldots, X_2 = \{x \notin X_n \cup X_{n-1} \cup \cdots \cup X_3\colon \mathrm{card}([x]_{R_2}) \leq n\}$, $X_1 = X \setminus (X_n \cup X_{n-1} \cup \cdots \cup X_2)$ and put $E_n = R_n \restriction X_n \cup R_{n-1} \restriction X_{n-1} \cup \cdots \cup R_2 \restriction X_2 \cup R_1 \restriction X_1$.

(2) $\Rightarrow$ (3). This is clear, since any finite Borel equivalence relation is smooth.

(3) $\Rightarrow$ (1). Let $E = \bigcup_{n=0}^{\infty} E_n$ with E_n smooth Borel equivalence relations, $E_n \subseteq E_{n+1}$, $E_0 = \Delta(X)$. Let s_n be a Borel selector for E_n, i.e., $s_n(x) E_n x$, $xE_n y \Rightarrow s_n(x) = s_n(y)$. Let $G_n = \{g_n^{(k)}\}_{k \in \mathbb{N}}$ be a countable group with $E_n = E_{G_n}$. Define the relation F_n on X by

$$\begin{aligned} xF_n y :\Leftrightarrow \exists m \leq n \{ &xE_m y \\ &\wedge \exists k_0, k_1, \ldots, k_m \\ &\leq n[x = g_0^{(k_0)} s_0 g_1^{(k_1)} s_1 \cdots g_m^{(k_m)} s_m(x)] \\ &\wedge \exists l_0, l_1, \ldots, l_m \leq n[y = g_0^{(l_0)} s_0 g_1^{(l_1)} s_1 \cdots g_m^{(l_m)} s_m(y)]\}. \end{aligned}$$

The proof will be complete from the following claims:

(a) $F_n \subseteq F_{n+1}$,

(b) $F_n \subseteq E_n$,

(c) $E \subseteq \bigcup_n F_n$,

(d) F_n is an equivalence relation,

(e) F_n is finite.

Claims (a), (b) are obvious. For (c), note that $\forall x \forall j \exists k_j(x)\ \ x = g_j^{(k_j(x))} s_j(x)$. If xEy, say xE_my, let $n = \max\{m, k_0(x), \dots, k_m(x), k_0(y), \dots, k_m(y)\}$. Then xF_ny. For (e), assuming (d), note that if $x \in [y]_{F_n}$, there is $m \leq n$ with xE_my and hence $s_m(x) = s_m(y)$. So x is completely determined by numbers $k_0, \dots, k_m \leq n$ and $s_m(y)$, so it can only take finitely many distinct values, i.e., $[y]_{F_n}$ is finite.

Proof of (d). Clearly F_n is symmetric and reflexive. We prove now that it is transitive. Assume xF_ny, yF_nz. Let $p, q \leq n$ be such that xE_py, yE_qz and there are $k_0, \dots, k_p, l_0, \dots, l_r$ $(r = \max(p, q))$ and $m_0, \dots, m_q$ such that

$$x = g_0^{(k_0)} s_0 \cdots g_p^{(k_p)} s_p(x), \quad y = g_0^{(l_0)} s_0 \cdots g_r^{(l_r)} s_r(y), \quad z = g_0^{(m_0)} s_0 \cdots g_q^{(m_q)} s_q(z).$$

Assume $p \leq q$. If $p = q$, we are done. So assume $p < q$. Let $v = g_{p+1}^{(m_{p+1})} s_{p+1} \cdots g_q^{(m_q)} s_q(y)$ and note that vE_py, since $y = g_0^{(m_0)} s_0 \cdots g_p^{(m_p)} s_p(v)$. Since also xE_py, we have $s_p(v) = s_p(x)$ and so

$$\begin{aligned} x &= g_0^{(k_0)} s_0 \cdots g_p^{(k_p)} s_p(v) \\ &= g_0^{(k_0)} s_0 \cdots g_p^{(k_p)} s_p g_{p+1}^{(m_{p+1})} s_{p+1} \cdots g_q^{(m_q)} s_q(y) \\ &= g_0^{(k_0)} s_0 \cdots g_p^{(k_p)} s_p g_{p+1}^{(m_{p+1})} s_{m+1} \cdots g_q^{(m_q)} s_q(x) \end{aligned}$$

since $s_q(x) = s_q(y)$. So xF_nz.

(1) $\Rightarrow$ (5). Assume $E = \bigcup_n E_n$ with E_n increasing finite Borel equivalence relations. We will find Borel assignments $C \mapsto <_C^n$ of linear orderings to each E_n-equivalence class C, which are increasing, i.e., if $C = [x]_{E_n}$, $D = [x]_{E_{n+1}}$ (so that $C \subseteq D$) then $<_C \subseteq <_D$ (i.e. for $x, y \in C$, $x <_C y \Leftrightarrow x <_D y$) and $<_D$ is an *end extension* of $<_C$, i.e., for $a \in D \setminus C$, either $c <_D a$, $\forall c \in C$ or $a <_D c$, $\forall c \in C$. Then we will put, for each E-equivalence class $C = [x]_E$, $<_C = \bigcup_n <_{C_n}^n$ where $C_n = [x]_{E_n}$. Then $<_C$ is a linear order of C and has order type finite or ω $(= \{0, 1, 2, \dots\})$ or ω^* $(= \{\dots, -2, -1, 0\})$ or $\mathbb{Z}$. In the two middle cases we can easily rearrange the order in a Borel way to make it also of order type $\mathbb{Z}$, so the proof is complete, modulo the definition of $<_C^n$.

Again without loss of generality we can take $X = \mathbb{R}$ and we let $<$ be the usual order of $\mathbb{R}$. For $n = 0$ let $<_C^0 = < \restriction C$. Assume now $<_C^n$ has been defined. Let C be an E_{n+1}-equivalence class and let $C_1, \dots, C_k$ be the E_n-equivalence classes contained in it, arranged in order so that (the $<$-least element of C_i) $<$ (the $<$-least element of C_j) iff $i < j$. Then let

$$x <_C^{n+1} y :\Leftrightarrow \exists k (x, y \in C_k \wedge x <_{C_k}^n y) \vee \exists k < l (x \in C_k \wedge y \in C_l).$$

(5) $\Rightarrow$ (1). Assume without loss of generality $X = 2^{\mathbb{N}}$ and each E-equivalence class is infinite, and hence ordered by $<_C$ in order type $\mathbb{Z}$. For each E-equivalence class C, let x_C be the lexicographically least element of the closure of C. The map $y \mapsto x_{[y]_E}$ is Borel. Put

$$S_n^C = \{x \in C : x \restriction n = x_C \restriction n\}.$$

Define now the equivalence relations $E_n \subseteq E$ as follows: If $(x, y) \in E$ with $[x]_E = [y]_E = C$ are such that (a) $x_C \in C$, then let

$$xE_ny :\Leftrightarrow [x = y \vee \text{ the distance of } x_C \text{ from } x, y \text{ in } <_C \text{ is at most } n].$$

If (a) fails, but (b) $\exists m[S_m^C$ is bounded below in $<_C]$, let m_0 be the least such m and z_C the $<_C$-least element of $S_{m_0}^C$. Then put

$$xE_ny :\Leftrightarrow [x = y \vee \text{ the distance of } z_C \text{ from } x, y \text{ in } <_C \text{ is at most } n].$$

If (a), (b) fail but (c) $\exists m[S_m^C$ is bounded above in $<_C]$ proceed as in (b). Finally, if (a), (b), (c) fail, so that $\{S_n^C\}$ form a decreasing sequence of subsets of C with S_n^C unbounded in both directions in $<_C$ and $\bigcap_n S_n^C = \varnothing$, then let

$$xE_ny :\Leftrightarrow \exists a \exists b[a, b \text{ are consecutive (in } <_C) \text{ members of } S_n^C \text{ and } a \leq_C x <_C b \text{ and } a \leq_C y <_C b].$$

It is now clear that the relations E_n are increasing finite Borel equivalence relations with $\bigcup_n E_n = E$, so E is hyperfinite. □

Here are also some basic closure properties of hyperfiniteness.

Proposition 5.2. *Let E, F be countable Borel equivalence relations on X, Y resp.*

(1) *$X = Y \wedge E$ is hyperfinite $\wedge F \subseteq E \Rightarrow F$ is hyperfinite;*
(2) *E is hyperfinite $\wedge F \leq E \Rightarrow F$ is hyperfinite;*
(3) *E is hyperfinite $\wedge A$ is* Borel *$\Rightarrow E \restriction A$ is hyperfinite;*
(4) *$A \subseteq X$ is Borel, full for E and $E \restriction A$ is hyperfinite $\Rightarrow E$ is hyperfinite;*
(5) *E, F are hyperfinite $\Rightarrow E \times F$ is hyperfinite.*

Proof. (1), (3), and (5) are immediate consequences of the definition. For (4), let $E = E_G$, with $G = \{g_n\}$ a countable group, and let for each $x \in X$, $N(x) =$ least n with $g_n \cdot x \in A$. Let $E \restriction A = \bigcup_n F_n$, with F_n finite Borel equivalence relations on A, $F_n \subseteq F_{n+1}$. Define E_n on X by

$$xE_ny :\Leftrightarrow [xEy \wedge N(x), N(y) < n \wedge g_{N(x)} \cdot xF_n g_{N(y)} \cdot y] \vee x = y.$$

Then $E_n \subseteq E_{n+1}$, $\bigcup_n E_n = E$, and the equivalence relations E_n are finite. Finally, for (2), let $f: Y \to X$ be a Borel map reducing F to E, so that f is countable-to-1. Then $A = f[Y] \subseteq X$ is Borel and $E \restriction A$ is hyperfinite (by (3)). Let g be a Borel inverse to f (i.e., $g: A \to Y$, $f(g(x)) = x$) and let $B = g[A]$. Then B is full for F and $E \restriction A \cong F \restriction B$, so $F \restriction B$ is hyperfinite and so is F by (4)). □

Before we proceed we would like to mention some open problems:

(1) Is the increasing union of hyperfinite Borel equivalence relations hyperfinite? (It is well known [FM, 4.2] that, if E_n are increasing Borel hyperfinite on X with $E = \bigcup_n E_n$ and μ is a measure on X, then there is Borel E-invariant $A \subseteq X$ with $\mu(X \setminus A) = 0$ and $E \restriction A$ hyperfinite.)

