TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 341, Number 1, January 1994

# REPRESENTATIONS OF THE SYMMETRIC GROUP IN DEFORMATIONS OF THE FREE LIE ALGEBRA

A. R. CALDERBANK, P. HANLON, AND S. SUNDARAM

ABSTRACT. We consider, for a given complex parameter $\alpha$, the nonassociative product defined on the tensor algebra of $n$-dimensional complex vector space by $[x, y] = x \otimes y - \alpha y \otimes x$. For $k$ symbols $x_1, \dots, x_k$, the left-normed bracketing is defined recursively to be the bracketing sequence $b_k$, where $b_1 = x_1$, $b_2 = [x_1, x_2]$, and $b_k = [b_{k-1}, x_k]$. The linear subspace spanned by all multilinear left-normed bracketings of homogeneous degree $n$, in the basis vectors $v_1, \dots, v_n$ of $\mathbb{C}^n$, is then an $S_n$-module $V_n(\alpha)$. Note that $V_n(1)$ is the Lie representation $\mathrm{Lie}_n$ of $S_n$ afforded by the $n$th-degree multilinear component of the free Lie algebra. Also, $V_n(-1)$ is the subspace of simple Jordan products in the free associative algebra as studied by Robbins [Ro]. Among our preliminary results is the observation that when $\alpha$ is not a root of unity, the module $V_n(\alpha)$ is simply the regular representation.

Thrall [T] showed that the regular representation of the symmetric group $S_n$ can be written as a direct sum of tensor products of symmetrised Lie modules $V_\lambda$. In this paper we determine the structure of the representations $V_n(\alpha)$ as a sum of a subset of these $V_\lambda$. The $V_\lambda$, indexed by the partitions $\lambda$ of $n$, are defined as follows: let $m_i$ be the multiplicity of the part $i$ in $\lambda$, let $\mathrm{Lie}_i$ be the Lie representation of $S_i$, and let $\iota_k$ denote the trivial character of the symmetric group $S_k$. Let $\iota_{m_i}[\mathrm{Lie}_i]$ denote the character of the wreath product $S_{m_i}[S_i]$ of $S_{m_i}$ acting on $m_i$ copies of $S_i$. Then $V_\lambda$ is isomorphic to the $S_n$-module

$$\left(\iota_{m_1}[\mathrm{Lie}_1] \otimes \cdots \otimes \iota_{m_i}[\mathrm{Lie}_i] \otimes \cdots\right) \uparrow_{S_{m_1}[S_1] \times \cdots \times S_{m_i}[S_i] \times \cdots}^{S_n} .$$

Our theorem now states that when $\alpha$ is a primitive $p$th root of unity, the $S_n$-module $V_n(\alpha)$ is isomorphic to the direct sum of those $V_\lambda$, where $\lambda$ runs over all partitions $\lambda$ of $n$ such that no part of $\lambda$ is a multiple of $p$.

## INTRODUCTION

This work is inspired by a recent renewal of interest in the free Lie algebra (see [R, G]). The papers [BS, Su], consider $S_n$-submodules generated by a specified sequence of bracketings. In this paper we consider a deformation of the free Lie algebra, defined for a complex parameter $\alpha$, by setting the bracket of $x$ and $y$ to be $[x, y] = xy - \alpha yx$. Let $J_\alpha$ denote the complex vector space spanned by all possible bracketings on some set of generators $\{x_i\}$; it is then a free nonassociative algebra with this bracket, on the generators $\{x_i\}$. Note

Received by the editors March 13, 1991 and, in revised form, October 19, 1991.
1991 *Mathematics Subject Classification*. Primary 17B10; Secondary 20C05, 05E10.
*Key words and phrases*. Deformations, free Lie algebra, representation, Poincaré-Birkhoff-Witt theorem.

that $J_1$ is simply the free Lie algebra, while $J_{-1}$ is a free Jordan algebra. Suppose $J_\alpha$ is freely generated by $n$ generators $x_1, \dots, x_n$. This paper studies the subspace spanned by multilinear left-normed bracketings of homogeneous degree $n$, corresponding to permutations of the generators $x_1, \dots, x_n$. In the terminology of Garsia [G], this is the subspace generated by bracketings corresponding to the left comb $L_n$, and may be explicitly defined as the span $V_n(\alpha)$ of all bracketings of the form $[\dots[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}], \dots], x_{\sigma(n-1)}], x_{\sigma(n)}]$, where $\sigma$ runs through all permutations in the symmetric group $S_n$. There is an obvious left action of $S_n$ on $V_n(\alpha)$.

The subspace $V_n(-1)$ of the free Jordan algebra on $n$ generators has already been considered in the work of David Robbins [Ro], who calls the elements of $V_n(-1)$ *simple* Jordan elements.[1] Our result on the structure of $V_n(\alpha)$ (Theorem 2.1) generalises Robbins' main theorem, which states that a basis for the space of simple Jordan elements (as a subspace of the free associative algebra) is obtained by taking symmetrised products of basis elements of the odd-degree subspace of the free Lie algebra [Ro, Theorem 6, p. 365]. Robbins then deduces formulas for the dimension and Frobenius characteristic of $V_n(-1)$ as an $S_n$-module [Ro, §10].

In §1, we use the observation that $V_n(\alpha)$ can be realised as a principal left ideal in the group algebra $\mathbf{C}S_n$ to compute the dimension of this representation, and in particular to show that these $\alpha$-deformations of the Lie bracket yield interesting representations only at roots of unity (Proposition 1.1). This fact suggests that there may be a connection with Hecke algebras at roots of unity, and therefore, by Schur-Weyl duality, with quantum groups.

The next section contains our main result describing the module $V_n(\alpha)$ as a direct sum of certain symmetrised Lie modules. The main tools used are an explicit basis of orthogonal idempotents for the group algebra found by Garsia and Reutenauer (see [GR]), and various bases for the descent algebra of the symmetric group. Finally, in §3 we use these results to compute a generating function for the character values of the representations $V_n(\alpha)$. We show that, in the case $\alpha = -1$, these characters are plethystic inverses of certain characters arising in the work of Calderbank, Hanlon, and Robinson [CHR], on the top homology of the lattice of partitions of $n$ into blocks of odd size.

## 1

We recall briefly some facts about the representation of $S_n$ afforded by the free Lie algebra on $n$ generators. Write $\mathrm{Lie}_n$ for this representation. It is well known that $\mathrm{Lie}_n$ can be realised in the group algebra of $S_n$ as the principal left ideal generated by the Dynkin element $\theta_n$ (see [J, G, Su]). The element $\theta_n$ is defined to be the expression $(1-\gamma_2)(1-\gamma_3)\cdots(1-\gamma_n)$, where $\gamma_i(1) = i$, $\gamma_i(j) = j-1$ for $j = 2, \dots, i$, and $\gamma$ fixes all letters greater than $i$. For the sake of completeness we review some facts which are discussed more extensively in [G]. Recall Garsia's observation that in the free Lie algebra, a particular sequence of bracketings of $n$ elements can be encoded by a labelled binary tree with $n$ leaves, the encoding being defined inductively as follows:

- the binary tree on two leaves, with left child labelled $x$ and right child labelled $y$, encodes the bracketing $[x, y] = xy - yx$;

[1] S. Sundaram would like to thank Michel Racine for bringing this reference to her attention.

- let $w(T)$ denote the bracketing encoded by a labelled tree $T$. If $T$ is a binary tree with $n$ leaves, with left subtree $T_1$ and right subtree $T_2$, then the bracketing encoded by $T$ is defined recursively by $w(T) = [w(T_1), w(T_2)]$.

Consider now trees whose leaves are labelled consecutively from left to right with the labels $1, \ldots, n$. We claim that the Dynkin element $\theta_n$ is in fact the expansion of the bracketing encoded by a particular tree $L_n$ (called the left comb in [G]), equipped with this canonical labelling, in the group algebra of $S_n$. The left comb $L_2$ is defined to be the unique binary tree on two leaves, while $L_n$ is defined recursively as the binary tree on $n$ leaves whose left subtree is the left comb $L_{n-1}$ and whose right subtree is a single leaf. It is clear that the expansion of the bracketing $w(L_n)$ of the left comb in the group algebra of $S_n$ satisfies the identity $w(L_n) = w(L_{n-1})(1-\gamma_n)$, where $\gamma_n$ is the $n$-cycle defined in the preceding paragraph. From the definition of the Dynkin element, it follows immediately that $w(L_n) = \theta_n$. The fact that this generates the entire free Lie algebra is now a simple consequence of the Jacobi identity, which, recast in terms of trees, shows that any bracketing can be written as a linear combination of suitably labelled left combs. (For a more precise statement see [Su, Proposition 2.1].)

