

NONSIMPLE, RIBBON FIBERED KNOTS

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ABSTRACT. The connected sum of an arbitrary knot and its mirror image is a ribbon knot, however the converse is not necessarily true for all ribbon knots. We prove that the converse holds for any ribbon fibered knot which is a connected sum of iterated torus knots, knots with irreducible Alexander polynomials, or cables of such knots. This gives a practical method to detect nonribbon fibered knots. The proof uses a characterization of homotopically ribbon, fibered knots by their monodromies due to Casson and Gordon. We also study when cable fibered knots are ribbon and results which support the following conjecture.

Conjecture. If a (p, q) cable of a fibered knot k is ribbon where $p (> 1)$ is the winding number of a cable in $S^1 \times D^2$, then $q = \pm 1$ and k is ribbon.

INTRODUCTION

Casson and Gordon [4] introduced the notion of “homotopically ribbon” for a knot in a homology sphere, which is an analog of “ribbon” for a knot in S^3 , and proved that a fibered knot in a homology sphere is homotopically ribbon if and only if its closed monodromy extends over a handlebody. (A ribbon knot is homotopically ribbon.) This reduces the study of ribbon fibered knots to that of automorphisms of a handlebody. In this spirit Casson and Long [5] developed an algorithm to determine if a surface automorphism is null-cobordant, and found a new example of an algebraically slice but nonribbon fibered knot. (Also see Bonahon [3].) In this paper we study ribbon fibered knots in light of the theorem of Casson-Gordon, but by a method different from Casson and Long’s. Our method is suitable for detecting nonsimple nonribbon fibered knots, while theirs is restricted to simple ones. (A knot is said to be simple if every incompressible torus in the knot exterior is boundary parallel.)

A closed monodromy f of a ribbon fibered knot K preserves a certain “pattern” h (a union of 2-submanifolds) induced by a decomposition of the knot exterior of K . Now f extends over a handlebody H . If (H, h) admits a characteristic submanifold in the sense of Johannson, then it is kept invariant by the extension of f . We thoroughly study the position of the characteristic submanifold in H to see the relation among the prime summands of K . Our working question is as follows:

Received by the editors June 20, 1991.

1991 *Mathematics Subject Classification.* Primary 57M25; Secondary 57N60.

Key words and phrases. Ribbon, homotopically ribbon, fibered knot, monodromy, Johannson’s characteristic submanifold.

Question 1. Suppose that K_1, \dots, K_n are prime fibered knots in homology spheres and $K_1 \# \dots \# K_n$ is homotopically ribbon. Under what circumstances can we pair $\{1, \dots, n\}$ into $\{i_s, j_s\}_{s=1}^m$ (so that $2m = n$) such that $K_{i_s} \cong -K_{j_s}$?

Theorem 8.6. *There is a class \mathcal{K} of fibered knots in homotopy spheres giving an affirmative answer to Question 1. The class \mathcal{K} contains the iterated torus knots, any fibered knot with an irreducible Alexander polynomial, and any iterated cable about such a knot.*

As a result of this theorem the following knots are algebraically sliced but not ribbon. (The algebraically-sliceness is verified by Livingston and Melvin [22] and Kawachi [20], respectively.)

- (1) Example 1 in §8.1, the connected sum of four iterated torus knots

$$(2, 13; 2, 3) \# (2, 15) \# (2, -15; 2, -3) \# (2, -13).$$

- (2) Example 2 in §8.1, a $(2, 1)$ cable of a $-$ amphicheiral fibered knot with an irreducible Alexander polynomial (e.g., the $(2, 1)$ cable of the figure eight knot).

By assuming $K_1 \cong \dots \cong K_n$ Question 1 relates to the famous question: does the classical knot concordance group contain any nontrivial elements of finite order other than 2? Replacing the assumption of sliceness by homotopically ribbon, we ask:

Question 2. Let K be a fibered knot in a homology sphere. If nK (n -fold connected sum of K) is homotopically ribbon, is $2K$ homotopically ribbon?

Theorem 5.8. *Question 2 has an affirmative answer if every slice knot in a homology sphere bounding a contractible 4-manifold is homotopically ribbon.*

Let J be a knot in $S^1 \times D^2 (\subset S^3)$, and k a knot in a homology sphere M . Then $J(k)$ denotes the knot $f(J)$ in M where $f: S^1 \times D^2 \rightarrow N(k) (\subset M)$ is the canonical framing of k which sends the core of $S^1 \times D^2$ to k preserving orientations. The knot J is said to be a (p, q) cable of $S^1 \times D^2$ if J is a $p(\text{longitude}) + q(\text{meridian})$ curve on $\partial(S^1 \times \frac{1}{2})$.

Theorem 8.5. *Let K be a (p, q) cable of a fibered knot k in a homology sphere where $p > 1$. If nK is homotopically ribbon for some $n > 0$, then (1) so is K , (2) $q = \pm 1$; k is homotopically ribbon if $p > 2$.*

There is a fibered knot k in a homology sphere such that k is not homotopically ribbon, but the $(2, 1)$ cable of k is (Example 3 in §8.1), so that the assumption $p > 2$ in (2) above is necessary. However, for a cable knot in a homotopy sphere the assumption $p > 2$ can be omitted if 3-manifold groups are residually finite (Theorem 8.5, part 3). It is easy to see a $(p, 1)$ cable of a homotopically ribbon fibered knot is homotopically ribbon. Thus, these results prove that only such a knot is a homotopically ribbon, cable fibered knot in a homotopy sphere if Thurston's geometrization conjecture is true.

Question 3. If a (p, q) cable of a classical knot k is ribbon (resp. slice) where $p > 1$, is k ribbon (resp. slice) and $q = \pm 1$?

This paper is organized as follows. Sections 1 through 4 are preliminary parts. Section 1 relates homotopically ribbon concordance and compression

bodies. The precise definition of the monodromy of a fibered knot in a homotopy sphere with unknotted holes is given in §2. We determine the fibered knots with periodic monodromies also in §2. Given a fibered knot in a homology sphere, a torus decomposition of the knot exterior breaks the monodromy into periodic or irreducible automorphisms. In §3, we study the properties of each automorphism, which are used to characterize the action of a closed monodromy on a boundary pattern h in §5.1. In §4, a “dual knot” J^* is defined for certain fibered knots J in a homotopy sphere with unknotted holes; then $J(J^*(0))$ is homotopically ribbon. Sections 5 through 7 are the main parts of our work. Theorems 5.2–5.4, stated in §5.1, are the fundamental results, which are proved in §§5, 6, 7 respectively. All results stated in the introduction are consequences of the fundamental theorems or the arguments used in the proofs. Section 8 deals with the specific cases, e.g., cable knots.

I would like to thank my supervisor Professor Cameron Gordon for helpful conversations and his encouragement. Thanks are also due to Professor Akio Kawauchi who informed me that the $(2, 1)$ cable of the figure eight knot is algebraically slice, but possibly nonslice.

1. HOMOTOPICALLY RIBBON CONCORDANCE

All manifolds including knots and surfaces are suitably oriented and smooth. For a manifold M , $-M$ is obtained from M by reversing its orientation. All diffeomorphisms are orientation preserving. For an automorphism f of a manifold X , $-f$ denotes the automorphism f on $-X$. The symbol \cong means diffeomorphic. The symbol \simeq means isotopic. Unlabeled homomorphisms are induced by inclusions unless otherwise stated.

Let K_i , $i = 0, 1$, be a knot in a homology 3-sphere M_i . We write $K_0 \cong K_1$ if $(M_0, K_0) \cong (M_1, K_1)$. We denote by $K_0 \# K_1$ the connected sum $(M_0, K_0) \# (M_1, K_1)$. The definitions below are based on a homotopy property of ribbon concordance between classical knots [13, Lemma 3.1]. We shall write $(M_1, K_1) \geq (M_0, K_0)$ or simply $K_1 \geq K_0$ if there are a 4-manifold X^4 with the homology groups of S^3 and an annulus A in X^4 such that

$$\begin{aligned} (\partial X^4, A \cap \partial X^4) &= (M_1, K_1) \amalg (-M_0, -K_0), \\ \pi_1(M_1 - K_1) \rightarrow \pi_1(X^4 - A) &\text{ is surjective, and} \\ \pi_1(M_0 - K_0) \rightarrow \pi_1(X^4 - A) &\text{ is injective.} \end{aligned}$$

We say K_1 is *homotopically ribbon concordant* to K_0 . A knot K_1 is *homotopically ribbon* if $K_1 \geq 0$, the unknot in S^3 .

Let K be a knot in a homology 3-sphere M . A knot K is *trivial* if K bounds a disk in M . A nontrivial knot K is *prime* if K cannot be written as $K_0 \# K_1$ where neither of K_i is a trivial knot in S^3 . Let F be a compact connected surface. We say that K is a *fibered knot* in M with *fiber* F if there is an automorphism $f: F \rightarrow F$ (the *monodromy* of K) such that $\partial F \cong S^1$, $f|_{\partial F} = \text{id}$, and the mapping torus $F \times I / f$ ($\cong F \times [0, 1] / ((f(x), 0) \sim (x, 1))$ for all x in F) is diffeomorphic to $\overline{M - N(K)}$ where $x_0 \times I / f$, $x_0 \in \partial F$, is a meridian of K . Note that $\overline{M - N(K)}$ is a Haken manifold.

Lemma 1.1. *Let $K_1 \geq K_2 \geq \dots$ be a nonincreasing sequence of fibered knots in homology spheres. Then there is some m such that $K_n = K_m$ for all $n \geq m$.*

Proof. Since the knot exterior of K_i is Haken, Lemmas 3.2 and 3.4 in Gordon [13] hold. Thus there is some m such that $\deg \Delta_m(t) = \deg \Delta_n(t)$ for all $n \geq m$, where $\Delta_i(t)$ denotes the Alexander polynomial of K_i . The commutator subgroup of a fibered knot group is free and hence transfinitely nilpotent [13, p. 157], hence Lemma 3.4 shows that $K_n = K_m$ for all $n \geq m$. \square

A *compression body* B is a cobordism $\text{rel } \partial$ between surfaces $\partial_+ B$ and $\partial_- B$ such that $B \cong \partial_+ B \times I \cup 2\text{-handles} \cup 3\text{-handles}$, where handles are attached to $\partial_+ B \times 1$, and $\partial_- B$ has no 2-sphere components. A compression body is *trivial* if $B = \partial_+ B \times I$.

Lemma 1.2. *Let f, F, K, M be as above. Suppose that f extends to an automorphism φ of a compression body B so that $\partial_+ B = F$, $\partial_- B$ is connected and $\varphi|_{\partial F \times I} = \text{id}$. Then there is a fibered knot K' in a homology 3-sphere M' such that $\varphi|_{\partial_- B}$ is its monodromy and $(M, K) \geq (M', K')$. Furthermore, $\Delta_1(t) \doteq \Delta_0(t)f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t]$ where Δ_1, Δ_0 denote the Alexander polynomials of K, K' respectively.*

Proof. Glue $S^1 \times D^2$ to the mapping torus of $\varphi|_{\partial_- B}: \partial_- B \rightarrow \partial_- B$ along their boundaries so that a simple loop $p \times I/\varphi$ is a meridian of $S^1 \times D^2$ for all $p \in \partial(\partial_- B)$. The resulting manifold M' say, is a homology 3-sphere containing a fibered knot $S^1 \times 0$ whose monodromy is $\varphi|_{\partial_- B}: \partial_- B \rightarrow \partial_- B$. Let $K' = S^1 \times 0 \subset M'$. Glue $S^1 \times D^2 \times I$ to $B \times I/\varphi$ so that for all $t \in I$ and $q \in \partial F$, $S^1 \times \partial D^2 \times t = \partial F \times t \times I/\varphi$ and the simple loop $q \times t \times I/\varphi$ is a meridian of $S^1 \times D^2 \times t$. Call this 4-manifold X^4 and let $A = S^1 \times 0 \times I \subset X^4$. Then

$$(\partial X^4, \partial A) = (M, K) \cup (M', K').$$

Since $\pi_1(\partial_+ B) \rightarrow \pi_1(B)$ is surjective and $\pi_1(\partial_- B) \rightarrow \pi_1(B)$ is injective, it follows that $\pi_1(M - K) \rightarrow \pi_1(X^4 - A)$ is surjective and $\pi_1(M' - K') \rightarrow \pi_1(X^4 - A)$ is injective.

The statement about Δ_0 and Δ_1 follows from the argument in the remark of [1, p. 541]. (Gilmer [10, Proposition 1.4] proves this inequality for ribbon concordance between classical knots.) \square

Lemma 1.3. *Let M, M' be as in Lemma 1.2. Suppose that 3-manifold groups are residually finite. Then $\pi_1(M) = 1$ implies $\pi_1(M') = 1$.*

Proof. Let B be as in Lemma 1.2 and let $\partial_- B = F'$; then $B = (F' \times I) \cup^m 1\text{-handles}$. Let y_1, \dots, y_m be elements of $\pi_1(B)$ corresponding to the 1-handles in B and let x_1, \dots, x_n be free generators of $\pi_1(F')$, so that $\pi_1(B) = \langle x_1, \dots, x_n, y_1, \dots, y_m \rangle$. Let φ_* denote the isomorphism of $\pi_1(B)$ induced by φ . Setting $s_i = tu_i t^{-1} \varphi_*(u_i)^{-1}$ where $u_i = x_i$ (for $1 \leq i \leq n$) or $= y_{i-n}$ (for $n+1 \leq i \leq n+m$), we then have

$$\pi_1(B \times I/\varphi) = (\pi_1(B) * \langle t \rangle) / N(s_1, \dots, s_{n+m})$$

where $N(\dots)$ is normal closure. Note that t corresponds to a loop $p \times I/\varphi$, $p \in \partial F'$.

Defining X^4 as in the proof of Lemma 1.2, we have the epimorphism from $\pi_1(B \times I/\varphi)$ to $\pi_1(X^4)$ induced by the quotient map, and have the group presentation

$$\pi_1(X^4) = (\langle x_1, \dots, x_n \rangle * \langle y_1, \dots, y_m \rangle) / N(\bar{s}_1, \dots, \bar{s}_{n+m})$$

where \bar{s}_i is the image of s_i . Since $\varphi(F') = F'$, $\varphi_*(x_i)$ is a word in $\{x_j\}$, so

$$\langle x_1, \dots, x_n \rangle / N(\bar{s}_1, \dots, \bar{s}_n) \cong \pi_1(M').$$

Let $G = \pi_1(M')$. We then have

$$\pi_1(X^4) = (G * \langle y_1, \dots, y_m \rangle) / N(\bar{s}_{n+1}, \dots, \bar{s}_{n+m}).$$

Since $\pi_1(F) \rightarrow \pi_1(B)$ is surjective, the first row homomorphism in the commutative diagram below is surjective. (The vertical arrows are induced by the quotient maps.) Hence, the homomorphism $\pi_1(M) \rightarrow \pi_1(X^4)$ is also surjective. It follows from $\pi_1(M) = 1$ that $\pi_1(X^4) = 1$.

$$\begin{array}{ccc} \pi_1(F \times I / \varphi) & \longrightarrow & \pi_1(B \times I / \varphi) \\ \downarrow & & \downarrow \\ \pi_1(M) & \longrightarrow & \pi_1(X^4) \end{array}$$

Let us show $G = 1$. Since $H_1(M')$ is trivial, $\{\bar{s}_{n+i}\}_{i=1}^m$ is an independent system of equations in $\{y_i\}$. By our assumption G is residually finite. Reference [9] proves that G is a subgroup of $\pi_1(X^4)$, thus $G = 1$. \square

Suppose that Thurston's geometrization conjecture holds. Then the Poincaré conjecture holds and 3-manifold groups are residually finite. Then, by Lemmas 1.2 and 1.3 the compression of the monodromy (in the way described in 1.2) of a fibered knot in S^3 gives rise to a homotopically ribbon concordance between classical knots.

2. FIBERED KNOTS IN HOMOTOPY SPHERES WITH UNKNOTTED HOLES

For a homotopy sphere P and a nonnegative integer s let $P(s)$ denote the exterior of an s -component trivial (possibly empty) link, $\bigcup_{i=1}^s l_i$, in P . We call $P(s)$ a *homotopy sphere with unknotted holes*. Let J be a simple loop in $P(s)$. We say that J is a *fibered knot* in $P(s)$ with *fiber* F if there is an automorphism f of F and a diffeomorphism $g: (F \times I)/f \rightarrow \overline{P(s)} - N(J)$ such that under this identification the following hold.

1. $(\partial F \times 0) \cap \partial N(J)$ is a preferred longitude of J in P ;
2. $(\partial F \times 0) \cap \partial P(s)$ consists of meridians of $\bigcup l_i$ in P .

Condition 2 shows that the fibration of $\overline{P(s)} - N(J)$ extends over $\overline{P} - N(J)$, i.e. we can regard J as a fibered knot in P . Let $\partial_* F$ be the components of ∂F such that $\partial_* F \times 0 = (\partial F \times 0) \cap N(J)$. Note that the definition of $\partial_* F$ depends on J . The automorphism f acts on the set of components of $\partial F - \partial_* F$ and divide it into orbits. The number of orbits is s ($= |\partial P(s)|$). Let $\partial_1 F, \dots, \partial_s F$ be representatives of the orbits and let $m(i)$ be the cardinality of the orbit containing $\partial_i F$, $1 \leq i \leq s$. We say that $f: (F, \partial_* F) \rightarrow (F, \partial_* F)$ is the *monodromy* of J if the following hold.

1. $f|_{\partial_* F} = \text{id}$, and $p \times I/f$ is a meridian of J , where $p \in \partial_* F$;
2. for $1 \leq i \leq s$ $f^{m(i)}|_{\partial_i F} = \text{id}$ and the simple loop $(\bigcup_{j=1}^{m(i)} f^j(p_i) \times I)/f$ is a preferred longitude of l_i where $p_i \in \partial_i F$.

Remark. If J is a fibered knot in $P(s) = \overline{P - \bigcup_{i=1}^s N(l_i)}$, then in the sense of Eisenbud and Neumann [8] the multilink $(P, 1J \cup 0l_1 \cup \dots \cup 0l_s)$ is fibered. Hence, as is pointed out in [8, p. 34], the fibration is unique up to isotopy. Similarly a fibration for a fibered knot in a homology sphere is unique up to isotopy.

Let J_i , $i = 0, 1$, be fibered knots in homotopy spheres with unknotted holes M_i . We write $J_0 \cong J_1$ as fibered knots if there is a diffeomorphism from (M_0, J_0) to (M_1, J_1) which preserves the fibrations. The knot $-J_i$ denotes the knot $(-M_i, -J_i)$.

Lemma 2.1. *Let $f_i: F_i \rightarrow F_i$ be the monodromy of a fibered knot J_i in a homotopy sphere with unknotted holes M_i , $i = 0, 1$. Suppose that $f_0 \amalg -f_1$ extends to an automorphism φ of a 3-manifold U such that*

1. $U \cong F_0 \times I$; this diffeomorphism restricts to $F_0 \xrightarrow{\text{id}} F_0 \times 0$ and $-F_1 \cong F_0 \times 1$,
2. φ is the identity on $\partial_* F_0 \times I$, and
3. $\partial_* F_0 \times 1 = \partial_* F_1$.

Then $J_0 \cong J_1$ as fibered knots.

Proof. The natural projection from $F_0 \times 0$ to $-(F_0 \times 1)$ induces a diffeomorphism $g: F_0 \rightarrow F_1$ such that $g(\partial_* F_0) = \partial_* F_1$ and $gf_0g^{-1} \simeq f_1 \text{ rel } \partial_* F_1$ [26, Lemma 3.5]. By condition 1 of the definition of monodromy, g induces a diffeomorphism from $\overline{M_0 - N(J_0)}$ to $\overline{M_1 - N(J_1)}$ carrying a meridian of J_0 to that of J_1 . It follows that $J_0 \cong J_1$ as fibered knots. \square

Lemma 2.2. *Let $f_i: F_i \rightarrow F_i$, J_i , M_i , $i = 0, 1$, be as in Lemma 2.1. Let b be a component of $\partial F_0 - \partial_* F_0$ and m be the smallest positive integer such that $f_0^m(b) = b$. Suppose that $f_0 \amalg -f_1$ extends to an automorphism φ of a 3-manifold U which satisfies conditions 1 and 2 of Lemma 2.1. Then, φ^m can be isotoped rel $F_0 \times \{0, 1\}$ so that φ^m is the identity on $(\partial_* F_0 \cup b) \times I$.*

Proof. Let x_0 be a point on b . By the definition of monodromy φ^m fixes $x_0 \times \{0, 1\}$ ($\subset F_0 \times I$). To prove the lemma it suffices to show that $\varphi^m(x_0 \times I)$ is rel $x_0 \times \{0, 1\}$ isotopic to $x_0 \times I$ in $b \times I$. Let n be the winding number of $\varphi^m(x_0 \times I) \text{ rel } x_0 \times \partial I$ in the annulus $b \times I$. We shall show $n = 0$. The natural projection from $F_0 \times 0$ to $-(F_1 \times 0)$ induces a diffeomorphism $g: \overline{M_0 - N(J_0)}$ to $\overline{M_1 - N(J_1)}$. Let T_0, T_1 be the components of $\overline{M_i - N(J_i)}$, $i = 0, 1$, containing $b, g(b)$ respectively. Let λ_i, μ_i be a longitude-meridian system on T_i determined by the trivial link removed from a homotopy sphere. We first assume $g(\partial N(J_0)) = \partial N(J_1)$, i.e., condition 3 of Lemma 2.1 holds. Then g sends a meridian of J_0 to that of J_1 , and hence g extends to a diffeomorphism from M_0 to M_1 . On the other hand g carries λ_0 , a longitude of T_0 , to a simple loop $\lambda_1 + n\mu_1$ in T_1 . Since λ_i is null-homologous (in fact, null-homotopic) in M_i , we obtain $n[\mu_1] = [\lambda_1 + n\mu_1] = [g(\lambda_0)] = 0 \in H_1(M_1)$. It follows that $n = 0$, proving the lemma.