(2) Is the notion of hyperfiniteness effective? More precisely suppose $X = \mathbb{N}^{\mathbb{N}}$ and $E \in \Delta_1^1$ is a hyperfinite equivalence relation. Is there a Δ_1^1 automorphism T of X inducing E?

(3) What is the complexity of the class of hyperfinite Borel equivalence relations, i.e. what is the complexity of the set

$$\{x: x \text{ codes a Borel equivalence relation on } \mathbb{N}^{\mathbb{N}} \text{ which is hyperfinite}\}.$$

It is clearly Σ_2^1. If (2) has a positive answer which relativizes, then it is Π_1^1.

6. Some examples

We will discuss here some examples of hyperfinite (and nonhyperfinite) Borel equivalence relations.

If T is a Borel automorphism of a standard Borel space X and we denote by E_T the equivalence relation generated by T, then, by 5.1(4), E_T is hyperfinite. Of particular importance will be the case of $T = s$, the shift on the space $2^{\mathbb{Z}}$ $(s(x)(n) = x(n+1))$. In this case $E_s = E(\mathbb{Z}, 2)$ in our notation of §1. It will turn out (see §9) that $E(\mathbb{Z}, 2)$ is *invariantly universal* for hyperfinite Borel E (with no finite equivalence classes), i.e., for every such E we have $E \sqsubseteq^i E(\mathbb{Z}, 2)$.

For any set Ω, recall that we defined in §2 the equivalence relations $E_0(\Omega)$, $E_t(\Omega)$ on $\Omega^{\mathbb{N}}$ by

$$xE_0(\Omega)y \Leftrightarrow \exists n \forall m \geq n (x_m = y_m),$$
$$xE_t(\Omega)y \Leftrightarrow \exists n \exists m \forall k (x_{n+k} = y_{m+k}).$$

Then, for any *countable* set Ω, $E_0(\Omega)$ and $E_t(\Omega)$ are hyperfinite. In fact, these are special cases of the following more general situation:

Let X be a standard Borel space and U a Borel map on X such that U is countable-to-1. Define the equivalence relations $E_0(U)$, $E_t(U)$ on X by

$$xE_0(U)y :\Leftrightarrow \exists n \geq 0\ [U^n(x) = U^n(y)]$$
$$xE_t(U)y :\Leftrightarrow \exists n \geq 0 \exists m \geq 0\ [U^n(x) = U^m(y)].$$

Then $E_0(\Omega)$, $E_t(\Omega)$ are just $E_0(U)$, $E_t(U)$ with $U : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ defined by $U(x)(n) = x(n+1)$ (i.e., the one-sided shift). Since $E_0(U) = \bigcup_n E_n$ with $xE_n y \Leftrightarrow U^n(x) = U^n(y)$, we see by 5.1(3) that $E_0(U)$ is hyperfinite. We will see in §8 that $E_t(U)$ is hyperfinite as well.

All the $E_{\mathbb{Z}}$ are hyperfinite. How about more general E_G? One has the following measure theoretic result.

Theorem 6.1 (Connes-Feldman-Weiss [CFW]). *Let X be a standard Borel space, E a countable Borel equivalence relation on X, and μ a measure on X. If E is of the form E_G with G a countable amenable group, or (more generally) if E is μ-amenable, then there is a Borel E-invariant set $Y \subseteq X$ with $\mu(X \setminus Y) = 0$ such that $E \upharpoonright Y$ is hyperfinite.*

It is an open problem (see Weiss [W2]) whether every E_G with G amenable is hyperfinite. This is already open in the case G is abelian, but has been proved for $G = \mathbb{Z}^n$ by Weiss.

There is an even stronger result in the case of category instead of measure. We need a definition first. Let E be a countable Borel equivalence relation on a Polish space X. We call E *generically ergodic* if every E-invariant Borel set is meager or comeager. We call E *generic* if for each Borel meager set A, $[A]_E$ is also meager.

Theorem 6.2 (Sullivan-Weiss-Wright [SWW]). *Let X be a perfect Polish space and E an arbitrary countable Borel equivalence relation on X. If E is generic and generically ergodic, then there is an E-invariant dense G_δ set $X_0 \subseteq X$ and there is an E_0-invariant dense G_δ set $Y_0 \subseteq 2^{\mathbb{N}}$ such that $E \upharpoonright X_0 \cong E_0 \upharpoonright Y_0$ by a continuous isomorphism. In particular, $E \upharpoonright X_0$ is hyperfinite.*

The following corollary is due to Woodin.

Corollary 6.3 (Woodin). *Let E be a countable Borel equivalence relation on a perfect Polish space X. Assume E is generically ergodic. Then there are a dense*

G_δ $X_0 \subseteq X$ and an E_0-invariant dense G_δ $Y_0 \subseteq 2^{\mathbb{N}}$ such that $E \restriction X_0 \cong E_0 \restriction Y_0$ by a continuous isomorphism. In particular, $E \restriction X_0$ is hyperfinite, and thus there is an E-invariant comeager set Z_0 (namely $[X_0]_E$) such that $E \restriction Z_0$ is hyperfinite.

Proof. Let $E = E_G$, with $G = \{g_n\}$ a countable group. Write $g_n(x) = g_n \cdot x$. Let $\{V_n\}$ be a basis for X. For each pair $(n, i) \in \mathbb{N}^2$ for which it is possible, choose a comeager-in-V_n set $P_{n,i}$ such that $g_i[P_{n,i}]$ is meager. Put $X_1 = X \setminus \bigcup_{n,i} g_i[P_{n,i}]$. Then X_1 is comeager, so let $X_2 \subseteq X_1$ be a dense G_δ.

Claim. $E \restriction X_2$ is generically ergodic and generic.

Then, by 6.2 (since $E \restriction X_2$ satisfies all its conditions), there are dense G_δ $X_0 \subseteq X_2$ and a dense G_δ E_0-invariant set $Y_0 \subseteq 2^{\mathbb{N}}$ such that $E \restriction X_0 \cong E_0 \restriction Y_0$ via a continuous isomorphism. So it remains to prove the claim.

Clearly $E \restriction X_2$ is generically ergodic. To show that it is generic, it is clearly enough to show that, if $B \subseteq X_2$ is meager in X_2 (or equivalently in X), then $[B]_{E \restriction X_2}$ is meager in X. For that, note that $[B]_{E\restriction X_2} = [B]_E \cap X_2$ and $[B]_E = \bigcup_i g_i[B]$, so it suffices to show that $g_i[B] \cap X_2$ is meager in X_2, i.e., $g_i[B]$ is meager in X. Let j be such that $g_j = g_i^{-1}$. We can of course assume that B is Borel. If $g_i[B]$ is not meager, let V_n be such that $g_i[B]$ is comeager in V_n. So there is a comeager-in-V_n set P such that $g_i^{-1}[P] = g_j[P]$ is meager. Then $P_{n,j}$ and $g_i[B]$ are both comeager in V_n, so they intersect. Let $x \in P_{n,j} \cap g_i[B]$. Then $g_j(x) \in B \cap g_j[P_{n,j}] = \varnothing$ (since $B \subseteq X_2 \subseteq X_1 = X \setminus \bigcup_{n,i} g_i[P_{n,i}]$). □

As an application, we see immediately that, for any countable group G, $E(G, 2)$ is *generically hyperfinite*, i.e. for some Borel comeager E-invariant $X \subseteq 2^G$, $E \restriction X$ is hyperfinite. (On the other hand $E(F_2, 2)$, for example, is not hyperfinite.) For that it is enough to check that $E(G, 2)$, for infinite G, is generically ergodic. By 1.1 of [SWW] it is enough to see that $E(G, 2)$ has a dense orbit, and this is easy to verify using the fact that G is infinite.

For another example, consider the equivalence relation $\equiv_T$ of Turing equivalence on $2^{\mathbb{N}}$ ($x \equiv_T y :\Leftrightarrow x$ is recursive in y and y is recursive in x). Again $\equiv_T$ is not hyperfinite (see below). However, $\equiv_T$ is easily generically ergodic, so $\equiv_T$ is generically hyperfinite.

Also, as was shown by Mycielski (see, e.g., [MU, I.6]), if $E(\mathbb{R}/\mathbb{Q})$ denotes the *Vitali* equivalence relation on $\mathbb{R}$ ($xE(\mathbb{R}/\mathbb{Q})y :\Leftrightarrow x - y \in \mathbb{Q}$) then (in our notation) $E(\mathbb{R}/\mathbb{Q}) \approx^* E_0$, so $E(\mathbb{R}/\mathbb{Q})$ is hyperfinite. This can also be seen by noticing that $E(\mathbb{R}/\mathbb{Q})$ is an increasing union of smooth Borel equivalence relations, namely $E(\mathbb{R}/(n!)^{-1}\mathbb{Z})$. (Actually, from the results in §9 it will follow that $E(\mathbb{R}/\mathbb{Q}) \cong E_t$.)

