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A WEAK CHARACTERISTIC PAIR FOR END-IRREDUCIBLE 3-MANIFOLDS

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ABSTRACT. This extends a weakened version of the Characteristic Pair Theorem of Jaco, Shalen, and Johannson to a large subclass of the class of end-irreducible 3-manifolds. The Main Theorem of this paper states that if (W, w) is a noncompact 3-manifold pair (where W is a noncompact 3-manifold that has an exhausting sequence with certain nice properties and where w is incompressible in W), then there is a Seifert pair (Σ, Φ) contained in (W, w) such that any 2-manifold that is strongly essential in (W, w) and each of whose components is a torus, an annulus, an open annulus, or a half-open annulus is isotopic in (W, w) into (Σ, Φ).

INTRODUCTION

In the introduction, some nonstandard terms are used. These terms, as well as others, will be defined in §1.

In [JS] and [Jo], it was proved that if (M, T) is a compact, sufficiently large 3-manifold pair, then there exists a characteristic pair for (M, T) that is unique up to isotopy. This Characteristic Pair Theorem is a beautiful and powerful result. However, the neophyte might find the above statement somewhat unrevealing. A workaday statement of the Theorem might be: If (M, T) is a compact 3-manifold pair, where M is Haken and T is incompressible in M, then there is a Seifert pair (Σ, Φ) perfectly embedded in (M, T) into which any essential annulus or torus of (M, T) can be isotoped; furthermore a canonical (Σ, Φ) can be chosen. The fact that there is a canonical choice is where the word is "characteristic" comes in.

It is this second statement on which the result of this paper is modeled.

Main Theorem. *Let W be a noncompact, orientable, connected, irreducible 3-manifold and let $w \subset \partial W$ be a compact 2-manifold that is incompressible in W. Furthermore, assume that there is an end-irreducible exhausting sequence $V = \{V_n\}$ for W such that no component of $\mathrm{Fr}(V_n; W)$ is a disk, a 2-sphere, or a torus for any $n \geq 0$. There exists a Seifert pair $(\Sigma, \Phi) \subset (W, w)$ into which any strongly essential torus, annulus, open annulus, or half-open annulus of (W, w) may be isotoped. Moreover $\mathrm{Fr}(\Sigma; W)$ is strongly essential in (W, w).*

The reader will notice the lack of uniqueness in this result. This is why the Main Theorem could be referred to as a "weak" characteristic pair theorem.

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Also note that some obvious modifications will have to be made in the usual definition of Seifert pair for this to make sense.

Let $(M, w \cup G)$ be a compact, irreducible 3-manifold pair, where G is a nonempty compact 2-manifold no component of which is a disk, torus, or a 2-sphere and $w \cap G = \varnothing$. Let $W = M \setminus G$. Then (W, w) satisfies the hypothesis of the Main Theorem. If (S, F) is a characteristic pair for $(M, w \cup G)$, then we may take $(\Sigma, \Phi) = (S \cap W, F \cap w)$.

In cases where (W, w) contains no strongly essential torus, annulus, open annulus, or half-open annulus (i.e., is atoroidal), then (Σ, Φ) is empty. If only open and half-open annuli are ruled out, then every component of (Σ, Φ) must be compact, even though (Σ, Φ) itself may still be noncompact.

The proof of the Main Theorem requires the existence of an exhaustion V such that

(1) V is end-irreducible,

(2) no component of $\mathrm{Fr}(V_n; W)$ is a disk, 2-sphere, or torus for any $n \geq 0$.

The need for (1) comes from the use of Theorem V.2.1 of [JS] in the proof of Theorem 3.12. The author sees no way to remove this requirement directly. However, the end-reduction methods of [BT] are powerful and offer hope for a more subtle approach in this case.

The need for (2) comes from technical requirements in proving some results about 2-manifolds, in particular, Corollary 2.6. Actually, the elimination of 2-sphere components of $\mathrm{Fr}(V_n; W)$ is vacuous if we assume $W \neq \mathbf{R}^3$ because W is irreducible. The author believes that the theorem would hold true without (2) except for special cases. In any case, the result here is strong enough to be used in [W], which is the author's main interest.

There are two ways that the Main Theorem falls short of what the author would like to have proved. The first of these is the lack of uniqueness. In the compact case, a unique object could be obtained by using the compact analog of the Main Theorem and taking (Σ, Φ) to have the minimum number of components. In the present case, however, (Σ, Φ) might have an infinite number of components. The author is optimistic that some form of uniqueness can be obtained.

The second shortcoming of the Main Theorem (at least in relation to the Characteristic Pair Theorem) is that it deals only with *embedded* 2-manifolds. The obstruction to this is in §3. In that section, a controlled version of the "workaday" characteristic pair theorem is proved, namely Theorem 3.12. If Theorem 3.12 could be extended to singular 2-manifolds, the author believes that a singular version of the Main Theorem would follow.

In this paper, the author uses what some refer to as the "layer" approach. The alternative to this is the "finite ambiguity" approach which has been used in [BBF] and [BF] to great advantage.

Let (W, w) and V be as in the statement of the Main Theorem.

A "finite ambiguity" approach to the Main Theorem might proceed as follows. For $n \geq 0$, let (Σ_n, Φ_n) be the characteristic pair of $(V_n, w \cup \mathrm{Fr}(V_n; W))$. For each $0 \leq k \leq n$, isotope Σ_n so that Σ_n meets V_k "nicely" in a 3-manifold $\Sigma_{n,k}$. Argue that for each k there are only finitely many topologically distinct $\Sigma_{n,k}$. Successively reindex $\{V_n\}$ so that there is only one type of $\Sigma_{n,0}, \Sigma_{n,1}, \ldots$ for each $n \geq 0$. Then $\Sigma = \bigcup_{n=0}^{\infty} \Sigma_{n,n}$ would presumably be the Σ sought in the Main Theorem.