Next assume $T = g(\partial N(J_0)) \neq \partial N(J_1)$. Then g sends a meridian of J_0 to a longitude of T , say λ . Glue $S^1 \times D^2$ to $\overline{M_1 - N(J_1)}$ along T so that λ is null-homotopic in $S^1 \times D^2$. Let M be the resulting manifold; g extends to a diffeomorphism from M_0 to M . Then $[\lambda_1 + n\mu_1] = [g(\lambda_0)] = 0 \in H_1(M)$. We claim that λ_1 is null-homologous in M , establishing Lemma 2.2 as above. In

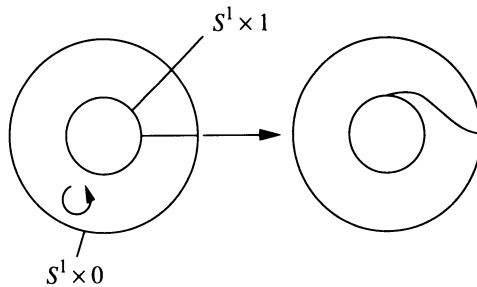


FIGURE 1

$\overline{M_1 - N(J_1)}$ the longitude λ_1 of T_1 is homologous to a multiple of a meridian of J_1 . However, since the fiber F_1 meets T in a single loop, a meridian of J_1 is homologous to λ in $\overline{M_1 - N(J_1)}$. Hence the claim follows from the fact that λ is null-homologous in M . \square

In the rest of this section we shall describe periodic monodromies of fibered knots in homotopy spheres with unknotted holes. In fact, these monodromies turn out to be those of torus knots in $S^1 \times D^2$ or S^3 .

Let g be a periodic automorphism of a compact surface G , and let $x \in G$ be a fixed point of g . Then there are a neighborhood B of x and a diffeomorphism h from B to the unit disk of R^2 such that $h(0) = 0$ and hgh^{-1} is the rotation of angle θ around 0. Define a *fixed point data* of x to be $\theta/2\pi$, which is determined up to addition of integers. Thus we consider that a fixed point data takes its value in Q/Z . [2] or [7] shows that if G is a closed surface and the set of fixed point data of g^n for all n are paired to cancel each other, then $g|G: G \rightarrow G$ is periodically null-cobordant. However, our monodromies are not necessarily defined on closed surfaces. We need the notion of an r -Dehn twist for a rational number r as follows.

Let A be an annulus with $\partial A = \alpha_0 \cup \alpha_1$. An automorphism ϕ of A is an *r -Dehn twist* from α_0 to α_1 if there is a diffeomorphism $h: S^1 \times I \rightarrow A$ such that (see Figure 1):

1. $h(S^1 \times i) = \alpha_i$ for $i = 0, 1$;
2. the orientation of $S^1 \times 0$ induced from $S^1 \times I$ is counterclockwise;
3. $h\phi h^{-1}(\theta, t) = (\theta + rt, t)$.

Lemma 2.3. *Let $f: F \rightarrow F$ be the nontrivial periodic monodromy of a fibered knot J in $P(s)$, where P is a homotopy sphere and $s \geq 0$. Then assertion 1 or 2 below holds for some relatively prime $p > 1$ and q .*

1. J is the (p, q) torus knot in S^3 with genus of $F = (p-1)(q-1)/2$ and $|\partial F| = 1$.
2. J is the (p, q) cable of $S^1 \times D^2 (\subset S^3)$ with genus of

$$F = (p-1)(q-1)/2 \quad \text{and} \quad |\partial F| = p+1.$$

Proof. Let $M = \overline{P(s) - N(J)}$. Recall that (1) $M = F \times I / f$ is a Seifert fibered manifold each of whose fiber is $(\bigcup_k f^k(x) \times I) / f$ for some $x \in F$, and (2) a meridian μ of J in $\partial N(J)$ meets F in a single point ($\in \partial_* F$). By (1) the Seifert fibration of M extends over $\overline{P - N(J)}$. If μ is isotopic in $\partial N(J)$

to a fiber of the fibration of M , then f has a fixed point in ∂F . Therefore $f = \text{id}$, a contradiction. The Seifert fibration of M thus extends over $P(s)$, and hence over P . It follows that $P = S^3$, $s = 0$ or 1 , and J is a fiber of some Seifert fibration of S^3 . This completes the proof. \square

Investigating the intersections of F and a singular fiber of S^3 and those of ∂F and a regular fiber of S^3 , we obtain the following results.

Lemma 2.4. *Let $f: F \rightarrow F$ be the monodromy of the (p, q) torus knot in S^3 in part 1 or the (p, q) cable of $S^1 \times D^2$ in part 2, where $p > 1$.*

1. *Take a collar neighborhood N of ∂F . Then after an isotopy $\text{rel } \partial F$ f satisfies the following.*

- (a) $f(N) = N$ and $f|_{F-N}$ is a periodic map of order $|pq|$; $f^k|_{F-N}$ has a fixed point only if k is a multiple of p or q .
- (b) $f|_N$ is a $1/pq$ -Dehn twist from ∂F to $\partial N - F$.
- (c) $f^p|_{F-N}$ has p fixed points of data p/q ($\in \mathbb{Q}/\mathbb{Z}$).
- (d) $f^q|_{F-N}$ has $|q|$ fixed points of data q/p .

2. *Take pairwise disjoint collar neighborhoods N_0, \dots, N_p of ∂F such that $N_0 \supset \partial_* F$. Then after an isotopy $\text{rel } \partial F$ f satisfies the following.*

- (a) $f(\bigcup N_i) = \bigcup N_i$ and $f|_{F-\bigcup N_i}$ is a periodic map of order $|pq|$; $f^k|_{F-\bigcup N_i}$ has a fixed point only if k is a multiple of p .
- (b) $f|_{N_0}$ is a $1/pq$ -Dehn twist from $\partial_* F$ to $\partial N_0 - \partial_* F$.
- (c) $f^q|_{F-\bigcup N_i}$ has $|q|$ fixed points of data q/p .
- (d) $f^p|_{N_i}$ is a $-p/q$ -Dehn twist from $\partial N_i \cap \partial F$ to $\partial N_i - \partial F$ for $1 \leq i \leq p$.
- (e) f permutes $\partial F - \partial_* F$ cyclically.

Lemma 2.5. *Let $f: F \rightarrow F$ be the monodromy of a torus knot or a cable of $S^1 \times D^2$. Let c be the component of ∂F invariant under f . Denote $F \cup_{c=\partial} D^2$ by G . Then the automorphism $g = f \cup \text{id}$ of G is isotopic to an irreducible periodic map.*

Proof. It suffices to check that g is irreducible. After an isotopy F/f is a disk with at most two cone points or an annulus with at most 1 cone point as an orbifold. Thus, g can be isotoped so that G/g is a sphere with at most 3 cone points or a disk with at most 2 cone points. Hence, g is irreducible. \square

3. DECOMPOSING MONODROMY OF FIBERED KNOTS

Let $f: F \rightarrow F$ be the monodromy of a fibered knot K in a homology sphere M . Let \mathcal{T} be a minimal (possibly empty) set of incompressible tori in $X = \overline{M - N(K)}$ which decomposes X into a Seifert fibered manifold and a simple manifold. By Jaco and Shalen [18] or Johannson [19], $\bigcup \mathcal{T}$ is determined up to isotopy. Isotop $\bigcup \mathcal{T}$ so that $|F \cap \bigcup \mathcal{T}|$ is minimal and let $C = F \cap \bigcup \mathcal{T}$, a set of essential simple loops in F . By a homological argument we see that C consists of separating loops in F . Thus for the closure of any component of $F - C$, S say, exactly one component of ∂S is equal to ∂F , or separates ∂F from the other components of ∂S . We call this component the *root component* of ∂S , and denote it by $\partial_r S$. Let us break up f into simpler monodromies by using C in F .

Lemma 3.1. *Let \mathcal{S} denote the set of closures of components of $F - C$. After an isotopy rel ∂F of f the monodromy f satisfies the following.*

1. $f(C) = C$.
2. *Let $S \in \mathcal{S}$ such that $\partial S \supset \partial F$. If K is a connected sum of n prime knots with $n \geq 2$, then S is a disk with n punctures and $f|_S = \text{id}$.*

For $S \in \mathcal{S}$ let f_S denote the automorphism $f^m|_S$ of S where m is the least positive integer such that $f^m(S) = S$. Call f_S the *adjoint automorphism* of S . The lemmas below describe the properties of f_S .

Lemma 3.2. *With the data of Lemma 3.1 f satisfies the following.*

3. *For any $S \in \mathcal{S}$ f_S is irreducible or isotopic to a periodic map.*
4. *Take $R, (\neq)S \in \mathcal{S}$. If f^k preserves $R \cup S$ for some $k > 0$ and $f^k|R \cup S$ is isotopic to a periodic map on $R \cup S$, then $R \cap S = \emptyset$.*

Proof of Lemma 3.1. Under the natural identification $\overline{M - N(K \cup F)} \cong F \times I$ the set of annuli $\bigcup \mathcal{F} - N(F)$ can be isotoped to $C \times I$ because of the minimality of $|F \cap \bigcup \mathcal{F}|$. This completes part 1.

Since K is a connected sum of n prime knots, there are $n - 1$ pairwise nonparallel, separating, essential annuli A_1, \dots, A_{n-1} in $\overline{M - N(K)} = X$ such that each component of ∂A_i is a meridian of K . Let C_n be the regular neighborhood of $\partial X \cup (\bigcup_{i=1}^{n-1} A_i)$ in X ; then $C_n \cong (\text{a disk with } n \text{ punctures}) \times S^1$ is an essential Seifert fibered space in $(X, \partial X)$ in the sense of [19]. By [19, 10.10] $(C_n, \partial X)$ can be isotoped into the characteristic submanifold of $(X, \partial X)$. Now $S \times I/f = N$ say, is the closure of the component of $X - \bigcup \mathcal{F}$ containing ∂X . We can regard $(N, \partial X)$ as a Seifert fibered manifold component of the characteristic submanifold of $(X, \partial X)$, so that $C_n \subset N$ after an isotopy rel ∂X . Isotop the incompressible tori $\partial C_n - \partial X$ in N so as to be saturated by the fibration of N . Since C_n has a unique Seifert fibration up to isotopy, the Seifert fibration of N is an extension of that of C_n . With respect to the fibration of C_n , $\partial S \cap \partial X$ (a preferred longitude of K) meets each S^1 -fiber on ∂X in a single point. Therefore each S^1 -fiber of N meets S in a single point, so that $f|_S \simeq \text{id rel } \partial F$. Since M does not contain nonseparating tori, S is a punctured disk. If S is a disk with more than n punctures, then K is a connected sum of more than n nontrivial knots, a contradiction. Hence, S is a disk with n punctures. \square

Proof of Lemma 3.2. If f_S is reducible, then $S \times I/f_S$ (the closure of a component of $X - \bigcup \mathcal{F}$) contains an essential torus, and hence is a Seifert fibered manifold. Thus f_S is isotopic to a periodic map, completing part 3. Suppose that part 4 is false. Without loss of generality assume $R \cup S = \partial_r S$. Then $T = \partial_r S \times I/f_R$ is an element of \mathcal{F} , and by the assumption of part 4 $(R \times I/f_R) \cup_T (S \times I/f_S)$ is a boundary irreducible Seifert manifold in X . Thus, $\bigcup \mathcal{F} - T$ divides X into a Seifert fibered manifold and a simple manifold, contradicting the minimality of \mathcal{F} . \square

If K is a fibered knot in a homotopy sphere, then we can obtain more information about f_S .

Lemma 3.3. *With the data of Lemma 3.2 assume that M is a homotopy 3-sphere. Then after an isotopy rel ∂F f satisfies the following (in addition to 1–4 above).*

5. For any $S \in \mathcal{S}$ f_S is either the identity on a punctured disk, or irreducible and not isotopic to the identity.
6. For any $S \in \mathcal{S}$ $f_S: (S, \partial_r S) \rightarrow (S, \partial_r S)$ is the monodromy of a fibered knot in a homotopy sphere with unknotted holes Y , (so that $\partial_r S = \partial_* S$). Moreover, if $M = S^3$, then Y is the exterior of a trivial link in S^3 .

Proof. Let $N = S \times I / f_S$, the closure of a component of $M - \bigcup \mathcal{S}$. Let $T_0 = \partial_r S \times I / f_S$, T_1, \dots, T_s be the boundary components of ∂N and let E_i , $0 \leq i \leq s$, be the closures of components of $M - T_i$ such that $E_i \cap \text{Int } N = \emptyset$. Then E_0 is a homotopy solid torus containing K , and for $i > 0$, E_i is the exterior of a nontrivial fibered knot in a homotopy sphere. Let μ be a simple loop on T_0 null-homotopic in E_0 . By a homological argument μ meets $\partial_r S$ in a single point (after an isotopy in T_0).

Proof of part 6. Glue $S^1 \times D^2$ to N along T_0 so that μ is null-homotopic in $S^1 \times D^2$. Then each T_i is compressible in $N \cup S^1 \times D^2$. Let $N' = N \cup S^1 \times D^2$ and $K' = S^1 \times 0 \subset N'$. There are simple loops $c_i \subset T_i$, $1 \leq i \leq s$, bounding disks in N' . By a cut-and-paste argument these disks can be taken to be pairwise disjoint. Then gluing solid tori $(S^1 \times D^2)_i$, $1 \leq i \leq s$, to N' along $\partial N'$ so that each component of $\partial S - \partial_r S$ bounds a disk in some $(S^1 \times D^2)_i$, we obtain a homotopy sphere M' , say. By a homological argument c_i is a longitude of $(S^1 \times D^2)_i$, so $\bigcup_{i=1}^s (S^1 \times 0)_i$ is a trivial link in M' . It follows that K' is a fibered knot in $M'(s) = N'$ with $\partial_* S = \partial_r S$. If $M = S^3$, then E_0 and $\overline{M - E_i}$, $1 \leq i \leq s$, are solid tori. Replacing E_1 by a solid torus in the manner above, we still have S^3 . Thus induction on s proves $M' \cong S^3$.

Now we show that f can be isotoped so that f_S is the monodromy of K' . Let $R \in \mathcal{S}$ be adjacent to S with $R \cap S = \partial_r S$. Assume that f_R is the monodromy of the fibered knot (Y, K'') (say) which is obtained from $R \times I / f_R$ in the same way as $(M'(s), K')$ from $S \times I / f_S$. Then by condition 2 of the definition of monodromy the simple loop $\mu' = (\bigcup_j (f_R)^j(p) \times I) / f_R$, where $p \in \partial_r S$, is null-homotopic in Y , and hence in E_0 . Since $\mu' = p \times I / f_S$, μ' is a meridian of K' in $M'(s)$. This implies that f_S satisfies condition 1 of the definition of monodromy. An isotopy of f_S in a neighborhood of $\partial S - \partial_r S$ can deform f_S to be the monodromy of K' of $M'(s)$. Let m be the least positive integer such that $f^m|_S = f_S$. Since $f^k(S) \cap S = \emptyset$ for $1 \leq k < m$, the isotopy of f_S can be induced by an isotopy of f in a neighborhood of $\partial S - \partial_r S$ in F . Thus part 6 of Lemma 3.3 is proved by deforming f on the closure of each component of $F - C$ beginning with the component containing ∂F .

Proof of part 5. If $f_S \simeq \text{id}$, then S is a punctured disk. By part 6 any simple loop $p \times I / f_S$, $p \in \partial S$, is null-homotopic in N' , the exterior of a trivial link in S^3 . Hence each such a loop is an S^1 -fiber of $N \cong S \times S^1$, so that $f_S \simeq \text{id} \text{ rel } \partial S$. If f_S is reducible, then N is a boundary irreducible Seifert fibered submanifold in $\overline{M - N(K)}$ by part 3 of Lemma 3.2. Since N contains an essential torus, by the proof of Lemma 2.1 in Swarup [24] N is diffeomorphic to (a punctured disk) $\times S^1$ such that the simple loop μ is an S^1 -fiber. Since $|\mu \cap \partial_r S| = 1$, each S^1 -fiber of N meets S in a single point, so that $f_S \simeq \text{id}$. Therefore, we can isotop f so that parts 5 and 6 hold. \square

Suppose J is a fibered knot in $P(n)$ where P is a homotopy sphere and $n > 0$, and k_i is a fibered knot in a homology sphere M_i , $1 \leq i \leq n$. Numbering the components of $\partial P(n)$ from 1 to n , we can uniquely define the fibered knot $J(k_1, \dots, k_n)$ as follows. Let M be the homology sphere obtained from $P(n)$ by gluing each $\overline{M_i - N(k_i)}$ to the i th component of $\partial P(n)$ so that longitude-meridian pairs of $\bigcup k_i$ correspond to meridian-longitude pairs of the solid tori $\overline{P - P(n)}$. Then let $J(k_1, \dots, k_n)$ be the knot J in M .

Conversely, we define a satellite presentation for a fibered knot K in a homotopy sphere M as follows. Let T_1, \dots, T_n be incompressible tori in $\overline{M - N(K)}$ such that no T_i separates T_j ($i \neq j$) from $\partial N(K)$ in M . By the proof of Lemma 3.3 $\bigcup T_i$ divides M into $M'(n)$ containing K where M' is a homotopy sphere, and the disjoint union of the exterior of some nontrivial fibered knots k_i , $1 \leq i \leq n$, in homotopy spheres. Let J denote the fibered knot K in $M'(n)$; then we write $K = J(k_1, \dots, k_n)$, a *satellite presentation* of K . Note that this presentation depends on the choice of incompressible tori and the numbering of the chosen tori.

4. J AND J^*

Let J_i , $i = 0, 1$, be fibered knots with fibers F_i in homotopy spheres with unknotted holes. Let $f_i: F_i \rightarrow F_i$ be the monodromy of J_i . We write $J_1 = J_0^*$ if there is a diffeomorphism $g: F_0 \rightarrow -F_1$ such that $g(\partial_* F_0) \neq \partial_* F_1$ and $g f_0 g^{-1} \simeq -f_1 \text{ rel } \partial_* F_1$. In general, J_0^* is not uniquely determined by J_0 . But if J_0^* exists, some component, say b , of $\partial F_0 - \partial_* F_0$ is fixed by f_0 such that $f_0(b) = b$ and J_0^* is equivalent to the fibered knot with monodromy $f_0: (F_0, b) \rightarrow (F_0, b)$. Hence, when f_0 leaves at most one component of $\partial F_0 - \partial_* F_0$ invariant (e.g., $J_0 \subset S^1 \times D^2$), J_0^* is uniquely determined if it exists. If $J_1 = J_0^*$, then f_0 leaves $g(\partial_* F_0)$ invariant, so Lemma 2.2 shows $g f_0 g^{-1} \simeq -f_1 \text{ rel } \partial_* F_1 \cup g(\partial_* F_0)$. Hence $J_0 \cong J_1^*$.

If J is a fibered knot in $S^1 \times D^2$ intersecting some meridional disk once, then $J^* \cong -J$. For a nontrivial example of a pair of J and J^* see [10].

Proposition 4.1. *Let J be a fibered knot in a homotopy solid torus V . If J^* exists, then the following hold.*

1. J represents a generator of $H_1(V)$.
2. Let $K = J_1(J(J^*(k)))$, where J_1, k are arbitrary fibered knots in a homotopy solid torus, a homology sphere, respectively. Then $K \geq J_1(k)$. In particular $J(J^*(0)) \geq 0$, where 0 is the unknot is S^3 .

Proof. Let F_0 (resp. F_1) be the fiber of J (resp. J^*) and let $f_0: F_0 \rightarrow F_0$ (resp. $f_1: F_1 \rightarrow F_1$) be the monodromy of J (resp. J^*). Let $g: F_0 \rightarrow -F_1$ be a diffeomorphism used to define J^* . Since the ambient space of J has only one boundary component, f_0 acts on the components of $\partial F_0 - \partial_* F_0$ cyclically. Since f_0 leaves $g^{-1}(\partial_* F_1)$ invariant, $\partial F_0 - \partial_* F_0$ has just one component, proving part 1. By the definition of J^* and Lemma 2.2 the automorphism $f_0 \amalg f_1$ of $F_0 \amalg F_1$ extends to an automorphism φ of $F_1 \times I$ such that $\varphi|_{F_1 \times 1} = f_1$, $\varphi|_{(F_1 \times 0)} = g f_0 g^{-1}$, $\varphi|_{(\partial_* F_1 \cup g(\partial_* F_0)) \times I} = \text{id}$. Let F be the fiber of $J_1(k)$ with monodromy f . Take a separating annulus N on F such that after an isotopy $f|_N = \text{id}$ and $f|_{\overline{F - N}}$ breaks up into the monodromies of k and J_1 . Glue $F_1 \times I$ to $F \times I$ by identifying $g(\partial_* F_0) \times I$ and N , and we get

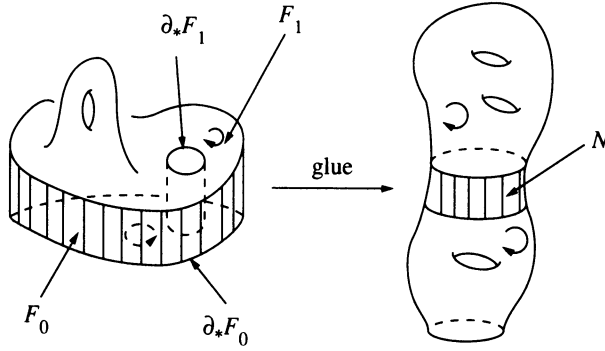


FIGURE 2

a compression body B (Figure 2). Then $\psi = \varphi \cup f \times \text{id}$ is an automorphism of B and $\psi|_{\partial_+ B}$ (resp. $\psi|_{\partial_- B}$) is the monodromy of K (resp. $J_1(k)$). Part 2 follows from Lemma 1.2. \square

Reference [10] defines J^* for certain (possibly nonfibered) knots in $S^1 \times D^2$, and proves that $J(J^*(0))$ is slice. It also shows that $J(J^*) \subset S^1 \times D^2 \times 1$ and the core of $S^1 \times D^2 \times 0$ bound an annulus in $S^1 \times D^2 \times I$ with no local maxima. Hence, if k is a knot in S^3 , then $J_1(J(J^*(k)))$ is ribbon concordant to $J(k)$, and in particular $J(J^*(0))$ is a ribbon knot.