This encoding obviously extends to our generalised bracketings, and it is clear that the representation $V_n(\alpha)$ in which we are interested may also be realised as a principal left ideal whose generator $\eta_n(\alpha)$ is the one obtained as follows: expand the $\alpha$-bracketing associated with the left comb labelled with $1, 2, \ldots, n$ from left to right. Thus one has

$$\eta_n(\alpha) = (1-\alpha\gamma_2)(1-\alpha\gamma_3)\cdots(1-\alpha\gamma_n),$$

the $\gamma_i$ being defined as above. (In particular, $\eta_n(1) = \theta_n$.)

Recall that the descent set of a permutation $\sigma$ in $S_n$ is defined to be $\{i : \sigma(i) > \sigma(i+1), 1 \le i \le n-1\}$. For any subset $S$ of $\{1, \ldots, n-1\}$, following [G] we denote by $D_{=S}$ the sum in the group algebra of the permutations in $S_n$ whose descent set equals $S$, with the convention that $D_\varnothing = 1$. Let $D_{=[1,k]}$ denote the sum (in the group algebra) of all the permutations in $S_n$ whose descent set equals the set $\{1, 2, \ldots, k\}$, for $k = 1, \ldots, n-1$. Then it is easy to see that

$$\eta_n(\alpha) = \sum_{k=0}^{n-1} (-\alpha)^k D_{=[1,k]}.$$

That is, the generator $\eta_n(\alpha)$ is in fact in the descent algebra of the symmetric group. This remark will be crucial in the next section.

In the rest of this section we derive some preliminary results about the representation $V_n(\alpha)$. Our first easy observation is the following.

**Proposition 1.1.** *If $\alpha$ is not an ith root of unity, for $1 \le i \le n$, the $S_n$-module $V_n(\alpha)$ is isomorphic to the regular representation.*

*Proof.* For each $i = 1, \ldots, n$, we have $(1-\alpha\gamma_i)(\sum_{k=0}^{i-1} \alpha^k {\gamma_i}^k) = 1-\alpha^i$. Hence, under the hypothesis, each factor in the product expression for $\eta_n(\alpha)$ is invertible in the group algebra, and consequently so is $\eta_n(\alpha)$. The result follows. □

For the remainder of this section we shall assume that $\alpha$ is a $p$th root of unity for some $p$. The argument of Proposition 1.1 in fact gives the following stronger result:

**Proposition 1.2.** *Let $n$ be an integer which is not a multiple of $p$. Then $V_n(\alpha)$ is isomorphic to the induced module $V_{n-1}(\alpha)\uparrow_{S_{n-1}}^{S_n}$.*

*Proof.* One has the following relation between the generators in the group algebra of $S_n$:

$$\eta_n(\alpha) = \eta_{n-1}(\alpha)(1-\alpha\gamma_n).$$

Since $n$ is not a multiple of $p$, the factor $(1-\alpha\gamma_n)$ is invertible in the group algebra $\mathbf{C}S_n$, and consequently it follows that the canonical projection from the left ideal generated by $\eta_{n-1}(\alpha)$ onto the left ideal generated by $\eta_n(\alpha)$ is an isomorphism. □

It follows that if $\alpha$ is a $p$th root of unity, then for $i=1,\ldots,p-1$ the dimension of $V_{kp+i}(\alpha)$ is $(kp+i)(kp+i-1)\cdots(kp+1)$ times the dimension of $V_{kp}(\alpha)$. The next result will determine these dimensions completely.

**Proposition 1.3.** *The restriction of $V_{kp}(\alpha)$ to $S_{kp-1}$ is isomorphic to the induced module $V_{kp-1}(\alpha)\downarrow_{S_{kp-2}}\uparrow^{S_{kp-1}}$. In particular the dimension of $V_{kp}(\alpha)$ is $(kp)!\prod_{i=1}^{k}\frac{ip-1}{ip}$.*

*Proof.* We shall need some classical properties of the Dynkin element $\theta_n$ which generates a left ideal isomorphic to the representation of $S_n$ on the free Lie algebra.

A result of Garsia [G, Theorem 2.1(iii)] states that

$$\theta_n D_{=S} = (-1)^{|S|}\theta_n,$$

for any subset $S$ of $\{1,\ldots,n-1\}$.

This allows us to compute (using the remarks at the beginning of this section)

$$\theta_{kp}\eta_{kp}(\alpha) = \theta_{kp}\sum_{i=0}^{kp-1}(-\alpha)^i D_{=[1,i]} = \theta_{kp}\sum_{i=0}^{kp-1}\alpha^i = 0,$$

the last equality being a consequence of the fact that $\alpha$ is a $p$th root of unity.

The idea of the proof is the same as the one used in [Su, Theorem 4.10]. It is convenient to think of the space $V_{kp}(\alpha)$ as the span of bracketings $\sigma(L_{kp})$ obtained by applying permutations $\sigma$ to the left comb $L_{kp}$, that is, as left combs with $kp$ leaves which are labelled $\sigma(1),\ldots,\sigma(kp)$ from left to right. (Note that the expansion of $\sigma(L_{kp})$ coincides with the product $\sigma\,\eta_{kp}(\alpha)$.) Let $W$ denote the subspace spanned by the labelled left combs $\tau(L_{kp})$, where $\tau$ ranges over all permutations such that $\tau(kp)=kp$. Let $G=S_{\{2,\ldots,kp\}}$ denote the subgroup of $S_{kp}$ which fixes '1'; let $H$ be the subgroup of $G$ which fixes '$kp$'. We consider the action of $G$ on the left ideal $V=\mathbf{C}\,\eta_{kp}(\alpha)$ and on $W$. Observe that

(i) there is an obvious $S_{kp-1}$-isomorphism between $W$ and $V_{kp-1}(\alpha)$;

(ii) $V=\bigoplus_{i=2}^{kp}(i,kp)W$. It is easy to see that the sum on the right is indeed direct: since $W$ must contain a basis consisting of left combs $L_{kp}$ labelled by permutations $\sigma$ which fix $kp$, any element $x$ of $(i,kp)W$ is distinguished by the fact that, in the expansion of $x$ (or in fact in the expansion of $(i,kp)\sigma(L_{kp})$) all permutations beginning with $kp$ must end in $i$. The equality of spaces follows by examining the identity $\theta_{kp}\,\eta_{kp}(\alpha)=0$. Rewriting this as a linear combination of permutations applied to the left comb $L_{kp}$, we have an identity of the form $\sum(\pm 1)\sigma(L_{kp}) = 0$, where the sum ranges over all permutations $\sigma$ appearing in the expansion of $\theta_{kp}$. Now observe that $\theta_n$ always contains a unique permutation $\tau$ such that $\tau(n) = 1$. It therefore follows that $V \subseteq \sum_{i=2}^{kp}(i, kp)W$. Since the reverse inclusion is clear, we have $V = \bigoplus_{i=2}^{kp}(i, kp)W$.

(iii) The subgroup of $G$ which leaves $W$ invariant is $H$. Hence the group $G$ transitively permutes the spaces $(i, kp)W$, for $i = 2, \ldots, kp$, among themselves, so clearly we have, as representations of $G$, that $V$ is isomorphic to $W\uparrow_H^G$. The result follows. □

Using Garsia's identity also gives the following

**Proposition 1.4.** *Suppose $\alpha$ is a pth root of unity, and $n$ an integer such that $p$ does not divide $n$. Then $\theta_n \in \mathbb{C}S_n\eta_n(\alpha)$.*

## 2

The main theorem of this section, Theorem 2.1, establishes the structure of the modules $V_n(\alpha)$ as a sum of symmetrised Lie modules.