The author feels that if he had been able to get the "finite ambiguity" approach to work, then this would have been a much shorter paper. However there were problems involving the construction of isotopies that he just could not overcome.

The "layer" approach taken in this paper may be described as follows. Let $M_0 = V_0$ and $m_0 = w \cup \mathrm{Fr}(M_0; W)$; for $n \geq 1$, let $M_n = \mathrm{cl}(V_n \setminus V_{n-1})$ and $m_n = \mathrm{Fr}(M_n; W)$. Each M_n is like a layer of an infinitely tall cake. Let (S_n, s_n) be the characteristic pair of (M_n, m_n). Since any strongly essential torus, annulus, open annulus, or half-open annulus can be made to intersect all of the M_n in essential annuli and tori (Lemma 7.5), it seems that it ought to follow that any strongly essential torus, annulus, etc., is isotopic in W into $\Sigma = \bigcup_{n=0}^{\infty} S_n$. There are problems however. It might be that $s_n \cap m_{n+1} \neq s_{n+1} \cap m_n$ for some n in which case the "layers" of Σ do not fit together. This is fixed by taking modified characteristic pairs (see Theorem 3.12) recursively with respect to $(M_n, m_{n,k})$ where $\{m_{n,k}\}$ is a particular sequence of 2-manifolds. It is shown in Corollary 2.6 that this sequence is eventually constant if one is careful about annulus components of the $m_{n,k}$. There are a number of possible ways to deal with this problem. In this paper there are dealt with by inserting "guides" around the characteristic pair of $(M_n, m_{n,k})$ for appropriate k. These "guides" also help to solve the more subtle problem of turning isotopies of the "layers" into a single isotopy of (W, w) (see Lemmas 4.5 and 5.2).

In pursuing the "layer" approach, the author has created two categories to simplify the exposition: stacks and admissible stacks. Much of the notational nightmare associated with noncompact 3-manifolds in general and the "layer" approach in particular has been eliminated (or at least lessened) in this environment. It seems to be a natural environment in which to work, and the author hopes that others will find this so too.

A brief outline of the paper is as follows. Section 1 provides the basic definitions for this paper, although other sections contain definitions as well. In §2 such results about 2-manifolds as we will need are proved.

A controlled version of the Characteristic Pair Theorem (for embedded 2-manifolds and without uniqueness) is proved in §3. In §4 some technical results about isotoping annuli and tori "nicely" into Seifert pairs are proved.

Isotopy lemmas in stack categories are proved in §5. In fact, Theorem 5.5 is a version of the Main Theorem in the category of admissible stacks. Section 6 provides results about Seifert stacks and round stacks. The Main Theorem, Theorem 7.6, is proved in §7.

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I dedicate this paper to the memory of my father.

1. Preliminary definitions

In this section, the basic vocabulary of this paper is defined.

In this paper, the author has found it necessary to work in a number of different categories. In each of these categories, there are certain analogous concepts. At times the same word has been used to denote a similar or equivalent concept in a different category. At other times, when the author thought the chance for confusion was too great, modified terms have been used. In any case, it is hoped that the understanding of the material has been helped rather than harmed.

Let X and Y be topological spaces. We let $\sharp(X)$ denote the number of components of X. If $f: X \to Y$ is a map, we say that f is *proper* if $f^{-1}(K)$ is compact for every compact $K \subset Y$. If $X \subset Y$, then we say that X is *proper* in Y if the inclusion map is proper. If $X \subset Y$ we denote the topological frontier of X in Y by $\mathrm{Fr}(X; Y)$. We let $\mathrm{cl}(X)$ denote the closure of X.

If F is an $(n-1)$-manifold and M is an n-manifold, we say that F is *properly embedded* in M if $F \subset M$, F is proper in M, and $F \cap \partial M = \partial F$.

Let $h_t: X \times I \to X$ be a proper map. Define $h_t: X \to X$ by $h_t(x) = h(x, t)$. We say that h_t is an *isotopy of* X if $h_0 = 1_X$ and h_t is a homeomorphism for every $t \in I$.

An *annulus* (*open annulus, half-open annulus*) is a space homeomorphic to $S^1 \times I$ ($S^1 \times \mathbf{R}$, $S^1 \times [0, \infty)$).

If R is a connected 2-manifold, we say that R is *round* if it is an annulus, open annulus, half-open annulus, or torus. If R is a 2-manifold, we say that R is *round* if each of its components is round.

If M is a μ-manifold and $m \subset \partial M$ is a $(\mu - 1)$-manifold that is proper in M, then we say that (M, m) is a *μ-manifold pair.*

Suppose that (M, m) and (N, n) are μ- and ν-manifold pairs, respectively. If $f: M \to N$ is a map such that $f(m) \subset n$, then we say that $f: (M, m) \to (N, n)$ is a *map of pairs*; if, in addition, f is a homeomorphism and $f|m: m \to n$ is a homeomorphism, then we say that f is a *homeomorphism of pairs.* If h_t is an isotopy of M such that $h_t: (M, m) \to (M, m)$ is a homeomorphism of pairs for every t, then we say that h_t is an *isotopy of* (M, m). If $M \subset N$ and $m \subset n$, then we write $(M, m) \subset (N, n)$. If M' is a component of M and $m' = m \cap M'$, then we say that (M', m') is a *component* of (M, m). If M is compact, then we say that (M, m) is *compact.*

Suppose that F is a $(\mu - 1)$-manifold. If there is a map of pairs f: $(F \times I, \partial F \times I)(M, m)$ that is an embedding and is proper, we say that $f(F \times 0)$ is *parallel in* (M, m) to $f(F \times 1)$ and that $(f(F \times I), f(\partial F \times I))$ is a *parallelism* in (M, m) from $f(F \times 0)$ to $f(F \times 1)$.