5

Given a homotopically ribbon, fibered knot, its closed monodromy extends over a handlebody H . In §5.1, by using a torus decomposition of the knot exterior a boundary pattern h is defined for H . Then, the fundamental results Theorems 5.2–5.4 are stated in terms of (H, h) . Lemma 5.10 reduces the proofs of the theorems to the case when h is a useful boundary pattern in the sense of Johansson [19]; then (H, h) has a characteristic submanifold. In §§5.2, 5.3 we shall study the position of a characteristic submanifold for (H, h) in the handlebody H , and prove Theorem 5.2. Sections 6 and 7 are devoted to the proofs of Theorems 5.3 and 5.4, respectively.

5.1. Fundamental theorems. Let K_i , $1 \leq i \leq n$, be prime fibered knots in homology spheres with fibers F_i . Let K be the fibered knot $K_1 \# \cdots \# K_n$ in a homology sphere M with monodromy $f: F \rightarrow F$. Suppose that K is homotopically ribbon. As in §3 define a set of simple loops, C , in F via a torus decomposition of $\overline{M - N(K)}$ and isotop $f \text{ rel } \partial F$ so as to satisfy Lemmas 3.1, 3.2 (and 3.3 if M is a homotopy sphere). If $n \geq 2$, by Lemma 3.1, part 2 the closure of the component of $F - C$ containing ∂F is a disk with n punctures, say D_n , and $f|_{D_n} = \text{id}$. The components of $F - D_n$ can be identified with the fibers F_1, \dots, F_n . Define \hat{F} to be the closed surface $F \cup_{\partial} D^2$ and $\hat{f} = f \cup \text{id}$, which is called the *closed monodromy* of K .

We define h_K , a set of 2-submanifolds in \hat{F} , as follows. Define B_{h_K} (or simply B_h) to be the closure of component of $\hat{F} - C$ containing ∂F and call this a *base* of h_K . In particular, when $n > 1$, $B_h = D_n \cup_{\partial} D^2$, so denote B_h

by \widehat{D}_n . (This is an abuse of notation; \widehat{D}_n is not a closed surface.) Then let

$$h_K = \begin{cases} \{ \text{the components of } N(C), \text{ the components of } \overline{F' - N(C)}, B_h \} & \text{if } B_h \text{ is not an annulus (e.g., } n > 2), \\ \{ \text{the components of } N(C - \partial B_h), \text{ the components of } \overline{F' - N(C - \partial B_h)}, B_h \} & \text{if } B_h \text{ is an annulus (e.g., } n = 2), \end{cases}$$

where $F' = \overline{F - B_h}$ and $N(\cdots)$ denotes a regular neighborhood in F' (Figure 3).

First isotop \hat{f} rel C so that $\hat{f}(h_K) = h_K$. By Casson and Gordon [4, Theorem 5.1] $\hat{f}: \widehat{F} \rightarrow \widehat{F}$ extends to an automorphism φ of a handlebody H_K , say. Then h_K is a complete boundary pattern of H_K and φ is an admissible automorphism of (H_K, h_K) in the sense of Johannson [19, p. 19].

For a component $S \in h_K - \{B_h\}$ define the *root component* $\partial_r S$ of ∂S to be the component of ∂S separating B_h from the other components of ∂S in ∂H_K . For $S \in h_K$ let m be the least positive integer such that $\hat{f}^m(S) = S$. We call $\hat{f}^m|_S: S \rightarrow S$ the *adjoint automorphism* of S , and denote it by \hat{f}_S . These are natural extensions of the definitions of a root boundary component and an adjoint automorphism for the closure of a component of $F - C$ in §3. Let us deform \hat{f} by using the lemmas in §3.

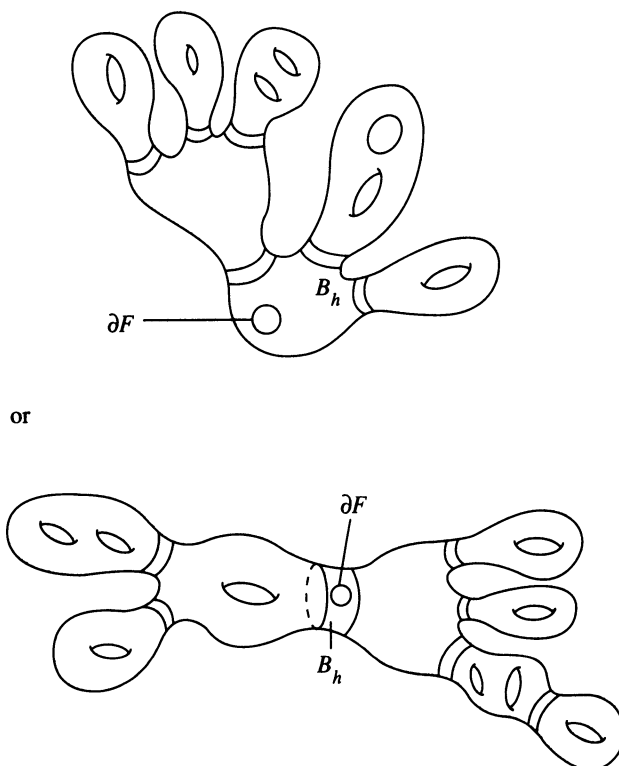


FIGURE 3. The boundary pattern of h

Proposition 5.1. *After an isotopy $\hat{f}: (\hat{F}, h_K) \rightarrow (\hat{F}, h_K)$ satisfies the following. Let M denote the ambient manifold of K .*

1. $\hat{f}(h_K) = h_K$; $\hat{f}_A = \text{id}$ for any annulus component $A \in h_K - \{B_h\}$.
2. $\hat{f}(B_h) = B_h$, $\hat{f}|_{\hat{D}_n} = \text{id}$.
3. For any $S \in h_K - \{B_h\}$ $\hat{f}_S: S \rightarrow S$ is irreducible or isotopic to a periodic map.
4. (a) Take $R, (\neq)S \in h_K - \{B_h\}$ adjacent to the same annulus component $A \in h_K - \{B_h\}$. If f^k preserves $R \cup A \cup S$ for some $k > 0$, then $f^k|R \cup A \cup S$ is not isotopic to a periodic map on $R \cup A \cup S$.
(b) If $n > 2$, then (a) holds also when R or $S = \hat{D}_n$.
5. If M is a homotopy sphere, then for any $S \in h_K - \{B_h\}$ \hat{f}_S is either the identity on a punctured disk, or irreducible and not isotopic to the identity.
6. If M is a homotopy sphere, then for any $S \in h_K - \{B_h\}$ $\hat{f}_S: (S, \partial_r S) \rightarrow (S, \partial_r S)$ is the monodromy of a fibered knot in a homotopy sphere with unknotted holes Y (say), so that $\partial_r S = \partial_* S$. Moreover, if $M = S^3$, then Y is the exterior of a trivial link in S^3 .

Proof. These are direct consequences of assertions 1–6 in Lemmas 3.1–3.3. We only remark on the exclusion of B_h . Let B'_h be the closure of the component of $F - C$ containing ∂F ; then $B_h = B'_h \cup D^2$. In general, even if $f|_{B'_h}$ is irreducible (resp. not isotopic to a periodic map), $\hat{f}|_{B_h}$ is possibly reducible (resp. isotopic to a periodic map). But both $f|_{B'_h}$ and $\hat{f}|_{B_h}$ are the identity when $n > 1$ (cf. part 4(b)). \square

Here we can state our fundamental theorems.

Theorem 5.2. *Let K_1, \dots, K_n ($n \geq 2$) be prime fibered knots in homology spheres such that $K = K_1 \# \dots \# K_n \geq 0$. Define H_K, \hat{D}_n as above.*

1. *Assume that $\partial H_K - \hat{D}_n$ is incompressible in H_K . Then $\{1, \dots, n\}$ can be paired into $\coprod_{s=1}^m \{i_s, j_s\}$ such that*

$$K_{i_s} \# K_{j_s} \geq 0 \quad \text{for all } s.$$

2. *There are (possibly composite or trivial) fibered knots \tilde{K}_i , $1 \leq i \leq n$, in homology spheres satisfying the following.*
 - (a) $K_i \geq \tilde{K}_i$ and $K' = \#_{i=1}^n \tilde{K}_i \geq 0$.
 - (b) K' satisfies the assumption in part 1, i.e., $\partial H_{K'} - \hat{D}_l$ is incompressible in $H_{K'}$ where l is the number of factors in the prime decomposition of K' .

For a composite fibered knot in a homotopy sphere, we can obtain finer results under a weaker assumption stated in terms of the boundary pattern h_K . Specifically, the assumption of Theorem 5.2 part 1 is stronger than needed in this case.

A component $S \in h_K$ is *reducible* if $\hat{f}_S: S \rightarrow S$ is reducible. If S is not an annulus and \hat{f}_S is isotopic to the identity, then S is called a *trivial* component of h . For nonannulus components S_0, S_1 of $h_K - \{B_h\}$ take the fibered knots J_0, J_1 with monodromies $\hat{f}_{S_i}: (S_i, \partial_r S_i) \rightarrow (S_i, \partial_r S_i)$. We write $S_1 = S_0^*$ if $J_1 \cong J_0^*$.

Theorem 5.3. *Let K_1, \dots, K_n ($n \geq 2$) be prime fibered knots in homotopy spheres such that $K = K_1 \# \dots \# K_n \geq 0$. Define F_i , $1 \leq i \leq n$, H_K , h_K , \widehat{D}_n as above. Assume that each nontrivial component of h_K is incompressible in H_K . Then the following hold.*

1. *For each i , $1 \leq i \leq n$, there are prime fibered knots K_j^i , $1 \leq j \leq n(i)$, in homotopy spheres (or S^3 if K_i is a knot in S^3) such that $K_i \geq \#_{j=1}^{n(i)} K_j^i$ and $\{K_j^i\}_{i,j}$ can be paired into $\coprod_{s=1}^m \{K_{j_s}^{i_s}, K_{j'_s}^{i'_s}\}$ such that $K_{j_s}^{i_s} \cong -K_{j'_s}^{i'_s}$.*
2. *Suppose that some K_{i_0} does not have a satellite presentation $J(k\# -k)$ or $J_1(J(J^*(k)))$ where $k \not\cong 0$ and $J \not\cong (S^1 \times D^2, S^1 \times 0)$. Then we may choose $n(i_0) = 1$ and $K_{i_0} = K_1^{i_0}$ in part 1. The second assumption can be replaced by the following: there are no $S_0, S_1 \in h_K \cap F_{i_0}$ such that S_0 and S_1 are adjacent via an annulus, and $S_1 = S_0^*$.*

Remark. It is not difficult to see $J(k\# -k) \geq J(0)$. Proposition 4.1 shows $J_1(J(J^*(k))) \geq J_1(k)$. (In each inequality, if the first knot is in S^3 , so is the second one and “ \geq ” is ribbon concordance [11].) These facts justify the assumptions in part 2. In fact, $\#_j K_j^i$ in part 1 is obtained from K_i by performing these two types of reductions repeatedly.

Theorem 5.4. *Let $K = J(k)$ be a fibered knot in a homotopy sphere such that $K \geq 0$, $k \not\cong 0$, and J is a cable of $S^1 \times D^2$. If each nontrivial component of h_K is incompressible in H_K , then $J \cong (S^1 \times D^2, S^1 \times 0)$, a trivial cable.*

Remark. Theorem 5.4 does not necessarily hold for a cable knot in a homology sphere. Let N be the homology sphere obtained by a $-\frac{1}{2}$ -Dehn surgery of S^3 on the figure eight knot. Let k denote the core of the glued solid torus, and $K (\subset N)$ the $(2, 1)$ cable of k . Then the closed monodromy of K extends over a handlebody with the assumption of Theorem 5.4 satisfied. For the proof see Example 3 in §8.1.

Proof of Theorem 5.2, part 2. Take a maximal compression body B of $\partial H_K - \widehat{D}_n$ in H_K . Now $\partial H_K - \widehat{D}_n$ is the disjoint union of the fibers of K_i , $1 \leq i \leq n$, so B consists of n compression bodies B_1, \dots, B_n such that $\partial_+ B_i = F_i$. Since B is maximal, $\partial_- B_i$ is incompressible in H_K , hence $\partial_- B_i$ has no closed surface components. This implies that $\partial_- B_i$ is connected for each i . Recall that we denote by φ an extension of the closed monodromy of K over H_K . The maximality of B implies that $\varphi(B) = B$ after an isotopy, so that $\varphi(B_i) = B_i$. Let \tilde{K}_i be the fibered knot in a homology sphere with monodromy $\varphi|_{\partial_- B_i}$. By Lemma 1.2 $\varphi|_{B_i}: B_i \rightarrow B_i$ induces a homotopically ribbon concordance $K_i \geq \tilde{K}_i$. Let $H' = \overline{H_K - B}$. Then $\varphi|_{H'}$ is an automorphism of the handlebody H' and $\varphi|_{\partial H'}$ is the closed monodromy of the fibered knot $\#_{i=1}^n \tilde{K}_i$, therefore $\#^n \tilde{K}_i \geq 0$ (assertion (a)). We have already shown that $\partial H' - \widehat{D}_n = \coprod_i \partial_- B_i$ is incompressible in H' , implying assertion (b). \square

Our main task is to prove Theorem 5.2, part 1, and Theorems 5.3 and 5.4. We first see that several results in the Introduction are immediate consequences of these theorems.

Theorem 5.5. *Let K_1, \dots, K_n be prime fibered knots in homology (resp. homotopy) spheres M_i such that $K = K_1 \# \dots \# K_n \geq 0$. If each K_i satisfies either condition 1 or 2 below, then $\{1, \dots, n\}$ can be paired into $\coprod_{s=1}^m \{i_s, j_s\}$ such that $K_{i_s} \# K_{j_s} \geq 0$ (resp. $K_{i_s} \cong -K_{j_s}$).*

1. K_i is minimal with respect to “ \leq ” among all fibered knots in homology spheres.
2. There is no $f(t) \in \mathbb{Z}[t] - \{t^k\}_k$ such that $f(t)f(t^{-1}) | \Delta_{K_i}(t)$.

Proof. By Lemma 1.2 each of conditions 1 and 2 implies that the monodromy of K_i does not extend over a nontrivial compression body B with $\partial_- B$ connected. From the proof of Theorem 5.2 part 2 $\partial H_K - \widehat{D}_n$ has a trivial maximal compression body, hence it is incompressible in H_K . Applying Theorem 5.2 part 1 to K proves Theorem 5.5 for knots in homology spheres. Now suppose all M_i are homotopy spheres. If each K_i is minimal, by the remark just after Theorem 5.3 K_i cannot have the satellite presentations precluded in the hypothesis of Theorem 5.3 part 2 for any i . Then, applying Theorem 5.3 part 2 to K gives the desired pairings. To complete the proof let us show condition 2 implies condition 1. Suppose K_i does not satisfy condition 1. Then K_i is homotopically ribbon concordant to some fibered knot L with $\deg \Delta_{K_i} > \deg \Delta_L$ (cf. the proof of Lemma 1.1). Hence by [1, p. 541] $\Delta_{K_i}(t)$ is divisible by a nontrivial Laurent polynomial $f(t)f(t^{-1})$, violating condition 1. \square

Theorem 5.6. *Let K be a (possibly composite) nontrivial fibered knot in a homology (resp. homotopy) sphere such that $nK \geq 0$ for some $n > 0$. If K satisfies either condition 1 or 2 in Theorem 5.5, then $2K \geq 0$ (resp. K is $-$ amphicheiral).*

Proof. Without loss of generality assume that n is even. (If necessary, take $2n$.) Let $K = k_1 \# \dots \# k_l$ be a prime decomposition of K . Then each k_i satisfies condition 1 or 2 in Theorem 5.5. Before applying Theorem 5.5 to nK , identify a set of l vertices $V = \{1, \dots, l\}$ with $\{k_1, \dots, k_l\}$. The pairings given by Theorem 5.5 yield a set of unoriented edges E for V by connecting i and $j \in V$ such that $k_i \# k_j \geq 0$; the degree of each vertex is n . In other words, (E, V) is an n -regular graph. By Petersen’s theorem [16, p. 36] even regular graphs are 2-factorable. This means that the $2l$ knots $k_1, k_1, k_2, k_2, \dots, k_l, k_l$ can be paired into $\coprod_{s=1}^l \{k_{i_s}, k_{j_s}\}$ such that $k_{i_s} \# k_{j_s} \geq 0$. Therefore $2K \geq 0$. Now assume that K is a knot in a homotopy sphere; then $k_{i_s} \cong -k_{j_s}$. If $i_s \neq j_s$ for some s , then K is not minimal, contradicting conditions 1 and 2. It follows $k_i \cong -k_i$ for $1 \leq i \leq l$, hence $K \cong -K$. \square

The following question, an analog of the ribbon-slice problem, is a slight modification of Question 4.22(B) in [21].

Question 5.7. Let K be a knot in a homology sphere M . Suppose that M bounds a contractible 4-manifold V and K bounds a disk in V . Then, is K homotopically ribbon?

An affirmative answer to this question improves Theorem 5.6 dramatically.

Theorem 5.8. *Let K be a fibered knot in a homology sphere such that $nK \geq 0$ for some $n > 0$. If Question 5.7 has an affirmative answer, then $2K \geq 0$.*

Corollary 5.9. *If Question 5.7 has an affirmative answer and the 4-dimensional Poincaré conjecture holds, then the order of a classical knot concordance class represented by a fibered knot is 1, 2, or ∞ .*

Proof of Theorem 5.8. Let M be the ambient manifold of K . By Lemma 1.1 there is a fibered knot K' in a homology sphere M' which is minimal among all fibered knots $\leq K$. Let V be a homology 4-ball such that $\partial V = nM$ and nK bounds a homotopically ribbon disk in V . Let W be a 4-manifold such that $\partial W = nM \amalg n(-M')$, $nK \geq nK'$ in W . Set $X^4 = -V \cup_{nM} W$. Since $(1 =) \pi_1(M) \rightarrow \pi_1(V)$ and $\pi_1(M) \rightarrow \pi_1(W)$ are surjective, X^4 is contractible. The fibered knot nK' bounds a smooth disk in $-X^4$. The theorem's assumption implies $nK' \geq 0$. Therefore, Theorem 5.8 follows from Theorem 5.6. \square

Our first step toward Theorems 5.2–5.4 is to prove the following lemma.

Lemma 5.10. *The proofs of Theorems 5.2, part 1, 5.3, and 5.4 reduce to the case when every component of h_K is incompressible in H_K .*

Proof. We first show the assertion about Theorem 5.3. Suppose that Theorem 5.3 holds for K if every component of h_K is incompressible in H_K . Let $\alpha(K) = |h_K|$. We then prove Theorem 5.3 by induction on $\alpha(K)$. Since $n \geq 2$, $\alpha(K) \geq 3$. When $\alpha(K) = 3$, h_K contains only nontrivial components, hence Theorem 5.3 holds in this case.

Take a fibered knot $K = K_1 \# \cdots \# K_n$ satisfying the assumption of Theorem 5.3. Assume that Theorem 5.3 holds for all K' such that $\alpha(K') < \alpha(K)$. Without loss of generality there is a trivial component, S , of h_K compressible in H_K . By Proposition 5.1 parts 4 and 5 trivial components cannot be adjacent to each other via annuli. Combining this with the theorem's assumption, we see that a boundary of a compression disk D for S is not parallel to ∂S in ∂H .

Case 1. $S = \widehat{D}_n$. The above observation implies $n > 2$. Without loss of generality, on ∂H_K the circle ∂D separates $\partial H_K - \widehat{D}_n = \amalg_{i=1}^n F_i$ into F_1, \dots, F_l and F_{l+1}, \dots, F_n , where $1 < l < n-1$. Isotop $\varphi \text{ rel } \partial H_K$ so as to fix D . Then the closure of components of $H_K - D$ consists of two handlebodies H_1 and H_2 . The induced automorphisms, φ_i , of ∂H_i are the closed monodromies of $\#^l K_i = L_1$ and $\#_{l+1}^n K_i = L_2$; note that $h_{L_i} - \{B_{L_i}\} \subset h_K - \{\widehat{D}_n\}$. Then for $i = 1, 2$ we have $L_i \geq 0$, $\alpha(L_i) < \alpha(K)$ and every nontrivial component of h_{L_i} is incompressible in H_{L_i} . By the induction hypothesis Theorem 5.3 holds for L_1 and L_2 , and hence for K .

Case 2. $S \neq \widehat{D}_n$. Let $s = |\partial S| - 1 (> 1)$. Without loss of generality $S \subset F_1$. Then K_1 has a satellite presentation $J_0(k_1 \# \cdots \# k_s)$ where k_i is a prime knot. Note that J_0 is prime, for K_1 is prime. Since $\{\partial \varphi^i(D)\}_i$ is a set of disjoint simple loops in ∂H_K , after an isotopy $\{\varphi^i(D)\}_i$ consists of disjoint disks in H_K . Cutting H_K along $\bigcup_i \varphi^i(D)$, we obtain handlebodies H_0, H_1, \dots, H_m such that $\varphi(H_0) = H_0$; then $\varphi^m(H_1) = H_1$. Without loss of generality we see that $\varphi|\partial H_0$ (resp. $\varphi^m|\partial H_1$) is the monodromy of $J_0(k_1 \# \cdots \# k_l) \# K_2 \# \cdots \# K_n = K'$, say, (resp. $k_{l+1} \# \cdots \# k_s = L$, say), where $1 < l < s$. Hence, by Casson and Gordon [4, Theorem 5.1] $K' \geq 0$ and $L \geq 0$. Since the boundary patterns $h_{K'}$, h_L are induced from h_K , it follows that $\alpha(K') < \alpha(K)$, $\alpha(L) < \alpha(K)$, and both $h_{K'}$ and h_L satisfy the assumption of Theorem 5.3.