For each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ of $n$ (we write $\lambda \vdash n$), let $W_\lambda(z)$ be the polynomial

$$W_\lambda(z) = \frac{\prod_{i=1}^{l}(1 - z^{\lambda_i})}{1 - z}.$$

Let $V_\lambda$ denote the $S_n$-module

$$(\iota_{m_1}[\mathrm{Lie}_1] \otimes \cdots \otimes \iota_{m_i}[\mathrm{Lie}_i] \otimes \cdots)\uparrow_{S_{m_1}[S_1]\times\cdots\times S_{m_i}[S_i]\times\cdots}^{S_n}.$$

Here $m_i$ is the multiplicity of the part $i$ in $\lambda$, $\mathrm{Lie}_i$ is the Lie representation of $S_i$, $\iota_k$ denotes the trivial character of the symmetric group $S_k$, while $\iota_{m_i}[\mathrm{Lie}_i]$ denotes the character of the wreath product $S_{m_i}[S_i]$ of $S_{m_i}$ acting on $m_i$ copies of $S_i$.

The induced modules $V_\lambda$ first appear in a paper of Thrall [T], who essentially showed that the regular representation of $S_n$ decomposes as a direct sum of the modules $\bigoplus_{\lambda\vdash n} V_\lambda$, where the sum runs over all partitions of $n$. (Viewed in terms of the general linear group $GL(V)$, this gives a decomposition of the tensor space $V^{\otimes n}$ which is simply a $GL(V)$-equivariant reformulation of the Poincaré-Birkhoff-Witt theorem for the free Lie algebra.)

**Theorem 2.1.** *Assume $\alpha \neq 1$. The module $V_n(\alpha)$ is isomorphic to the direct sum of those $V_\lambda$ for which $W_\lambda(\alpha)$ is nonzero. In particular, $V_n(\alpha)$ is the regular representation unless $\alpha$ is a pth root of unity for some $p \leq n$. If $\alpha$ is a primitive pth root of unity, $V_n(\alpha)$ is isomorphic to the direct sum $\bigoplus_{\lambda\vdash n\,:\,\lambda_i\not\equiv 0(p)} V_\lambda$.*

The proof of Theorem 2.1 is based on recent work of Garsia and Reutenauer [GR]. To describe their results, we first define a composition $p$ of $n$, written $p \models n$, to be a sequence of positive integers $p = (p_1, \ldots, p_l)$ whose sum is $n$. For instance $(2, 1, 1)$ and $(1, 2, 1)$ are two distinct compositions of 4. If $p \models n$ we write $\Lambda(p)$ for the unique partition of $n$ determined by rearranging the parts of $p$ in weakly decreasing order. (So $\Lambda(1, 1, 2) = (2, 1, 1)$.) If $\lambda$ is a partition write $m_i(\lambda)$ for the number of times $i$ occurs as a part of $\lambda$. Also write $s(\lambda)$ for the product $\prod_i m_i(\lambda)!$.

The descent set of a permutation $\sigma \in S_n$ was defined in the preceding section. Recall that we wrote $D_{=S}$ for the sum of all permutations in $S_n$ with descent set $S$. Let $\Sigma_n$ denote the linear span of the $D_{=S}$. A theorem of Solomon [So] asserts that $\Sigma_n$ is closed under multiplication in the group algebra, hence is a subalgebra of $\mathbf{C}S_n$ of dimension $2^{n-1}$. In their paper, Garsia and Reutenauer construct some remarkable bases for the descent algebra $\Sigma_n$. Following the notation in [GR], given a composition $p = (p_1, \dots, p_{l-1}) \models n$, we write $S(p)$ for the subset of $\{1, 2, \dots, n-1\}$ defined by $S(p) = \{p_1, p_1+p_2, \dots, p_1+p_2+\dots+p_{l-1}\}$. The map $p \mapsto S(p)$ is clearly a bijection between compositions of $n$ and subsets of $\{1, \dots, n-1\}$. Also the largest part of $S(p)$ is $M(p) = n - p_l$.

The following partial order on the set of compositions of $n$ is defined in [GR]: Let $p = (p_1, \dots, p_l)$ and $(q_1, \dots, q_m)$ be two compositions of $n$. We say $p$ is a refinement of $q$, written $p \propto q$, if there are integers $0 < a_1 < a_2 < \dots < a_m = l$ such that

$$\begin{aligned} q_1 &= p_1 + \dots + p_{a_1}, \\ q_2 &= p_{a_1+1} + \dots + p_{a_2}, \\ &\vdots \\ q_m &= p_{a_{m-1}+1} + \dots + p_{a_m}. \end{aligned}$$

If $p \propto q$ we define $F(p, q) = a_1!(a_2 - a_1)! \cdots (a_m - a_{m-1})!$.

Two bases of the descent algebra, indexed by compositions of $n$, are now defined in [GR] as follows:

$$B_p = D_{\subseteq S(p)}, \qquad B_q = \sum_{p \propto q} \frac{1}{F(p, q)} I_p.$$

Note that by triangularity the second equation uniquely determines the basis of $I_p$'s.

We now state one of the key results of [GR].

**Theorem 2.2** [GR, Theorems 3.1, 4.4]. *There exists a set $\{E_\lambda\}_{\lambda \vdash n}$ of pairwise orthogonal idempotents in the descent algebra $\Sigma_n$ which decompose the identity, and hence $\mathbf{C}S_n = \bigoplus_{\lambda \vdash n} \mathbf{C}S_n E_\lambda$. These idempotents have the following properties:*

(a) *The left ideal $\mathbf{C}S_n E_\lambda$ generated by $E_\lambda$ is isomorphic to the $S_n$-module $V_\lambda$ of Theorem 2.1, i.e., to*

$$(\iota_{m_1}[\mathrm{Lie}_1] \otimes \dots \otimes \iota_{m_i}[\mathrm{Lie}_i] \otimes \cdots) \uparrow_{S_{m_1}[S_1] \times \dots \times S_{m_i}[S_i] \times \cdots}^{S_n}.$$

(b) *Choose an ordered basis for each left ideal $\mathbf{C}S_n E_\lambda$ and a total ordering of the partitions of $n$ which extends reverse lexicographic order. One now has a basis for the group algebra with respect to which the matrix for right multiplication by $I_q$ is block upper triangular (the blocks correspond to the subspaces $\mathbf{C}S_n E_\lambda$) with diagonal block $s(\Lambda(q))I$, where $I$ is the identity matrix for the block corresponding to $\mathbf{C}S_n E_{\Lambda(q)}$.*

The next corollary follows immediately.

**Corollary 2.3.** *Let $z \in \Sigma_n$, $z = \sum_{q \models n} a_q I_q$. Let $R_z$ denote the matrix for right multiplication by $z$ in the group algebra. Then $R_z$ is semisimple. Let*

$\psi_\lambda = s(\lambda)\sum_{q\models n\,:\,\Lambda(q)=\lambda} a_q$. *Then for each eigenvalue* $y$ *of* $R_z$, *the eigenspace of* $R_z$ *corresponding to* $y$, *as a left* $S_n$*-module, is isomorphic to* $\sum_{\lambda\vdash n\,:\,\psi_\lambda=y} V_\lambda$.

**Proof of Theorem 2.1.** We observed in §1 that $\eta_n(\alpha)\in\Sigma_n$. We apply Corollary 2.3 to the element $z=(1-\alpha)\eta_n(\alpha)$. We are going to show that the eigenvalues of $R_z$ are given by

$$(1-\alpha)W_\lambda(\alpha)=\prod_i(1-\alpha^{\lambda_i}),$$

for each partition $\lambda$ of $n$. (See also [Ro, Theorem 8], which is the special case of this result when $\alpha=-1$.) Since right multiplication by $R_z$ commutes with the left action in $\mathbf{C}S_n$, and, by the corollary, the matrix $R_z$ is semisimple, it follows that the module $V_n(\alpha)$ is a direct sum of those eigenspaces of $W_n(\alpha)$ corresponding to the nonzero eigenvalues. This clearly proves Theorem 2.1.