Finally, in some sense, all equivalence relations induced by flows, i.e., Borel actions of $\mathbb{R}$, are "hyperfinite". More precisely, let X be a standard Borel space and $g \cdot x$ a Borel action of $\mathbb{R}$ on X. Let $E_{\mathbb{R}}$ be the corresponding equivalence relation. Then by Wagh [Wa] (see also [K3]) $E_{\mathbb{R}}$ is Borel and there is a Borel set $A \subseteq X$ such that A is full for $E_{\mathbb{R}}$ and $E_{\mathbb{R}} \restriction A$ is hyperfinite. (Thus, using also [Mos, 4F.6], $E_{\mathbb{R}} \approx^* E_{\mathbb{R}} \restriction A$.)

We conclude this section with a couple of examples of nonhyperfinite countable Borel equivalence relations. The standard example is $E(F_2, 2)$ (for a proof see, e.g., [K1]). Since (identifying 2^{F_2} with $2^{\mathbb{N}}$) $E(F_2, 2) \subseteq \equiv_T$, it follows that $\equiv_T$ is not hyperfinite as well.

7. The bi-embeddability of nonsmooth hyperfinite relations

Our goal is to classify hyperfinite Borel equivalence relations up to the equivalences $\approx^*$, $\approx$, $\cong$. We will deal with $\approx^*$, $\approx$ here and with $\cong$ in §9.

First notice that it is easy to classify smooth relations up to $\approx^*$, $\approx$. Indeed, given a countable Borel equivalence relation E, let for $1 \leq n \leq \aleph_0$,

$$c_n(E) = \text{card}\{[x]_E\colon \text{card}([x]_E) = n\}.$$

Then we have for any smooth countable Borel E, F on X, Y resp.

(A) $E \approx^* F$ iff $\text{card}(X/E) = \text{card}(Y/F)$.

(B) $E \approx F$ iff $\forall n[\sum_{m\geq n} c_m(E) = \sum_{m \geq n} c_m(F)]$.

So it is enough to look at nonsmooth hyperfinite Borel equivalence relations. For those the answer is given by the next result.

Theorem 7.1. *Let E, F be nonsmooth hyperfinite Borel equivalence relations. Then $E \approx F$ (and thus $E \approx^* F$).*

Proof. It is enough, of course, to show that, for every nonsmooth hyperfinite Borel equivalence relation, we have $E \approx E_0$. That $E_0 \sqsubseteq E$ follows from 3.4. We will prove below that $E \sqsubseteq E_0$. Recall from 1.2 that $E = E_{\mathbb{Z}} \sqsubseteq E(\mathbb{Z}, 2^{\mathbb{N}})$. So it is enough to prove that $E(\mathbb{Z}, 2^{\mathbb{N}}) \sqsubseteq E_0$. Denote below $E(\mathbb{Z}, 2^{\mathbb{N}})$ by E.

Call an E-invariant Borel set $X \subseteq (2^{\mathbb{N}})^{\mathbb{Z}}$ *smooth* if $E \restriction X$ is smooth, i.e., there is a Borel selector on X for E. We claim that it is enough to show tht $E \sqsubseteq E_0$ modulo smooth sets, i.e., that there is smooth X such that $E \restriction (2^{\mathbb{N}})^{\mathbb{Z}} \backslash X \sqsubseteq E_0$. Indeed let $Y = (2^{\mathbb{N}})^{\mathbb{Z}} \backslash X$ and $f\colon Y \to 2^{\mathbb{N}}$ embed $E \restriction Y$ into E_0. Let $g\colon X \to 2^{\mathbb{N}}$ be Borel such that $xEy \Leftrightarrow g(x) = g(y)$. Also let $h\colon X \to \mathbb{N}$ be Borel such that $h \restriction [x]_E$ is injective for any $x \in X$. Finally let $p\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be Borel such that $x \neq y \Rightarrow \neg(p(x)E_0p(y))$. Then define $F\colon (2^{\mathbb{N}})^{\mathbb{Z}} \to 2^{\mathbb{N}}$ by

$$\begin{aligned} F(x) &= \langle f(x), 1^\infty \rangle, && \text{if } x \in Y \\ &= \langle pg(x), 1^{h(x)}{}^\wedge 0^\infty \rangle, && \text{if } x \in X \end{aligned}$$

where for $\alpha, \beta \in 2^{\mathbb{N}}$, $\langle \alpha, \beta \rangle \in 2^{\mathbb{N}}$ is given by

$$\langle \alpha, \beta \rangle := (\alpha(0), \beta(0), \alpha(1), \beta(1), \dots)$$

and $i^\infty := (i, i, i, \dots)$. Clearly F embeds E into E_0.

So in the proof below we will ignore smooth sets. Note that the collection of smooth sets is closed under countable unions.

For each $n \in \mathbb{N}$, $n \geq 1$, let $(2^n)^n$ be the set of all sequences $\langle w_0, \dots, w_{n-1} \rangle$ with $w_i \in 2^n$, i.e., $w_i = (w_i(0) \cdots w_i(n-1))$ is a binary sequence of length n. If $w \in (2^n)^n$ and $m \leq n$ let $w \restriction m$ denote the sequence $\langle w_0 \restriction m, w_1 \restriction m, \dots, w_{m-1} \restriction m \rangle$. For each $n \in \mathbb{N}$, $n \geq 1$, fix an ordering $<_n$ of $(2^n)^n$ such that, given $w, v \in (2^{n+1})^{n+1}$,

$$w \restriction n <_n v \restriction n \Rightarrow w <_{n+1} v.$$

For $w \in (2^n)^n$ and $x \in (2^{\mathbb{N}})^{\mathbb{Z}}$, we say that w *occurs in* x at $k \in \mathbb{Z}$ if, for all $i < n$ $w_i = x_{k+i} \restriction n$. We say that w *occurs in* x, in symbols $w \subseteq x$, if w occurs in x at some k.

Note first that the set of $x \in (2^{\mathbb{N}})^{\mathbb{Z}}$ for which there is $w \in \bigcup_n (2^n)^n$ occurring in x but for which $\{k \in \mathbb{Z}: w \text{ occurs in } x \text{ at } k\}$ is bounded above or below is E-invariant and smooth, so it can be neglected, i.e., we can assume we work on the set $X \subseteq (2^{\mathbb{N}})^{\mathbb{Z}}$ of all $x \in (2^{\mathbb{N}})^{\mathbb{Z}}$ with the property that, whenever $w \subseteq x$, w occurs in x at k for unboundedly many $k \in \mathbb{Z}$ in both directions.

Put now, for $x \in X$,

$$f_n(x) := \text{the } <_n\text{-least element of } (2^n)^n \text{ occurring in } x.$$

Then note that $f_{n+1}(x) \restriction n = f_n(x)$. Note that $xEy \Rightarrow f_n(x) = f_n(y)$. So we can define $f: X \to (2^{\mathbb{N}})^{\mathbb{N}}$ by $f(x) = (u_0, u_1, \dots)$ with $(u_0, u_1, \dots, u_{n-1}) \restriction n = f_n(x)$. Note again that $xEy \Rightarrow f(x) = f(y)$.

Given $u \in (2^{\mathbb{N}})^{\mathbb{N}}$, we say that u *occurs in* $x \in (2^{\mathbb{N}})^{\mathbb{Z}}$ at $k \in \mathbb{Z}$ if $u_i = x_{k+i}$, $\forall i \in \mathbb{N}$, and that u *occurs in* x if it occurs in x at some k.

The set of x for which $f(x)$ occurs in x and $\{k \in \mathbb{Z}: f(x) \text{ occurs in } x \text{ at } k\}$ is bounded below is clearly E-invariant and smooth, so it can be neglected. If $f(x)$ occurs in x and $\{k \in \mathbb{Z}: f(x) \text{ occurs in } x \text{ at } k\}$ is unbounded below, then x is periodic, so $[x]_E$ is finite. These x's form an E-invariant and smooth set as well and can be neglected.

So we can assume that we work on the set $Y \subseteq X$ of all $x \in X$ for which $f(x)$ does not occur in x. For such $x \in Y$ and $n \in \mathbb{N}$, define $k_n^x \in \mathbb{Z}$ as follows:

$$k_0^x = 0,$$
$$k_{2n+1}^x = \text{the least } k \text{ such that } k > k_{2n}^x \text{ and } f_{2n+1}(x) \text{ occurs in } x \text{ at } k,$$
$$k_{2n+2}^x = \text{the largest } k \text{ such that } k < k_{2n+1}^x \text{ and } f_{2n+2}(x) \text{ occurs in } x \text{ at } k.$$

Note that if either $\{k_{2n}^x\}$ or $\{k_{2n+1}^x\}$ is bounded, $f(x)$ occurs in x. Also note that k_{2n+1}^x is not between k_{2n-1}^x and k_{2n}^x, since this would contradict the definition of k_{2n}^x. Similarly for k_{2n+2}^x. So we have

$$\cdots \le k_4^x \le k_2^x \le k_0^x = 0 < k_1^x \le k_3^x \le \cdots$$

and $k_{2n}^x \to -\infty$, $k_{2n+1}^x \to +\infty$.

Instead of working with $2^{\mathbb{N}}$ below we will work with $P(\mathbb{N}) = \{A: A \subseteq \mathbb{N}\}$, identifying $A \subseteq \mathbb{N}$ with its characteristic function. Under this identification, the equivalence relation E_0 on $2^{\mathbb{N}}$ corresponds to the equivalence relation (also denoted by E_0) on $P(\mathbb{N})$ given by

$$AE_0B :\Leftrightarrow A \triangle B \text{ is finite}.$$

Fix also a bijection $\langle\ ,\ \rangle: \mathbb{N} \times \mathbb{M} \to \mathbb{N}$ where $\mathbb{M} :=$ the set of all finite sequences $(u_0, \dots, u_{m-1})$ with each u_i a finite binary sequence.