A regular neighborhood, B , of $F_1 \cup (\bigcup_{i=1}^m H_i)$ in H_K is a compression body such that $\varphi(B) = B$, $\partial_+ B = F_1$, and $\partial_- B$ is the fiber of a prime knot

$J_0(k_1 \# \cdots \# k_l) = K'_1$, say. It follows from Lemma 1.2 that $K_1 \geq K'_1$. Applying Theorem 5.3 to K' shows that Theorem 5.3, part 1 holds for K . Next suppose that K satisfies the assumption of Theorem 5.3, part 2; then so does $L = k_{l+1} \# \cdots \# k_s$. By applying Theorem 5.3, part 2 to L , we obtain $-k_s \cong k_j$ for some $l+1 \leq j \leq s$. Therefore K_1 has a presentation $J(k_s \# -k_s)$ for some J , a contradiction. Theorem 5.3, part 2 holds for K .

The assertion about Theorem 5.2, part 1 and Theorem 5.4 can be proved by the similar arguments. To prove the reduction of Theorem 5.2, part 1 (resp. Theorem 5.4) it suffices to consider only Case 1 (resp. Case 2) above. This completes the proof of Lemma 5.10. \square

Therefore, to prove Theorems 5.2–5.4 we may assume that every component of h_K is incompressible in H_K . Let us see that h_K is a useful, complete boundary pattern in the sense of [19]. Suppose not; then there is an i -faced disk D , $1 \leq i \leq 3$, in (H_K, h_K) . Since each boundary component of a surface in h_K is separating in ∂H_K , D is 1-faced or 2-faced. Recalling that every nonannulus component of h_K is adjacent to an annulus in h_K , isotop D so that ∂D is contained in a component of h_K . Then such a component is compressible in H_K , a contradiction.

Since h_K is useful, by [19, 9.4] (H_K, h_K) has a characteristic submanifold X_K , which consists of essential I -bundles and essential Seifert fibered manifolds. Define U_K to be the union of I -bundle components of X_K not homeomorphic to $S^1 \times D^2$, and let $V_K = X_K - U_K$. We shall see that V_K consists of essential Seifert fibered manifolds. Since φ is an admissible automorphism of (H_K, h_K) , $\varphi(X_K)$ is also a characteristic submanifold for (H_K, h_K) . By [19, 10.9] we can isotop φ admissibly with respect to h_K so that $\varphi(X_K) = X_K$. After this isotopy, we have $\varphi(U_K) = U_K$, $\varphi(V_K) = V_K$. For simplicity let H, h, X, U, V denote H_K, h_K, X_K, U_K, V_K respectively.

In the next two subsections we shall study the position of U and V in the handlebody H . We shall use the terminology of Johannson [19]. Refer to [19] for the definitions of essential surfaces, admissible I -bundles, admissibly parallel, complete manifolds, etc.

5.2. Position of U and V (1). Let K be a fibered knot in a homology sphere such that $K \geq 0$. Define $n, f, \varphi, B_h, \widehat{D}_n, H, h, X, U, V$ as in §5.1; in particular, we assume that h is a useful boundary pattern. At the end of this subsection the proofs of Theorems 5.2–5.4 reduce to the case when U contains no annulus components of $h - \{B_h\}$.

Lemma 5.11. 1. *Each component of $\text{Fr } X$ is an essential annulus in (H, h) such that each boundary component separates ∂H . Furthermore, each component of $\text{Fr } X$ separates H .*

2. *V consists of essential Seifert fibered manifolds in (H, h) .*

Proof. Let M be a component of X . If M is an essential Seifert fibered manifold, then $\text{Fr } M$ consists of essential annuli or essential tori by the definition. Since a handlebody contains no essential tori, each component of $\text{Fr } M$ is an essential annulus. Suppose that M is an I -bundle. If $\text{Fr } M$ contains a component other than an annulus, then for some S_1 in $h \cap \partial M$ contains a square component connecting the lids of M . Let S_2 be a component of h containing one of the lids of M ; then S_2 cannot be an annulus. Since S_2 is adjacent to

S_1 , S_1 is an annulus. By part 2 of 10.6 of [19] the characteristic submanifold X contains the annulus component of h entirely, hence S_1 is not an annulus, a contradiction. Thus $\text{Fr } X$ consists of annuli.

Let N denote the ambient manifold of K . Let c be a component of $\partial(\text{Fr } X)$, an essential simple loop in ∂H . Since $\varphi(X) = X$, for some positive integer k we have $\varphi^k(c) = c$. Thus c is a component of the intersection of F and the torus $(\bigcup_{i=1}^k \varphi^i(c) \times I) / \varphi$ in $N - N(K)$. Since this torus separates N , the union of simple loops $\bigcup_{i=1}^k \varphi^i(c)$ separates F . If $k = 1$, the simple loop c separates F . If $k > 1$ and c is nonseparating, then homology calculation shows $H_1(N)$ has a nontrivial torsion element, contradicting the fact that N is a homology sphere. Thus c separates F .

Let A be a component of $\text{Fr } X$. Then $\partial H - \partial A$ has three components. Let G be the union of two components of $\partial H - \partial A$ each of which has only one boundary component. If A did not separate H , then there would be a loop in H meeting the closed surface $G \cup A$ in a single point. Since $H \subset S^3$, this is a contradiction. This completes the proof of part 1.

Let M be an I -bundle component of X homeomorphic to $S^1 \times D^2$. By part 1 $M \cap \partial H$ consists of tori or annuli. Thus we may regard M as an essential Seifert fibered manifold in (H, h) , completing the proof of part 2. \square

Lemma 5.12. *If T is a component of the lids of U , then T is contained in a nonannulus component, S , of h such that T is isotopic to S in ∂H , or T is contained in a reducible component of h .*

Proof. By the definitions of U and an admissible I -bundle each component of the lids of U is contained in a nonannulus component of h . Suppose that S is an irreducible component of h , and T is one of the lids of U contained in S . Then T is a component of $X \cap S$. Note that $\bigcup_i (\hat{f}_S)^i(\partial T)$ consists of pairwise disjoint, essential simple loops in S and is kept invariant under the adjoint automorphism \hat{f}_S . Hence $\bigcup_i (\hat{f}_S)^i(\partial T)$ is isotopic to ∂S , so that T is isotopic to S in ∂H . \square

Lemma 5.13. *All components of U are admissible product I -bundles.*

Proof. Suppose that a component U_0 of U is a twisted I -bundle. Let T be the lid of U_0 . Then $\overline{(U_0 \cap \partial H) - T}$ is a (possibly empty) set of annuli each of which connects different components of ∂T . If there is an annulus, then some component of ∂T is nonseparating in ∂H , contradicting Lemma 5.11. It follows that $U_0 \cap \partial H = T$, which is connected. Since each component of ∂T separates ∂H , each component of $\text{Fr } U_0$ is separating in $\partial(\overline{H - U_0})$. Hence $U_0 (\subset H)$ satisfies the hypothesis of the lemma below.

Lemma 5.14. *Let M be a 3-manifold in a handlebody H such that $\text{Fr } M$ consists of separating incompressible annuli each of which does not separate ∂M but $\partial(\overline{H - M})$. Let $\alpha_1, \dots, \alpha_m$ be the cores of components of $\text{Fr } M$. Then M is a handlebody such that $\{\alpha_1, \dots, \alpha_m\}$ is primitive, i.e., there are disjoint disks D_1, \dots, D_m in M such that $|\alpha_i \cap D_j| = \delta_{ij}$.*

The proof of Lemma 5.14 is deferred until after the proof of Lemma 5.13 we are in. Identify U_0 with a twisted I -bundle $S \times_\tau I$ where S is a connected nonorientable surface. It follows from Lemma 5.14 that the set of cores of

$\text{Fr } U_0$ is primitive for the handlebody U_0 . Then each component of $\partial S \times_\tau \{\frac{1}{2}\}$ represents a generator of $H_1(S \times_\tau I)$, absurd. Lemma 5.13 is proved. \square

Lemma 5.14 is one of the key lemmas in this paper. It is unnecessarily strong for proving Lemma 5.13, but we shall need it essentially in §5.3.

Proof of Lemma 5.14. Let A_1, \dots, A_m be the components of $\text{Fr } M$ such that the core of A_i is α_i . Identifying A_i with $(S^1 \times I)_i$ (a copy of $S^1 \times I$), let $B_i = (\gamma \times I)_i$ where γ is a half circle of S^1 . Let M_1, \dots, M_m be the components of $\overline{H - M}$. Denote the copy of A_i in $\overline{H - M}$ by A'_i ; then A'_i separates ∂M_i . Also denote the copy of B_i in M_i by B'_i ($1 \leq i \leq m$). Let H' be the 3-manifold obtained from M and M_i , $1 \leq i \leq m$, by pasting B_i to B'_i for all i naturally. Then $H \cong H' \cup \bigcup_{i=1}^m h_i$ where h_i is a 2-handle with $h_i \cap H' = \overline{A_i - B_i} \cup \overline{A'_i - B'_i}$. Note that for any subset C of $\{1, \dots, m\}$ the manifold $H' \cup \bigcup_{i \in C} h_i$ is homotopy equivalent to the disk sum of $H'' = \overline{H - \bigcup_{i \notin C} M_i}$ and all M_i , $i \notin C$. Since A_i is incompressible in H , the groups $\pi_1(H'')$, $\pi_1(M_i)$ and $\pi_1(M)$ are subgroups of the free group $\pi_1(H)$. Hence $H' \cup (\bigcup_{i \in C} h_i)$ is a handlebody for any C , and so is M .

Let c_1, \dots, c_m ($\subset \partial H'$) denote the attaching circles of the 2-handles h_i such that $|c_i \cap B_j| = 2\delta_{ij}$. Then, by [14, Theorem 1] we see that $\{c_1, \dots, c_m\}$ is primitive for the handlebody H' . Take disjoint disks D_1, \dots, D_m in H' such that $|c_i \cap \partial D_j| = \delta_{ij}$, $c_i \cap \partial D_j \subset M$, and $|(\bigcup B_i) \cap (\bigcup D_j)|$ is minimal among all sets of such disks. Let us show $\bigcup D_j \subset M$, which completes the proof. Suppose not; then there is an arc β , a component of $B_i \cap D_j$ for some i, j . Assume that β bounds a disk Δ in B_i with $\Delta \cap c_i = \emptyset$. Compressing $\bigcup D_j$ in H' via Δ yields a set of dual disks for $\{c_i\}$ which has fewer intersection with $\bigcup B_i$ than $\bigcup D_j$ has, a contradiction. Hence, after an isotopy of $\bigcup D_j$, $B_i \cap D_j$ consists of arcs $p \times I$ ($\subset B_i$) where $p \in \gamma$ (cf. Figure 4).

Without loss of generality let $(\bigcup B_i) \cap D_1 \neq \emptyset$. Consider the components of

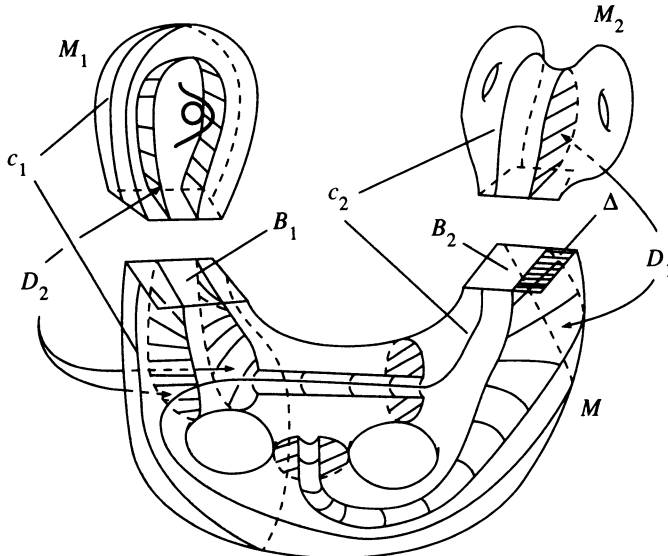


FIGURE 4

$D_1 - \bigcup B_i$ in D_1 and let E be the closure of an outermost one in D_1 such that

$$(*) \quad c_i \cap \partial E = \emptyset \quad \text{for all } i.$$

The existence of E is guaranteed by $|(\bigcup c_i) \cap \partial D_1| = 1$. The arc $\partial E - \partial D_1$ is contained in some B_{i_0} ; it follows that $|\alpha_j \cap \partial E| = \delta_{i_0 j}$ for all j . Note that $E \subset M$ or $E \subset M_{i_0}$. However the latter implies that the components of $\partial A'_{i_0}$ are connected by the arc $\partial E \cap \partial D_1$ in ∂M_{i_0} , a contradiction. Hence, $E \subset M$. Let E' be the closure of the component of $D_1 - \bigcup B_i$ adjacent to E . Then $E' \subset M_{i_0}$ and $\partial E' - \partial D_1 \subset B_{i_0}$. Since $c_1 \cap D_1 \subset M$, $\partial E' - \partial D_1$ consists of more than one arcs in B_{i_0} . Therefore some component of $\partial E' - \partial D_1$ does not bound an outermost disk of $D_1 - \bigcup B_i$ with the property $(*)$ above. In other words we have shown the following.

Assertion. *Let \mathcal{C} be the set of arcs $B_{i_0} \cap (\bigcup D_j)$. Let \mathcal{C}' be the subset of \mathcal{C} such that each $\beta \in \mathcal{C}'$ bounds an outermost disk of $D_j - \bigcup B_i$ in some D_j which does not meet any c_i . Then $\mathcal{C}' \neq \emptyset$ and $\mathcal{C} - \mathcal{C}' \neq \emptyset$.*

We shall deduce a contradiction from Assertion. Under an identification $B_{i_0} \cong I \times I$ each arc in \mathcal{C} is $x \times I$ for some x and $\alpha_{i_0} \cap B_{i_0} = I \times \frac{1}{2}$. It follows from Assertion that there are $\beta_i = x_i \times I \in \mathcal{C}$, $i = 1, 2$, such that $\beta_1 \in \mathcal{C}'$, $\beta_2 \in \mathcal{C} - \mathcal{C}'$, and no elements of \mathcal{C} are in $(x_1, x_2) \times I$ (without loss of generality $x_1 < x_2$). Set $D = (x_1, x_2) \times I$. Suppose that D_{j_k} contains β_k for $k = 1, 2$. Let E_1 be the closure of the outermost disk of $D_{j_1} - \bigcup B_i$ bounded by β_1 such that $(\bigcup c_i) \cap E_1 = \emptyset$. Let E_2 be the closure of a component of $D_{j_2} - \beta_2$ meeting c_{j_2} . Let $D'_{j_2} = E_1 \cup D \cup E_2$ and push it off D_{j_1} and D . Then $|(\bigcup B_i) \cap D'_{j_2}| < |(\bigcup B_i) \cap D_{j_2}|$. Define $D'_i = D_i$ if $i \neq j_2$. The disjoint disks D'_i , $1 \leq i \leq m$, satisfy $c_i \cap \partial D'_j = c_i \cap \partial D_j$ for all i, j and have fewer intersections with $\bigcup B_i$ than $\bigcup D_j$ has. This is a contradiction, completing the proof of Lemma 5.14. \square

Lemma 5.15. *The Seifert fibered manifold V in (H, h) satisfies the following.*

1. V consists of solid tori with $V \cap \partial H$ longitudinal annuli in V .
2. No pair of components of $V \cap \partial H$ are isotopic to each other in ∂H .
3. Let T be a component of $V \cap \partial H$. Then one of the following holds.
 - (a) T is a regular neighborhood of an annulus component of h .
 - (b) T is contained in a reducible component, S , of h such that the core of T is not parallel to ∂S in ∂H .

Proof. Let V_0 be a component of V . Each component of $\text{Fr } V_0$ is nonseparating in ∂V_0 . Since each component of $\partial(\text{Fr } V_0)$ separates ∂H , there is a component N of $\overline{H - V_0}$ such that the annulus $A_0 = N \cap V_0$ separates ∂N . Note that $V_0 \cup N$ is a handlebody, and V_0 satisfies the hypothesis on M in Lemma 5.11. It follows that V_0 is a solid torus with A_0 longitudinal in V_0 , completing the proof of part 1.

Assume that there are two components T_1, T_2 of $V \cap \partial H$ whose cores are parallel in ∂H . Let A be an annulus component of $\overline{\partial H - T_1 \cup T_2}$. If $A \cap U \neq \emptyset$, A contains at least one of the lids of U . Then some component of U is an I -bundle over an annulus, a contradiction to the definition of U . Hence $A \cap U = \emptyset$. With an adequate boundary pattern, $V' = V \cup N(A)$ (where $N(\dots)$ is a regular neighborhood in H) can be an essential Seifert fibered manifold in

(H, h) which is disjoint from U . Since $X = U \cup V$ is full [19, p. 85], so is $U \cup V'$. However the complexity [19, p. 84] of $U \cup V'$ is greater than that of X , contradicting X being a characteristic submanifold. This completes the proof of part 2.

Since $\varphi(V) = V$, $\partial(V \cap \partial H)$ is an invariant 1-submanifold of ∂H under φ . If $S (\in h)$ is not a reducible component, then the adjoint automorphism of S is irreducible. Hence $\partial(V \cap \partial H) \cap S$ consists of simple loops in S parallel to ∂S . Part 3 of Lemma 5.15 follows from the observation above and claim below. \square

Claim. Let A be an annulus component of h , and T be a component of $V \cap \partial H$ isotopic to A in ∂H . Then T is a regular neighborhood of A .

Proof. Assume that $T \not\supset A$. By part 2 of Proposition 10.6 of [19] $X \supset A$, so that $A \cap T = \emptyset$. As in the proof of part 2 we have just proved, if $A \cap U \neq \emptyset$, then U has a component homeomorphic to $S^1 \times D^2$, a contradiction. It follows that $A \subset V$. Then T is isotopic to the component of $V \cap \partial H$ which contains A , contradicting part 2. \square

The following lemmas show that the position of U is closely related to V .

Lemma 5.16. *Let α, β be components of $\text{Fr } U$, $\text{Fr } V$ respectively. If there is an annulus A in ∂H such that ∂A consists of a component of $\partial \alpha$ and a component of $\partial \beta$, and $A \cap V \subset \partial A$, then α is admissibly parallel to β .*

Proof. As in the proof of Lemma 5.15 we have $A \cap U = \emptyset$, hence $A \cap X = \partial A$. Pushing A into $\text{Int } H$ with $A \cap X = \partial A$, we obtain an essential annulus in (H', h') , where $H' = \overline{H} - \overline{X}$ and h' is the boundary pattern of H' induced from h (cf. [19, p. 90]). By the completeness of a characteristic submanifold [19, 10.10] α and β cobound (an annulus) $\times I$. \square

Lemma 5.17. *Assume that U does not contain any annulus components of $h - \{B_h\}$.*

1. $V \cap \partial H$ contains all annulus components of $h - \{B_h\}$.
2. Let α be a component of $\text{Fr } U$ such that a component of $\partial \alpha$ is contained in an irreducible component of h . Then α is admissibly parallel to $\text{Fr } V$.

Proof. Part 1 follows from the fact that every annulus component of h is contained in U or V . We obtain part 2 by using Lemma 5.16 with an aid of part 1 and Lemma 5.12. \square

The following general lemma seems to be folklore. The proof given below exploits a uniqueness of Seifert fibration.

Lemma 5.18. *Let S be connected, orientable surface. Let $S_1 (\neq D^2, S^1 \times I)$ be a connected 2-submanifold of S such that no component of ∂S_1 bounds a disk in S . Let f be an automorphism of S which preserves S_1 and is periodic up to isotopy. If there is an isotopy $g_t: S_1 \rightarrow S_1$ of $f|_{S_1}$ such that g_1 is periodic, then g_t extends to an isotopy f_t of f such that f_1 is periodic.*

Proof. Without loss of generality $S \neq S_1$; then the Euler number of S is negative. Let h_t be an isotopy of f such that h_1 is periodic. Set $M = S \times I / f$

and $M_1 = S_1 \times I / (f|S_1)$. Give M (resp. M_1) the Seifert fibration induced by the flow $(h_t(x), t)_t$ (resp. $(g_t(x), t)_t$). Deform the fibration of M so that ∂M_1 consists of saturated tori. Then the Seifert fibration of M is (after isotopy) an extension of that of M_1 , for a Seifert fibration of M_1 is unique up to isotopy. We may assume each S^1 -fiber of M is transverse to each $S \times t$. Then the fibration of M gives the claimed isotopy. \square

The lemmas below reduce the proofs of Theorems 5.3 and 5.4 to the case when U does not contain any annulus components of $h_K - \{B_h\}$; then Lemma 5.17 is valid.