Suppose that (M, m) is a 3-manifold pair. If F is a connected 2-manifold that is properly embedded in M, we say that F is *essential in* (M, m) whenever $\partial F \subset m$, F is incompressible in M, and F is not parallel in (M, m) to a 2-manifold in m. If F is a 2-manifold that is properly embedded in (M, m), we say that F is *essential in* (M, m) if each of its components is essential in (M, m).

Let F be a connected 2-manifold that is proper in M. We say that $K \subset M$ *traps* F in (M, m) if there is no proper map of pairs $h: (F \times I, \partial F \times I) \to (M, m)$ with $h(x, 0) = x$ for every $x \in F$ and $h(F \times I) \cap K = \varnothing$. We say that F is *strongy essential* in (M, m) if F is essential in (M, m) and is trapped in (M, m) by some compact subset of M.

We say that a properly embedded 2-manifold is *strongly essential* in (M, m) if each of its components is.

We say that (M, m) is *irreducible* if M is irreducible and m is incompressible in M. We say that (M, m) is *good* if M is orientable, (M, m) is irreducible, and no component of m is a disk, a 2-sphere, or a torus.

Let (S, s) be a 3-manifold pair. If there is a Seifert fibration of S in which s is a union of fibers, then we say that (S, s) is an *S^1-pair.* (We do not demand that S be compact.) If there exists an I-bundle ($[0, \infty)$-bundle, $\mathbf{R}$-bundle) $p: S \to F$ for some 2-manifold F such that $p|s: s \to F$ is a ∂I-bundle ($\partial[0, \infty)$-bundle, $\partial\mathbf{R}$-bundle), then we say that (S, s) is an *I-pair* (*$[0, \infty)$-pair,* *$\mathbf{R}$-pair*). Of course in the last case $s = \varnothing$.

If (S, s) is connected and is an S^1-pair, an I-pair, a $[0, \infty)$-pair, or an $\mathbf{R}$-pair, then we say that (S, s) is a *Seifert pair.* If (S, s) is a 3-manifold pair and each component of (S, s) is a Seifert pair, then we say that (S, s) is a *Seifert pair.* This is broader than [JS], for example.

Suppose that $(S, s) \subset (M, m)$ are 3-manifold pairs and that (S, s) is a Seifert pair. If S is proper in M, if $s = S \cap \partial M$, and if $\operatorname{Fr}(S; M)$ is incompressible in M, then we say that (S, s) is *well-embedded in* (M, m). If (S, s) is well-embedded in (M, m) and there is no component (S', s') of (S, s) that is isotopic in (M, m) into $(S \backslash S', s \backslash s')$ and $\operatorname{Fr}(S; M)$ is essential in (M, m), then we say that (S, s) is *perfectly embedded in* (M, m).

Let $\mu \geq 1$ be an integer and let M be a compact μ-manifold. Suppose that m and m' are disjoint, compact, perhaps empty, $(\mu - 1)$-manifolds in ∂M. Then (M, m, m') is called a *μ-triple.* We refer to m' as the *top* of M and to m as the *bottom* of M. Note that $(M, m \cup m')$ is a compact μ-manifold pair.

Suppose that (A, a, a') and (B, b, b') are an α-triple and a β-triple, respectively. A *map of triples* $f: (A, a, a') \to (B, b, b')$ is a map $f: A \to B$ such that $f(a) \subset b$ and $f(a') \subset b'$; if in addition $f: (A, a \cup a') \to (B, b \cup b')$ is a homeomorphism of pairs, then f is a *homeomorphism of triples.* If h_t is an isotopy of $(A, a \cup a')$ such that $h_t: (A, a, a') \to (A, a, a')$ is a homeomorphism of triples for every $t \in I$, then h_t is an *isotopy of* (A, a, a'). We write $(A, a, a') \subset (B, b, b')$ if $A \subset B$, $a \subset b$, and $a' \subset b'$.

Suppose that (M, m, m') is a 3-triple. We say that (M, m, m') is a(n) *Seifert* (*S^1-, I-, irreducible, good*) *triple* if $(M, m \cup m')$ is a Seifert (S^1-, I-, irreducible, good) pair.

Suppose that (R, r, r') is a 2-triple such that R is a round 2-manifold and $r \cup r' = \partial R$. Then we say that (R, r, r') is a *round triple.* Note that all round triples are compact.

Let (M, m, m') be a 3-triple and let (S, s, s') be a Seifert triple. We say that (S, s, s') is *well-embedded* (*perfectly embedded*) in (M, m, m') if $(S, s \cup s')$ is well-embedded (perfectly embedded) in $(M, m \cup m')$. If F is a 2-manifold that is essential in $(M, m \cup m')$, then F is *essential* in (M, m, m').

Let $\nu \geq 1$ be an integer. For $n \geq 0$ let (M_n, m_n, m_n') be a ν-triple. Let $M = \{(M_n, m_n, m_n')\} = \{(M_n, m_n, m_n') | n \geq 0\}$. Then M is said to be a ν-stack, and M_n is said to be the nth *layer* of M. Suppose that, for $n \geq 0$, $\mu: m_n' \to m_{n+1}$ is a homeomorphism. Then $\mu = \{\mu_n | n \geq 0\}$ is said to be a *gluing sequence for* M. If μ is a gluing sequence for M, then (M, μ) is said to be an *admissible ν-stack.*

For each $\nu \geq 1$, let $\mathscr{S}_\nu$ be the class of all ν-stacks and let $\mathscr{A}_\nu$ be the class of all admissible ν-stacks. Put $\mathscr{S} = \bigcup \mathscr{S}_\nu$ and $\mathscr{A} = \bigcup \mathscr{A}_\nu$.