We define now $G: Y \to P(\mathbb{N})$ as follows:

For each n, let $t_n^x = |k_{n+1}^x - k_n^x| + 1$ and let $r_n^x = ((r_n^x)_0, \dots, (r_n^x)_m)$, where $m = t_n^x - 1$, be the sequence given by

$$(r_n^x)_i = x_{\min\{k_n^x, k_{n+1}^x\}+i} \restriction n$$

for $0 \le i < t_n^x$. Put $G(x) = \{\langle n, r_n^x \rangle: n \in \mathbb{N}\} \subseteq \mathbb{N}$.

Note that G is injective, since knowing $G(x)$ we can easily reconstruct x. We will show that G embeds $E \restriction Y$ into E_0.

Assume xEy and say, without loss of generality, $m > 0$ is such that $x_{m+i} = y_i$, for all i. Let n_0 be such that $k_{2n_0+1}^x > m$. Then, since the functions

f_n are E-invariant, we have $k^x_{2n_0+1} = m + k^y_{2n_0+1}$, so, for all $n > 2n_0$, $k^x_n = m + k^y_n$. It follows that $G(x)\Delta G(y)$ is finite, i.e., $G(x)E_0G(y)$. The converse, i.e., $G(x)E_0G(y) \Rightarrow xEy$, is even easier, and we leave it to the reader. □

An immediate consequence of 7.1 and 2.3 (using the fact that E_t is compressible and, as shown in 8.2 below, hyperfinite) is the following strengthening of 3.4.

Corollary 7.2. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*
(1) *E is not smooth,*
(2) *$E_t \sqsubseteq^i E$.*

It follows also from 7.1 that, for any two nonsmooth Borel hyperfinite equivalence relations E, F on X, Y resp., there is a "Borel" bijection between the quotient spaces X/E and Y/F. (Here "Borel" can be interpreted in terms of the quotient Borel structures on these quotient spaces, but of course more is true.)

Applying this in particular to E_0, $E(\mathbb{Z}, 2)$ provides a solution to a problem of Mycielski; see [MU, I.6].

Another corollary is that the structure of measure classes of nonatomic, quasi-invariant, (ergodic) probability measures for any nonsmooth hyperfinite Borel equivalence relation is the same (using 2.6 and 4.2).

Finally, in view of the fact that all equivalence relations induced by Borel $\mathbb{R}$-actions have full Borel sets on which their restrictions are hyperfinite (see the end of §6), it follows that any two nonsmooth such equivalence relations are bi-reducible. (Actually, it turns out that, if they have uncountable equivalence classes, they are Borel isomorphic.)

8. Tail equivalence relations

We are going to use here ideas similar to that of the proof of 7.1 to study the tail equivalence relations $E_t(\Omega)$. It will be convenient to introduce a further notion here.

Let E be a (not necessarily countable) Borel equivalence relation on X. We call E *hypersmooth* if $E = \bigcup_n E_n$, $E_n \subseteq E_{n+1}$, E_n smooth Borel equivalence relations. Thus, for *countable* E, hypersmooth = hyperfinite. Moreover, if $E \leq F$ with F hyperfinite, then E is hypersmooth.

Consider $E_0(2^{\mathbb{N}})$ (as defined in §2). Then easily $E_0(2^{\mathbb{N}})$ is hypersmooth, but it is well known (see, e.g., [K3]) that $E_0(2^{\mathbb{N}}) \not\leq F$ for any countable Borel F. It is also easy to see that every hypersmooth E is embeddable in $E_0(2^{\mathbb{N}})$. Indeed, let $E = \bigcup_n F_n$ witness that E is hypersmooth with $E_0 = \Delta(X)$, and let $f_n\colon X \to 2^{\mathbb{N}}$ be such that $xF_ny \Leftrightarrow f_n(x) = f_n(y)$. Then, fixing a Borel isomorphism $\langle\ \rangle\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$, the Borel map $f(x) = \langle f_n(x)\rangle$ embeds E to $E_0(2^{\mathbb{N}})$.

The following result has been proved recently by A. S. Kechris and A. Louveau (unpublished): If E is Borel hypersmooth, then exactly one of the following holds: (I) $E \approx E_0(2^{\mathbb{N}})$ or (II) $E \leq E_0$. (For results along this line in a measure theoretic context, see [Ve2].)

Consider now an arbitrary Borel map $U\colon X \to X$ and define the equivalence relation $E_0(U)$ as in §6, and similarly for the tail equivalence relation $E_t(U)$.

The relations $E_0(2^{\mathbb{N}})$, $E_t(2^{\mathbb{N}})$ are obtained by taking: $U\colon (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ to be the shift $U(x)_n = x_{n+1}$. Clearly $E_0(U)$ is hypersmooth.

The next theorem extends results of Vershik [Ve1], Connes-Feldman-Weiss [CFW, Corollary 13].

Theorem 8.1. *Let $U\colon X \to X$ be Borel. Then $E_t(U)$ is hypersmooth.*

Corollary 8.2. *If $U\colon X \to X$ is Borel and countable-to-1, then $E_t(U)$ is hyperfinite. In particular, $E_t(\Omega)$ is hyperfinite for any countable Ω.*

Proof. First note that $E_t(U) \leq E_t(2^{\mathbb{N}})$. To see this, assume that $X = 2^{\mathbb{N}}$ and define $f\colon 2^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ by $f(x) = (x, U(x), U^2(x), \ldots)$. Then f reduces $E_t(U)$ to $E_t(2^{\mathbb{N}})$. So it is enough to show that $E_t(2^{\mathbb{N}})$ can be reduced to $E_0(2^{\mathbb{N}})$. As in the proof of 7.1, it is easy to see that we can neglect smooth sets for $E_t(2^{\mathbb{N}})$.

Using again the terminology and notation of the proof of 7.1, given any $x = (x_n) \in (2^{\mathbb{N}})^{\mathbb{N}}$, let

$$s_n^x = \text{the } <_n\text{-least } s \in (2^n)^n \text{ which occurs infinitely often in } x.$$

Then $xE_t(2^{\mathbb{N}})y \Rightarrow s_n^x = s_n^y$. Let

$$k_n^x = \text{least } k \text{ such that } s_n^x \text{ occurs in } x \text{ at } k.$$

Clearly $k_n^x \leq k_{n+1}^x$. The set of x's for which there is $yE_t(2^{\mathbb{N}})x$ with k_n^y eventually constant is smooth, since, for such x, $s_x = (u_0, u_1, \ldots)$, where $(u_0, \ldots, u_{n-1}) \restriction n = s_n^x$, is a Borel selector.

So we can work in the $E_t(2^{\mathbb{N}})$-invariant set $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ of x's such that, for all $yE_t(2^{\mathbb{N}})x$, $k_n^y \to \infty$. Let $\langle\ \rangle\colon \bigcup_n (2^{\mathbb{N}})^n \to 2^{\mathbb{N}}$ be a Borel bijection (with $\bigcup_n (2^{\mathbb{N}})^n$ having the obvious "direct sum" Borel structure). For $x \in X$, let $\overline{\alpha}_n(x) = \langle x_i, x_{i+1}, \ldots, x_j\rangle$, where $i = k_{n-1}^x$, $j = k_n^x - 1$, with $k_{-1}^x = 0$, and put

$$g(x) = (\overline{\alpha}_0(x), \overline{\alpha}_1(x), \ldots) \in (2^{\mathbb{N}})^{\mathbb{N}}.$$

Clearly g is a Borel injection and $g(x)E_0(2^{\mathbb{N}})g(y) \Rightarrow xE_t(2^{\mathbb{N}})y$. Conversely, assume that $xE_t(2^{\mathbb{N}})y$. Let p, q be such that $x_{p+n} = y_{q+n}$, $\forall n \in \mathbb{N}$. Let r be such that $k_r^x > p$, $k_r^y > q$. Then, for $t \geq r$, $k_t^x - p = k_t^y - q$ and so $\overline{\alpha}_t(x) = \overline{\alpha}_t(y)$ for $t > r$; therefore $g(x)E_0(2^{\mathbb{N}})g(y)$. $\square$

We have just seen that $E_t(2^{\mathbb{N}}) \sqsubseteq E_0(2^{\mathbb{N}})$. It is easy to see also that $E_0(2^{\mathbb{N}}) \sqsubseteq E_t(2^{\mathbb{N}})$. (Just send $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ to $(\langle \overline{0}, x_0\rangle, \langle \overline{1}, x_1\rangle, \ldots) \in (2^{\mathbb{N}})^{\mathbb{N}}$ where $\overline{i} = 0^{i\wedge}1^{\infty}$.) Thus $E_0(2^{\mathbb{N}}) \approx E_t(2^{\mathbb{N}})$.

We conclude this section with an open problem concerning hypersmooth relations:

Let E be a hypersmooth Borel equivalence relation on X and $U\colon X \to X$ a Borel map, such that $xEy \Rightarrow U(x)EU(y)$. Define the equivalence relation

$$xE_Uy :\Leftrightarrow \exists n \geq 0 \exists m \geq 0 (U^n(x)EU^m(y)).$$

Is E_U hypersmooth? In particular, if E is hyperfinite and U is countable-to-1, is E_U hyperfinite? In [CFW], such a result is proved in the measure theoretic context.

Let us point out some consequences of an affirmative answer to this particular case of the problem. So, for the following remarks, assume that all such E_U are hyperfinite.

