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THE H^p-CORONA THEOREM FOR THE POLYDISC

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ABSTRACT. Let $H^p = H^p(D^n)$ denote the usual Hardy spaces on the polydisc D^n. We prove in this paper the following theorem: *Suppose* $f_1, f_2, \dots, f_n \in H^\infty$, $\|f_j\|_{H^\infty} \le 1$, *and* $\sum_{j=1}^m |f_j(z)| \ge \delta > 0$. *Then for every* g *in* H^p, $1 < p < \infty$, *there are* H^p *functions* $g, g, \dots, g_m$ *such that* $\sum_{j=1}^m f_j(z)g_j(z) = g(z)$. *Moreover, we have* $\|g_j\|_{H^p} \le c(m, n, \delta, p)\|g\|_{H^p}$. (When $p = 2$, $n = 1$, this theorem is known to be equivalent to Carleson's corona theorem.)

Let Ω be a domain in $\mathbb{C}^n$ and $H^\infty = H^\infty(\Omega)$ the algebra of bounded analytic functions on Ω equipped with the sup norm, $\|f\|_{H^\infty} = \sup_{z\in\Omega}|f(z)|$. The corona problem is the following: Given $f_1, f_2, \dots, f_m \in H^\infty$, $\|f_j\|_{H^\infty} \le 1$, and $\sum_{j=1}^m |f_j(z)| \ge \delta > 0$ on Ω, are there H^∞ functions $g_1, g_2, \dots, g_n$ such that $\sum_{j=1}^m f_j g_j \equiv 1$? When $n = 1$ and $\Omega = D =$ the unit disc in $\mathbb{C}$, the answer is affirmative and that is Carleson's famous corona theorem [3]. For other planar domains see the extensive bibliography [9]. The deepest result is due to Garnett and Jones [6] who proved the corona theorem for Denjoy domains. In higher dimensions ($n > 1$), only negative results are known, see N. Sibony's articles [11, 12] and B. Cole's example in [7]. However, there are some partial results. Henkin [8] and Varopoulos [13] (with $m = 2$) showed that when $\Omega = B^n =$ the unit ball in $\mathbb{C}^n$, there are corona solutions $g_1, g_2, \dots, g_m$ in all H^p, $0 < p < \infty$. The similar result for the polydisc was proved by Varopoulos [14], Chang [4], and Lin [10]. Moreover, E. Amar [1] (with $m = 2$) and M. Andersson [2] (with $p = 2$) have recently proved the following H^p-corona theorem for the ball B^n: Suppose $f_1, f_2, \dots, f_m \in H^\infty = H^\infty(B^n)$, $\|f_j\|_{H^\infty} \le 1$, and $\sum_{j=1}^m |f_j(z)| \ge \delta > 0$ on B^n. Then for every g in $H^p(B^n)$, $1 \le p < \infty$, there are H^p functions $g_1, g_2, \dots, g_n$ such that $\sum f_j g_j \equiv g$. This theorem can be viewed as an evidence that the corona problem may be true for the ball, since when $n = 1$, $p = 2$, it is equivalent to Carleson's original theorem. The purpose of this paper is to prove the polydisc version of the above theorem (except for the ranges of p):

Theorem. *Suppose* $f_1, f_2, \dots, f_m \in H^\infty(D^n)$, $\|f_j\|_{H^\infty} \le 1$, *and* $\sum_{j=1}^m |f_j(z)| \ge \delta > 0$ *on* D^n. *Then for every* g *in* $H^p(D^n)$, $1 < p < \infty$, *there are* H^p *functions* $g_1, g_2, \dots, g_m$ *such that* $\sum_{j=1}^m f_j g_j \equiv g$. *Moreover, we have* $\|g_j\|_{H^p} \le c(m, n, \delta, p)\|g\|_{H^p}$.

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(Here, $H^p(D^n)$ are the usual Hardy spaces, namely, f is in $H^p(D^n)$ if and only if f is analytic and

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{(2\pi)^n} \int_{T^n} |f(rz)|^p \, d\sigma(z) < \infty,$$

$d\sigma$ is the Lebesgue measure on the distinguished boundary T^n.) The methods used here are quite different from those in the ball case. We will make a detailed study of the Koszul complex and $\overline{\partial}$-equations, as in the articles of Chang [4] and Lin [10]. As a mater of fact, the analysis here is quite similar to those papers. We will prove the theorem for the case $n = 2$ only. The cases $n \geq 3$ are not much more difficult, see [10]. We also remark that the proof below does not apply to the case $p = 1$, since the Hilbert transform fails to be bounded on L^1.