To compute the eigenvalues we need to write $z$ in terms of the basis $I_q$. Suppose $z=\sum_{q\models n} a_qI_q$. It suffices to show that, for each partition $\lambda=(\lambda_1,\dots,\lambda_l)$ of $n$,

$$s(\lambda)\sum_{q\models n\,:\,\Lambda(q)=\lambda} a_q=\prod_{i=1}^{l}(1-\alpha^{\lambda_i}). \tag{1}$$

Write $D_{\subseteq S}=\sum_{T\subseteq S}D_{=T}$. By inclusion-exclusion, $D_{=S}=\sum_{T\subseteq S}(-1)^{|S|-|T|}D_{\subseteq T}$. From the remarks in §1, one has

$$\begin{aligned}
z&=(1-\alpha)\sum_{k=0}^{n-1}(-\alpha)^kD_{=[1,k]}=(1-\alpha)\sum_{k=0}^{n-1}(-\alpha)^k\sum_{T\subseteq[1,k]}(-1)^{k-|T|}D_{\subseteq T}\\
&=(1-\alpha)\sum_T(-1)^{|T|}D_{\subseteq T}\sum_{i\geq\max\,T}\alpha^i\\
&=(1-\alpha)\sum_{q=(q_1,\dots,q_l)\models n}(-1)^{l-1}B_q\left(\sum_{i=n-q_l}^{n-1}\alpha^i\right)\quad\text{by definition of the basis } B_q
\end{aligned}$$

(recall that the largest element of $S(q)$ is $n-q_l$, and that $|S(q)|=l-1$).

$$\begin{aligned}
&=\sum_{q=(q_1,\dots,q_l)\models n}(-1)^{l-1}B_q(\alpha^{n-q_l}-\alpha^n)\\
&=\sum_{p\models n}\sum_{q\,:\,p\propto q}\frac{(-1)^{l(q)-1}}{F(p,q)}I_p(\alpha^{M(q)}-\alpha^n),
\end{aligned}$$

where $l(q)$ denotes the number of parts of $q$, and $M(q)$ is the largest part of $S(q)$.

The identity (1) is therefore equivalent to

$$s(\lambda)\sum_{p\,:\,\Lambda(p)=\lambda}\sum_{q\,:\,p\propto q}\frac{(-1)^{l(q)-1}}{F(p,q)}(\alpha^{M(q)}-\alpha^n)=\prod_i(1-\alpha^{\lambda_i}). \tag{2}$$

To prove this, we first show that the terms on the left can be grouped together so that multiplying by $s(\lambda)$ clears the denominators in each group, thereby reducing the expression on the left to a sum of polynomials in $\alpha$. Assume $\lambda=(\lambda_1,\dots,\lambda_l)$ has $l$ parts. We associate to $\lambda$ a sequence $\tilde\lambda$ which differs from $\lambda$ in that the occurrences of each part $i$ in $\lambda$ are now distinguished from each other by labelling them $i^{(1)}, i^{(2)}, \ldots,$ in order from left to right. For example if $\lambda = (3, 3, 2, 1, 1, 1)$, then $\tilde{\lambda} = (3^{(1)}, 3^{(2)}, 2^{(1)}, 1^{(1)}, 1^{(2)}, 1^{(3)})$.

The outer summation in (2) is over sequences $p = (p_1, \ldots, p_l)$ which are a reordering of the parts of $\lambda$. Clearly, there are $l!/s(\lambda)$ such reorderings. We shall rewrite the outer summation as a sum over all $l!$ reorderings $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_l)$ of the parts of $\tilde{\lambda}$. Note that each $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_l)$ uniquely determines a composition $p = (p_1, \ldots, p_l)$; simply erase the superscript in $\tilde{p}_i$ to get $p_i$. Write $U(\tilde{p})$ for the composition $p$ determined by the rearrangement $\tilde{p}$.

Next consider the inner summation on the left-hand side of (2) corresponding to a fixed composition $p = (p_1, \ldots, p_l)$ (which is of course a rearrangement of $\lambda$). Each $q$ in the inner sum satisfies $p \propto q$, and therefore corresponds to an ordered collection of disjoint subsets $(A_1, \ldots, A_m)$ of the form

$$A_1 = \{1, \ldots, a_1\},\ A_2 = \{1 + a_1, \ldots, a_2\},\ \ldots,\ A_m = \{1 + a_{m-1}, \ldots, a_m\},$$

where $q_1 = p_1 + \cdots + p_{a_1}$, $q_2 = p_{a_1+1} + \cdots + p_{a_2}$, $\ldots$, $q_m = p_{a_{m-1}+1} + \cdots + p_{a_m}$.

Let $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_l)$ be a reordering of $\tilde{\lambda}$, with $U(\tilde{p}) = p$. A collection of subsets $(A_1, \ldots, A_m)$ of the set $\{\tilde{p}_1, \ldots, \tilde{p}_l\}$ will be called a dominator of $\tilde{p}$ of type $q = (q_1, \ldots, q_m)$ if

$$A_1 = \{\tilde{p}_1, \ldots, \tilde{p}_{a_1}\}, \ldots, A_m = \{\tilde{p}_{1+a_{m-1}}, \ldots, \tilde{p}_{a_m}\},$$

where, as before $q_1 = p_1 + \cdots + p_{a_1}$, $\ldots$, $q_m = p_{a_{m-1}+1} + \cdots + p_{a_m}$.

We adopt the following notation: If $Q = A_1, \ldots, A_m$ is a dominator of $\tilde{p}$ of type $q$, write

(i) $\tilde{p} \propto Q$;
(ii) $l(Q) = m = l(q)$ for the length or the number of parts;
(iii) $M(Q) = M(S(q)) = n - q_m$ for the largest part of $S(q)$;
(iv) $F(\tilde{p}, Q) = F(p, q)$;
(v) $U(Q) = q$.

The following facts are then clear.

1. Let $U(\tilde{p}) = p$. Then

$$\sum_{Q\,:\,\tilde{p}\propto Q} \frac{(-1)^{l(Q)-1}}{F(\tilde{p}, Q)} (\alpha^{M(Q)} - \alpha^n) = \sum_{q\,:\,p\propto q} \frac{(-1)^{l(q)-1}}{F(p, q)} (\alpha^{M(q)} - \alpha^n),$$

and hence

2. for fixed $p$,

$$\sum_{\tilde{p}\,:\,U(\tilde{p}=p} \sum_{Q\,:\,\tilde{p}\propto Q} \frac{(-1)^{l(Q)-1}}{F(\tilde{p}, Q)} (\alpha^{M(Q)} - \alpha^n) = s(\lambda) \sum_{q\,:\,p\propto q} \frac{(-1)^{l(q)-1}}{F(p, q)} (\alpha^{M(q)} - \alpha^n).$$

Consequently the identity (2) is equivalent to

$$\sum_{\tilde{p}} \sum_{Q\,:\,\tilde{p}\propto Q} \frac{(-1)^{l(Q)-1}}{F(\tilde{p}, Q)} (\alpha^{M(Q)} - \alpha^n) = \prod (1 - \alpha^{\lambda_i}), \tag{3}$$

where the sum on the left ranges over all reorderings $\tilde{p}$ of $\tilde{\lambda}$.

Interchanging the order of summation on the left, we have

$$(4)\qquad \sum_{Q=(A_1,\ldots,A_m)} \sum_{\tilde{p}\propto Q} \left(\frac{1}{F(\tilde{p}, Q)}\right) (-1)^{l(Q)-1}(\alpha^{M(Q)} - \alpha^n) = \prod (1-\alpha^{\lambda_i}),$$

where the $Q$ range over all ordered partitions of the set of parts in $\tilde{\lambda}$.