First note that, if $E_{0,U}$ is defined by

$$xE_{0,U}y \Leftrightarrow \exists n[U^n(x)EU^n(x)],$$

then $E_{0,U} \subseteq E_U$, so $E_{0,U}$ is hyperfinite as well. From this we can derive that the increasing union of a sequence of hyperfinite equivalence relations is hyperfinite (see problem (1) at the end of §5). Indeed, if $\{E_n\}$ is such a sequence on the space X, consider the equivalence relation E on $X \times \mathbb{N}$ given by

$$(x, n)E^*(y, m) \Leftrightarrow n = m \wedge xE_ny.$$

Then E^* is the "disjoint union" of the E_n, so it is hyperfinite too. Define U on $X \times \mathbb{N}$ by $U(x, n) = (x, n+1)$. Then clearly U is injective and $(x, n)E^*(y, m) \Rightarrow U(x, n)E^*U(y, m)$, since the E_n are increasing. So $E^*_{0,U}$ is hyperfinite. But if $E = \bigcup_n E_n$, we have $xEy \Leftrightarrow (x, 0)E^*_{0,U}(y, 0)$, so E is hyperfinite.

Further, it follows that any E_G with G countable abelian is hyperfinite (see remarks following 6.1). Indeed, $G = \bigcup G_n$ with G_n finitely generated, $G_n \subseteq G_{n+1}$, so $E_G = \bigcup_n E_{G_n}$, an increasing union, and thus it is enough to show, by the preceding remarks, that each E_G with G finitely generated abelian is hyperfinite. We proceed by induction on the number of generators. It is clear for one generator. Assume it is true for n generators and let G have $n+1$ generators, say $a_1, \dots, a_{n+1}$. Let $H \subseteq G$ be the subgroup generated by $a_1, \dots, a_n$ and put $E = E_H$. If U is the automorphism corresponding to the generator a_{n+1}, then $xE_Hy \Rightarrow U(x)E_HU(y)$ and $E_G = (E_H)_U$, so E_G is hyperfinite.

9. Classification up to isomorphism

The main result of this section is the following classification theorem for aperiodic, nonsmooth hyperfinite Borel equivalence relations up to Borel isomorphism.

Theorem 9.1. *Let E, F be aperiodic, nonsmooth hyperfinite Borel equivalence relations. Let $\mathscr{E}_0(E)$, $\mathscr{E}_0(F)$ be their sets of invariant, ergodic probability measures. Then*

$$E \cong F \Leftrightarrow \operatorname{card}(\mathscr{E}_0(E)) = \operatorname{card}(\mathscr{E}_0(F)).$$

(Note that because E is aperiodic, invariant probability measures are nonatomic.)

This classification was conjectured by M. G. Nadkarni (see [CN2]), who proved first (see [N3]) the case when $\operatorname{card}(\mathscr{E}_0(E)) = \operatorname{card}(\mathscr{E}_0(F))$ is countable, on the basis of his result (3.5 in this paper) and Theorem 7.1.

Before we proceed to the proof of 9.1, we would like to mention some corollaries.

First, for each countable Borel equivalence relation E, let

$$c_n(E) := \operatorname{card}\{C \in X/E \colon \operatorname{card}(C) = n\}, \qquad 1 \le n \le \aleph_0;$$

$$s(E) := 0, \quad \text{if } E \text{ is smooth; } 1, \text{ otherwise;}$$

$$t(E) := \operatorname{card}(\mathscr{E}_0(E)).$$

Corollary 9.2. *The sequence $\{\langle c_n\rangle_{1\le n\le\aleph_0}, s, t\}$ is a complete list of invariants for Borel isomorphism of hyperfinite Borel equivalence relations.*

Note that c_n, t can take only the values $0, 1, 2, \dots, \aleph_0, 2^{\aleph_0}$.

Denote next by $\Delta(n)$ the equality relation on a set of cardinality n, $1 \leq n \leq \aleph_0$ and by $E^*(\mathbb{Z}, 2)$ the restriction of $E(\mathbb{Z}, 2)$ to its aperiodic part, i.e., to the $E(\mathbb{Z}, 2)$-invariant Borel set $\{x \in 2^{\mathbb{N}}: s^n(x) \neq x, \forall n \neq 0\}$, where s is the shift map. Let also

$$E \sqsubset^i F :\Leftrightarrow E \sqsubseteq^i F \wedge \neg(F \sqsubseteq^i E) \qquad (\Leftrightarrow E \sqsubseteq^i F \wedge \neg(E \cong F)).$$

Corollary 9.3. *Any aperiodic, nonsmooth hyperfinite Borel equivalence relation is Borel isomorphic to exactly one of*

$$E_t, E_0 \times \Delta(n)(1 \leq n \leq \aleph_0), E^*(\mathbb{Z}, 2).$$

Moreover, we have

$$E_t \sqsubset^i E_0 \sqsubset^i E_0 \times \Delta(2) \sqsubset^i \cdots \sqsubset^i E_0 \times \Delta(\aleph_0) \sqsubset^i E^*(\mathbb{Z}, 2).$$

Proof of 9.3. Recall that E_t is compressible, so $\mathscr{E}_0(E_t) = \varnothing$. It is easy to check that E_0 is uniquely ergodic, i.e., $\text{card}(\mathscr{E}(E_0)) = 1$, and therefore

$$\text{card}(\mathscr{E}_0(E_0 \times \Delta(n))) = n$$

for $1 \leq n \leq \aleph_0$. Finally, $\text{card}(\mathscr{E}_0(E^*(\mathbb{Z}, 2))) = 2^{\aleph_0}$ (take for example the product measures on $2^{\mathbb{Z}}$ with each coordinate having the $(p, 1-p)$-measure, $0 < p < 1$). Since $\mathscr{E}_0(E)$ is a Borel set in a standard Borel space, $\text{card}(\mathscr{E}_0(E))$ is always one of $0, 1, 2, \ldots, \aleph_0, 2^{\aleph_0}$ so the first result follows from 9.1.

That $E_t \sqsubset^i E_0$ follows from 7.2, and $E_0 \times \Delta(n) \sqsubset^i E_0 \times \Delta(m)$ for $n < m$ is obvious. That $E_0 \times \Delta(\aleph_0) \sqsubset^i E^*(\mathbb{Z}, 2)$ will follow immediately from the ergodic decompostion, see 9.5 below. □

For the proof of Theorem 9.1 we will need two further results from ergodic theory: Dye's Theorem (see Dye [D], Sutherland [Su], Weiss [W1], or Hamachi-Osikawa [HO]) and the Ergodic Decomposition Theorem. Let us state first Dye's Theorem.

Dye's Theorem 9.4. *Let E, F be hyperfinite Borel equivalence relations on X, Y resp., and let $\mu \in \mathscr{E}_0(E)$, $\nu \in \mathscr{E}_0(F)$. Then there are invariant Borel sets $X_0 \subseteq X$, $Y_0 \subseteq Y$ such that $\mu(X_0) = \nu(Y_0) = 1$ and $E \restriction X_0 \cong F \restriction Y_0$ via a Borel isomorphism that sends μ to ν.*

Next we will state the Ergodic Decomposition Theorem as formulated in Varadarajan [Va]. (The result actually holds, as proved in [Va], even for Borel actions of arbitrary second countable locally compact groups.)

Let E be a countable Borel equivalence relation on X. Denote by $\mathscr{I}(E)$, $\mathscr{E}(E)$ the sets of E-invariant, resp. E-invariant ergodic, probability measures. Since $E = E_G$ for some Borel action of a countable group G, $\mathscr{I}(E)$ and $\mathscr{E}(E)$ are Borel sets (see [Va]). It is also proved in [Va] that $\mathscr{I}(E) \neq \varnothing$ iff $\mathscr{E}(E) \neq \varnothing$. If E is aperiodic, then clearly $\mathscr{I}(E) = \mathscr{I}_0(E)$, $\mathscr{E}(E) = \mathscr{E}_0(E)$. We have now:

Ergodic Decomposition Theorem 9.5 (Varadarajan [Va]). *Let E be a countable Borel equivalence relation on a standard Borel space X. Assume $\mathscr{I}(E) \neq \varnothing$ (thus $\mathscr{E}(E) \neq \varnothing$). Then there is a Borel surjection $x \mapsto e_x$ from X onto $\mathscr{E}(E)$ such that:*

(1) $xEy \Rightarrow e_x = e_y$.

(2) *If* $X_e := \{x\colon e_x = e\}$ *for any* $e \in \mathscr{E}(E)$, *so that* X_e *is Borel and* E-*invariant by* (1), *then* $e(X_e) = 1$ *and* e *is the unique invariant, ergodic probability mesure for* $E \restriction X_e$ (*i.e.,* $\mathscr{E}(E \restriction X_e) = \{e\}$).

(3) *For each* $\mu \in \mathscr{I}(E)$, *we have*

$$\mu(A) = \int e_x(A)\, d\mu(x)$$

for $A \subseteq X$ *Borel.*

Proof of Theorem 9.1. We consider first the case $\operatorname{card}(\mathscr{E}_0(E)) = \operatorname{card}(\mathscr{E}_0(F)) = 0$. Then by Nadkarni's Theorem 3.5 and the fact that $\mathscr{I}_0(E) = \mathscr{I}_0(F) = \varnothing$, we have that E, F are compressible. Then, by 7.1 and 2.3, $E \cong F$.