Proof of the theorem $n = 2$. We begin with the case $m = 2$, i.e. two generators f_1 and f_2, and may assume f_1, f_2 and g are analytic across T^2, by a normal families argument. Let

$$\varphi_j = \frac{g\overline{f}_j}{|f_1|^2 + |f_2|^2}.$$

Then $f_1\varphi_1 + f_2\varphi_2 = g$, and $\|\varphi_j\|_{L^p(T^2)} \leq c(\delta)\|g\|_{H^p}$. But φ_j need not be analytic. To rectify that, we put

$$g_1 = \varphi_1 + bf_2, \qquad g_2 = \varphi_2 - bf_1. \tag{1}$$

Then we still have $f_1 g_1 + f_2 g_2 = g$. The requirement on the analyticity of g_j imposes a $\overline{\partial}$-equation on b, namely,

$$(2) \quad \begin{cases} (2.1) \quad \dfrac{\partial b}{\partial \overline{z}_1} = \dfrac{1}{g}\left[\varphi_1 \dfrac{\partial \varphi_2}{\partial \overline{z}_1} - \varphi_2 \dfrac{\partial \varphi_1}{\partial \overline{z}_1}\right] = \dfrac{g\left(\overline{f}_1 \dfrac{\partial \overline{f}_2}{\partial \overline{z}_1} - \overline{f}_2 \dfrac{\partial \overline{f}_1}{\partial \overline{z}_1}\right)}{|f_1|^2 + |f_2|^2)^2} = G_1, \\[2ex] (2.2) \quad \dfrac{\partial b}{\partial \overline{z}_2} = \dfrac{1}{g}\left[\varphi_1 \dfrac{\partial \varphi_2}{\partial \overline{z}_2} - \varphi_2 \dfrac{\partial \varphi_1}{\partial \overline{z}_2}\right] = \dfrac{g\left(\overline{f}_1 \dfrac{\partial \overline{f}_2}{\partial \overline{z}_2} - \overline{f}_2 \dfrac{\partial \overline{f}_1}{\partial \overline{z}_2}\right)}{(|f_1|^2 + |f_2|^2)^2} = G_2. \end{cases}$$

The advantage of dealing with two generators is that the $\overline{\partial}$-system above is automatically $\overline{\partial}$-closed, i.e., $\partial G_1/\partial \overline{z}_2 = \partial G_2/\partial \overline{z}_1$. To have $g_j \in H^p(D^2)$ we need to solve (2) with $\|b\|_{L^p(T^2)} \leq C(p, \delta)\|g\|_{H^p}$.

We consider the equation (2.1) first, and will show, by one-variable technique, that it has a solution b_1 so that $\|b_1\|_{L^p(T^2)} \leq C\|g\|_{H^p}$. To do this, we fix z_2 in D_2 (we think of $D^2 = D_1 \times D_2$), and take a C^∞ solution to (2,1), say, for example,

$$b_0(z_1, z_2) = c \int_{D_1} \frac{G_1(\xi_1, z_2)}{\xi_1 - z_1} \, d\xi_1 \wedge d\overline{\xi}_1.$$

Then functions of the form $b_0 + k$, where k is analytic in z_1, solve (2.1). We are concerned with $\|b_0 + k\|_{L^p(\partial D_1)}$. By duality (one-variable), $L^p/H^p = (H_0^q)^*$,

$\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}\inf_{k\in H^p}\|b_0+k\|_{H^p(D_1)}&=\sup\left\{c\left|\int_{\partial D_1}b_0h\,dz_1\right|:h\in H_0^q(D_1),\ \|h\|_{H^q(D_1)}\le 1\right\}\\&=\sup\left\{c\left|\int_{D_1}G_1h\,dz_1\wedge d\bar z_1\right|:h\in H_0^q(D_1),\ \|h\|_{H^q(D_1)}\le 1\right\},\end{aligned}$$

by Stokes' theorem. But Wolff's proof of the corona theorem [5] has shown that

$$\begin{aligned}\left|\int_{D_1}G_1h\,dz_1\wedge d\bar z_2\right|&\le c(\delta)\|g^h\|_{H^1(D_1)}\le c(\delta,p)\|g\|_{H^p(D_1)}\|h\|_{H^q(D_1)}\\&\le c(\delta,p)\|g\|_{H^p(D_1)}.\end{aligned}$$