But for a fixed ordered partition $Q = (A_1, \ldots, A_m)$, it is clear that the number of reorderings $\tilde{p}$ of $\tilde{\lambda}$ such that $\tilde{p} \propto Q$ is $\prod_i |A_i|!$. By definition of $F(\tilde{p}, Q)$, it follows that the inner sum evaluates to 1. Hence proving (4) reduces to showing that

$$(5)\qquad \sum_{Q=(A_1,\ldots,A_m)} (-1)^{m-1}(\alpha^{M(Q)} - \alpha^n) = \prod (1-\alpha^{\lambda_i}).$$

Let us first show that the coefficients of $\alpha^n$ on each side of (5) are equal. The coefficient on the left is

$\sum_{Q=(A_1,\ldots,A_m)} (-1)^{m-1}(-1)$, the sum ranging over all ordered set partitions of $\tilde{\lambda}$,

$= \sum_{B_1/\cdots/B_m} (-1)^m m!$,

summing over partitions of an $l$-set into $m$ nonempty blocks,

$= \sum_{m=1}^{l} S(l, m)\,(-1)^m\, m!$,

$S(l, m)$ being the Stirling number of the second kind,

$= (-1)^l$,

which is precisely the coefficient of $\alpha^n$ on the right. Note that the last equality is a well-known identity of Frobenius, and follows easily from the elementary identity

$$x^l = \sum_{m=1}^{l} S(l, m) \binom{x}{m} m!.$$

Now consider the other coefficients $\alpha^i$, for $0 \le i \le n-1$. Letting $C_{\alpha^i}$ denote the coefficient of $\alpha^i$, we need to show that

$$(6)\qquad C_{\alpha^i}\left(\sum_{Q=(A_1,\ldots,A_m)} (-1)^{m-1}\alpha^{M(Q)}\right) = C_{\alpha^i}\left(\prod_{i=1}^{l}(1-\alpha^{\lambda_i})\right),$$

the sum on the left ranging over all ordered partitions of $\tilde{\lambda}$, as usual.

Let $\mathscr{M}$ be a nonempty subset of $\tilde{\lambda}$. We can think of $\mathscr{M}$ as indexing the term $\prod_{\lambda_i \notin \mathscr{M}} (-\alpha)^{\lambda_i}$ in the right side of (6). Note that this term comes from taking parts of $\tilde{\lambda}$ which are in the complement of $\mathscr{M}$.

It is therefore enough to show that

$$(7)\qquad \sum_{Q=(A_1,\ldots,A_m)} (-1)^{m-1}\alpha^{M(Q)} = \prod_{\lambda_i \notin \mathscr{M}} (-\alpha)^{\lambda_i},$$

the sum on the left now ranging over all ordered partitions of $\tilde{\lambda}$, such that $A_m = \mathscr{M}$.

Note first that each $Q = (A_1, \ldots, A_m)$ with $A_m = \mathscr{M}$ satisfies

$$M(Q) = \sum_{\lambda_i \notin \mathscr{M}} \lambda_i .$$

(This is because $M(Q)$ equals $n$ minus the sum of the parts in $A_m = \mathscr{M}$, and the parts of $\tilde{\lambda}$ add up to $n$.) Thus the left side of (7) is a scalar multiple of the same monomial as the right-hand side. Next note that each $Q$ in the sum consists simply of an ordered partition $(A_1, \ldots, A_{m-1})$ of the complement of $\mathscr{M}$ in $\tilde{\lambda}$, followed by $\mathscr{M}$. Hence (7) reduces to the identity

$$\sum_{A_1, \ldots, A_{m-1}} (-1)^{m-1} = (-1)^{|\tilde{\lambda} \setminus \mathscr{M}|},$$

where the sum on the left now ranges over all partitions of the complement of $\mathscr{M}$ into $(m-1)$ nonempty blocks. But this follows as before from the identity of Frobenius involving Stirling numbers. This completes the proof of Theorem 2.1. □

## 3

We now translate Theorem 2.1 into a statement about Frobenius characteristics (or, equivalently, cycle indicators), and thereby obtain a generating function for the character of the representations afforded by the $\eta_n(\alpha)$.

Following [M], we denote by ch the Frobenius characteristic map from the character ring of the symmetric group to the ring of symmetric functions in the variables $\{x_1, x_2, \ldots\}$. Under this map, the trivial character of the symmetric group $S_n$ maps to the homogeneous symmetric function $h_n$. The characteristic of an induction product is simply the ordinary product of characteristics in the ring of symmetric functions. If $f$ and $g$ are respectively the characteristics of the representations $V_f$ of $S_m$ and $V_g$ of $S_n$, then it is well known that the plethysm $f[g]$ is the characteristic of the wreath product representation $V_f[V_g] \uparrow_{S_m[S_n]}^{S_{mn}}$. Write $\ell_n$ for the characteristic of the Lie representation afforded by the left ideal $\mathbb{C}S_n\theta_n$.

Now Theorem 2.1 says that for $\alpha$ equal to a primitive $p$th root of unity,

$$\text{(A)} \qquad \sum_{n\geq 0} \operatorname{ch}(\eta_n(\alpha)) = \sum_{m\geq 0} h_m \left[ \sum_{n\geq 1,\ n\not\equiv 0\,(p)} \ell_n \right].$$

One also has the following two identities:

(i) (see [M])

$$\sum_{m\geq 0} h_m = \exp \sum_{i\geq 1} \frac{p_i}{i},$$

where $p_i(x_1, x_2, \ldots) = x_1^i + x_2^i + \cdots$ is the $i$th power sum symmetric function. This is simply a standard fact about the cycle indicator of the trivial representation of the symmetric group. Note that $h_0 = p_0 = 1$.

(ii) (See [B])

$$\ell_n = \frac{1}{n} \sum_{n\equiv 0\,(d)} \mu(d)\, p_d^{n/d}(x_1, x_2, \ldots),$$

where $p_d$ is the ($d$th) power sum, as before. This is a theorem of Brandt, generalising Witt's dimension formula (see [J]) for the $n$th-degree subspace of the free Lie algebra.

Using these two formulas and simplifying, we have

**Theorem 3.1.** *Let $\alpha$ be a primitive $q$th root of unity, $q \neq 1$. Then*

$$\sum_{n\geq 0} \mathrm{ch}(\eta_n(\alpha)) = \exp\left(\sum_{n\geq 1}\ \sum_{i\geq 1,\, n\not\equiv 0\,(q)} \frac{1}{in} \sum_{d,\, n\equiv 0\,(d)} \mu(d)\, p_{id}^{n/d}\right).$$

We can now deduce the next corollary.

**Corollary 3.2.** *Let $q$ be a prime and $\alpha$ a primitive $q$th root of unity. Then*

$$\sum_{n\geq 0} \mathrm{ch}(\eta_n(\alpha)) = \prod_{k\geq 0} \frac{(1-p_{q^k}^q)^{1/q^{k+1}}}{(1-p_{q^k})^{1/q^k}}.$$

*In particular, for every $\sigma$ in $S_n$, the character of $\sigma$ acting on $\mathbb{C}S_n\eta_n(\alpha)$ is zero unless every cycle length of $\sigma$ is a power of the prime $q$.*

*Proof.* Substituting $n = dr$ in the formula of Theorem 3.1, and observing that since $q$ is prime, $q$ does not divide $n = dr$ iff $q$ does not divide either of $d$ or $r$, one has

$$\begin{aligned} \sum_{n\geq 0} \mathrm{ch}(\eta_n(\alpha)) &= \exp\left(\sum_{i\geq 1}\ \sum_{d\geq 1,\, d\not\equiv 0\,(q)}\ \sum_{r\geq 1,\, r\not\equiv 0 (q)} \frac{1}{idr}\mu(d)\, p_{id}^r\right) \\ \text{(B)} \qquad &= \exp\left[\sum_{i\geq 1}\ \sum_{d\geq 1,\, d\not\equiv 0\,(q)} \frac{1}{id}\mu(d)\left\{\sum_{r\geq 1}\frac{p_{id}^r}{r} - \sum_{\substack{s\geq 1\\ (r=sq)}} \frac{1}{qs}(p_{id}^s)^q\right\}\right] \\ &= \exp\left\{F(1) - \frac{1}{q}F(q)\right\}, \end{aligned}$$

where $F(x) = \sum_{i,r\geq 1}(1/ir)\sum_{d\geq 1,\, d\not\equiv 0\,(q)}(p_{id}^x)^r\, \mu(d)/d$. We compute $F(x)$ by summing first over all possible $d$ and then subtracting the sum over multiples $d = kq$ of $q$. Since $\sum_{r\geq 1}(p_m^x)^r/r = \log(1-p_m^x)^{-1}$, this gives

$$\begin{aligned} F(x) &= \sum_{i,d\geq 1} \frac{1}{id}\mu(d)\ \log(1-p_{id}^x)^{-1} \\ &\quad - \frac{1}{q}\sum_{i,k\geq 1} \frac{1}{ik}\mu(kq)\ \log(1-p_{ikq}^x)^{-1} \\ &= \sum_{s\geq 1} \log(1-p_s^x)^{-1}\left(\frac{1}{s}\sum_{d,s\equiv 0\,(d)} \mu(d)\right) \\ &\quad - \frac{1}{q}\sum_{s\geq 1}\log(1-p_{qs}^x)^{-1}\left(\frac{1}{s}\sum_{k,s\equiv 0\,(k)} \mu(kq)\right). \end{aligned}$$

But for $q$ prime, one also has

$$\sum_{k,s\equiv 0\,(k)} \mu(kq) = \begin{cases} -1, & \text{if } s = q^t,\ t \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$F(x) = \log(1-p_1{}^x)^{-1} + \frac{1}{q}\frac{1}{q^t}\log\prod_{t\geq 0}(1-p_{t+1}^x)^{-1} = \log\prod_{u\geq 0}(1-p_{q^u}^x)^{-1/q^u}.$$

Substituting in (B) now gives the desired expression. □

Note that if $q$ is not prime, one cannot obtain an expression as simple as (B). For $q = 4$, direct computation yields the following generating function.