Let $A, B \in \mathscr{S}$ with $A = \{(A_n, a_n, a_n')\}$ and $B = \{(B_n, b_n, b_n')\}$. Suppose that, for each $n \geq 0$, $f_n: (A_n, a_n, a_n') \to (B_n, b_n, b_n')$ is a map of triples. Let

$f = \{f_n\}$. Then we say that $f\colon A \to B$ is an *$\mathcal{S}$-map*. Suppose that α and β are gluing sequences for A and B, respectively. We say that $f\colon (A, \alpha) \to (B, \beta)$ is an *$\mathcal{A}$-map* if $f_{n+1}\alpha_n = \beta_n f_n$ for all $n \geq 0$. It is easy to check that ($\mathcal{S}$, $\mathcal{S}$-maps) and ($\mathcal{A}$, $\mathcal{A}$-maps) are categories.

Let M be a ν-stack. Suppose, for each $n \geq 0$, that $(h_n)_t$ is an isotopy of (M_n, m_n, m'_n). For each $t \in I$, let $H_t\colon M \to M$ be the $\mathcal{S}$-map $H_t = \{(h_n)_t\}$. Then we say that H_t is an *$\mathcal{S}$-isotopy of M*. If in addition (M, μ) is an admissible ν-stack and $H_t\colon (M, \mu) \to (M, \mu)$ is an $\mathcal{A}$-map for each $t \in I$, then we say that H_t is an *$\mathcal{A}$-isotopy of (M, μ)*.

Suppose that M is a 3-stack. If each layer of M is a Seifert (S^1-, I-, good) triple the M is said to be a *Seifert (S^1-, I-, good) stack*.

Suppose that $A, B \in \mathcal{S}$. We say that A is a *substack* of B if $(A_n, a_n, a'_n) \subset (B_n, b_n, b'_n)$ for every $n \geq 0$. If (B, β) is an admissible stack, we say that A is an *admissible substack of* (B, β) if A is a substack of B and $(A, \{\beta_n | a'_n\})$ is an admissible stack.

Let M be a 3-stack. If F is a 2-stack that is a substack of M, then we say that F is *essential in* M if (F_n, f_n, f'_n) is essential in (M_n, m_n, m'_n) for each $n \geq 0$.

We shall define a functor $\mathcal{Q}$ from $\mathcal{A}$ to the category of noncompact PL-manifolds with proper maps as follows. Let $(A, \alpha) \in \mathcal{A}$. Let $\mathcal{Q}(A)$ be the space obtained from the disjoint union of the elements of $\{A_n\}$ by identifying x with $\alpha_n(x)$ for all $x \in a'_n$ and $n \geq 0$. The space $\mathcal{Q}(A)$ is given the quotient topology. If (A, α) and (B, β) are in $\mathcal{A}$ and if $f\colon A \to B$ is an $\mathcal{A}$-map, let $\mathcal{Q}(f)\colon \mathcal{Q}(A) \to \mathcal{Q}(B)$ be the unique map induced by $\bigcup_{n=1}^{\infty} f_n\colon \bigcup_{n=0}^{\infty} A_n \to \bigcup_{n=0}^{\infty} B_n$. We shall refer to $\mathcal{Q}$ as the *quotient functor*. This is to be distinguished from the *quotient map*, $p\colon \bigcup_{n=0}^{\infty} A_n \to \mathcal{Q}(A)$.

If M and N are ν-stacks such that M is a substack of N, then $\mathrm{Fr}(M)$ denotes the stack $\{(F_k, \partial F_k \cap n_k, \partial F_k \cap n'_k)\}$, where $F_k = \mathrm{Fr}(M_k; N_k)$.

Let W be a connected, noncompact 3-manifold and let $w \subset \partial W$ be a compact 2-manifold.

An *exhaustion for* W is a set $V = \{V_n\} = \{V_n | n \geq 0\}$ of compact 3-submanifolds of W such that $W = \bigcup_{n=0}^{\infty} V_n$ and $V_n \subset V_{n+1} \setminus \mathrm{Fr}(V_{n+1}; W)$ for $n \geq 0$. We say that V is an *exhaustion for* (W, w) if in addition $w \subset V_0 \setminus \mathrm{Fr}(V_0; W)$.

An exhaustion V for W is *end-irreducible* if $\mathrm{Fr}(V_n; W)$ is incompressible in W for $n \geq 0$. If V is an exhaustion for (W, w), we say that V is *very good* if $(V_n, w \cup \mathrm{Fr}(V_n; W))$ is a good 3-manifold pair for every $n \geq 0$.

If there is an end-irreducible (very good) exhaustion for W $((W, w))$, then we say that W $((W, w))$ is *end-irreducible* (*very good*).

Observe that if V is very good, then $(\mathrm{cl}(V_{n+1} \setminus V_n), \mathrm{Fr}(V_{n+1}; W) \cup \mathrm{Fr}(V_n; W))$ is good for every $n \geq 0$.

2. Some results about 2-manifolds

This section is concerned primarily with proving a number of results about 2-manifolds which will be of use in the rest of the paper. The proofs are not difficult. They are included mainly to show why some of the technical conditions assumed are actually needed.

For the rest of the section, we will assume that F is a compact, orientable

2-manifold no component of which is a disk, a 2-sphere, or a torus.

Suppose that G is a compact 2-manifold contained in $F\backslash\partial F$. We say that G is *hard* in F if the inclusion-induced map $\pi_1(G') \to \pi_1(F')$ is monic and nontrivial for each component G' of G, where F' is the component of F which contains G'. Note that if G is hard in F, then no component of G or $\mathrm{cl}(F\backslash G)$ is a disk, 2-sphere, torus, Möbius band, or projective plane.

We say that an annulus component A of $\mathrm{cl}(F\backslash G)$ that meets two components G' and G'' of G in components of ∂A is a *reducing annulus of G in F with respect to G' and G''*.

We say that G is *irredundant in F* provided there is no reducing annulus for G in F with respect to G' and G'' such that G' and G'' are both annuli. We say that G is *strongly irredundant* in F if there is no reducing annulus for G in F with respect to G' and G'' such that either G' or G'' is an annulus.