Thus there is a solution b_1 to (2.1) such that $\|b_1\|_{L^p(\partial D_1)}\le c(\delta,P)\|g\|_{H^p(D_1)}$. Notice that the argument above is done in such a way that the constant $c(\delta,p)$ is independent of z_2 in D_2. Therefore, after integrating over $z_2\in\partial D_2$, we obtain $\|b_1\|_{L^p(T^2)}\le c\|g\|_{H^p(D^2)}$. Similarly, there is a solution b_2 to (2.2) so that $\|b_2\|_{L^p(T^2)}\le c(p,\delta)\|g\|_{H^p}$. However, we need a common solution to (2.1) and (2.2). To this end, we have to find h_1, h_2 in $L^p(T^2)$ with h_j analytic in z_j, so that $b_1+h_1=b_2+h_2$. Put

$$h_1(z_1,z_2)=c\int_{\partial D_1}\frac{(b_2-b_1)(\xi_1,z_2)}{\xi_1-z_1}\,d\xi_1,$$

and $h_2=h_1+b_1-b_2$. Clearly, h_1 is analytic in z_1, and h_2 is analytic in z_2, for

$$\begin{aligned}\frac{\partial h_2}{\partial\bar z_2}&=\frac{\partial h_1}{\partial\bar z_2}+\frac{\partial}{\partial\bar z_2}(b_1-b_2)\\&=c\int_{\partial D_1}\frac{\partial(b_2-b_1)(\xi_1,z_2)/\partial\bar z_2}{\xi_1-z_1}\,d\xi_1+\frac{\partial}{\partial\bar z_2}(b_1-b_2)\\&=\frac{\partial}{\partial\bar z_2}(b_2-b_1)+\frac{\partial}{\partial\bar z_2}(b_1-b_2)\\&=0\end{aligned}$$

since $\frac{\partial}{\partial\bar z_2}(b_2-b_1)$ is analytic in z_1, by the $\bar\partial$-closedness of (2).

Moreover, $\|h_j\|_{L^p(T^2)}\le c(\|b_1\|_{L^p(T^2)}+\|b_2\|_{L^p(T^2)})\le c\|g\|_{H^p}$. Hence there is a solution b to (2) with $\|b\|_{L^p(T^2)}\le c\|g\|_{H^p}$, and the proof of the case $m=2$ is complete.

When the number m of generators is greater than two, the algebraic trick (1) is not available, and higher-order $\bar\partial$-equations are needed. We refer the readers to the paper [10] for the details of the Koszul complex which reduces the corona problem to a set of $\bar\partial$-equations. The system of $\bar\partial$-equations we need to solve here is

$$\begin{aligned}\frac{\partial^2 d_{j,k,l}}{\partial\bar z_1\partial\bar z_2}=\frac{1}{g^2}\Big[&\varphi_j\frac{\partial\varphi_k}{\partial\bar z_1}\frac{\partial\varphi_l}{\partial\bar z_2}-\varphi_j\frac{\partial\varphi_l}{\partial\bar z_1}\frac{\partial\varphi_k}{\partial\bar z_2}+\varphi_1\frac{\partial\varphi_j}{\partial\bar z_1}\frac{\partial\varphi_k}{\partial\bar z_2}\\&-\varphi_l\frac{\partial\varphi_k}{\partial\bar z_1}\frac{\partial\varphi_j}{\partial\bar z_2}+\varphi_k\frac{\partial\varphi_l}{\partial\bar z_1}\frac{\partial\varphi_j}{\partial\bar z_2}-\varphi_k\frac{\partial\varphi_j}{\partial\bar z_1}\frac{\partial\varphi_l}{\partial\bar z_2}\Big]\equiv\Phi\end{aligned}\tag{3}$$