**Example 3.3.** ($\alpha$ is a primitive fourth root of unity.) One can check that

$$\sum_{n\geq 0} \operatorname{ch}(\eta_n(\alpha)) = \frac{(1-p_1^4)^{1/4}}{1-p_1} \prod_{k\geq 1} \frac{(1-p_{2^k}^4)^{1/2^{k+2}}}{(1-p_{2^k}^2)^{1/2^{k+1}}}.$$

We end this section with a calculation for the case when $\alpha$ is a square root of unity (the case of the free Jordan algebra). A curious plethystic relation exists between the representations $V_n(-1)$ and certain characters of the symmetric group discovered in [CHR], in connection with the partition lattice $\Pi_n$. It is well known that the character of the symmetric group $S_n$ acting on the top homology of the Cohen-Macaulay lattice $\Pi_n$ is the same as the Lie representation $\mathrm{Lie}_n$ tensored with the sign. Let $\Pi_n^1$ denote the subposet of the partition lattice $\Pi_n$ (of an $n$-element set) consisting of those partitions all of whose blocks are odd. The work of Björner shows that the poset $\Pi_n^1$ is Cohen-Macaulay. In [CHR], the authors compute generating functions for the character $\beta_n$ of the symmetric group $S_n$ on the nonvanishing top homology of the poset $\Pi_n^1$. The following generating functions completely determine these character values. We continue to denote characteristic by ch.

**Theorem 3.4** [CHR, Theorem 4.7]. *One has the generating functions*:

(C) $$\sum_{n\geq 0} h_{2n+1}\left[\sum_{m\geq 0}(-1)^m \operatorname{ch}(\beta_{2m+1})\right] = p_1;$$

(D) $$\sum_{n\geq 0} h_{2n}\left[\sum_{m\geq 0}(-1)^m \operatorname{ch}(\beta_{2m+1})\right] = -\sum_{m\geq 0}(-1)^m \operatorname{ch}(\beta_{2m}).$$

It follows from this result that the dimension of the representation $\beta_n$ is

$$\begin{aligned} &[(2m-3)(2m-5)\cdots 5\cdot 3\cdot 1]^2, && n = 2m-1, \\ &(2m-1)[(2m-3)(2m-5)\cdots 5\cdot 3\cdot 1]^2, && n = 2m, \end{aligned}$$

and also that $\beta_{2m}\downarrow_{S_{2m-1}} = \beta_{2m-1}\downarrow_{S_{2m-2}}\uparrow^{S_{2m-1}}$. It is easy to see from Propositions 1.2–1.3 that the dimension of $V_n(-1)$ is

$$\begin{aligned} &(2m-1)[(2m-3)(2m-5)\cdots 5\cdot 3\cdot 1]^2, && n = 2m-1, \\ &[(2m-1)(2m-3)\cdots 5\cdot 3\cdot 1]^2, && n = 2m. \end{aligned}$$

In view of the induction-restriction property also shared by the $V_n(-1)$, it is not unreasonable to expect that there is some connection with the $\beta_n$. In fact, we have

**Proposition 3.5.** *Let $\alpha$ be a primitive square root of unity. Then the sum of the characters of the representation $V_n(-1)$, for $n$ odd, is the plethystic inverse of the alternating sum of the $\beta_{2m}$ restricted to $S_{2m-1}$. That is,*

$$\sum_{n\geq 0} \mathrm{ch}(\eta_{2n+1}(-1)) \left[ \sum_{m\geq 0} (-1)^m \frac{\partial}{\partial p_1} \mathrm{ch}(\beta_{2m}) \right] = p_1 .$$

*Proof*. We remind the reader that restricting a character $\chi$ from $S_n$ to $S_{n-1}$ is equivalent to taking the first partial derivative of $\mathrm{ch}(\chi)$ with respect to the power sum $p_1$, while inducing from $S_n$ to $S_{n+1}$ corresponds to multiplying by $p_1$. Using the identity (A) for the case $\alpha = -1$ $(p = 2)$ and comparing parity of degrees, we obtain for $i = 0, 1$,

$$\text{(E)} \qquad \sum_{n\equiv i\,(2)} \mathrm{ch}(\eta_n(-1)) = \sum_{n\equiv i\,(2)} h_n \left[ \sum_{n \text{ odd}} \ell_n \right],$$

We now compute

$$\begin{aligned} &\sum_{m\geq 0} (-1)^m \mathrm{ch}(\beta_{2m}) \left[ \sum_{n \text{ odd}} \mathrm{ch}(\eta_n(-1)) \right] \\ &\quad = \sum_{m\geq 0} (-1)^m \mathrm{ch}(\beta_{2m}) \left[ \sum_{n \text{ odd}} h_n \left[ \sum_{n \text{ odd}} \ell_n \right]\right] \\ &\quad = \left[ \sum_{m\geq 0} (-1)^m \mathrm{ch}(\beta_{2m}) \left[ \sum_{n \text{ odd}} h_n \right]\right] \left[ \sum_{n \text{ odd}} \ell_n \right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by associativity of plethysm)} \\ &\quad = \sum_{n\geq 0} h_{2n} \left[ \sum_{n \text{ odd}} \ell_n \right], \end{aligned}$$

where the last line is obtained by substituting from equations (C) and (D), and using the fact that $p_1$ is the identity for the plethysm operation, so that $f[g] = p_1 \Leftrightarrow g[f] = p_1$. We have just shown that

$$\sum_{m\geq 0} (-1)^m \mathrm{ch}(\beta_{2m}) \left[ \sum_{n \text{ odd}} \mathrm{ch}(\eta_n(-1)) \right] = \sum_{n \text{ even}} \mathrm{ch}(\eta_n(-1)).$$

Differentiating this with respect to $p_1$, we have

$$\begin{aligned} &\sum_{m\geq 0} (-1)^m \frac{\partial}{\partial p_1} \mathrm{ch}(\beta_{2m}) \left[ \sum_{n\geq 0} \mathrm{ch}(\eta_{2n+1}(-1)) \right] \sum_{n\geq 0} \frac{\partial}{\partial p_1} \mathrm{ch}(\eta_{2n+1}(-1)) \\ &\quad = \sum_{n\geq 0} \frac{\partial}{\partial p_1} \mathrm{ch}(\eta_{2n}(-1)). \end{aligned}$$

But Proposition 1.3 says that

$$\frac{\partial}{\partial p_1}\operatorname{ch}(\eta_{2n}(-1)) = p_1 \frac{\partial}{\partial p_1}\operatorname{ch}(\eta_{2n-1}(-1)).$$

Substituting in the preceding equation now gives the desired result. □

We remark that the analogous equation does not hold for $\alpha$ an arbitrary $p$th root of unity, and characters of the top homology of the poset of partitions in which all blocks are congruent to 1 modulo $p$.

## 4

In conclusion, we present two sets of tables, one for the character values for the $V_n(\alpha)$, and the other for the multiplicities of its decomposition into $S_n$-irreducibles.