Given a polyhedron A, we let $\chi(A)$ denote the Euler characteristic of A. Recall that

$$\chi(A \cap B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

and $\chi(S^1) = 0$ where S^1 is the 1-sphere.

Lemma 2.1. *Suppose that G and H are hard in F and $H \subset G\backslash\partial G$. Then H is hard in G.*

Proof. The proof is left as an exercise. □

Lemma 2.2. *If G and H are compact 2-manifolds in F with $H \subset G\backslash\partial G$ such that H is hard in G, then $\chi(G) \leq \chi(H)$.*

Proof. Note $\chi(G) = \chi(H) + \chi(G\backslash H)$.

Since H is hard in G, no component of $\mathrm{cl}(G\backslash H)$ is a disk, 2-sphere, or projective plane. Therefore $\chi(G\backslash H) \leq 0$. It follows that $\chi(G) \leq \chi(H)$. □

For a compact 2-manifold $G \subset F\backslash\partial F$, let $\alpha(G)$ denote the number of annulus components of G and let the ordered triple $c(G) = (-\chi(G), \alpha(G), -\#G)$ be the *complexity of G*. We put dictionary order of the set of complexities of homeomorphism classes of 2-manifolds in F.

Lemma 2.3. *Suppose that G and H are 2-manifolds that are hard and strongly irredundant in F such that $H \subset G\backslash\partial G$.*

(1) *If $\chi(G) = \chi(H)$ and A is a component of G, then $\chi(A) = \chi(A \cap H)$.*
(2) *If $\chi(G) = \chi(H)$ and A is a component of G which contains an annulus component B of H, then A is a regular neighborhood of B.*
(3) *If $\chi(G) = \chi(H)$ and $\alpha(G) = \alpha(H)$, then $\#(G) \leq \#(H)$.*
(4) *Consequently, $c(H) \leq c(G)$.*

Proof. Let $n = \#(G)$ and let $A_1, \dots, A_n$ be the distinct components of G. It follows by Lemmas 2.1 and 2.2 that $\chi(A_k \cap H) \leq \chi(A_k)$ for $k = 1, \dots, n$, i.e., $\chi(A_k) - \chi(A_k \cap H) \geq 0$.

Note that

$$\sum_{k=1}^{n}(\chi(A_k) - \chi(A_k \cap H)) = \chi(G) - \chi(H).$$

Hence $\chi(A_k) = \chi(A_k \cap H)$, for $k = 1, \dots, n$, if $\chi(G) = \chi(H)$. This proves (1).

To prove (2), suppose $\chi(G) = \chi(H)$. Let B be an annulus component of H and let A be the component of G which contains B. Since $\chi(A) = \chi(A \cap H)$, it follows that $\chi(A\backslash H) = 0$. Therefore any component of $\mathrm{cl}(A\backslash H)$ is an annulus. Let J' and J'' be the components of ∂B and let A' and A'' be the components of $\mathrm{cl}(A\backslash H)$ that contain J' and J'', respectively. Since H is strongly irredundant in F, it follows that $\partial A'\backslash J'$ and $\partial A''\backslash J''$ are components of ∂A. Therefore $A = A' \cup B \cup A''$ and the conclusion of (2) follows.

To prove (3), suppose that $\chi(G) = \chi(H)$, $\alpha(G) = \alpha(H)$, and A is a component of G. We claim that $A \cap H \neq \varnothing$. To get a contradiction, suppose $A \cap H = \varnothing$. Then $\chi(A \cap H) = 0$. By (1) it follows that $\chi(A) = 0$. Therefore A is an annulus. Since every annulus component of H is contained in an annulus component of G by (2), it follows that $\alpha(G) > \alpha(H)$. This is a contradiction. Therefore A contains at least one component of H. Therefore $\#(G) \leq \#(H)$.

Part (4) now follows easily from Lemmas 2.1, 2.2, and parts (2) and (3). □

Lemma 2.4. *Suppose that H is a compact 2-manifold that is hard and irredundant in F. Let Φ be the maximum value of $-3\chi(F') + \#(\partial F')$ for any component F' of F. Then*

(1) $\alpha(H) \leq \frac{1}{2}\#(F)\Phi$ *and*

(2) $\#(H) \leq -\chi(F) + \frac{1}{2}\#(F)\Phi$.

Proof. Let A be the union of the annulus components of H. Then $\#(A) = \alpha(H)$.

Let F' be a component of F and let $A' = A \cap F'$. Put $B' = \mathrm{cl}(F' \setminus A')$. Then $\chi(F') = \chi(B')$.

Let n be the number of nonannulus components of B'. Then $n \leq -\chi(B')$. Observe that $n \geq \#(B') - \#(\partial F')$ because H is strongly irredundant in F. Therefore $\#(B') \leq -\chi(B') + \#(\partial F') = -\chi(F)' + \#(\partial F')$.

Let $p = \#(B')$ and let $B'_1, \dots, B'_p$ be the components of B'. Then $\chi(B'_i) \leq 2 - \#(\partial B'_i)$ for $1 \leq i \leq p$. Consequently $\chi(B') \leq 2\#(B') - \#(\partial B')$. Since $\chi(B') = \chi(F')$ and $\#(\partial B') = \#(\partial F') + 2\#(A')$, it follows that $2\#(B') \geq \chi(F') + \#(\partial F') + 2\#(A')$. Consequently, $\#(A') \leq \#(B') - \frac{1}{2}(\chi(F') + \#(\partial F'))$. Therefore

$$\begin{aligned}\#(A') &\leq \#(B') - \tfrac{1}{2}(\chi(F') + \#(\partial F')) \\ &= -\chi(F') + \#(\partial F') - \tfrac{1}{2}(\chi(F') + \#(\partial F')) \\ &= \tfrac{1}{2}(\#(\partial F') - 3\chi(F')).\end{aligned}$$

The inequality of (1) follows by the Pigeonhole Principle.