Lemma 5.19. *Let K_1, \dots, K_n be prime fibered knots in homotopy spheres such that (1) $K = K_1 \# \dots \# K_n \geq 0$; (2) if $n = 1$, K_1 is a (p, q) cable of a nontrivial knot, where $p > 1$. Assume that h_K is useful, and U_K contains an annulus component of $h_K \cap F_{i_0} (\subset h_K)$, where $F_{i_0} (\subset \partial H_K)$ is the fiber of some K_{i_0} . Without loss of generality assume $i_0 = 1$. Then the following hold.*

1. *There are $S_0, S_1 \in h_K \cap F_1$ such that S_0 and S_1 are adjacent via an annulus and $S_1 = S_0^*$. If Theorem 5.3 holds for fibered knots L with $\alpha(L) < \alpha(K)$, then K_1 has a satellite presentation $J(k\# - k)$ or $J_1(J(J^*(k)))$ where $J \not\cong (S^1 \times D^2, S^1 \times 0)$ and $k \not\cong 0$.*
2. *There is a nontrivial (possibly composite) fibered knot K'_1 such that $K' = K'_1 \# K_2 \# \dots \# K_n \geq 0$ and $K_1 \geq K'_1$. If $n = 1$, K'_1 is a (p, q) cable of a nontrivial knot.*
3. *With K' as above each nontrivial component of $h_{K'}$ is incompressible in $H_{K'} (\subset H_K)$.*
4. *With K' as above $\alpha(K) > \alpha(K')$, where $\alpha(K) = |h_K|$.*

Lemma 5.20. *The proofs of Theorems 5.2, part 1, 5.3, and 5.4 reduce to the case when U_K contains no annulus components of $h_K - \{B_h\}$.*

Proof of Lemma 5.19. Let U_0 be a component of U containing an annulus component, A , of $h \cap F_1$. Let S_0 and S_1 be the two components of h adjacent to A such that $S_1 \cap A = \partial_r S_1$. Then $S'_i = U_0 \cap S_i$, $i = 0, 1$, are the lids of U_0 . Let m be the least positive integer such that $\varphi^m(S_1) = S_1$; then m is also the least positive integer such that $\varphi^m(S_0) = S_0$. For simplicity let $\psi = \varphi^m$.

Claim. For each i (a) $S_i \neq B_h$, (b) S'_i is isotopic to S_i in ∂H .

Proof. Suppose $S_0 = B_h$. When $n = 1$, it follows from the irreducibility of $\varphi|B_h$ (Lemma 2.5) that $S_0 = S'_0$, and hence $\varphi(U_0) = U_0$. This implies that $\varphi(S_1) = S_1$ and $p = |\partial B_h| = 1$, a contradiction. When $n > 1$, $\varphi|S_0 = \text{id}$, so that $\varphi|U_0 \simeq \text{id}$. By Lemma 5.18 $\varphi|S_0 \cup A \cup S_1 \simeq \text{id}$, contradicting Proposition 5.1, part 4. This proves (a). If some S'_i is not isotopic to S_i , then S_i is a trivial component (Lemma 5.12, Proposition 5.1, part 5). Hence, $\psi|U_0 \simeq \text{id}$. A contradiction follows as above, so the claim is proved.

Proof of parts 1, 2. Let $s+1 = |(\partial S_0 - \partial_r S_0)/\psi|$ and \tilde{J}_i be the fibered knot in a homotopy sphere with s unknotted holes with monodromy $\psi|(S_i, \partial_r S_i)$, $i = 0, 1$. Identify U_0 with $S_1 \times I$ so that $S_i \cap U_0 = S_1 \times i$ for $i = 0, 1$ and $A = \partial_r S_1 \times I$. Then by part 1 of Proposition 5.1 $\psi|A = \text{id}$. Isotop φ in a neighborhood of $\partial(S_0 \cup A \cup S_1)$ so that $\psi|S_i \cap U_0$ is the monodromy of \tilde{J}_i for each i . The natural projection from $S_1 \times 0$ to $-(S_1 \times 1)$ gives a diffeomorphism $g: S_0 \rightarrow -S_1$ such that (1) $g(\partial_r S_0) \neq \partial_r S_1$ and (2)

$g \circ (\psi|_{S_0}) \circ g^{-1} \simeq -\psi|_{S_1 \text{ rel } \partial_r S_1}$. Since $\partial_* = \partial_r$ by part 6 of Proposition 5.1, we have $\tilde{J}_1 \cong \tilde{J}_0^*$, i.e., $S_1 = S_0^*$.

Note that φ permutes the set of handlebodies $\overline{H - \bigcup_j \varphi^j(U_0)}$. Let H_i , $0 \leq i \leq s$, be representatives of the orbits such that $B_h \subset H_0$. Then $F_1 \cap H_i$ consists of two components for each $i > 0$. For each $0 \leq i \leq s$ let T_i be the component of $F_1 \cap H_i$ such that $T_i \cap S_1 \neq \emptyset$. For $1 \leq i \leq s$ let T_{s+i} be $(F_1 \cap H_i) - T_i$ (Figure 5). Let m_i ($1 \leq i \leq s$) be the least positive integers such that $\varphi^{m_i}(H_i) = H_i$, and let $m_0 = m$. For $0 \leq i \leq 2s$ define k_i to be the (possibly composite) fibered knot in a homotopy sphere with monodromy $\varphi^{m_i}|_{T_i}$. Define J_1 to be the fibered knot in a homotopy solid torus with monodromy $\varphi|(F_1 \cap H_0 - \bigcup \varphi^j(T_0), \partial F_1)$. It follows from $S_i \neq B_h$ that when $n = 1$, the fiber of J_1 is not an annulus. Hence, $J_1(k_0)$ is a (p, q) cable of a nontrivial knot when $n = 1$. Now K_1 has a satellite presentation

$$K_1 = J_1(\tilde{J}_0(\tilde{J}_0^*(k_0, \dots, k_s), k_{s+1}, \dots, k_{2s})).$$

By Lemma 2.2 we may assume $\varphi^{m_i}|_{\text{Fr } H_i} = \text{id}$. Hence $\varphi^{m_i}|\partial H_i$, $i \geq 1$, is the closed monodromy of $k_i \# k_{s+i}$, which is ≥ 0 . Also $\varphi|\partial H_0$ is the closed monodromy of $K' = J_1(k_0) \# K_2 \# \dots \# K_n$, which is ≥ 0 . Let $F'_1 \subset \partial H_0$ be the fiber of $J_1(k_0)$, i.e., the component of $\partial H_0 - \widehat{D}_n$ containing T_i for all i . By Lemma 5.14 the core of each component of the m annuli $\text{Fr } H_0$ is primitive for $\overline{H - H_0}$, which consists of m handlebodies. Hence, a regular neighborhood of $F'_1 \cup \overline{H - H_0}$ in H is a compression body with $\partial_- \cong F'_1$, $\partial_+ \cong F_1$. This compression body induces $K_1 \geq J_1(k_0)$, so $K'_1 = J_1(k_0)$ is the desired knot in part 2.

Let us show the second statement of part 1. Since $\alpha(k_i \# k_{s+i}) < \alpha(K)$, we can apply Theorem 5.3 to $k_i \# k_{s+i}$ for each i . Although k_i, k_{s+i} may not be prime

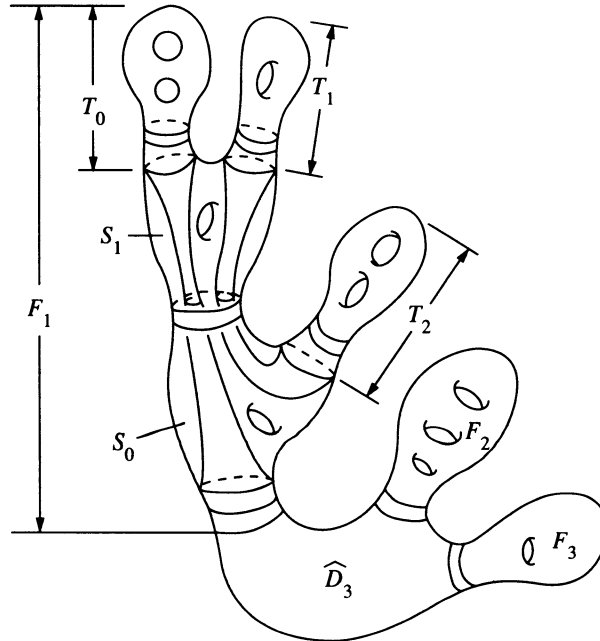


FIGURE 5. $s = m = m_1 = 1$ and the fiber of J_1 is an annulus

knots, it follows that (1) $k_i \cong -k_{s+i}$ or (2) at least one of these knots has a satellite presentation in Lemma 5.19, part 1. (2) implies that K_1 also has such a presentation. If (1) is the case, let J be the fibered knot whose monodromy is the restriction of ψ to $(S_0 \cup (\bigcup_j \bigcup_{i=s+1}^{2s} \phi^j(T_i)), \partial_r S_0)$, a surface with two boundary components. Then $K_1 = J_1(J(J^*(k_0)))$ as stated in part 1.

Proof of parts 3, 4. Write $H' = H_{K'} (= H_0$ in the proof of part 2). Let us consider the boundary pattern $h_{K'}$. It is clear that every component of h_K in F_i , $i > 1$, is contained in $h_{K'}$, and hence incompressible in H' . To go further choose $G_0, G_1 \in h_K \cap H'$ as follows. Let G_1 be the nonannulus component of $h_K \cap H'$ such that $\partial_r G_1$ is parallel to (a component of) ∂S_1 in ∂H . If B_h is an annulus and $\partial_r S_0 \subset B_h$, let $G_0 = B_h$. Otherwise, let G_0 be the nonannulus component of $h_K \cap H'$ such that $\partial_r S_0$ is parallel to ∂G_0 . Let A' be the annulus in $\partial H'$ bounded by G_1 and G_0 . Note that $\psi|_{A'} = \text{id}$. Let k be the least positive integer such that $\phi^k(G_0) = G_0$. Let $T = G_0 \cup \bigcup_i \phi^{ki}(A' \cup G_1)$. Note that T is connected, and either $\phi^i(T) = T$ or $\phi^i(T) \cap T = \emptyset$ for each i . Each component of h_K in $\partial H' - \bigcup_i \phi^i(T)$ is a component of $h_{K'}$ and incompressible in H' . Let us determine $h_{K'} \cap T$.

Case 1. $T \notin h_{K'}$. If G_0 is an annulus, then $h_{K'} \cap T$ consists of G_0 and the components of $\phi^i(A' \cup G_1)$, $i \geq 0$. Otherwise, $h_{K'} \cap T$ consists of the components of $\phi^i(G_0)$, $\phi^i(A')$, $\phi^i(G_1)$, $i \geq 0$. It follows that $\alpha(K) > \alpha(K')$ and each nontrivial component of $h_{K'}$ is incompressible in H' .

Case 2. $T \in h_{K'}$. We claim that T is a trivial component of $h_{K'}$. Since G_1 is not an annulus, the adjoint automorphism of T , $\phi^k|_T$, is reducible. Let T' be T minus the disk bounded by $\partial F'$ where $F' (\subset \partial H')$ is the fiber of K' . (T' is possibly equal to T .) Since $\phi^k|_T$ is reducible, so is $\phi^k|_{T'}$. Then, Lemma 3.3 part 5 implies that $\phi^k|_{T'} \simeq \text{id}$, so that T is a trivial component of $h_{K'}$. Hence, each nontrivial component of $h_{K'}$ is incompressible in H' and $\alpha(K') < \alpha(K)$. This completes the proof of parts 3 and 4. \square

Proof of Lemma 5.20. Let K be a composite fibered knot in a homology sphere satisfying the assumption of Theorem 5.2. Suppose that h_K is useful and some component U_0 of U_K contains an annulus A in $h_K - \{B_h\}$. If A is not adjacent to B_h , then $\partial U_0 \cap (\partial H_K - B_h)$ consists of the two lids of U_0 and (a nonempty set of) annuli connecting them. It follows that $\partial H_K - B_h$ is compressible in H_K , contradicting the assumption of Theorem 5.2, part 1. Hence A is adjacent to $B_h = \hat{D}_n$, so that $\phi|_{U_0} \simeq \text{id}$. By Lemma 5.18 this contradicts Proposition 5.1 part 4. Hence, U_K contains no annulus components of $h_K - \{B_h\}$.

Let $K = K_1 \# \dots \# K_n$ be a fibered knot in a homotopy sphere satisfying the assumption of Theorem 5.3. Suppose that if h_K is useful and U_K contains no annulus component of $h_K - \{B_h\}$, then Theorem 5.3 holds for K . We prove Theorem 5.3 by induction on $\alpha(K)$ just as in the proof of Lemma 5.10. Without loss of generality assume that h_K is useful and U_K contains an annulus component of $h_K \cap F_1$ where F_1 is the fiber of K_1 in ∂H_K . Assume that Theorem 5.3 holds for every L with $\alpha(L) < \alpha(K)$. Let K'_1 be the knot obtained by applying Lemma 5.19, part 2 to K . Then $K' = K'_1 \# K_2 \# \dots \# K_n$ satisfies the assumption of Theorem 5.3 and $\alpha(K') < \alpha(K)$. By the induction hypothesis Theorem 5.3 holds for K' . Since $K_1 \geq K'_1$, this implies that Theorem 5.3,

part 1 holds for K even if K'_1 is composite. Let us see that Theorem 5.3, part 2 holds for K . Assume that for some i_0 either $h_K \cap F_{i_0}$ or K_{i_0} satisfies the assumption of part 2 of Theorem 5.3. It follows from Lemma 5.19, part 1 that $i_0 \neq 1$. Then, applying Theorem 5.3, part 2 to K' proves that the theorem holds for K .

The reduction of Theorem 5.4 can also be proved by induction on $\alpha(K)$. We leave this to the reader as an exercise. This completes the proof of Lemma 5.20. \square

5.3. Position of U and V (2). For a homotopically-ribbon fibered knot K in a homology sphere, define $n, f, \varphi, H, h, U, V, B_h, \widehat{D}_n$ as in §5.1; in particular, h is a useful boundary pattern. We do not necessarily assume that (1) U contains no annulus components of $h - \{B_h\}$. We first show that if $n > 2$, then (2) component of $\text{Fr } V$ is parallel to an annulus component of h . If $n > 2$, assertions (1) and (2) contradict each other (Lemma 5.22). Hence the proof of Theorem 5.3 reduces to the case $n = 2$, and the proof of Theorem 5.2 part 1 is completed. We then study the position of $\text{Fr } V$ in H , which almost determines that of $\text{Fr } U$ by Lemmas 5.16 and 5.17.

Let us recall one result of [19]. By the completeness of the characteristic submanifold X , $\overline{H - X}$ consists of simple 3-manifolds and $(\text{annuli}) \times I$ (cf. Example 3 of [19, p. 158]). An admissible automorphism of a simple 3-manifold is admissibly isotopic to a periodic map [19, Proposition 27.1].

Lemma 5.21. *Let α be a component of $\text{Fr } V$. If α is parallel to an annulus $A \in h$, then the following assertions 1 and 2 hold.*

1. φ^k preserves $S_1 \cup A \cup S_2$ for some k where $S_1 (\neq) S_2 \in h$ are adjacent to A , and $\varphi^k|_{S_1 \cup A \cup S_2}$ is isotopic to a periodic map.
2. Either $A = B_h$, or A is adjacent to B_h and $n = 1$.

Proof. Let V_0 be the component of V containing α ; then V_0 is a regular neighborhood of A in H .

Claim. No component of $\partial(\text{Fr } U)$ is parallel to a component of $\partial\alpha$ in ∂H .

Proof of Claim. Suppose that there is a component of U , U_0 say, such that some component of $\text{Fr } U_0$, β say, has a boundary component parallel to $\partial\alpha$ in ∂H . Then such a component of $\partial\beta$ is in S_1 or S_2 . By Lemma 5.16 α and β cobounds 3-manifold $W \cong \beta \times I$ in H . Now $U_0 \cup W \cup V_0$ is an essential I -bundle in (H, h) which cannot be admissibly isotoped into X , a contradiction. Claim is proved.

Let $\partial_i \alpha = \partial\alpha \cap S_i$ for $i = 1, 2$. Let S'_i be the component of $\overline{S_i - X}$ containing $\partial_i \alpha$. Then S'_i is not an annulus from Claim above and the fact that no component of $\partial(\text{Fr } V)$ but $\partial_2 \alpha$ is parallel to $\partial_1 \alpha$. Set $S' = S'_1 \cup \alpha \cup S'_2$. Since $S' \subset \overline{H - X}$ and S'_i is not an annulus, S' is contained in the boundary of a simple manifold component of $\overline{H - X}$. Hence, for some k , $\varphi^k(S') = S'$ and $\varphi^k|_{S'}$ is isotopic to a periodic map leaving S'_1, α , and S'_2 fixed as sets respectively. Let $A' = V_0 \cap \partial H$. Since α is parallel to A' , the above statement about φ^k remains true with S' (resp. α) replaced by $S'_1 \cup A' \cup S'_2$ (resp. A'). Lemma 5.18 then implies that $\varphi^k|_{S_1 \cup A \cup S_2}$ is isotopic to a periodic map as claimed in part 1.

Proposition 5.1, part 4 implies that $A = B_h$, or $n = 1$ and A is adjacent to B_h . This completes the proof of part 2. \square

Lemma 5.22. *At least one of the following holds.*

1. U contains an annulus component of h .
2. Some component of $\text{Fr } V$ is parallel to an annulus component of h .

Proof. Assume that neither 1 nor 2 above holds. Take a compressing disk D for ∂H in H such that $|D \cap \text{Fr } V|$ is minimal among all such disks. Let Δ be an outermost component of $D - \text{Fr } V$ in D . If the disk Δ is contained in a component V_0 , say, of V , then $\text{Fr } V_0$ is an annulus parallel to $V_0 \cap \partial H$. The annulus $V_0 \cap \partial H$ is either a regular neighborhood of some annulus component of h or contained in a component of h (Lemma 5.15 part 3). The former implies assertion 2; the latter contradicts $\text{Fr } V$ being essential in (B, h) . Hence $\Delta \cap \text{Int } V = \emptyset$. Since assertion 1 is false, $V \cap \partial H$ contains all annulus components of h (Lemma 5.17, part 1). It follows that the arc $\partial \Delta \cap \partial D$ is contained in some component of h , contradicting the fact that $\text{Fr } V$ is essential. \square

Proposition 5.23. *If U_K contains no annulus components of $h_K - \{B_h\}$, then $n = 1$ or 2. In particular, the proof of Theorem 5.3 reduces to the case when $n = 2$, h_K is useful and U_K contains no annulus components of $h_K - \{\widehat{D}_2\}$.*

Proof. Suppose $n > 2$; then B_h is not an annulus. Thus, assertion 1 in Lemma 5.22 is not satisfied. Lemma 5.21 implies that assertion 2 is not satisfied. This contradicts Lemma 5.22. \square

The lemma below can be proved by a similar argument, so the proof is omitted.

Lemma 5.24. *Suppose that U contains no annulus components of $h - \{B_h\}$. If $n = 2$ and $\widehat{D}_2 \subset V$, then the component of V containing \widehat{D}_2 is a regular neighborhood of \widehat{D}_2 .*

Proof of Theorem 5.2, part 1. The results follow from Lemma 5.20 and Proposition 5.23. \square

For a separating, simple loop l in ∂H let $[l]$ denote the (possibly empty) component of $\partial H - l$ which is disjoint from B_h . Let l_1, l_2 be essential separating simple loops in ∂H . We define a relation “ $<$ ” as follows: $l_1 < l_2$ if l_i is isotopic to some l'_i , $i = 1, 2$, in ∂H such that $l'_2 \subset [l'_1]$.

Lemma 5.25. *Let A_1 and A_2 be disjoint, essential, separating annuli in H . Let $\partial A = c_1 \cup c_2$ and $\partial A_2 = c_3 \cup c_4$. If c_3 separates c_1 from c_2 in ∂H , then so does c_4 . In particular, if $c_1 < c_3 < c_2$, then $c_1 < c_4 < c_2$.*

Proof. Suppose for a contradiction that c_4 does not separate c_1 and c_2 . Then there is an arc a_1 in ∂H which connects c_1 and c_2 , meets c_3 transversely in a single point, and $a_1 \cap c_4 = \emptyset$. Take an arc a_2 in A_1 with $\partial a_1 = \partial a_2$ and set $c = a_1 \cup a_2$. Push the simple loop c slightly into $\text{Int } H$. Then c meets the closed surface $S \cup A_2$ transversely in a single point, where S is the submanifold of ∂H bounded by c_3 and c_4 . This is absurd, for $S \cup A_2$ and c can be considered to be in S^3 . \square

Assuming that U contains no annulus components of $h - \{B_h\}$, introduce the following notation. Let S be a nonannulus component of h . Let

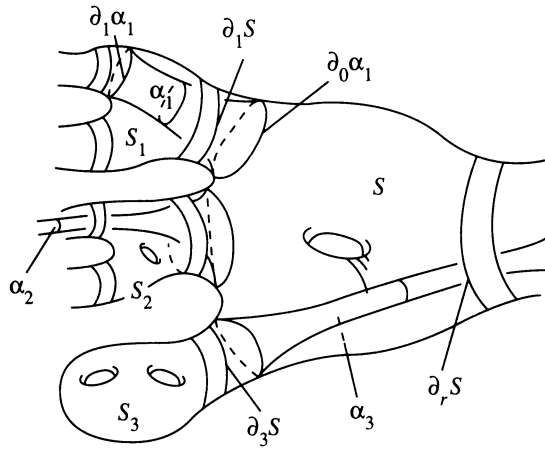


FIGURE 6

$\{\partial_1 S, \dots, \partial_p S\}$ denote the set of components of ∂S if $S = B_h$, or the set of components of $\partial S - \partial_r S$ if $S \neq B_h$. For every i , $1 \leq i \leq p$, there is exactly one component α_i , say, of $\text{Fr } V$ such that one component of $\partial \alpha_i$ is in S and parallel to $\partial_i S$ (Lemmas 5.15 and 5.17). Let $\partial_0 \alpha_i$ denote such a component, and $\partial_1 \alpha_i$ the other component of $\partial \alpha_i$. We say that α_i is *associated* to $\partial_i S$. Let S_i , $1 \leq i \leq p$, denote the nonannulus component of h such that $\partial_r S_i$ is parallel to $\partial_i S$. See Figure 6.

Lemma 5.26. Assume that (1) U contains no annulus components of $h - \{B_h\}$, (2) either $S \neq B_h$ or no component of $\text{Fr } V$ is parallel to an annulus in h . With the notation above the following hold.