Each of Tables 1–7 corresponds to a particular value of $n$ and contains character values of the representation $\mathbb{C}S_n\eta_n(\alpha)$, as $\alpha$ ranges from 1 to the $n$th root of unity. Recall that, by Proposition 1.1, for fixed $n$, once $\alpha$ is a $p$th root of unity for $p > n$, $V_n(\alpha)$ stabilises to the regular representation of $S_n$. The rows of the table are indexed by the conjugacy classes ${x_1}^{m_1}{x_2}^{m_2}\cdots{x_i}^{m_i}\cdots$, indicating the class with $m_i$ cycles of length $i$.

Note in particular the zero values in the column headed $\alpha^p = 1$, as predicted by Corollary 3.2, when $p$ is prime and the cycle lengths are not powers of $p$. Also, by Proposition 1.2, if $\alpha^n \neq 1$, one computes that the character value $\chi_{n,\alpha}(\sigma)$ on a permutation $\sigma$ of cycle type $1^{m_1}\mu$ is equal to the number of fixed points $m_1$ of $\sigma$ times the value $\chi_{n-1,\alpha}(\sigma')$ in the preceding table, where $\sigma'$ is the permutation in $S_{n-1}$ of cycle type $1^{m_1-1}\mu$.

Tables 8–14 contain, for each $n = 2, \ldots, 8$, the multiplicities of the decomposition into irreducibles of the representations $V_n(\alpha)$. The irreducibles are labelled by partitions $\lambda = (n^{m_n}\cdots 2^{m_2}1^{m_1})$. (For example, $(2^3\,1^2)$ denotes the partition whose 5 parts are 2, 2, 2, 1, 1.) One notes that for $n \geq 3$ and $\alpha$ equal to a $p$th root of unity, the trivial representation $(n)$ appears in $V_n(\alpha)$ for all $p \geq 2$, while the sign representation $(1^n)$ appears for all $p \geq 3$. We can prove this directly, as well as the fact that $V_n(e^{2\pi i/n})$ is isomorphic to the complement of the Lie representation $V_n(1)$ in the group algebra. (So the first and last columns in the tables add up to the regular representation.)

TABLE 1. Character values, $n = 2$:

| class | $\alpha = 1$ | $\alpha^2 = 1$ |
|---|---|---|
| $x_1^2$ | 1 | 1 |
| $x_2$ | −1 | 1 |

TABLE 2. Character values, $n = 3$ :

| class | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ |
|---|---|---|---|
| $x_1^3$ | 2 | 3 | 4 |
| $x_1x_2$ | 0 | 1 | 0 |
| $x_3$ | −1 | 0 | 1 |

TABLE 3. Character values, $n = 4$ :

| class | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ |
|---|---|---|---|---|
| $x_1^4$ | 6 | 9 | 16 | 18 |
| $x_1^2 x_2$ | 0 | 1 | 0 | 0 |
| $x_2^2$ | −2 | 1 | 0 | 2 |
| $x_1 x_3$ | 0 | 0 | 1 | 0 |
| $x_4$ | 0 | 1 | 0 | 0 |

TABLE 4. Character values, $n = 5$ :

| class | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ |
|---|---|---|---|---|---|
| $x_1^5$ | 24 | 45 | 80 | 90 | 96 |
| $x_1^3 x_2$ | 0 | 3 | 0 | 0 | 0 |
| $x_1 x_2^2$ | 0 | 1 | 0 | 2 | 0 |
| $x_1^2 x_3$ | 0 | 0 | 2 | 0 | 0 |
| $x_2 x_3$ | 0 | 0 | 0 | 0 | 0 |
| $x_1 x_4$ | 0 | 1 | 0 | 0 | 0 |
| $x_5$ | −1 | 0 | 0 | 0 | 1 |

TABLE 5. Character values, $n = 6$ :

| class | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ | $\alpha^6 = 1$ |
|---|---|---|---|---|---|---|
| $x_1^6$ | 120 | 225 | 400 | 540 | 576 | 600 |
| $x_1^4 x_2$ | 0 | 9 | 0 | 0 | 0 | 0 |
| $x_1^2 x_2^2$ | 0 | 1 | 0 | 4 | 0 | 0 |
| $x_2^3$ | −8 | 9 | 0 | 0 | 0 | 8 |
| $x_1^3 x_3$ | 0 | 0 | 4 | 0 | 0 | 0 |
| $x_1 x_2 x_3$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_3^2$ | −3 | 0 | 4 | 0 | 0 | 3 |
| $x_1^2 x_4$ | 0 | 1 | 0 | 0 | 0 | 0 |
| $x_2 x_4$ | 0 | 1 | 0 | 0 | 0 | 0 |
| $x_1 x_5$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $x_6$ | 1 | 0 | 0 | 0 | 0 | −1 |

TABLE 6. Character values, $n = 7$ :

| class | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ | $\alpha^6 = 1$ | $\alpha^7 = 1$ |
|---|---|---|---|---|---|---|---|
| $x_1^7$ | 720 | 1575 | 2800 | 3780 | 4032 | 4200 | 4320 |
| $x_1^5x_2$ | 0 | 45 | 0 | 0 | 0 | 0 | 0 |
| $x_1^3x_2^2$ | 0 | 3 | 0 | 12 | 0 | 0 | 0 |
| $x_1x_2^3$ | 0 | 9 | 0 | 0 | 0 | 8 | 0 |
| $x_1^4x_3$ | 0 | 0 | 16 | 0 | 0 | 0 | 0 |
| $x_1^2x_2x_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2^2x_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_3^2$ | 0 | 0 | 4 | 0 | 0 | 3 | 0 |
| $x_1^3x_4$ | 0 | 3 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2x_4$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $x_3x_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_5$ | 0 | 0 | 0 | 0 | 2 | 0 | 0 |
| $x_2x_5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_6$ | 0 | 0 | 0 | 0 | 0 | −1 | 0 |
| $x_7$ | −1 | 0 | 0 | 0 | 0 | 0 | 1 |

TABLE 7. Character values, $n = 8$ :

| class | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ | $\alpha^6 = 1$ | $\alpha^7 = 1$ | $\alpha^8 = 1$ |
|---|---|---|---|---|---|---|---|---|
| $x_1^8$ | 5040 | 11025 | 22400 | 26460 | 32256 | 33600 | 34560 | 35280 |
| $x_1^6x_2$ | 0 | 225 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^4x_2^2$ | 0 | 9 | 0 | 36 | 0 | 0 | 0 | 0 |
| $x_1^2x_2^3$ | 0 | 9 | 0 | 0 | 0 | 16 | 0 | 0 |
| $x_2^4$ | −48 | 33 | 0 | 12 | 0 | 0 | 0 | 48 |
| $x_1^5x_3$ | 0 | 0 | 80 | 0 | 0 | 0 | 0 | 0 |
| $x_1^3x_2x_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_2^2x_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_3^2$ | 0 | 0 | 8 | 0 | 0 | 6 | 0 | 0 |
| $x_2x_3^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^4x_4$ | 0 | 9 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_2x_4$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2^2x_4$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_3x_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_4^2$ | 0 | 1 | 0 | 4 | 0 | 0 | 0 | 0 |
| $x_1^3x_5$ | 0 | 0 | 0 | 0 | 6 | 0 | 0 | 0 |
| $x_1x_2x_5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_3x_5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1^2x_6$ | 0 | 0 | 0 | 0 | 0 | −2 | 0 | 0 |
| $x_2x_6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_1x_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $x_8$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE 8. Decomposition into irreducibles, $n = 2$ :

| irrep | $\alpha = 1$ | $\alpha^2 = 1$ |
|---|---|---|
| (2) | 0 | 1 |
| $(1^2)$ | 1 | 0 |

TABLE 9. Decomposition into irreducibles, $n = 3$ :

| irrep | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ |
|---|---|---|---|
| (3) | 0 | 1 | 1 |
| (2 1) | 1 | 1 | 1 |
| $(1^3)$ | 0 | 0 | 1 |

TABLE 10. Decomposition into irreducibles, $n = 4$ :

| irrep | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ |
|---|---|---|---|---|
| (4) | 0 | 1 | 1 | 1 |
| (3 1) | 1 | 1 | 2 | 2 |
| $(2^2)$ | 0 | 1 | 1 | 2 |
| $(2\,1^2)$ | 1 | 1 | 2 | 2 |
| $(1^4)$ | 0 | 0 | 1 | 1 |