To prove (2), let $m = \#(H)$ and let $H_1, \dots, H_m$ be the distinct components of H. By Lemma 2.2, $\chi(F) \leq \chi(H) = \sum_{k=1}^{m} \chi(H_k) \leq (-1)(\#(H) - \alpha(H))$, i.e., $-\chi(F) + \alpha(H) \geq \#(H)$. Therefore by (1), the inequality of (2) follows. □

Theorem 2.5. *If G and H are compact 2-manifolds that are hard and strongly irredundant in F such that $H \subset G\backslash\partial G$ and $c(G) = c(H)$, then G is a regular neighborhood of H.*

Proof. Recall $(-\chi(G), \alpha(G), -\#(G)) = (-\chi(H), \alpha(H), -\#(H))$. Let $n = \#(G)$. Let $A_1, \dots, A_n$ be the components of G. For $k = 1, \dots, n$, let $B_k = A_k \cap H$. By Lemma 2.3(2), it follows that A_k is a regular neighborhood of B_k whenever a component of B_k is an annulus. Hence if a component of B_k is an annulus, then B_k is connected and A_k is an annulus. Since $\alpha(G) = \alpha(H)$, we may choose notation so that A_k and B_k are annuli iff $1 \leq k \leq \alpha(G)$.

Suppose that $k \geq \alpha(G)+1$. By Lemma 2.3(1), it follows that $\chi(A_k) = \chi(B_k)$. Since $\chi(A_k) \neq 0$, it follows that $B_k \neq \varnothing$. Since $\#(G) = \#(H)$, it follows that A_k contains exactly one component of H, i.e., B_k is connected. Note that each component of $\mathrm{cl}(A_k \backslash B_k)$ is an annulus because $\chi(A_k \backslash B_k) = 0$. And since B_k is connected, it follows that A_k is a regular neighborhood of B_k. This ends the proof. □

Corollary 2.6. *Suppose that, for $i = 0, 1, 2, \ldots$, G_i is a nonempty compact 2-manifold that is hard and irredundant in F. If $G_{i+1} \subset G_i \setminus \partial G_i$ for $i \geq 0$, then there is an integer N such that G_N is a regular neighborhood of G_i for $i \geq N+1$.*

Proof. Let $c_i = c(G_i)$ for $i \geq 0$. By Lemma 2.3(4), $c_0 \geq c_1 \geq \cdots$, and by Lemmas 2.2 and 2.4, the sequence $\{c_i\}$ is bounded below. Therefore, there is an integer N such that $c_i = c_N$ for $i \geq N$. Applying Theorem 2.5 ends the proof. □

Corollary 2.7. *Let G and H be compact 2-manifolds that are hard and strongly irredundant in F such that $H \subset G \backslash \partial G$. If there is an isotopy $h_t \colon F \to F$ such that $h_1(G) \subset H \backslash \partial H$, then G is a regular neighborhood of H.*

Proof. Recall that by Lemma 2.2, $\chi(G) \leq \chi(H)$ and $\chi(H) \leq \chi(h_1(G))$. Hence $\chi(H) = \chi(G)$. It follows similarly by Lemma 2.3(2) that $\alpha(G) = \alpha(H)$ and by Lemma 2.3(3) that $\#(G) = \#(H)$. Applying Theorem 2.5 ends the proof. □

Let $G \subset G^* \subset F \backslash \partial F$ be compact 2-manifolds. If every component of $\mathrm{cl}(G^* \backslash G)$ is a disk, we say that G^* is obtained from G by *plugging in F*. If every component of $\mathrm{cl}(G^* \backslash G)$ is a reducing annulus of G in F with respect to G' and G'' such that both G' and G'' are annuli, then G^* is said to be obtained from G by *caulking in F*.

Suppose that G is the union of the nonannulus components of G^*. If J is a simple closed curve in F that is isotopic in F to a component of ∂G, then we say that J is *bewildered in F with respect to G^**. We say that G^* is *guided in F* if for every component J of ∂G^* there is an annulus component of G^* whose core is isotopic in F to J. If α is a closed 1-manifold in F such that every component of α bewildered in F with respect to G^* is contained in an annulus component of G^*, we say that α is *safe in F with respect to G^**.

The terms above will be seen again with different but parallel meaning in a later section.

Lemma 2.8. *Suppose that $J \subset F \backslash \partial F$ is a simple closed curve that is noncontractible in F and suppose that $K \subset F \backslash \partial F$ is a closed 1-manifold each of whose components is noncontractible in F. Suppose that J meets K transversely and that $J \cap K \neq \varnothing$. If there is a homotopy $f \colon J \times I \to F$ such that $f(z, 0) = z$ and $f(J \times 1) \cap K = \varnothing$, then there is a disk $D \subset F$ such that $\partial D = \alpha \cup \beta$ where $\alpha = \partial D \cap J$ and $\beta = \partial D \cap K$.*

Proof. We may assume that f is in general position with respect to K. We may assume that the homotopy class of f is chosen so that $\#(f^{-1}(K))$ is minimal.

Now $f^{-1}(K)$ is a compact 1-manifold that is properly embedded in $J \times I$ and $\partial f^{-1}(K) \subset J \times 0$, in fact. Since $\#(f^{-1}(K))$ is minimal and each component of K is noncontractible in F, every simple closed curve component of $f^{-1}(K)$ is noncontractible in $J \times I$. Since $J \cap K \neq \varnothing$, there is a component $\tilde{\beta}$ of $f^{-1}(K)$ that is an arc. Since $\partial \tilde{\beta} \subset J \times 0$, there is a disk $\tilde{D} \subset J \times I$ that is separated off by $\tilde{\beta}$. We may choose $\tilde{\beta}$ so that $\tilde{D} \cap f^{-1}(K) = \tilde{\beta}$. Put $\tilde{\alpha} = \tilde{D} \cap (J \times 0)$.