1. There is no i , $1 \leq i \leq p$, such that $\partial_1 \alpha_i > \partial_i S$.
2. If $S = B_h$ (nonannulus), then $\partial_1 \alpha_i \subset S$ for all $1 \leq i \leq p$.

Proof of part 1. For $S \in h$ let $N_S = \#\{G \in h - \{S\} \mid \partial_r G > \text{a component of } \partial S\}$. We proceed by induction on N_S . Take an arbitrary nonannulus component $S \in h$. Suppose that Lemma 5.26 part 1 is true for any nonannulus component $S' \in h$ such that $N_{S'} < N_S$. Assume for a contradiction that $\partial_1 \alpha_{i_0} > \partial_{i_0} S$ for some i_0 . For simplicity set $\alpha = \alpha_{i_0}$. Suppose that $\partial_1 \alpha$ is not parallel to $\partial_0 \alpha$ in ∂H . Let G_1 be the component of h containing $\partial_1 \alpha$. There is a collection of nonannulus components of h , $\{G_1, \dots, G_s, S\}$, such that G_j is, via an annulus component of h , adjacent to G_{j+1} if $j+1 \leq s$, and to S if $j = s$ (thus $G_s = S_{i_0}$). Denote

$$B = \{b \mid b \text{ is a component of } \partial(\text{Fr } V) \text{ with } \partial_0 \alpha < b < \partial_1 \alpha\}.$$

Using Lemma 5.25 repeatedly, we can find b_1 and b_2 in B such that

1. $b_1 < b_2$,
2. $b_1 \cup b_2$ is a boundary of a component of $\text{Fr } V$,
3. no element b in B satisfies $b_1 < b < b_2$.

Let β be the component of $\text{Fr } V$ such that $\partial \beta = b_1 \cup b_2$. Let $b_1 \subset G_\lambda$, $b_2 \subset G_\mu$, where $1 \leq \lambda, \mu \leq s$. There are two possibilities about λ and μ .

Case 1. $\lambda \neq \mu$. It follows from Lemma 5.17 part 1 that $\mu = \lambda + 1$ and b_1 is parallel to b_2 in ∂H . This contradicts either the lemma's assumption (2) or Lemma 5.21 part 2. Thus Case 1 does not occur.

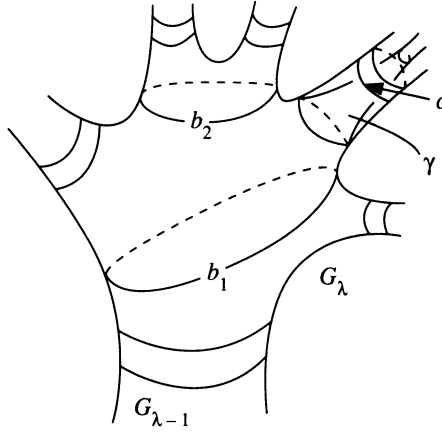


FIGURE 7

Case 2. $\lambda = \mu$.

Claim. There is a 3-submanifold M of H such that $M \cap \partial H \subset \text{Int } G_{\lambda}$, $\text{Fr } M \subset \text{Fr } V$, and $M \cap \partial H$ is connected.

Proof of Claim. Let M_1 be the closure of the outermost component of $H - \beta$, i.e., $M_1 \cap \partial H$ is connected. If $\partial G_{\lambda} \cap M_1 = \emptyset$, then M_1 is the desired manifold. So suppose not. Let c be a component of ∂G_{λ} contained in M_1 . Since $c > b_1$, $c \neq \partial_r G_{\lambda}$. Let γ be the component of $\text{Fr } V$ associated to c , so $\partial_0 \gamma \subset \text{Int } G_{\lambda}$ is parallel to c (Figure 7).

Since $N_{G_{\lambda}} < N_S$, by the induction hypothesis we can apply part 1 of Lemma 5.26 to $\partial_1 \gamma$. It follows that $\partial_1 \gamma \not\propto c$. Thus the 2-submanifold of ∂H bounded by $\partial_0 \gamma$ and $\partial_1 \gamma$ contains $b_1 \cup b_2$ but not c . Let M_2 be the closure of the outermost component of $M_1 - \gamma$. Then $M_2 \cap G_{\lambda}$ is nonempty, connected and does not contain c , so that $|M_2 \cap \partial G_{\lambda}| < |M_1 \cap \partial G_{\lambda}|$. If $M_2 \cap \partial G_{\lambda} = \emptyset$, then M_2 is the claimed manifold M . Otherwise, construct M_3 from M_2 just as we obtain M_2 from M_1 . In a finite number of steps we obtain the claimed 3-manifold M in H .

However, the following lemma precludes the existence of this manifold. Thus Case 2 does not occur. Lemma 5.26, part 1 is proved. \square

Lemma 5.27. *There is no 3-submanifold M of H such that $M \cap \partial H$ is connected and contained in a component of h , and $\text{Fr } M \subset \text{Fr } X$.*

Proof. Apply Lemma 5.14 to M ; then M is a handlebody and the set of cores of $\text{Fr } M$ is primitive for the handlebody M . Hence there is a 2-faced disk D in M such that

$$D \cap \text{Fr } M = \text{a cocore of a component of } \text{Fr } M,$$

$$D \cap \partial H = \text{an arc in a component of } h.$$

This implies that some component of $\text{Fr } X$ is not essential in (H, h) , a contradiction. \square

Proof of Lemma 5.26, part 2. We proceed in a manner similar to Case 2 in the proof of Lemma 5.26, part 1. Define 3-submanifolds M_1, M_2, \dots of H as follows, and finally obtain a 3-manifold violating Lemma 5.27. Let M_1 be the

closure of the outermost component of $H - \alpha_1$. Let us see that $M_1 \cap B_h$ is connected and $M_1 \cap \partial_1 B_h = \emptyset$. If $\partial_1 \alpha_1 \subset B_h$, this is obvious. So assume $\partial_1 \alpha_1 \subset \partial H - B_h$. Since $\partial B_h = \bigcup_{i=1}^n \partial_i B_h$, $\partial_1 \alpha_1 > \partial_0 \alpha_j$ for some $1 \leq j \leq n$. By part 1 of Lemma 5.26 $j \neq 1$. Thus, $M_1 \cap B_h$ is connected and $M_1 \cap \partial_1 B_h = \emptyset$, as desired. If $M_1 \cap \partial B_h = \emptyset$, then M_1 is the desired manifold, i.e., $M_1 \cap B_h \subset \text{Int } B_h$, $M_1 \cap B_h$ is connected and $\text{Fr } M_1 \subset \text{Fr } V$. Otherwise, there is a component of ∂B_h , say $\partial_k B_h$ ($k \neq 1$), contained in $M_1 \cap B_h$. Let M_2 be the closure of the outermost component of $M_1 - \alpha_k$. The same argument as above shows that $M_2 \cap B_h$ is nonempty, connected and $M_2 \cap \partial_i B_h = \emptyset$, $i = 1, k$. Repeating this finally gives a 3-manifold M contradicting Lemma 5.27. \square

Lemma 5.26 puts a strong restriction on the position of $\text{Fr } V$. Furthermore, we shall prove that there are no i, j such that $\partial_1 \alpha_i > \partial_j S$ for a knot in a homotopy sphere. In the next section this is done by using the Cyclic Surgery Theorem.

6. PROOF OF THEOREM 5.3

Let K_i , $i = 1, 2$, be a prime fibered knot with fiber F_i in a homotopy sphere M_i such that $K_1 \# K_2 \geq 0$. For $K = K_1 \# K_2$, define $f, \varphi, \widehat{D}_2, h, H, U, V$ as in §5.1. Throughout this section assume that U contains no annulus components of $h - \{\widehat{D}_2\}$. By Proposition 5.23 the proof of Theorem 5.3 is reduced to proving Theorem 5.3 for our K . Therefore it suffices to show Proposition 6.1 below.

Proposition 6.1. *With the above assumption the following hold.*

1. *There are (possibly composite or trivial) fibered knots \tilde{K}_i , $i = 1, 2$, such that $K_i \geq \tilde{K}_i$ and $\tilde{K}_1 \cong -\tilde{K}_2$.*
2. *\tilde{K}_i is obtained from K_i by successive reductions of type $J(k\# - k) \geq J(0)$. In particular, if K_i does not have a satellite presentation $J(k\# - k)$ where $k \not\cong 0$, then $K_i = \tilde{K}_i$.*

Let \mathcal{T}_i be a minimal collection of decomposing tori in $\overline{M_i - N(K_i)}$ in the sense of Jaco and Shalen, and Johannson. If $\mathcal{T}_i \neq \emptyset$, then denote by

$$K_i = J_i(k_1^i, \dots, k_{s_i}^i), \quad i = 1, 2,$$

the satellite presentation of K_i obtained via the subset of \mathcal{T}_i consisting of all the tori closest to K_i (§3). Note that each k_j^i is nontrivial, but possibly composite. Let Y_i denote the ambient manifold of J_i , so that $|\partial Y_i| = s_i$. For notational convenience if $\mathcal{T}_i = \emptyset$, denote $J_i = K_i$, $s_i = 0$, and $Y_i = M_i$.

Then Proposition 6.1 follows from the proposition below.

Proposition 6.2. 1. $s_1 = s_2$ and $(Y_1, J_1) \cong (-Y_2, -J_2)$ as fibered knots.

2. *There is a permutation σ of $\{1, \dots, s_1\}$ such that*

- (a) $k_j^1 \# k_{\sigma(j)}^2 \geq 0$ for $1 \leq j \leq s_1$.
- (b) *Set $k = k_j^1 \# k_{\sigma(j)}^2$. Then H_k is a submanifold of H_K such that $\text{Fr } H_k$ consists of essential annuli, and $h_k - \{B_{h_k}\} \subset h_K - \{\widehat{D}_2\}$.*

Proof of Proposition 6.1. We proceed, assuming Proposition 6.2. For simplicity let $s = s_1$, $\sigma(j) = j$ for all $1 \leq j \leq s$. Let $k_j^i = \#_{r=1}^{m(i)} L_r^i$ be the prime

decomposition where $i = 1, 2$; then $(\#_{r=1}^{m(1)} L_r^1) \# (\#_{r=1}^{m(2)} L_r^2) = k \geq 0$. As in the proof of Lemma 5.10, compressing B_{h_k} in H_k yields a handlebody $H_L (\subset H_k)$ where $L = \#_{(i,r) \in S} L_r^i$ and $S \subset \{(i, r) | i = 1, 2, 1 \leq r \leq m(i)\}$; then $|S| > 1$ and $\{B_{h_L}\} = h_L - h_k$. By compressing B_{h_k} maximally each component of h_L is incompressible in H_L , and hence in H_k . If there is an essential I -bundle in (H_L, h_L) containing an annulus in $h_L - \{B_{h_L}\}$, then U_K contains an annulus in $h_K - \{\widehat{D}_2\}$, a contradiction. It follows that U_L contains no annulus in $h_L - \{B_{h_L}\}$, so that $|S| = 2$ by Proposition 5.23. Hence, $\{L_r^i\}_{i,r}$ can be paired into $\coprod \{L_p^u, L_q^v\}$ so that $L = L_p^u \# L_q^v \geq 0$ and h_L is useful. We also see that U_L does not contain any annulus in $h_L - \{\widehat{D}_2\}$.

Let us show Proposition 6.1 by induction on $|h_K|$. Note that for each pair $\{L_p^u, L_q^v\}$ we have $|h_L| < |h_K|$ where $L = L_p^u \# L_q^v$. Applying Proposition 6.1 to L , we obtain fibered knots $\widetilde{L}_p^u, \widetilde{L}_q^v$ such that $L_p^u \geq \widetilde{L}_p^u$, $L_q^v \geq \widetilde{L}_q^v$, and $\widetilde{L}_p^u \cong -\widetilde{L}_q^v$. Furthermore, \widetilde{L}_p^u (resp. \widetilde{L}_q^v) is obtained from L_p^u (resp. L_q^v) by successive reductions of type $J(k\# - k) \geq J(0)$. It follows that $k_j^i \geq \#_{r=1}^{m(i)} \widetilde{L}_r^i$ for $i = 1, 2$. For $i = 1, 2$ let N_i be a (possibly empty) subset of $\{1, \dots, m(i)\}$ which is minimal among all subsets N'_i such that $\{1, \dots, m(i)\} - N'_i$ can be paired into $\coprod_{t=1}^m \{p_t, q_t\}$ with $\widetilde{L}_{p_t}^i \cong -\widetilde{L}_{q_t}^i$. Recalling that L_r^i, \widetilde{L}_r^i and N_i depend on j , define \widetilde{k}_j^i for $i = 1, 2, 1 \leq j \leq s$, as follows.

$$\widetilde{k}_j^i = \begin{cases} \#_{r \in N_i} \widetilde{L}_r^i & \text{if } N_i \neq \emptyset, \\ 0, \text{ the unknot in } S^3 & \text{if } N_i = \emptyset. \end{cases}$$

Then $\widetilde{k}_j^1 \cong -\widetilde{k}_j^2$, and \widetilde{k}_j^i is obtained from k_j^i by successive reductions of type $J(k\# - k) \geq J(0)$. Therefore, $J_i(\widetilde{k}_1^i, \dots, \widetilde{k}_s^i) = \widetilde{K}_i$ (say) is also obtained from K_i by such reductions, and $\widetilde{K}_1 \cong -\widetilde{K}_2$. This establishes Proposition 6.1. \square

Proof of Proposition 6.2. Let $h = h_K$. Let R (resp. S) be the component of h which is adjacent to \widehat{D}_2 and contained in F_1 (resp. F_2), a fiber of K_1 (resp. F_2). Let $\partial_i R, 1 \leq i \leq p_1$ (resp. $\partial_j S, 1 \leq j \leq p_2$) be the components of $\partial R - \partial_r R$ (resp. $\partial S - \partial_r S$). Let $R_i, 1 \leq i \leq p_1$ (resp. $S_j, 1 \leq j \leq p_2$) be the components of h such that $\partial_r R_i$ (resp. $\partial_r S_j$) is parallel to $\partial_i R$ (resp. $\partial_j S$) in ∂H . We denote by $\alpha_i, 1 \leq i \leq p_1$ (resp. $\beta_j, 1 \leq j \leq p_2$) the components of $\text{Fr } V$ associated to $\partial_i R$ (resp. $\partial_j S$). Note that s_1 (resp. s_2) is equal to the number of orbits $|\{\partial_i R\}_i / \varphi|$ (resp. $|\{\partial_j S\}_j / \varphi|$).

Denote by A_i (resp. B_j) the annulus in ∂H bounded by $\partial_i R$ and $\partial_0 \alpha_i$ (resp. $\partial_j S$ and $\partial_0 \beta_j$). Isotop $\varphi \text{ rel } \partial H - R \cup S$ so that $\varphi[\overline{R} - \bigcup A_i]$ (resp. $\varphi[\overline{S} - \bigcup B_j]$) is the monodromy of J_1 (resp. J_2). Then, if $\varphi^m(A_i) = A_i$ for some m , then $\varphi^m|_{A_i} = \text{id}$ (the same is true of B_j).

We divide the proof into the two case $\widehat{D}_2 \subset U$ and $\widehat{D}_2 \subset V$. Note that $\varphi|R$ and $\varphi|S$ are irreducible.

Case 1. $\widehat{D}_2 \subset U$. Let U_0 be the component of U containing \widehat{D}_2 . Then $R \cap U_0$ and $S \cap U_0$ are the lids of U_0 . By Lemma 5.16 $\text{Fr } U_0$ is parallel to $\bigcup \alpha_i$. Isotop $\text{Fr } U_0$ to $\bigcup \alpha_i$; then $R \cap U_0 = \overline{R} - \bigcup A_i$, $S \cap U_0 = \overline{S} - \bigcup B_j$. Identify U_0 with $R \times I$ such that $R \cap U_0 = R \times 0$, $S \cap U_0 = R \times 1$, and $\widehat{D}_2 = \partial_r R \times I$.

Then $\varphi|_{U_0}$ gives a diffeomorphism of $R \times I$ such that $\varphi|(R, \partial_r R) \times \{i\}$ is the monodromy of J_{i+1} , $i = 0, 1$, and $\varphi|\partial_r R \times I = \text{id}$. Since $\partial_r S = \partial_r R \times 1$, Lemma 2.1 shows that $(Y_1, J_1) \cong (-Y_2, -J_2)$ as fibered knots and $s_1 = s_2$. This completes the proof of Proposition 6.2, part 1.

Note that there is a permutation ρ of $\{1, \dots, p_1\}$ such that $\alpha_i = \beta_{\rho(i)}$. By the action of φ on $\{\alpha_i\}_i$, ρ induces the permutation σ , say, of $\{1, \dots, s_1\}$. For simplicity assume that σ and ρ are the identities. Take any i such that $1 \leq i \leq s_1$, and let m be the least positive integer such that $\varphi^m(R_i) = R_i$. Let H' be the closure of the component of $H - \alpha_i$ containing R_i . By Lemma 2.2 $\varphi^m|_{\alpha_i} \simeq \text{id rel } \partial\alpha_i$. Thus $\varphi^m|\partial H'$ is the closed monodromy of $k_i^1 \# k_i^2 = k$. Lastly consider part 2(b). Let A be the annulus on $\partial H'$ bounded by R_i and S_i . If k_i^1 (resp. k_i^2) is a composite knot, then R_i (resp. S_i) is a trivial component of h_K . Thus the base, B , of h_k is equal to either A , $R_i \cup A$, $A \cup S_i$, or $R_i \cup A \cup S_i$, so that $\{B\} = h_k - h_K$, completing the proof in Case 1.

Case 2. $\widehat{D}_2 \subset V$. Let V_0 be the component of V containing \widehat{D}_2 . By Lemma 5.24 V_0 is a regular neighborhood of \widehat{D}_2 . Then $g = \varphi|R \cup \widehat{D}_2 \cup S$ is admissibly isotopic to a periodic map (Lemma 5.21). It follows from Lemma 2.3 that J_i is a torus knot in S^3 or a cable of $S^1 \times D^2$ of type (p_i, q_i) , say, where $p_1, p_2 > 1$. In what follows we suppose that J_1 is the (p_1, q_1) cable of $S^1 \times D^2$. The case when both of J_i are torus knots is considered at the end of Subcase 1 below. Lemma 2.4 implies that $g^{p_1 q_1 p_2 q_2}$ is isotopic to a $(p_1 q_1 + p_2 q_2)$ -Dehn twist along a core of \widehat{D}_2 . Hence $(*)$ $p_1 q_1 + p_2 q_2 = 0$, for g is periodic up to isotopy. Consider the position of $\partial_1 \alpha_1$. Note that φ permutes $\{\partial_i R\}_{i=1}^{p_1}$ cyclically and hence $s_1 = 1$. If $\partial_1 \alpha_1 > \partial_i R$ for some $i \neq 1$, then $\partial_1 \alpha_1 > \partial_j R$ for some $j \neq i$, and $\partial_j R$ separates $\partial_1 \alpha_i$ from $\partial_0 \alpha_i$. This is impossible. Hence, by Lemma 5.26, part 1 $\partial_1 \alpha_1 \subset R$ or $\subset F_2$.

Subcase 1. $\partial_1 \alpha_1 \subset F_2$. If $\partial_1 \alpha_1 \subset F_2 - S$, then $\partial_j S < \partial_1 \alpha_1$ for some j , $1 \leq j \leq p_2$. Since $\partial_0 \beta_j$ separates $\partial_0 \alpha_1$ from $\partial_1 \alpha_1$, so does $\partial_1 \beta_j$. Thus $\partial_1 \beta_j$ is parallel to $\partial_r S$. This contradicts Lemma 5.24, hence $\partial_1 \alpha_1 \subset S$. Therefore, $\partial_1 \alpha_1 = \partial_0 \beta_j$ for some $1 \leq j \leq p_2$. This is true of each $\partial_1 \alpha_i$, $1 \leq i \leq p_1$, therefore there is a permutation ρ of $\{1, \dots, p_1\}$ such that $\alpha_i = \beta_{\rho(i)}$. It follows that $p_1 = p_2$. Recalling $(*)$, we obtain $q_1 = -q_2$ and hence $J_1 \cong -J_2$. This proves Proposition 6.2, part 1.

For simplicity set $p_1 = p$, $q_1 = q$. Let M be a submanifold of H such that $\partial M = \bigcup \alpha_i \cup \overline{R} - \bigcup \overline{A_i} \cup \widehat{D}_2 \cup \overline{S} - \bigcup \overline{B_i}$. Then M is a simple manifold, so that $\varphi|M: M \rightarrow M$ is admissibly isotopic to a periodic map. In particular, $\varphi^{pq}|\partial M$ is periodic. By Lemma 2.4 $\varphi^{pq}|M \cap \partial H$ is isotopic rel $\bigcup \partial \alpha_i$ to the product of τ_i^p, η_i^{-p} , $1 \leq i \leq p$, where τ_i (resp. η_i) is a Dehn twist along a simple loop parallel to $\partial_0 \alpha_i$ (resp. $\partial_1 \alpha_i$). Hence $\varphi^p|_{\alpha_i} \simeq \text{id rel } \partial \alpha_i$ for each i . The proof of Proposition 6.2 part 2 is now completed by the argument similar to the last paragraph in Case 1. [If both J_i are torus knots, then (H, h) is a simple manifold, so that φ is periodic up to isotopy. By considering the fixed point data of $\varphi|\partial H$ we obtain $J_1 \cong -J_2$.]