So we can assume that $\operatorname{card}(\mathscr{E}_0(E)) = \operatorname{card}(\mathscr{E}_0(F)) > 0$. Since $\mathscr{E}_0(E)$, $\mathscr{E}_0(F)$ are Borel sets in standard Borel spaces and have the same cardinality, there is a Borel bijection $e \mapsto e'$ from $\mathscr{E}_0(E)$ onto $\mathscr{E}_0(F)$. Let $x \mapsto e_x$, $y \mapsto f_y$ be ergodic decompositions of E, F resp. by 9.5. Let X_e, Y_f be the corresponding sets. Then, by Dye's Theorem 9.4, we can find invariant Borel sets $\widehat{X}_e$, $\widehat{Y}_{e'}$ with $\widehat{X}_e \subseteq X_e$, $\widehat{Y}_{e'} \subseteq Y_{e'}$, $e(\widehat{X}_e) = e'(\widehat{Y}_{e'}) = 1$ and Borel isomorphisms h_e of $E \restriction \widehat{X}_e$, $F \restriction \widehat{Y}_{e'}$. Since the proof of Dye's Theorem is "effective" enough, the following uniform version actually holds:

($*$) If $\widehat{X} := \bigcup_e \widehat{X}_e$, $\widehat{Y} := \bigcup_e \widehat{Y}_{e'}$, then $\widehat{X}$, $\widehat{Y}$ are Borel (and of course invariant) and also $h := \bigcup_e h_e$ is Borel (and provides a Borel isomorphism of $E \restriction \widehat{X}$, $F \restriction \widehat{Y}$).

Granting ($*$), we have of course that $E \restriction \widehat{X} \cong F \restriction \widehat{Y}$. Consider now $E \restriction (X \setminus \widehat{X})$, $F \restriction (Y \setminus \widehat{Y})$. By shrinking $\widehat{X}_{e_0}$, $\widehat{Y}_{e_0'}$ for some fixed e_0 if necessary, we can assume that $E \restriction (X \setminus \widehat{X})$, $F \restriction (Y \setminus \widehat{Y})$ are nonsmooth. (We are using here the fact that if R is a nonsmooth countable Borel equivalence relation on Z and $\mu \in \mathscr{E}_0(R)$, then there is a Borel R-invariant set $W \subseteq Z$ with $\mu(W) = 1$ and $R \restriction (Z \setminus W)$ nonsmooth. This follows, for example, from 7.2.) By property (3) of the ergodic decomposition, $E \restriction (X \setminus \widehat{X})$, $F \restriction (Y \setminus \widehat{Y})$ have no invariant probability measures, so by the argument in the beginning of the proof, $E \restriction (X \setminus \widehat{X}) \cong F \restriction (Y \setminus \widehat{Y})$. So $E \cong F$.

We now make some comments on the proof of ($*$). The proof of Dye's Theorem on which it is based (see, e.g., Sutherland [Su] or Hamachi-Osikawa [HO]) proceeds by showing that, if λ is the canonical product measure on $2^{\mathbb{N}}$ (with each coordinate having the $(1/2, 1/2)$-measure), then for any hyperfinite Borel equivalence relation E on X and $\mu \in \mathscr{E}_0(E)$ there are invariant Borel sets $X_0 \subseteq X$, $Y_0 \subseteq 2^{\mathbb{N}}$ such that $\mu(X_0) = \lambda(Y_0) = 1$ and $E \restriction X_0 \cong E_0 \restriction Y_0$ (via a Borel isomorphism sending μ to λ—which is of course the unique element of $\mathscr{E}_0(E_0)$). By going in detail through the proof of Dye's Theorem, one has a parametrized version which in the case we are interested in can be formulated as follows.

Lemma 9.6. *Let* E *be an aperiodic, hyperfinite Borel equivalence relation with* $\mathscr{E}_0(E) \neq \varnothing$. *Let* $x \mapsto e_x$ *be an ergodic decomposition of* E *with* X_e *the corresponding sets. Then there are Borel invariant sets* $\widetilde{X}_e \subseteq X_e$, $\widetilde{Y}_e \subseteq 2^{\mathbb{N}}$ *with* $e(\widetilde{X}_e) = \lambda(\widetilde{Y}_e) = 1$ *and Borel isomorphisms* g_e *of* $E \restriction \widetilde{X}_e$ *with* $E_0 \restriction \widetilde{Y}_e$ *sending* (*necessarily*) e *to* λ *such that* $\widetilde{X} = \bigcup_e X_e$ *is Borel, and* $g\colon \mathscr{E} \times \widetilde{X} \to 2^{\mathbb{N}}$ *given by* $g(e, x) = g_e(x)$ *if* $x \in \widetilde{X}_e$, 0^∞ *if* $x \notin \widetilde{X}_e$ *is Borel as well.*

In earlier handwritten circulated versions of this paper we have carried out the detailed calculations needed to extract this lemma from a proof of Dye's Theorem. Mercifully, however, we found out since then that these have been written up in the literature in Krieger [Kr3], §2 and also [HO], II-4 (in somewhat different formulations), so we will refer the reader to these papers for the detailed proof.

From Lemma 9.6 it is easy to derive now $(*)$: Let $\widetilde{X}_e$, p_e work for E and $\widetilde{Y}_f$, q_f work for F according to Lemma 9.6. Put $p_e[\widetilde{X}_e] \cap q_{e'}[\widetilde{Y}_{e'}] = Z_e \subseteq 2^{\mathbb{N}}$, so that Z_e is E_0-invariant and $\lambda(Z_e) = 1$. Put $\widehat{X}_e = p_e^{-1}[Z_e]$, $\widehat{Y}_{e'} = q_{e'}^{-1}[Z_e]$. Again these are invariant and $e(\widehat{X}_e) = e'(\widehat{Y}_{e'}) = 1$. Finally $h_e = q_{e'}^{-1} \circ (p_e \upharpoonright \widehat{X}_e)$ is a Borel isomorphism of $E \upharpoonright \widehat{X}_e$ with $F \upharpoonright \widehat{Y}_{e'}$. Let $\widehat{X} = \bigcup_e \widehat{X}_e$, $\widehat{Y} = \bigcup_e \widehat{Y}_{e'}$, $h = \bigcup_e h_e$. We will check that $\widehat{X}$, $\widehat{Y}$, h are Borel. Take first $\widehat{X}$ ($\widehat{Y}$ being similar). We have

$$\begin{aligned} x \in \widehat{X} &\Leftrightarrow x \in \widetilde{X} \wedge p_{e_x}(x) \in q_{(e_x)'}[\widehat{Y}_{(e_x)'}] \\ &\Leftrightarrow x \in \widetilde{X} \wedge \exists y[f_y = (e_x)' \wedge y \in \widetilde{Y} \wedge p(e_x, x) = q(f_y, y)] \\ &\Leftrightarrow x \in \widetilde{X} \wedge \exists! y[f_y = (e_x)' \wedge y \in \widetilde{Y} \wedge p(e_x, x) = q(f_y, y)]. \end{aligned}$$

Since the 1-1 projection of a Borel set is Borel, $\widehat{X}$ is Borel. Finally,

$$h(x) = y \Leftrightarrow x \in \widehat{X} \wedge y \in \widehat{Y} \wedge f_y = (e_x)' \wedge p(e_x, x) = q(f_y, y).$$

So h is Borel as well. □

Theorem 9.1 also has the following corollary, which gives a variant of 5.1 (2):

Corollary 9.7. *Let E be a hyperfinite, aperiodic Borel equivalence relation on X. Then for any sequence $\{m_n\}$ with $m_n \geq 1$, $m_{n+1}/m_n \in \mathbb{N}$, $m_n \to \infty$, there is a sequence $\{E_n\}$ of finite Borel equivalence relations such that $E_n \subseteq E_{n+1}$, $\bigcup_n E_n = E$ and every E_n-equivalence class has cardinality m_n.*

Proof. We can assume that $m_{n+1}/m_n \geq 2$. Identify k_n with $\{0, 1, \dots, k_n - 1\}$, and consider the compact product space $Y = \prod k_n \subseteq \mathbb{N}^{\mathbb{N}}$ and the equivalence relation $F = E_0(\mathbb{N}) \upharpoonright Y$. The product measure μ on Y, where each k_n has the uniform measure, is the unique invariant probability measure for F. If now $\text{card}(\mathscr{E}_0(E)) \leq 1$, then $E \cong F \upharpoonright Z$, with Z an F-invariant Borel subset of Y. Since the conclusion of the corollary holds trivially for F (let $xF_n y \Leftrightarrow \forall m \geq n(x_m = y_m)$), it holds for E as well. In the case when $\text{card}(\mathscr{E}_0(E)) > 1$, let Δ be the equality relation on a standard Borel space of cardinality equal to $\text{card}(\mathscr{E}_0(E))$. Then $\text{card}(\mathscr{E}_0(F \times \Delta)) = \text{card}(\mathscr{E}_0(E))$, so $F \times \Delta \cong E$. Again, the conclusion of the corollary is clear for $F \times \Delta$ (using $F_n \times \Delta$), so the proof is complete. □

Choosing a Borel set X_n that meets each E_n-equivalence class at exactly one point in the above corollary, we can find a Borel automorphism $T_n \in [[E]]$ such that X_n, $T_n[X_n]$, $\dots$, $T_n^{m_n - 1}[X_n]$ is a decomposition of X. Moreover, we can of course take $X_1 \supseteq X_2 \supseteq \cdots$, and make sure that for $x \in X_{n+1}$, $T_{n+1}^k(x) = T_n^k(x)$, for $k < m_n$. Thus 9.7 can be viewed as a Rohlin-type lemma for aperiodic hyperfinite Borel equivalence relations. (For a discussion of the classical Rohlin lemma, see for example [W1].)