Put $\alpha = f(\tilde{\alpha})$ and $\beta = f(\tilde{\beta})$. Then $\alpha \cup \beta$ is a simple closed curve that is contractible in F. Let D be the disk that is bounded by $\alpha \cup \beta$. □

Suppose that G and H' are compact 2-manifolds that are hard in F such that $H' \subset G \setminus \partial G$. Suppose that J is a closed 1-manifold in F such that each component of J is noncontractible in F. Suppose that there is an isotopy of F which takes J into $G \setminus \partial G$. We may assume that J is chosen within its isotopy class in F so that $\#(J \cap \partial H')$ is minimal. Let H be a regular neighborhood of $J \cup H'$ in $F \setminus \partial F$.

Lemma 2.9. *H is isotopic in F into G.*

Proof. Move J by an isotopy of F which is the identity on H' so that $\#(J \cap \partial G)$ is minimal.

We claim that $J \cap \partial G = \varnothing$. Suppose that this is not so and let J' be a component of J such that $J' \cap \partial G \neq \varnothing$. By Lemma 2.8, there is a disk $D \subset F$ such that $\partial D = \alpha \cup \beta$ where α and β are arcs in J' and ∂G, respectively.

We claim that $D \cap H' = \varnothing$. First suppose that $\partial D \cap \partial H' \neq \varnothing$ and let γ be an arc component of $\partial H' \cap D$. Then $\partial \gamma \subset \alpha$ since $H' \subset G \setminus \partial G$. Let D_γ be the disk in D which is separated off by γ. By pushing along D_γ, we may reduce $\#(J \cap \partial H')$, which is a contradiction. So $\partial D \cap \partial H' = \varnothing$. So either $\partial H' \subset D \setminus \partial D$ or $\partial H' \subset F \setminus D$. The first alternative is prohibited by the fact that H' is hard in F. Consequently, $D \cap H' = \varnothing$ because $H' \subset G \setminus \partial G$ and $(D \setminus \partial D) \cap \partial G \neq \varnothing$.

Since $D \cap H' = \varnothing$, we can use D to reduce $\#(J \cap \partial G)$ by an isotopy fixed on H' and thereby obtain a contradiction. We may conclude that $J \cap \partial G = \varnothing$.

It is not difficult to see that if J' is a component of J such that $J' \cap G = \varnothing$, then J' must be parallel in F to a component of ∂G. By an isotopy fixed on the union of H' and the components of J that meet H', we may assume that $J \subset G \setminus \partial G$.

Now push H into G by using the product structure of $H \setminus (H' \cup J)$. □

Lemma 2.10. *Suppose G and L are compact 2-manifolds in $F \setminus \partial F$ such that $L \subset G$ and G is hard in F. If L^* is obtained from L by plugging in F, then $L^* \subset G$.*

Proof. Let $D = \mathrm{cl}(L^* \setminus L)$. Suppose that L^* is not contained in G. Then there is a component K of ∂G which is contained in $D \setminus \partial D$. This contradicts that G is hard in F. □

Lemma 2.11. *Suppose that G and L are compact 2-manifolds in $F \setminus \partial F$ such that $L \subset G \setminus \partial G$ and G is hard in F. If L^* is obtained from L by caulking in F, then L^* is isotopic in F into G.*

Proof. Let $A_1, \ldots, A_n$ be a set of annulus components of L such that for $1 \leq k \leq n-1$ there is a reducing annulus B_k for L in F with respect to A_k

and A_{k+1} which is contained in L^*. Suppose that $\partial A_1 \backslash \partial B_1$ and $\partial A_n \backslash \partial B_{n-1}$ are components of ∂L^*.

Suppose that B_i contains a component of ∂G for some i. Let $A' = \bigcup_{k=1}^{n-1}(A_{k+1} \cup B_k)$. Now $A_1 \subset G \backslash \partial G$, in particular. We may push A' into the component of G which contains A_1. This will reduce the number of components of ∂G contained in L^*. It follows that L^* can be isotoped in F into $G \backslash \partial G$. □

For the rest of the section, G will be a compact 2-manifold that is hard in F. Let G^* be a compact 2-manifold which is a union of components of G such that G is isotopic in F into $G^* \backslash \partial G^*$ and such that $\#(G^*)$ is minimal. We shall refer to G^* as a *strongly irredundant associate of* G. Lemma 2.12 vindicates the use of this term.

Lemma 2.12. *G^* is strongly irredundant in F.*

Proof. Suppose that G^* is not strongly irredundant. Let A be a reducing annulus for G^* in F with respect to G' and G'' such that G' is an annulus. We may isotop G^* into $G^* \backslash G'$. This contradicts the minimality of $\#(G^*)$. □

Lemma 2.13. *Suppose that H are G are compact 2-manifolds that are hard, guided, and irredundant in F. Let H^* and G^* be the strongly irredundant associates of H and G, respectively. If H^* is isotopic in F to G^*, then H is isotopic in F to G.*

Proof. We may assume that $H^* = G^*$. Let $\tilde{H}$ and $\tilde{G}$ be the union of the nonannulus components of H and G respectively. Since $H^* = G^*$, it follows that $\tilde{H} = \tilde{G}$.

Since G and H are guided and irredundant in F, for each isotopy class $[J]$ of a component J of $\partial \tilde{H} = \partial \tilde{G}$ there are unique annulus components $H_{[J]}$ and $G_{[J]}$ of H and G, respectively, whose cores are parallel in F to J. Perform an isotopy fixed on H^* to take $\bigcup_{[J]} \partial H_{[J]}$ to $\bigcup_{[J]} G_{[J]}$. This ends the proof. □

3. Compact 3-manifold pairs

The aim of this section is to prove Theorem 3.12, which is a version of the Characteristic Pair Theorem with more control. This theorem serves as the chief cornerstone of the entire paper. This section owes much to [JS].