Subcase 2. $\partial_1 \alpha_1 \subset R$. We shall show that Subcase 2 is not the case. To do this we shall need the Cyclic Surgery Theorem. Note that $\partial_1 \alpha_1 = \partial_0 \alpha_i$ for some $1 < i \leq p_1$, i.e., $\alpha_1 = \alpha_i$. It follows that $\{\partial_0 \alpha_1, \dots, \partial_0 \alpha_{p_1}\}$ are paired via the annuli α_j . Therefore, p_1 is even. Without loss of generality assume that $\alpha_1 = \alpha_2$. Set $p = p_1$, $q = q_1$, and $\psi = \varphi^p$. First isotop $\varphi \text{ rel } \partial H - R$

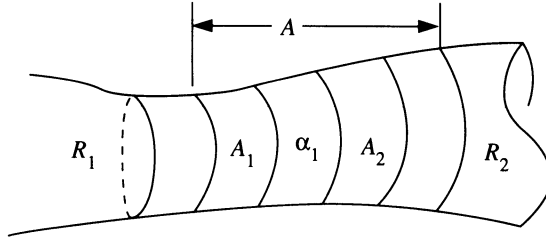


FIGURE 8

so that φ satisfies the assertions (a)–(e) of Lemma 2.4, part 2 with N_i , $i > 0$, replaced by A_i , $1 \leq i \leq p_1$. Then $\varphi|_{\overline{R - N_0 \cup (\bigcup A_i)}}$ is a periodic map of order $|pq|$, and $\psi|_{A_i}$ is a $-p/q$ -Dehn twist from $\partial_i R$ to $\partial_0 \alpha_i$.

Claim 1. $\psi|_{\alpha_1} \simeq \text{id rel } \partial \alpha_1$.

Proof. Let $T = \overline{R - N_0 \cup (\bigcup A_i) \cup (\bigcup \alpha_i)}$, a surface with one puncture. Since $T \subset \overline{H - X}$, $\psi|_T: T \rightarrow T$ is isotopic to a periodic map. Since $\psi|_{T - \bigcup \alpha_i}$ is periodic, $\psi|_{\alpha_1}$ is isotopic rel $\partial \alpha_1$ to a periodic map. Hence the rotational angle of $\psi|_{\partial_0 \alpha_1}$ is equal to that of $\psi|_{\partial_1 \alpha_1}$. It follows that $p/q \equiv -p/q \pmod{1}$, thus $|q| = 1, 2$. Since p is even, we have $q = \pm 1$. This implies that $\psi|_{\partial_0 \alpha_1} = \text{id}$, so that $\psi|_{\alpha_1} = \text{id}$. Claim 1 is proved.

Let H' be the closure of the component of $H - \alpha_1$ containing R_1 and R_2 . Let A be the annulus in $\partial H'$ bounded by $\partial_1 R$ and $\partial_r R_2$. Since $A \supset A_1 \cup A_2$, $\psi|_A$ is a $-2p/q$ -Dehn (full-)twist (Figure 8). Let F'_i , $i = 1, 2$, be the components of $\overline{\partial H' - A}$ containing R_i . Recall that $K_1 = J_1(k_1^1)$ and $\psi|_{F'_i}$ is the monodromy of k_1^1 for each i . For simplicity let $k_1 = k_1^1$. Then $\psi|_{A \cup F'_2}$ is the monodromy of a fibered knot k'_1 in a homology sphere defined as follows. Let N be the ambient manifold of k_1 , and let N' be the homology sphere obtained from N after a $2p/q$ -Dehn surgery along k_1 . The knot k'_1 in N' is the core of the glued solid torus. Thus $\psi|_{\partial H'}$ is the closed monodromy of $K' = k_1 \# k'_1$, a knot in the homology sphere $N \# N'$.

Claim 2. $k_1 = k_1^1$ is not prime.

Proof. Suppose for a contradiction that k_1 is prime. Then the boundary pattern $h_{K'}$ for H' is

$$h_{K'} = \{A \cup R_2, \text{ the components of } h_K \cap \overline{\partial H' - A \cup R_2}\}.$$

Here the annulus component of $h_{K'}$ bounded by R_1 and A plays the role of \tilde{D}_2 . We call this annulus \tilde{D}_2 . The usefulness of h_K for H guarantees that $h_{K'}$ is a useful boundary pattern for H' . We first show that $U_{K'}$ contains no annulus components of $h_{K'} - \{\tilde{D}_2\}$. If there is such a component, U' , of $U_{K'}$, then U' is not a component of U_K . Therefore U' intersects a component of $h_{K'}$ which does not belong to h_K , so that $(A \cup R_2) \cap U'$ is a lid of U' . However, U' is disjoint from \tilde{D}_2 which is adjacent to A . Since A is an annulus, this implies that U' can be admissibly isotoped off A . Thus U' is an essential I -bundle in (H_K, h_K) . Then U' is admissibly isotoped to some component of U_K , a contradiction. Therefore, $U_{K'}$ contains no annulus in $h_{K'} - \{\tilde{D}_2\}$.

Let us show $\tilde{D}_2 \subset U_{K'}$. Suppose not. As in the beginning of Case 2, Lemmas 5.24 and 5.21 imply that $\psi|_T$ is isotopic to a periodic map where $T = R_1 \cup \tilde{D}_2 \cup A \cup R_2$. $\psi|_{R_1}$ and $\psi|_{R_2}$ are isotopic to the same periodic map of order

m , say. Then $\psi^m|T \simeq \tau^{2-2mp/q}$, where τ is a Dehn twist along the core of A . Since $q \neq mp$ and the core of A is not parallel to ∂T , $\psi^m|T$ is not isotopic to a periodic map, a contradiction. Hence $\tilde{D}_2 \subset U_{K'}$. Let U_0 be the component of $U_{K'}$ containing \tilde{D}_2 . Following the argument in Case 1, identify U_0 with $R_1 \times I$ such that $U_0 \cap R_1 = R_1 \times 0$, $U_0 \cap (A \cup R_2) = R_1 \times 1$, and $\tilde{D}_2 = \partial_r R_1 \times I$. The natural projection $R_1 \times 0 \rightarrow -(R_1 \times 1)$, then, induces a diffeomorphism $g: R_1 \rightarrow -R_2$ such that

$$g^{-1} \circ (\psi| - R_2) \circ g = \tau^{2p/q} \circ (\psi|R_1)$$

where $\tau: R_1 \rightarrow R_1$ is a Dehn twist along a simple loop parallel to $\partial_r R_1$.

Let J be the fibered knot in a homotopy sphere with unknotted holes, P , with monodromy $\psi|R_1$. Set $M = \overline{P - N(J)}$, which is not $S^1 \times D^2$. Then g induces a diffeomorphism $g': M \rightarrow -M$ such that $g'(\mu) = \mu\lambda^r$, $g'(\lambda) = \lambda^{-1}$, where $r = -2p/q$ and (μ, λ) is a pair of a meridian and a longitude of J . It follows that the Dehn filling $-M(\mu\lambda^r) \cong M(\mu) = P$. Since $p > 1$ and $q = \pm 1$, $|r| \geq 4$. If P is a homotopy sphere, then by the Cyclic Surgery Theorem [6] M is a Seifert fibered manifold. It follows that $P = S^3$ and J is a nontrivial torus knot. Since torus knots have Property P, $M(\mu\lambda^r) \not\cong S^3$, a contradiction. Hence $\partial P \neq \emptyset$. Then ∂P compresses in $M(\mu\lambda^r)$ and $M(\mu)$. By [6, Theorem 2.4.4] $M (\neq T^2 \times I)$ is a cable space. Thus, $P = S^1 \times D^2$ and J is a nontrivial cable of P . However, this contradicts the fact that $M(\mu\lambda^r) = S^1 \times D^2$ [12, Lemma 7.2]. Claim 2 is proved.

Claim 3. k_1 is not a composite knot.

Proof. Recall that $K_1 = J_1(k_1)$ and $K' = k_1 \# k'_1$. Assume for a contradiction that k_1 is composite; $\psi|R_1 = \text{id}$ and R_1 is a punctured disk with $|\partial R_1| > 2$. Then the boundary pattern $h_{K'}$ for H' is the same as the one when k_1 is prime. That is,

$$h_{K'} = \{A \cup R_2, \text{ the components of } h_K \cap \overline{\partial H' - A \cup R_2}\}.$$

This time R_1 plays the role of \hat{D}_l where l is the number of factors in the prime decomposition of K' . It follows that $l > 2$. The usefulness of h_K shows that $h_{K'}$ is useful for H' . By Proposition 5.23 $U_{K'}$ contains an annulus component of $h_{K'}$. The argument about U' in Claim 2 shows that $U_{K'}$ contains the annulus in $h_{K'}$ bounded by A and R_1 . However, $U_{K'}$ does not contain an annulus component adjacent to a trivial component, a contradiction.

Since Claims 2 and 3 contradict each other, Subcase 2 ($\partial_1 \alpha_1 \subset R$) is impossible. This completes the proof of Proposition 6.2. \square

7. PROOF OF THEOREM 5.4

Let $K = J(k)$ be a fibered knot in a homotopy sphere such that $K \geq 0$, $k \not\equiv 0$, and J is the (p, q) cable of $S^1 \times D^2$ with $p > 1$. As in Theorem 5.4 assume that each nontrivial component of h_K is incompressible in H_K . By Lemma 5.20 it suffices to deduce a contradiction by assuming that U_K contains no annuli in $h_K - \{B_h\}$. Keep in mind that B_h is the union of the fiber of J and a disk. Because a boundary irreducible Seifert fibered manifold in the exterior of K is a cable space or (a punctured disk) $\times S^1$ [24, Lemma 2.1].

Case 1. No component of V is a regular neighborhood of an annulus in $h_K - \{B_h\}$.

As in the proof of Proposition 5.23, we can deduce a contradiction by assuming that $|B_h| > 2$. Hence, K is the $(2, \pm 1)$ cable of k . Note that $\hat{f}^2|_{B_h}$ is isotopic rel ∂B_h to a ± 4 -Dehn twist (Lemma 2.4 part 2(d)). If k is a prime knot, then a contradiction follows from the proof of Claim 2 in Theorem 5.3. If k is composite, then a contradiction follows from the proof of Claim 3 in Theorem 5.3. Hence, Case 1 is impossible.

Case 2. Some component of V is a regular neighborhood of an annulus in $h_K - \{B_h\}$.

Then, by Lemma 5.21, part 2 such an annulus is adjacent to B_h . Let A_1, \dots, A_p be the annuli in h_K adjacent to B_h ($p = |\partial B_h|$) and let S_i , $1 \leq i \leq p$, be the components of $h_K - \{B_h\}$ adjacent to A_i . Lemma 5.21, part 1, then, says that $\varphi|_S$ is admissibly isotopic to a periodic map where

$$S = B_h \cup \bigcup_{i=1}^p (A_i \cup S_i).$$

Hence the adjoint automorphism of S_i is the monodromy of a cable of $S^1 \times D^2$ or a torus knot in S^3 of type (r, s) , say. From Lemma 2.4 $\varphi^{pqrs}|_S$ is isotopic to a product of $(q - prs)$ -Dehn twists along the cores of A_i 's. Since $q - prs \neq 0$ for any r, s , the automorphism $\varphi|_S$ is not isotopic to a periodic map, a contradiction. Theorem 5.4 is proved. \square

8

8.1. Iterated torus knots, cable knots. Throughout this subsection (p, q) denotes the (p, q) torus knot, $J_{(p,q)}$ the (p, q) cable of $S^1 \times D^2$, and $\sigma(\dots)$ the signature of a knot in a homology sphere. When S is a closed surface, the following lemma is the same as Lemma 4.2 of [2]. A similar proof applies to a bounded surface S , so we omit the proof.

Lemma 8.1. *Let $g: S \rightarrow S$ be a periodic map. Suppose that g extends over a nontrivial compression body B . Then there is an essential simple loop C in $\partial_+ B$ such that C bounds a disk in B , and for each k , $f^k(C) = C$ or $f^k(C) \cap C = \emptyset$. In particular, if g is irreducible, then C is parallel to ∂S .*

A knot $K \subset S^3$ is a *graph knot* if the knot exterior of K is a graph manifold. Every graph knot is obtained by repeatedly taking a cable or connected sum starting from the unknot, and hence it is a fibered knot.

Theorem 8.2. *Let K_1, \dots, K_n be prime graph knots such that $K_1 \# \dots \# K_n \geq 0$. Then there are a partition $\{1, \dots, n\} = \coprod_{s=1}^m \{i_s, j_s\} \coprod_{t=1}^l \alpha_t$ (hence $n = 2m + l$) and graph knots K'_i , $1 \leq i \leq n$, satisfying the following properties.*

1. $K'_{i_s} \cong K'_{j_s}$ for $1 \leq s \leq m$, and $K'_{\alpha_t} = 0$ for $1 \leq t \leq l$.
2. Each K_i is ribbon concordant to K'_i by successive reductions of type $J(k\# - k) \geq J(0)$. In particular, K'_i is the unknot or a prime graph knot.

As immediate consequences we obtain

Corollary 8.3. *A graph knot K is homotopically ribbon if and only if we can obtain the unknot from K by successive reductions of type $J(k\# - k) \geq J(0)$.*

Corollary 8.4. *Let K_1, \dots, K_n be iterated torus knots such that $K_1 \# \dots \# K_n \geq 0$. Then $\{1, \dots, n\}$ can be paired into $\coprod_{s=1}^m \{i_s, j_s\}$ (hence $n = 2m$) such that $K_{i_s} \cong -K_{j_s}$.*

Remark. Corollary 8.4 shows that an iterated torus knot is not ribbon. In fact, it is not difficult to see that an iterated torus knot is not slice by showing that the Alexander polynomial is not of the form $f(t)f(t^{-1})$.

Proof of Theorem 8.2. Let $K = K_1 \# \dots \# K_n$. Let the automorphism φ of H_K be an extension of the closed monodromy of K . Note that the adjoint automorphism of a nontrivial component of $h_K - \{B_h\}$ is the monodromy of a cable of a solid torus or a torus knot.

We prove the theorem by induction on $|h_K|$. If $|h_K| = 3$, then $n = 2$ and $K = K_1 \# K_2$, a connected sum of torus knots. By Lemma 8.1 and the fact that a torus knot is not slice, each component of h_K is incompressible in H_K , so that $K_1 \cong -K_2$ by Theorem 5.3 or Proposition 6.1. Suppose Theorem 8.2 is true for all K' such that $|h_{K'}| < |h_K|$.

Case 1. There is a nontrivial component of h_K compressible in H_K .

Let $S \in h_K$ be a compressible nontrivial component such that $\partial_r S > \partial_r R$ for every such $R \in h_K$. Without loss of generality S is contained in the fiber of K_1 . Let m be the least positive integer such that $\varphi^m(S) = S$. Let J_1 be the fibered knot with monodromy $\varphi^m|_S$; then J_1 is a nontrivial cable of $S^1 \times D^2$ or a nontrivial torus knot. Note that either $K_1 = J_0(J_1(L))$ or $K_1 = J_0(J_1)$ for some $J_0 (\subset S^1 \times D^2)$ and nontrivial graph knot L . Take a maximal compression body B for S in H_K ; after an isotopy $\varphi^m(B) = B$. Let D' be a component of $\overline{\partial B} - \partial_+ B$ such that $\partial D'$ is a component of $\partial S - \partial_r S$. Note that $D' = \emptyset$ if J_1 is a torus knot.

Subcase 1. D' is not a disk.

We show this is absurd. Let D be the component of $\overline{\partial B} - \partial_+ B$ such that $\partial D = \partial_r S$. Since D' is not a disk, by Lemma 8.1 D is a disk. Let $S' = S \cup_{\partial_r S} D$. By Lemma 2.5 $\varphi^m|_{S'}$ is irreducible. Regard B as a compression body of S' . Then, by Lemma 8.1 B is a trivial compression body of S' , so that S' is incompressible in H . Hence S' is not a closed surface, implying $D' \cong S'$. Since $D' \neq \emptyset$, J_1 is not a torus knot. Let H' be the component of $\overline{H - B}$ containing D' . The automorphism $\varphi^m|_{\partial H'}$ is the closed monodromy of $J_1(L)$, hence $k = J_1(L) \geq 0$. Since each nontrivial component of h_k is incompressible in H' , applying Theorem 5.4 to k shows that J_1 is a trivial cable, a contradiction.

Subcase 2. D' is a disk.

Then J_1 is a nontrivial cable of $S^1 \times D^2$, and K_1 has a presentation $J_0(J_1(L))$. Let T be the nonannulus component of h_K such that $\partial_r T$ is parallel to $\partial D'$ in ∂H ; then T is a trivial component by the maximality of $\partial_r S$. Let l be the least positive integer such that $\varphi^l(T) = T$, and H'' be the closure of the component of $H - D'$ containing T . Then L is the composite graph knot with closed monodromy $\varphi^l|_{\partial H''}$, so that $L \geq 0$. Since $|h_L| < |h_K|$, the induction hypothesis implies L is ribbon concordant to the unknot by successive reductions of type $J(k\# - k) \geq J(0)$. Hence K_1 is ribbon concordant to $K'_1 = J_0(J_1(0))$ by reductions of this type, and we have $K' = K'_1 \# K_2 \# \dots \# K_n \geq 0$. Since $|h_{K'}| < |h_K|$, Theorem 8.2 holds for K' . Note that if J_0, J_1 are a $(p, 1)$ cable and a $(q, 1)$ cable for some p, q , then

$K'_1 = 0$; otherwise, K'_1 is a *prime* graph knot. Hence Theorem 8.2 also holds for K .

Case 2. Every nontrivial component of h_K is incompressible in H_K .

First we show that (*) there are no $S_0, S_1 \in h_K - \{\widehat{D}_n\}$ such that S_0, S_1 are adjacent via an annulus and $S_1 = S_0^*$. If there is a nonannulus component $S \in h_K - \{B_h\}$ such that the fibered knot with monodromy f_S has a dual knot, then f_S leaves some component of $\partial S - \partial_r S$ invariant. Then, S is a trivial component. Since two trivial components cannot be adjacent via an annulus, we obtain (*).

If $n = 1$, K_1 cannot be a nontrivial cable knot (Theorem 5.4), a contradiction. So suppose $n \geq 2$. From the proof of Lemma 5.10 there are a partition \mathcal{P} of $\{1, \dots, n\}$ and prime graph knots K'_i , $1 \leq i \leq n$, such that $K_i \geq K'_i$, and for each $P \in \mathcal{P}$, $K' = K'_P = \#_{i \in P} K'_i \geq 0$, and $h_{K'}$ is useful for $H_{K'}$. As in Subcase 2 of Case 1 above the induction hypothesis implies K_i is ribbon concordant to K'_i by successive reductions of type $J(k\# - k) \geq J(0)$. Obviously $h_{K'}$ also satisfies the condition (*) above, hence $U_{K'}$ does not contain any annulus components of $h_{K'}$ by Lemma 5.19, part 1. Since $|h_K| \geq |h_{K'}|$, we may assume $|\mathcal{P}| = 1$ and $K_i = K'_i$. It follows from Proposition 5.23 that $n = 2$. By Proposition 5.1 there are \tilde{K}_1, \tilde{K}_2 such that $\tilde{K}_1 \cong -\tilde{K}_2$ and \tilde{K}_i is obtained from K_i by successive reductions of type $J(k\# - k) \geq J(0)$. Thus K_i is ribbon concordant to \tilde{K}_i , and \tilde{K}_i is the unknot or a prime graph knot. This completes the proof of Theorem 8.2. \square

Example 1. The connected sum of four iterated torus knots below is algebraically slice but not homotopically ribbon.

$$(2, 13; 2, 3)\#(2, 15)\#(2, -15; 2, -3)\#(2, -13)$$

where $(p_1, q_1; p_2, q_2)$ denotes the (p_1, q_1) cable knot about the (p_2, q_2) torus knot. (This iterated torus knot is denoted by $(q_1, p_1; q_2, p_2)$ in [22].) Livingston and Melvin [22] prove that the knot is algebraically slice. The fact that it is nonribbon is an immediate consequence of Corollary 8.4.

Any $(p, 1)$ cable of a (homotopically) ribbon fibered knot is (homotopically) ribbon. The following theorem hence shows that all homotopically ribbon, cable fibered knots in homotopy spheres are such knots if Thurston's geometrization conjecture is true.

Theorem 8.5. *Let $J_{(p,q)}(k)$ be a (p, q) cable of a nontrivial fibered knot k in a homology sphere M where $p > 1$. If $nJ_{(p,q)}(k) \geq 0$ for some $n > 0$, then the following hold.*

1. $J_{(p,q)}(k) \geq 0$.
2. $q = \pm 1$; if $p > 2$, then $k \geq 0$.
3. If M is a homotopy sphere and 3-manifold groups are residually finite, then $k \geq 0$ and $q = \pm 1$.

Proof. If $k \geq 0$, then the assumption $nJ_{(p,q)}(k) \geq 0$ implies that $n(p, q)$ is algebraically slice, hence $\sigma((p, q)) = 0$. It follows that $q = \pm 1$ and $J_{(p,q)}(k) \geq (p, \pm 1) = 0$. Therefore, we may assume $k \not\geq 0$.

Proof of part 1. Let n be the least positive integer such that $nJ_{(p,q)}(k) \geq 0$. Assume for a contradiction that $n > 1$. Let $J_i = J_{(p,q)}$, $1 \leq i \leq n$, with fiber F_i . Let $k_i = k$, $K_i = J_i(k_i)$ for $1 \leq i \leq n$; then $K_1 \# \cdots \# K_n = K \geq 0$. The closed monodromy of K extends to an automorphism φ of a handlebody H_K . If $n > 2$, let A_i , $1 \leq i \leq n$, be the annuli in h_K adjacent to \widehat{D}_n . Otherwise, let $A_i = \emptyset$. We may regard F_i as the component of $h_K - \{\widehat{D}_n\}$ adjacent to A_i . Let B_j , $1 \leq j \leq np$, be the annuli in $h_K - \{\bigcup A_i\}$ adjacent to F_i , $1 \leq i \leq n$, such that $B_i \cap F_i \neq \emptyset$ for $1 \leq i \leq n$. Let S be the union of the np components of $\partial H_K - \bigcup_{j=1}^{np} B_j$ which are disjoint from \widehat{D}_n . Take a maximal compression body B for S in H_K , and set $H' = \overline{H_K - B}$, a handlebody. Since a maximal compression body is unique up to isotopy, we may assume $\varphi(H') = H'$.