TABLE 11. Decomposition into irreducibles, $n = 5$ :

| irrep | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ |
|---|---|---|---|---|---|
| (5) | 0 | 1 | 1 | 1 | 1 |
| (4 1) | 1 | 2 | 3 | 3 | 3 |
| (3 2) | 1 | 2 | 3 | 4 | 4 |
| $(3\,1^2)$ | 1 | 2 | 4 | 4 | 5 |
| $(2^2\,1)$ | 1 | 2 | 3 | 4 | 4 |
| $(2\,1^3)$ | 1 | 1 | 3 | 3 | 3 |
| $(1^5)$ | 0 | 0 | 1 | 1 | 1 |

TABLE 12. Decomposition into irreducibles, $n = 6$ :

| irrep | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ | $\alpha^6 = 1$ |
|---|---|---|---|---|---|---|
| (6) | 0 | 1 | 1 | 1 | 1 | 1 |
| (5 1) | 1 | 2 | 3 | 4 | 4 | 4 |
| (4 2) | 1 | 4 | 5 | 7 | 7 | 8 |
| $(4\,1^2)$ | 2 | 3 | 6 | 7 | 8 | 8 |
| (3 3) | 1 | 1 | 3 | 4 | 4 | 4 |
| (3 2 1) | 3 | 5 | 8 | 12 | 13 | 13 |
| $(3\,1^3)$ | 1 | 3 | 6 | 7 | 8 | 9 |
| $(2^3)$ | 0 | 2 | 3 | 4 | 4 | 5 |
| $(2^2\,1^2)$ | 2 | 2 | 5 | 7 | 7 | 7 |
| $(2\,1^4)$ | 1 | 1 | 3 | 4 | 4 | 4 |
| $(1^6)$ | 0 | 0 | 1 | 1 | 1 | 1 |

TABLE 13. Decomposition into irreducibles, $n = 7$ :

| irrep | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ | $\alpha^6 = 1$ | $\alpha^7 = 1$ |
|---|---|---|---|---|---|---|---|
| (7) | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| (6 1) | 1 | 3 | 4 | 5 | 5 | 5 | 5 |
| (5 2) | 2 | 6 | 8 | 11 | 11 | 12 | 12 |
| $(5\,1^2)$ | 2 | 5 | 9 | 11 | 12 | 12 | 13 |
| (4 3) | 2 | 5 | 8 | 11 | 11 | 12 | 12 |
| (4 2 1) | 5 | 12 | 19 | 26 | 28 | 29 | 30 |
| $(4\,1^3)$ | 3 | 6 | 12 | 14 | 16 | 17 | 17 |
| (3 3 1) | 3 | 6 | 11 | 16 | 17 | 17 | 18 |
| (3 2 2) | 3 | 7 | 11 | 16 | 17 | 18 | 18 |
| $(3\,2\,1^2)$ | 5 | 10 | 19 | 26 | 28 | 29 | 30 |
| $(3\,1^4)$ | 2 | 4 | 9 | 11 | 12 | 13 | 13 |
| $(2^3\,1)$ | 2 | 4 | 8 | 11 | 11 | 12 | 12 |
| $(2^2\,1^3)$ | 2 | 3 | 8 | 11 | 11 | 11 | 12 |
| $(2\,1^5)$ | 1 | 1 | 4 | 5 | 5 | 5 | 5 |
| $(1^7)$ | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

TABLE 14. Decomposition into irreducibles, $n = 8$ :

| irrep | $\alpha = 1$ | $\alpha^2 = 1$ | $\alpha^3 = 1$ | $\alpha^4 = 1$ | $\alpha^5 = 1$ | $\alpha^6 = 1$ | $\alpha^7 = 1$ | $\alpha^8 = 1$ |
|---|---|---|---|---|---|---|---|---|
| (8) | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| (7 1) | 1 | 3 | 5 | 5 | 6 | 6 | 6 | 6 |
| (6 2) | 2 | 8 | 12 | 14 | 16 | 17 | 17 | 18 |
| $(6\,1^2)$ | 3 | 7 | 13 | 14 | 17 | 17 | 18 | 18 |
| (5 3) | 4 | 9 | 16 | 19 | 22 | 24 | 24 | 24 |
| (5 2 1) | 8 | 20 | 36 | 42 | 51 | 53 | 55 | 56 |
| $(5\,1^3)$ | 4 | 10 | 21 | 22 | 28 | 29 | 30 | 31 |
| (4 4) | 1 | 5 | 8 | 10 | 11 | 12 | 12 | 13 |
| (4 3 1) | 9 | 20 | 38 | 46 | 56 | 58 | 60 | 61 |
| (4 2 2) | 6 | 17 | 30 | 37 | 45 | 47 | 48 | 50 |
| $(4\,2\,1^2)$ | 12 | 24 | 50 | 58 | 72 | 75 | 77 | 78 |
| $(4\,1^4)$ | 4 | 9 | 21 | 22 | 28 | 30 | 30 | 31 |
| $(3^2\,2)$ | 6 | 11 | 22 | 28 | 34 | 35 | 36 | 36 |
| $(3^2\,1^2)$ | 6 | 15 | 30 | 37 | 45 | 46 | 48 | 50 |
| $(3\,2^2\,1)$ | 9 | 18 | 38 | 46 | 56 | 59 | 60 | 61 |
| $(3\,2\,1^3)$ | 8 | 15 | 36 | 42 | 51 | 53 | 55 | 56 |
| $(3\,1^5)$ | 3 | 4 | 13 | 14 | 17 | 18 | 18 | 18 |
| $(2^4)$ | 1 | 4 | 8 | 10 | 11 | 12 | 12 | 13 |
| $(2^3\,1^2)$ | 4 | 6 | 16 | 19 | 22 | 23 | 24 | 24 |
| $(2^2\,1^4)$ | 2 | 4 | 12 | 14 | 16 | 16 | 17 | 18 |
| $(2\,1^5)$ | 1 | 1 | 5 | 5 | 6 | 6 | 6 | 6 |
| $(1^8)$ | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

## REFERENCES


[BS] H. Barcelo and S. Sundaram, *On some submodules of the action of the symmetric group on the free Lie algebra*, J. Algebra **154** (1993), 12–26.

[B] A. J. Brandt, *The free Lie ring and Lie representations of the full linear group*, Trans. Amer. Math. Soc. **56** (1944), 528-536.

[CHR] A. R. Calderbank, P. Hanlon, and R. W. Robinson, *Partitions into even and odd block size and some unusual characters of the symmetric groups*, Proc. London Math. Soc. (3) **53** (1986), 288–320.

[G] A. M. Garsia, *Combinatorics of the free Lie algebra and the symmetric group*, Analysis: Research Papers Published in Honour of Jürgen Moser's 60th Birthday (P. H. Rabinowitz and E. Zehnder, eds.), Academic Press, San Diego, California, 1990, pp. 309–382.

[GR] A. M. Garsia and C. Reutenauer, *A decomposition of Solomon's descent algebra*, Adv. Math. **77** (1989), 189–262.

[J] N. Jacobson, *Lie algebras*, Dover, New York, 1979.

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press, New York, 1979.

[R] C. Reutenauer, *Theorem of Poincaré-Birkhoff-Witt and symmetric group representations of degrees equal to Stirling numbers*, Lecture Notes in Math., vol. 1234, Springer-Verlag, New York and Berlin, 1986.

[Ro] D. P. Robbins, *Jordan elements in the free associative algebra*. I, J. Algebra **19** (1971), 354–378.

[So] L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), 255–264.

[Su] S. Sundaram, *Decompositions of $S_n$-submodules in the free Lie algebra*, J. Algebra (1990) (to appear).

[T] R. M. Thrall, *On symmetrized Kronecker powers and the structure of the free Lie ring*, Amer. J. Math. **64** (1942), 371–388.

AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974
*E-mail address*: `rc@research.att.com`

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109
*E-mail address*: `hanlon@math.lsa.umich.edu`

Département de Mathématiques et d'Informatique, Université du Québec à Montréal, Montréal H3C 3P8, Québec, Canada

S. Sundaram: Department of Mathematics and Computer Science, University of Miami, Coral Gables, Florida 33124
*E-mail address*: `sheila@paris-gw.cs.miami.edu`