Let us conclude this section by pointing out that Theorem 9.1, even in the case $\operatorname{card}\mathscr{E}_0(E)) = 1$, can be viewed as a purely descriptive set theoretic version of Dye's Theorem (which of course was one of the key ingredients in its proof). Indeed, given E, F, μ, ν as in Dye's Theorem 9.4, let by the Ergodic Decomposition 9.5 $X_0 \subseteq X$, $Y_0 \subseteq Y$ be invariant Borel sets of measure 1 for the corresponding measures, such that μ is the only E-invariant measure with $\mu(X_0) = 1$ and similarly for Y_0, ν. Of course we can also assume that $E \restriction X_0$, $F \restriction Y_0$ are aperiodic. Thus, by 9.1, $E \restriction X_0 \cong F \restriction Y_0$. But any isomorphism must send μ to ν by the uniqueness of these measures, so we have recovered Dye's Theorem.

10. An example—Lipschitz automorphisms of $2^{\mathbb{N}}$

In order to illustrate some of the ideas involved in §9, we will analyze a class of examples of hyperfinite Borel equivalence relations, namely those induced by Lipschitz automorphisms of $2^{\mathbb{N}}$.

As usual, 2^n denotes the set of binary sequences of length n. Given permutations π, ρ of 2^n, 2^m resp. with $n \leq m$, we write $\pi \leq \rho$ if

$$\rho(s) \restriction n = \pi(s \restriction n)$$

for any $s \in 2^m$. If $\pi_1 \leq \pi_2 \leq \pi_3 \leq \cdots$, where π_n is a permutation of 2^n, then $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $f(x) = \bigcup_n \pi_n(x \restriction n)$ is a homemorphism of $2^{\mathbb{N}}$. (Note that π_n is uniquely determined by f.) These are called the *Lipschitz automorphisms* of $2^{\mathbb{N}}$.

We will analyze a Lipschitz automorphism f in terms of its *orbit tree* T_f. Given n and an orbit $\mathscr{O}$ of π_n on 2^n, say $\mathscr{O} = \{s_1, \dots, s_k\}$ with $\pi_n(s_i) = s_{i+1}$ if $i < k$ and $\pi_n(s_k) = s_1$, we say that an orbit $\mathscr{O}'$ of π_{n+1} on 2^{n+1}, say $\mathscr{O}' = \{s'_1 \dots s'_m\}$, *extends* $\mathscr{O}$ if $s'_i \restriction n \in \mathscr{O}$ for $i = 1, \dots, m$. Then it is easy to check that either $\mathscr{O}$ has one extension $\mathscr{O}'$ of twice the size of $\mathscr{O}$ or else $\mathscr{O}$ has two extensions $\mathscr{O}'$, $\mathscr{O}''$ of the same size as $\mathscr{O}$. (In particular, every orbit $\mathscr{O}$ has size a power of 2.) The tree T_f is the tree consisting of the orbits (of the π_n's), where the children of each orbit of 2^n are its extensions in 2^{n+1}. More precisely, T_f has a root (by convention). The children of the root are the (1 or 2) orbits of 2^1. The children of each one of these are the (1 or 2) orbits of 2^2 extending it, etc. We denote by $[T_f]$ the set of all infinite paths through T_f, i.e., $a \in [T_f]$ iff $a = (\mathscr{O}_1, \mathscr{O}_2, \dots)$ where $\mathscr{O}_n$ is an orbit in 2^n and $\mathscr{O}_{n+1}$ extends $\mathscr{O}_n$.

Given any $x \in 2^{\mathbb{N}}$, denote by a_x the unique $a \in [T_f]$ for which $x \restriction n \in a_x(n)$, $\forall n \geq 1$. Finally, for each $a \in [T_f]$, let

$$X_a = X_a^f := \{x \in 2^{\mathbb{N}} \colon a_x = a\}.$$

Note that X_a is invariant under f. There are two possibilities for each X_a:

(1) From some point on, say n_0, the orbits $a(n)$ have two extensions, so $\operatorname{card}(a(n)) = \operatorname{card}(a(n+1))$ for $n \geq n_0$. Say $\operatorname{card}(a(n_0)) = 2^{m_0}$. Then $\operatorname{card}(X_a) = 2^{m_0}$ and X_a is a finite orbit of f.

(2) For infinitely many n, $a(n)$ has only one extension, $a(n+1)$; thus in particular $\operatorname{card}(a(n+1)) = 2\operatorname{card}(a(n))$. Then X_a is a perfect set.

In case (1), denote by $\mu_a = \mu_a^f$ the unique f-invariant probability measure living on X_a. We claim that in case (2) there is also a unique f-invariant probability measure on X_a which we also denote by $\mu_a = \mu_a^f$: Indeed, let T_a be the tree of the closed set X_a, i.e. $\{s \in 2^n : n \in \mathbb{N}, s \in a(n)\}$. If $s \in T_a \cap 2^n$, define $\mu_a(N_s) = 2^{-p}$, where $p = \text{card}(a(n))$ and $N_s = \{x \in 2^{\mathbb{N}} : x \restriction n = s\}$. For $s \notin T_a$, let $\mu_a(N_s) = 0$. Since clearly we have that $\mu_a(N_s) = \mu_a(N_{s^\frown 0}) + \mu_a(N_{s^\frown 1})$, this defines a probability measure on $2^{\mathbb{N}}$. It is clearly f-invariant and easily unique.

Now put for each $x \in 2^{\mathbb{N}}$, $e_x = e_x^f := \mu_{a_x}^f$.

Proposition 10.1. *The map $x \mapsto e_x$ is an ergodic decomposition for E_f (= the equivalence relation induced by f) whose components are the sets $X_e := X_a$, where $\mu_a = e$.*

Proof. We have to verify the properties stated in 9.5. First we will check that $x \mapsto e_x$ is surjective.

Let $e \in \mathscr{E}(E_f)$. For each orbit $\mathscr{O}$ of some 2^n, let $N_{\mathscr{O}} = \bigcup_{s \in \mathscr{O}} N_s$. Thus $N_{\mathscr{O}}$ is an f-invariant clopen set, so $e(N_{\mathscr{O}}) = 0$ or 1. So e determines a unique path $a \in [T_f]$ with $e(N_{a(n)}) = 1$, $\forall n$. Since $X_a = \bigcap_n N_{a(n)}$, we have $e(X_a) = 1$, so $e = \mu_a$.

Clearly (1), (2) of 9.5 hold. We want now to verify (3). So let μ be f-invariant, $A \subseteq 2^{\mathbb{N}}$ Borel. We want to show $\mu(A) = \int \mu_{a_x}(A)\, d\mu(x)$. For that it is enough to show that $\mu(N_s) = \int \mu_{a_x}(N_s)\, d\mu(x)$ for each $s \in 2^n$, $n \in \mathbb{N}$. Now s belongs to a unique orbit $\mathscr{O}$ of 2^n. So clearly

$$\mu(N_s) = \frac{\mu(N_{\mathscr{O}})}{\text{card}(\mathscr{O})}.$$

Also

$$\begin{aligned}\int \mu_{a_x}(N_s)\, d\mu(x) &= \int_{N_{\mathscr{O}}} \mu_{a_x}(N_s)\, d\mu(x) \\ &= \int_{N_{\mathscr{O}}} \text{card}(\mathscr{O})^{-1}\, d\mu(x) = \frac{\mu(N_{\mathscr{O}})}{\text{card}(\mathscr{O})}. \quad \square\end{aligned}$$

Let us denote now by $C_f(2^n)$ the cardinality of the set $\{a \in [T_f] : \text{card}(X_a) = 2^n\}$ and by $C_f(\infty)$ the cardinality of the set $\{a \in [T_f] : X_a \text{ is perfect}\}$. Finally, for any two Borel automorphisms f, g on standard Borel spaces X, Y resp., let $f \cong g$ iff there is a Borel bijection $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. We now have:

Theorem 10.2. *Let f, g be Lipschitz automorphsims of $2^{\mathbb{N}}$. Then the following are equivalent:*

(1) $f \cong g$;

(2) $E_f \cong E_g$;

(3) $C_f(2^n) = C_g(2^n)$, $C_f(\infty) = C_g(\infty)$ *for all* n;

(4) *f, g have the same cardinality of finite orbits of any given size and the same cardinality of nonatomic, ergodic, invariant probability measures.*

Proof. By 10.1, (3), (4) are equivalent. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (4). We prove now that (3) $\Rightarrow$ (1).

Let P_f, P_g be the sets of periodic points for f, g resp. These are Borel invariant sets and by (3) $f \restriction P_f \cong g \restriction P_g$. Since $K_f := \{a \in [T_f] : X_a^f$ is perfect$\}$ and K_g are Borel and have the same cardinality, there is a Borel bijection $a \mapsto a'$ between them. It is thus enough to find, in a uniform Borel way from a, a Borel isomorphism of $f \restriction X_a^f$ with $g \restriction X_{a'}^g$.

Fix n_0, n_0' such that $\text{card}(a(n_0)) = \text{card}(a'(n_0')) = 2$. Let also π_n, ρ_n be the permutations of 2^n determining f, g resp. Let $n_0 < n_1 < n_2 < \cdots$ be the numbers for which $\text{card}(a(n_k)) = 2^{k+1}$ for the first time, and similarly $n_0' < n_1' < \cdots$ for a'. By induction we can then easily define bijections $T_k \colon a(n_k) \to a'(n_k)$ such that $T_k \circ \pi_{n_k} = \rho_{n_k'} \circ T_k$ and $T_l(s) \restriction k = T_k(s \restriction k)$ for $l \geq k$, $s \in 2^l$. Then the map $x \mapsto \bigcup_k T_k(x \restriction n_k)$ is a Borel isomorphism of $f \restriction X_a^f$ and $g \restriction X_{a'}^g$. □

In the preceding result, one actually obtains a classification of Lipschitz automorphisms themselves up to Borel isomorphisms. The general problem of classifying arbitrary Borel automorphisms up to Borel isomorphism is open (see, e.g., Weiss [W2]).