For $1 \leq i \leq n$ the compression body B induces a homotopically ribbon concordance $k_i \geq k'_i$ where k'_i is a nontrivial fibered knot in a homology sphere. Note that the fibered knots k'_i , $1 \leq i \leq n$, are not necessarily of the same knot type. The $\varphi|_{\partial H'}$ is the closed monodromy of $K' = J_1(k'_1) \# \cdots \# J_n(k'_n)$ (≥ 0).

Let us define a boundary pattern h' of H' . Take a 1-submanifold C of $\text{Fr } B$ with the minimal number of components such that $\varphi(C) = C$, and $\varphi|_{(\text{Fr } B - C)}$ is a union of irreducible automorphisms and periodic ones up to isotopy. Define

$$h' = \{\text{the components of } h_K \cap H', N(C), \text{ and } \overline{\text{Fr } B - N(C)}\}.$$

Since the ambient manifold of $J_i(k'_i)$ is not necessarily a homotopy sphere, $\varphi^p|_{F_i \cup B_i \cup S}$ is possibly isotopic to a periodic map, where $S (\neq F_i)$ is a component of h' adjacent to B_i . That is, h' is not necessarily equal to $h_{K'}$.

Claim. h' is a useful boundary pattern for H' .

Proof. By the maximality of B , $\text{Fr } B$ is incompressible in H' . Hence each component of $h' \cap \text{Fr } B$ is incompressible. By the minimality of n , \widehat{D}_n is incompressible in H' . Since $\varphi|_{F_i}$ is isotopic to an irreducible periodic map, if F_i is compressible, so is ∂F_i in H' (Lemma 8.1). This contradicts the incompressibility of \widehat{D}_n or $\text{Fr } B$. Therefore each component of h' is incompressible in H' , so that h' is a useful boundary pattern.

Thus (H', h') has a characteristic submanifold X . Define the subsets U and V of X as in §5.1. It is easily verified that Lemmas 5.11–5.17 hold for these U and V . For any component of h' the adjoint automorphism is irreducible or isotopic to a periodic map (cf. part 3 of Proposition 5.1). Thus Lemma 5.21, part 1 holds for V .

Let us deduce a contradiction by assuming $n > 2$. Since $\varphi|_{F_i} \neq \text{id}$, each A_i is not contained in U but in V . By Lemma 5.21, part 1 we also see that no component of $\text{Fr } V$ is parallel to A_i . Let α_i be the component of $\text{Fr } V$ associated to the simple loop $\partial A_i \cap \partial \widehat{D}_n$. Since $\partial_0 \alpha_i$ is invariant under φ , so is $\partial_1 \alpha_i$. No components of $\partial F_i - \partial_r F_i$ are invariant under φ , so that $\partial_1 \alpha_i \subset \partial H' - \text{Fr } B$. It follows that $\partial_1 \alpha_i \subset \widehat{D}_n$ or $\partial_1 \alpha_i \subset F_j$ ($i \neq j$) (and $\partial_1 \alpha_i$ is parallel to $\partial_r F_j$). In either case we get a contradiction by constructing a 3-manifold violating Lemma 5.27. (If $\partial_1 \alpha_{i_0} \subset F_j$ for some $i_0 \neq j$, follow the proof of Lemma 5.26, part 2. If $\partial_1 \alpha_i \subset \widehat{D}_n$ for all i , the closure of the outermost component of $H' - \bigcup \alpha_i$ contradicts Lemma 5.27.) Thus it follows that $n = 2$.

Corollary 3.2 of [25] shows that $(S^1 \times D^2, J_1) \not\cong (-S^1 \times D^2, -J_1)$, hence

$\widehat{D}_2 \not\subset U$, i.e., $\widehat{D}_2 \subset V$. Again, using the fact that no component of $\partial F_i - \partial_r F_i$ are invariant under φ , we see that some component of V is a regular neighborhood of \widehat{D}_2 . By Lemma 5.21, part 1 $\varphi|_{F_1 \cup \widehat{D}_2 \cup F_2}$ is then isotopic to a periodic map. However, on this set φ^{pq} is isotopic to a ± 2 -Dehn twist along the core of \widehat{D}_2 , which is not isotopic to a periodic automorphism of $F_1 \cup \widehat{D}_2 \cup F_2$. This is absurd, completing the proof of part 1.

Proof of parts 2 and 3. Recall that if $k \geq 0$, then the stated results follow. We shall deduce a contradiction by assuming $k \not\geq 0$. We proceed similarly to the proof of part 1. As usual let φ denote an automorphism of a handlebody H_K , an extension of the closed monodromy of K . Set $F' = B_{h_K}$, a union of the fiber of the (p, q) cable of $S^1 \times D^2$ and a disk. Let A_1, \dots, A_p be the annuli components of h_K adjacent to F' , and set $S = \overline{\partial H_K - F' \cup \bigcup_{i=1}^p A_i}$. Then S has p components and φ^p restricts to the monodromy of k on each component of S . Take a maximal compression body B for S in H_K and set $H' = \overline{H} - \overline{B}$. Without loss of generality $\varphi(H') = H'$. Let $k' \not\geq 0$ be the fibered knot in a homology sphere whose monodromy is the restriction of φ to a component of $\text{Fr } B$. Then $k \geq k'$ and $\varphi|_{\partial H'}$ is the closed monodromy of $K' = J_{(p,q)}(k') (\geq 0)$. Let C be a 1-submanifold of $\text{Fr } B$ defined as in the proof of part 1. Define

$$h' = \{F', A_1, \dots, A_p, \text{ the components of } \overline{\text{Fr } B - N(C)}\}.$$

As in part 1 h' is a useful boundary pattern of H' . Here we first prove part 3. If k is a knot in a homotopy sphere and 3-manifold groups are residually finite, then k' and hence K' are knots in homotopy spheres (Lemma 1.2). By Swarup [24, Lemma 2.1] h' equals $h_{K'}$. Hence we can apply Theorem 5.4 to K' , so that $p = 1$, a contradiction. This establishes the proof of part 3.

Let us prove part 2. When $p = 2$, by [23, Theorem 9] or [14, Theorem 12] $\sigma((2, q)) = \sigma(J_{(2,q)}(k))$, which is equal to 0. Hence, we obtain the desired result $q = \pm 1$. We may assume $p > 2$. As in §5.1 define U, V for a characteristic submanifold X of (H', h') .

Claim 1. U contains no annulus components of h' .

Proof. Suppose not. Let U_0 be a component of U containing an annulus component of h' . First suppose that U_0 contains some A_i . Since $\varphi|_{F'}$ is irreducible, $U_0 \cap F'$, a lid of U_0 , is isotopic to F' . In particular, $U_0 \cap F'$ is invariant under φ . However, the other lid of U_0 , L say, is contained in some components of $\text{Fr } B$. The automorphism φ permutes the components of $\text{Fr } B$ cyclically, hence L is not invariant under φ , a contradiction. Thus U_0 contains an annulus in $h' \cap \text{Fr } B$. Then $\partial U_0 \cap \text{Fr } B$ consists of the two lids of U_0 and (a nonempty set of) annuli connecting them, so that $\text{Fr } B$ is compressible in H' . This is a contradiction.

Claim 2. If $q \neq \pm 1$, no component of $\text{Fr } V$ is parallel to an annulus in h' .

Proof. Assume for a contradiction that a component of $\text{Fr } V$ is parallel to some A_i . Then, there are the components V_1, \dots, V_p of V such that V_i is a regular neighborhood of A_i . Let M be the components of $\overline{H' - X}$ containing $\bigcup_{i=1}^p \text{Fr } V_i$. Then M is a simple manifold with a certain boundary pattern, in particular the automorphism $\varphi|_M$ is admissibly isotopic to a periodic map. The boundary pattern of M induced from h' consists of $\overline{F' - \bigcup_{i=1}^p V_i}$, $\text{Fr } V_1, \dots, \text{Fr } V_p$, submanifolds T_1 of $\text{Fr } B$, and properly embedded annuli T_2 such that

$\partial T_2 \subset \text{Fr } B$. Since φ permutes the components of $\text{Fr } B$ cyclically, the automorphism $\varphi|_{T_1}$ cannot be isotoped to have fixed points. The automorphism $\varphi|_{T_2}$ is not isotopic to a periodic map with fixed points; otherwise, φ acts on some (annulus) component of T_2 as involution, contradicting $|\text{Fr } B| = p > 2$. On the other hand, since $q \neq \pm 1$, $\varphi|_{F'}$ is isotopic to a periodic map with just one fixed point. Therefore, the fixed point datum of $\varphi|_{\partial M}$ cannot cancel. It follows that $\varphi|_{\partial M}$ is not null-cobordant, a contradiction.

Now Claims 1 and 2 contradict Lemma 5.22, hence we have $q = \pm 1$. Claim 3 below shows that Claim 2 is true even if $q = \pm 1$. We again get a contradiction by Lemma 5.22, completing the proof of Theorem 8.5. \square

Claim 3. No component of $\text{Fr } V$ is parallel to an annulus in h' .

Proof. We may assume $q = \pm 1$. Thus F' is a sphere with p punctures and φ permutes the components of $\partial F'$ cyclically. Let S_i , $1 \leq i \leq p$, be the component of $h' - \{F'\}$ which is adjacent to A_i , and set $S = F' \cup \bigcup_{i=1}^p (A_i \cup S_i)$. For a contradiction assume Claim 3 is false. Isotop φ admissibly so that $\varphi|_S$ is a periodic map (Lemma 5.21). Then $\varphi|_S$ is of period p and has two fixed points of type $1/p$. Since $\varphi^p|_{S_i} = \text{id}$, S_i and hence S are punctured disks. (Otherwise, the knot exterior of k' contains a nonseparating torus.) This shows that the closed monodromy of k' is that of a connected sum of certain m prime fibered knots where $m = |\partial S_i| - 1$. Remove F' , A_i and S_i , $1 \leq i \leq p$, from h' and add S . We then have the new boundary pattern \tilde{h} for H' . If S is incompressible in H' , \tilde{h} is useful. Define U' and V' for a characteristic submanifold of (H', \tilde{h}) . We can easily verify Claims 1 and 3 hold for U' , V' . Lemma 5.22 then gives a contradiction.

Suppose S is compressible in H' . By Lemma 8.1 a maximal compression body of S in H' can be obtained by compressing a set of pairwise disjoint, essential simple loops which is invariant under the action of φ . Let $\{l_j\}_j$ be such a set such that l_j bound pairwise disjoint disks D_j in H' invariant under φ . Let us consider $\bar{l}_j = q(l_j)$ in $\bar{S} = S/\varphi$ where $q: S \rightarrow \bar{S}$ is the natural projection. Note that \bar{S} is a disk with m punctures having two cone points. Suppose that some \bar{l}_j bounds a disk in \bar{S} containing the two cone points of \bar{S} . Then $S - \bigcup_{k=1}^p \varphi^k(l_j)$ consists of a disk with p punctures and p copies of a disk with m punctures, say, S'_1, \dots, S'_p . Let H_1 be the closure of a component of $H' - \bigcup_{k=1}^p \varphi^k(D_j)$ containing S'_1 ; then $\varphi^p(H_1) = H_1$. Since $\varphi^p|_{S'_1} = \text{id}$, we see that $\varphi^p|_{\partial H_1}$ is the closed monodromy of k' . It follows $k \geq k' \geq 0$, a contradiction. Thus there is a component C of $\bar{S} - \bigcup \bar{l}_j$ containing a component of $\partial \bar{S}$ and a cone point. Define the handlebody H'' to be the closure of the component of $H' - \bigcup_j D_j$ containing the punctured disk $q^{-1}(C)$; then $\varphi(H'') = H''$. Let s' be the union of $q^{-1}(C)$ and the disks D_j contained in $\partial H''$, so that $|\partial S'| (\neq 0)$ is a multiple of p . Since S' is incompressible in H'' , the boundary pattern $h'' = (\overline{\partial H'' - S'} \cap h') \cup \{S'\}$ for H'' is useful. Just like $\varphi|_S$, $\varphi|_{S'}$ is of period p . Hence, defining U' , V' for a characteristic submanifold of (H'', h'') , we see that Claims 1 and 3 hold for these U' and V' . This gives a contradiction as above, completing the proof of Claim 3. \square

Example 2. Let k be a $-$ amphicheiral fibered knot in a homotopy sphere with an irreducible Alexander polynomial $\Delta(t)$. Let $K_m = J_{(m,1)}(k)$. Then for any $n > 0$, nK_{2m} , $2nK_{2m+1}$ are algebraically slice but not homotopically ribbon.

Remark. The proof of algebraically-sliceness is a slight generalization of the proof of Example 4.2 in Kawauchi [20]. The arguments show that K_{2m} is algebraically slice if a knot k has an irreducible Alexander polynomial satisfying the condition $(*)$ below.

Proof. (1) Algebraically-sliceness. Since k is $-$ amphicheiral, by [17] $\Delta(t)$ satisfies

$$(*) \quad \Delta(t^2) = f(t)f(-t), \quad f(t^{-1}) \doteq f(-t) \text{ for some } f(t) \in \mathbb{Z}[t].$$

It is well known that the Alexander polynomial of K_m is $\Delta(t^m)$. We claim that $f(t^m)$ and $f(-t^m)$ are relatively prime in $\mathbb{Z}[t]$. It suffices to show that $f(t)$ and $f(-t)$ are relatively prime in $\mathbb{Z}[t]$. Let $g(t) \in \mathbb{Z}[t]$ be a g.c.d. of $f(t)$ and $f(-t)$. Then $h(t) = g(t)g(-t)$ divides $f(t)f(-t)$ and hence $\Delta(t^2)$. Since $h(t)$ is an even function, $\Delta(t)$ has a factor $h(t^{1/2}) \in \mathbb{Z}[t]$. Without loss of generality we may assume that $\deg g = \deg f$; then $f(t) = \pm f(-t)$. It follows that $t^{1/2}f(t^{1/2})$ or $f(t^{1/2})$ is a \mathbb{Z} -polynomial, contradicting the irreducibility of $\Delta(t)$. Thus $f(t^m)$ and $f(-t^m)$ are relatively prime, so that [20, Lemma 4.1(3)] shows that K_{2m} is algebraically slice.

Let us show that $2K_{2m+1}$ is algebraically slice. By [20, Proposition 4.1(3)] $2K_{2m+1}$ is algebraically cobordant to $J_{(2m+1, 1)}(2k)$. Since $2k \cong k\#-k$ is slice, $2K_{2m+1}$ is algebraically slice.

(2) If $nK_m \geq 0$ for some $n > 0$, then $K = K_m \geq 0$ by Theorem 8.5, part 1. As in the proof of Theorem 5.5 (§5.1) the irreducibility of $\Delta(t)$ implies that $\partial H_K - B_h$ is incompressible in H_K . An application of Theorem 5.4 to K gives a contradiction. \square

Theorem 8.5, parts 2 and 3 lead us to the question: if the $(2, 1)$ cable of a knot k is homotopically ribbon, is $k \geq 0$? The example below shows that the answer is “no” for a $(2, 1)$ cable knot in a *homology* sphere. Hence, Theorem 5.4 and Example 2 cannot be generalized to a fibered knot in a homology sphere.

Example 3. Let N be the homology sphere obtained by a $-\frac{1}{2}$ -Dehn surgery of S^3 along the figure eight knot k . Let k' denote the core of the glued solid torus. Then $J_{(2, 1)}(k')$ is a homotopically ribbon fibered knot in N .

Proof. Let $\varphi: F \rightarrow F$ be the closed monodromy of $K = J_{(2, 1)}(k)$. Let F_1, F_2 be the two components of $\overline{F - B_{h_K}}$. Then $\varphi^2|F_1: F_1 \rightarrow F_1$ is the monodromy of k and $\varphi^2|B_h$ is a -4 -Dehn twist.

Claim. There is a diffeomorphism $G: F_1 \rightarrow -F_1$ such that $G^2 = \varphi^2|F_1$.

Proof of claim. Let x, y be simple loops on F_1 intersecting in a single point, and τ_1, τ_2 denote the Dehn twists along x, y respectively. Without loss of generality the monodromy $\varphi^2|F_1 = \tau_1 \circ \tau_2^{-1}$. Hence, letting r be an orientation reversing diffeomorphism of F_1 exchanging x and y , we obtain $(r \circ \tau_2^{-1})^2 = \tau_1 \circ \tau_2^{-1}$. Hence, $r \circ \tau_2^{-1}$ is the claimed diffeomorphism.

Denote by τ a Dehn twist along a simple loop on F_1 parallel to ∂F_1 . Then $\tau^2 \circ (\varphi^2|F_1)$ is the monodromy of k' in N , and $\tau^2 \circ \varphi$ is the closed monodromy of $J_{(2, 1)}(k')$. Let $\psi = \tau^2 \circ \varphi$. Identify $F_1 \cup F_2$ and $F_1 \times \{0, 1\}$ by $\text{id}: F_1 \rightarrow F_1 \times 0$ and $G \circ (\varphi^{-1}|F_2): F_2 \rightarrow F_1 \times 1$. Under this identification we have

$$\begin{aligned} \psi(x, 1) &= (\tau^2 \circ \varphi^2 \circ G^{-1}(x), 0), \\ \psi(x, 0) &= (G(x), 1), \quad \text{where } x \in F_1. \end{aligned}$$

We can identify B_{h_K} and $\partial F_1 \times I$ so that

$$\psi|_{\partial F_1 \times I} = \eta^{-2} \circ r,$$

where η is a Dehn twist along the core of $\partial F_1 \times I$, and r is an involution of $\partial F_1 \times I$ such that $r(x, t) = (G(x), 1 - t)$. Since $\varphi^2|_{F_1} = G^2$ and $\tau^2 \circ \eta^{-2} \simeq \text{id}$ in $\partial(F_1 \times I)$, after an isotopy we obtain

$$\psi(x, t) = (G(x), 1 - t) \quad \text{in } \partial(F_1 \times I).$$

It is easy to see ψ extends over the handlebody $F_1 \times I$, hence $J_{(2,1)}(k') \geq 0$. \square

Remark. As pointed out in the proof the monodromies of $J_{(2,1)}(k) (\subset S^3)$ and $J_{(2,1)}(k') (\subset N)$ differ only by a Dehn twist along a null-homologous loop. Therefore the two monodromies induce equivalent isomorphisms on the homology groups.

8.2. A class \mathcal{K} . Define \mathcal{K} to be a class of prime fibered knots in homotopy spheres consisting of $J_1(\cdots(J_m(k))\cdots)$ and k where J_i and k satisfy the following conditions.

1. k is a torus knot or a prime fibered knot in homotopy spheres such that no nontrivial Laurent polynomial $f(t)f(t^{-1})$ divides $\Delta_k(t)$.
2. Each J_i is a nontrivial cable of $S^1 \times D^2$.

Theorem 8.6. *If $K_1, \dots, K_n \in \mathcal{K}$ and $K_1 \# \cdots \# K_n \geq 0$, then $\{1, \dots, n\}$ can be paired into $\coprod_{s=1}^m \{i_s, j_s\}$ such that $K_{i_s} \cong -K_{j_s}$.*

Proof of Theorem 8.6. First we consider the case $n > 1$. Set $K = K_1 \# \cdots \# K_n$. Let an automorphism φ of H_K be an extension of the closed monodromy of K . For each $1 \leq i \leq n$ let $K_i = J_1^i(\cdots(J_{m(i)}^i(k_i))\cdots)$ be a satellite presentation satisfying the definition of \mathcal{K} . Let S_j^i, S^i denote the fiber of J_j^i, k_i respectively. Let \mathcal{T} be a minimal set of tori in the complement of K which decompose it into a Seifert fibered manifold and a simple manifold. Isotoping \mathcal{T} so that $|F \cap (\bigcup \mathcal{T})|$ is minimal where F is the fiber of K , and set $C = F \cap (\bigcup \mathcal{T})$. Note that each nonannulus component of $h_K - \{\widehat{D}_n\}$ is a copy of S_j^i or contained in a copy of S^i for some i, j .

Case 1. Every nontrivial component of h_K is incompressible in H_K .

First note that condition 1 in the definition of \mathcal{K} implies that k_i is minimal with respect to " \leq " among all fibered knots. If K_i has presentation $J(k\# - k)$ or $J_1(J(J^*(k)))$ where $k \not\cong 0$ and $J \not\cong (S^1 \times D^2, S^1 \times 0)$, then so does k_i . This implies k_i is not minimal, contradicting the observation above. Then, Theorem 8.6 follows from Theorem 5.4, part 2.

Case 2. There is a nontrivial component of h_K compressible in H_K .

We shall deduce a contradiction in this case. As in the proof of Theorem 8.2 take a nontrivial compressible component $R \in h_K$ such that $\partial_r R > \partial_r R'$ for every such a component $R' \in h_K$. The assumption on k_i implies that the fiber S^i of k_i is incompressible in H_K . Hence, R is (a copy) of $S_{j_0}^{i_0}$ for some i_0, j_0 , so that there is a compression disk D in H_K such that $\partial_r R = \partial D$ (this is because the monodromy of a cable of $S^1 \times D^2$ is irreducible). Let H' be the closure of the component of $H_K - D$ containing R ; then $\varphi^m(H') = H'$. Set $\psi = \varphi^m|_{H'}$. Then, the closed monodromy of $L = J_{j_0}^{i_0}(J_{j_0+1}^{i_0}(\cdots(k_{i_0})\cdots))$

is $\psi|_{\partial H'}$, which extends over the handlebody $H' (= H_L)$. Since L is a non-trivial cable knot and h_L is a useful boundary pattern for H' , this contradicts Theorem 5.4.

To complete the proof of Theorem 8.6 we need to consider the case $n = 1$. However, just as in the arguments in Case 2 above we can deduce a contradiction. \square

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