$$(4)\quad \begin{cases} (4.1) & \dfrac{\partial b_{j,k}}{\partial \overline{z}_1} = \dfrac{1}{g}\left[\varphi_j \dfrac{\partial \varphi_k}{\partial \overline{z}_1} - \varphi_k \dfrac{\partial \varphi_j}{\partial \overline{z}_1}\right] - \sum_1 f_1 \dfrac{\partial d_{j,k,l}}{\partial \overline{z}_1}, \\ (4.2) & \dfrac{\partial b_{j,k}}{\partial \overline{z}_2} = \dfrac{1}{g}\left[\varphi_j \dfrac{\partial \varphi_k}{\partial \overline{z}_2} - \varphi_k \dfrac{\partial \varphi_j}{\partial \overline{z}_2}\right] - \sum_1 f_1 \dfrac{\partial d_{j,k,l}}{\partial \overline{z}_2}, \end{cases}$$

where

$$\varphi_j = \frac{g\overline{f}_j}{|f_1|^2 + |f_2|^2 + \cdots + |f_m|^2}.$$

Then the functions g_j defined by $g_j = \varphi_j - \sum_k f_k b_{j,k}$ will be analytic and satisfy $f_1 g_1 + f_2 g_2 + \cdots + f_m g_m = g$. To have g_j in H^P we will have to solve (3) and (4) so that $b_{j,k}$ is in $L^P(T^2)$.

We start with the equation (3), $\partial^2 d/\partial \overline{z}_1 \partial \overline{z}_2 = \Phi$, and will solve it so that $\|d\|_{L^p(T^2)} \le c(\delta, P, m)\|g\|_{H^p}$. (The indices j, k, l are suppressed.) A solution to (3) is

$$d_0(z_1, z_2) = c\int_{D^2} \frac{\Phi(\xi_1, \xi_2)}{(\xi_1 - z_1)(\xi_2 - z_2)}\, d\xi_1 \wedge d\overline{\xi}_1 \wedge d\xi_2 \wedge d\overline{\xi}_2,$$

and functions of the form $d_0 + k$, where $\partial^2 k/\partial \overline{z}_1 \partial \overline{z}_2 = 0$, are also solutions to (3). By duality,

$$\begin{aligned} &\inf\left\{ \|d_0 + k\|_{L^p(T^2)} : \frac{\partial^2 k}{\partial \overline{z}_1 \partial \overline{z}_2} = 0 \right\} \\ &= \sup\left\{ c\left| \int_{T^2} d_0 h\, dz_1\, dz_2 \right| : h \in H_0^q(D^2),\ \|h\|_{H^q} \le 1 \right\}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \\ &= \sup\left\{ c\left| \int_{D^2} \Phi h \right| : h \in H_0^q(D^2),\ \|h\|_{H^q} \le 1 \right\}, \quad \text{by Stokes' theorem.} \end{aligned}$$

But the arguments in the papers of [4 and 10], which involves some difficult estimates on Carleson measures on the bidisc, already showed that

$$\left| \int_{D^2} \Phi h \right| \le c\|gh\|_{H^1} \le c\|g\|_{H^p}\|h\|_{H^q} \le c\|g\|_{H^p}.$$

Hence there is indeed a solution d to (3) with $\|d\|_{L^p(T^2)} \le c\|g\|_{H^p}$.

We now turn to equation (4). (We will suppress the indices j and k again.) For (4.1), one-variable result gives a solution $b_{1,1}$ with $\|b_{1,1}\|_{L^p(T^2)} \le c(p, \delta, m)\|g\|_{H^p}$, to

$$\frac{\partial b_{1,1}}{\partial \overline{z}_1} = \frac{1}{g}\left[\varphi_j \frac{\partial \varphi_k}{\partial \overline{z}_1} - \varphi_k \frac{\partial \varphi_j}{\partial \overline{z}_j}\right].$$

Thus the function $b_1 = b_{1,1} - \sum_1 f_1\, d_{j,k,l}$ is a solution to (4.1) with the correct L^p bound. Similarly, we have a solution b_2 to (4.2). Now the argument presented earlier gives a common solution $b_{j,k}$ to the equation (4) with the desired $L^p(T^2)$ estimate. We have completed the proof of the theorem.

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