It is clear that $E_0 = E_f$, "except on eventually constant sequences," where f has an orbit tree of the form o–o–o–o $\cdots$, i.e., every orbit has only one extension. It is easy to construct orbit trees of aperiodic f, which have any prescribed cardinality $c_f(\infty) \in \{1, 2, \ldots, \aleph_0, 2^{\aleph_0}\}$, and thus represent any noncompressible, aperiodic, nonsmooth hyperfinite E up to Borel isomorphism by such an E_f.

11. Miscellanea

We would like to collect here various remarks related to the results in this paper and other work in the literature.

(1) Let E be a hyperfinite Borel equivalence relation on X and $\mu \in \mathscr{E}\mathscr{I}_0(E)$. Then it is well known (see e.g. Weiss [W1, p. 93] or Schmidt [S1, 8.15]) that there is $\nu \in \mathscr{E}\mathscr{I}_0(E_0)$ and a Borel isomorphism π of $E \restriction X_0$, $E_0 \restriction Y_0$ where X_0, Y_0 are invariant Borel sets with $\mu(X_0) = \nu(Y_0) = 1$, such that $\pi\mu = \nu$.

This can also be seen as follows: If there is $\mu' \sim \mu$ with $\mu' \in \mathscr{I}_0(E)$, then we are done by Dye's Theorem. Otherwise, by the Hopf Theorem (see, e.g., [Fr, 3.2]), there is Borel E-invariant X_0 of μ-measure 1 on which $E \restriction X_0$ is compressible. Then, by Theorem 7.1 and Proposition 2.3, $E \restriction X_0 \cong E_0 \restriction Y_0$, where Y_0 is an E_0-invariant Borel subset of $2^{\mathbb{N}}$. If π is the Borel isomorphism, put $\nu = \pi\mu$.

(2) If G is a countable group and $g \cdot x$ a Borel action of G on a standard Borel space X, we say that the action is *free* if $\forall x \in X \forall g \in G (g \neq 1 \Rightarrow g \cdot x \neq x)$.

There are countable Borel E which cannot be represented as E_G for a free action of a countable group G; see Adams [A]. However, it is not known whether, for any countable Borel E and any $\mu \in \mathscr{E}\mathscr{I}_0(E)$, we can write $E \restriction X = E_G$ for a free Borel action of a group G on a Borel invariant set X with $\mu(X) = 1$ (ergodicity of μ is important here by the example in [A]). We also do not know whether any compressible countable Borel equivalence relation can be represented as E_G for a free action of a countable group G. Also, even if a compressible E can be so represented, it is not clear for what countable groups G we can write E as E_G for a free action of G. Here is one relevant fact (whose proof uses an argument due to Mackey).

Proposition 11.1. *Let E be a compressible countable Borel equivalence relation. If $E = E_G$ for a free Borel action of a group G and $G \subseteq H$ (H countable), then $E = E_H$ for a free Borel action of H.*

Proof. On the space $X \times H$ consider the following action of G:

$$g \cdot (x, h) = (g \cdot x, gh).$$

Denote by $\sim_G$ the corresponding equivalence relation. Note that $(x, h) \mapsto h$ is injective in each $\sim_G$-equivalence class, so $\sim_G$ is smooth and thus we can consider $(X \times H)/\sim_G$ as a standard Borel space. Consider then the following action of H on $(X \times H)/\sim_G$:

$$f \cdot [(x, h)]_{\sim_G} = [(x, hf^{-1})]_{\sim_G}.$$

It is easy to check that it is free. Now $T(x) = [(x, 1)]_{\sim_G}$ is injective and $xEy \Leftrightarrow T(x) \sim_H T(h)$, where $\sim_H$ is the equivalence relation induced by the action of H on $(X \times H)/\sim_G$. Thus $E \sqsubseteq \sim_H$ so, as E is compressible, $E \sqsubseteq^i \sim_H$ and hence E is induced by a free Borel action of H. $\square$

In the case where E is compressible hyperfinite, one can actually induce E by a free action of an arbitrary infinite group.

Proposition 11.2. *Let E be a compressible hyperfinite Borel equivalence relation. Let G be any infinite countable group. Then E is of the form E_G for a free Borel action of G.*

Proof. Consider $E(G, 2)$ and let $X \subseteq 2^G$ be the free part of the action of G on 2^G, i.e., $X = \{x \in 2^G : \forall g \in G(g \neq 1 \Rightarrow g \cdot x \neq x)\}$. First it is easy to check that X is a dense G_δ. Since $E(G, 2)$ is generically ergodic (see §6), it follows that $E(G, 2) \upharpoonright X$ is not smooth, so by 7.1 $E \sqsubseteq E(G, 2) \upharpoonright X$, so $E \sqsubseteq^i E(G, 2) \upharpoonright X$, thus E can be induced by a free Borel action of G. $\square$

This fact provides an affirmative answer to a question mentioned in [S2, p. 16]: Does every countable infinite group have a free, nonsingular, ergodic hyperfinite action? In our terminology, this asks whether for any infinite countable G there is a hyperfinite $E = E_G$ induced by a free Borel action of G, which has an ergodic, quasi-invariant measure. Since the existence of such a measure is equivalent to the nonsmoothness of E, this follows from 11.2.

(3) Our final remarks deal with the concept of a generator for a Borel automorphism. Given a standard Borel space X and a Borel automorphism T of X, a *generator* for T is a partition $\{A_i\}_{i \in I}$ of X into Borel sets such that $\{T^n[A_i]\}_{i \in I, n \in \mathbb{Z}}$ generates the Borel sets, i.e., the Borel sets form the smallest σ-algebra containing A_i and closed under T, T^{-1}. Weiss [W3] has shown that every aperiodic T has a countable generator (i.e. $I = \mathbb{N}$) modulo smooth sets (i.e., $\{T^n[A_i]\}$ generates the Borel sets modulo smooth sets), and in fact $\{T^n[A_i]\}_{i \in \mathbb{N}, n \in \mathbb{N}}$ suffices (this is sometimes called a *strong* generator). We can prove here (by a different method) the following:

Proposition 11.3. *Let T be a Borel automorphism of a standard Borel space X. Then T has a countable generator iff T has only countably many finite orbits.*

Proof. If T has a countable generator $\{A_i\}_{i \in \mathbb{N}}$, then the map $\varphi : X \to \mathbb{N}^{\mathbb{Z}}$ given by $\varphi(x)(i) = n \Leftrightarrow T^i(x) \in A_n$ is a Borel injection, and if s is the shift on $\mathbb{N}^{\mathbb{Z}}$ then $\varphi \circ T = s \circ \varphi$. So φ shows that T is Borel isomorphic to the restriction of s onto a Borel invariant subset of $\mathbb{N}^{\mathbb{Z}}$. Since s on $\mathbb{N}^{\mathbb{Z}}$ has only countably many finite orbits, so does T.

Conversely, assume T has only countably many finite orbits. Let $X_0 = \{x \in X : \text{the orbit of } x \text{ is finite}\}$, $X_1 = X \setminus X_0$. Clearly $T \restriction X_0$ has a countable generator. So it is enough to show that $T \restriction X_1$ has a countable generator. By 9.3, if $E_1 = E_{T \restriction X_1}$, the equivalence relation induced by T on X_1, then there is a shift-invariant Borel set $Y_1 \subseteq 2^{\mathbb{Z}}$ with $E_1 \cong E \restriction Y_1$, where $E = E(\mathbb{Z}, 2)$. Note that the shift on $2^{\mathbb{Z}}$ (and thus its restriction to Y_1) has a 2-generator (i.e., a generator with $I = 2$). It will therefore be enough to prove the following lemma.

Lemma 11.4. *Let Z be a standard Borel space and V, U two Borel automorphisms of Z. If V, U induce the same equivalence relation E and U has a countable generator, then V has a countable generator.*

Proof. Partition Z into Borel sets $\{B_i\}_{i \in \mathbb{N}}$ such that $U \restriction B_i = V^{N(i)} \restriction B_i$ and $U^{-1} \restriction B_i = V^{M(i)} \restriction B_i$, for some $N, M \colon \mathbb{N} \to \mathbb{N}$. Consider a generator $\{A_n\}_{n \in \mathbb{N}}$ for U and the countable partition $\{C_k\}_{k \in \mathbb{N}} = \{B_i \cap A_n\}_{i,n \in \mathbb{N}}$. It is enough to show that $U^j[A_n]$ belongs to the smallest σ-algebra $\mathscr{B}$ containing all the C_k and closed under V, V^{-1}. Without loss of generality, assume $j \geq 0$, and proceed by induction on j. For $j = 0$, $U^0[A_n] = A_n = \bigcup_i (B_i \cap A_n)$. Assume $U^j[A_n] = B \in \mathscr{B}$. Then $U^{j+1}[A_n] = U[B] = U[\bigcup_i (B \cap B_i)] = \bigcup_i U[B \cap B_i] = \bigcup_i V^{N(i)}[B \cap B_i] = \bigcup_i [V^{N(i)}[B] \cap \bigcup_n V^{N(i)}[B_i \cap A_n]]$. □

Note that by the preceding, if T is any aperiodic Borel automorphism, there is a Borel automorphism T' generating the same equivalence relation which has a 2-generator. As pointed out in Weiss [W3, p. 324], there are aperiodic T which have invariant probability measures with no finite generators. But, as mentioned in the same paper, it is not known whether every T with no invariant probability measures has a 2-generator.

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