Throughout the rest of this section, (M, m) will be a good 3-manifold pair. Suppose that $(S, s) \subset (M, m)$ is a Seifert pair.

Let B be a component of $\text{cl}(M \backslash S)$ and let $b = B \cap \partial M$. Suppose that $b \subset m$.

If (B, b) is homeomorphic as a pair to $(D^2 \times I, D^2 \times \partial I)$, then (B, b) is called a *plug for* (S, s) *in* (M, m).

If (B, b) is homeomorphic as a pair to $(S^1 \times I \times I, S^1 \times I \times 0)$, then (B, b) is called a *caulk for* (S, s) *in* (M, m).

If (B, b) is homeomorphic to $(D^2 \times S^1, \varnothing)$, then (B, b) is called a *filling for* (S, s) *in* (M, m).

Suppose that (C, c) is a 3-manifold pair. Let $(S', s') = (S \cup C, s \cup c)$. If each component of (C, c) is a plug (caulk, filling) for (S, s) in (M, m), then we say that (S', s') is obtained from (S, s) by *plugging* (*caulking, filling*) *in* (M, m). Note that in this case s' is obtained from s by plugging (caulking) in (M, m) as defined in §2.

We now fix F and G to be compact 2-manifolds in $m\backslash\partial m$ that are hard in m. We say that a subset H of $m\backslash\partial m$ is *isotopic separately in* m *into* F *and* G if there are isotopies f_t and g_t of m such that $f_1(H) \subset F\backslash\partial F$ and $g_1(H) \subset G\backslash\partial G$.

For the remainder of this section, let (S, s) be a Seifert pair that is perfectly embedded in (M, m) such that s is hard in m and is isotopic separately in m into F and G.

Let A be a round 2-manifold that is essential in (M, m) such that ∂A is isotopic in m separately into F and G. Let

$$c(A) = (\#(\partial A \cap \partial s), \#(A \cap \mathrm{Fr}(S; M))).$$

We choose A within its (M, m) isotopy class so that $c(A)$ is minimal in dictionary order. It will simplify the exposition if we assume that S does not contain a component of A. Let U_A be a regular neighborhood of A in M such that $U_A \cap \partial M \subset m\backslash\partial m$ and such that $U_A \cap S$ is a regular neighborhood of $A \cap S$ in S and $U_A \cap \mathrm{Fr}(S; M)$ is a regular neighborhood of $A \cap \mathrm{Fr}(S; M)$ in $\mathrm{Fr}(S; M)$. Let $S_A = S \cup U_A$ and let $s_A = S_A \cap m$.

Lemma 3.1. *Let* α *be a component of* $A \cap \mathrm{Fr}(S; M)$, *let* A' *be the component of* A *that contains* α, *and let* R *be the component of* $\mathrm{Fr}(S; M)$ *that contains* α. *Then either* α *is a simple closed curve that is noncontractible in both* A' *and* R *or* A' *and* R *are both annuli and* α *is a spanning arc of both* A' *and* R.

Proof. In the case that α is a simple closed curve, it follows by the usual arguments involving the incompressibility of A' and R in M, the irreducibility of M, and the minimality of $c(A)$ that α is noncontractible on both A' and R.

Now suppose that α is an arc. Then A' and R are both annuli. Suppose that α is a separating arc of A'. Let D' be the disk in A' that is separated off by α. There is an arc component β of $D' \cap \mathrm{Fr}(S; M)$ which separates off a disk $D \subset D'$ such that $D \cap \mathrm{Fr}(S; M) = \beta$. Since $\mathrm{Fr}(S; M)$ is essential in (M, m) and since $D \cap \partial M \subset m$, it follows that β is not a spanning arc of $\mathrm{Fr}(S; M)$; thus there is a disk $E \subset \mathrm{Fr}(S; M)$ that is separated off by β. Note that $D \cup E$ is a disk with $\partial(D \cup E) \subset m$. Since m is incompressible in M, there is a disk $E' \subset m$ with $\partial E' \subset (D \cup E)$. We can use E' to reduce $\#(\partial A \cap \partial s)$ by an isotopy of (M, m). This reduces $c(A)$, which is a contradiction. Therefore α is not a separating arc of A'. A similar argument shows that α is a spanning arc of R. □

Let A' be a component of $A \cap S$ and let (S', s') be the component of (S, s) such that $A' \subset S'$. Again recall that $A' \cap \mathrm{Fr}(S; M) \neq \varnothing$. Note that A' is either an annulus which has at least one boundary component contained in $\mathrm{Fr}(S; M)$ or a disk which meets $\mathrm{Fr}(S; M)$ in two disjoint arcs contained in $\partial A'$.

Lemma 3.2. *If* A' *is a component of* $A \cap S$, *then* A' *is not parallel in* S *into* ∂S.

Proof. Otherwise $c(A)$ can be reduced by an isotopy of (M, m). □

Lemma 3.3.

(1) *If A' is an annulus and (S', s') is an I-pair, then there is a Seifert fibration of S' such that (S', s') is an S^1-pair.*

(2) *If A' is a disk and (S', s') is an S^1-pair, then S' is a solid torus in which A' is a meridian disk and there is an I-bundle $p: S' \to \sigma$ such that $p|s': s' \to \sigma$ is a ∂I-bundle, where σ is either an annulus or a Möbius band.*

Proof. Part (1) follows by Lemma 3.2 and Corollary II.7.7 and Remark II.7.8 of [JS].

We now prove (2). Recall that by Lemma 2.2, A' is not boundary parallel. It follows that A' is a compressing disk for $\partial S'$. Therefore S' is a solid torus. Recall that $A' \cap \mathrm{Fr}(S; M)$ consists of a pair of disjoint arcs. Since $\partial S'$ is a torus and $\mathrm{Fr}(S; M) \neq \varnothing$, it follows that each component of s' is an annulus. Therefore, s' has at most two components. If s' has two components, then there is a homeomorphism from (S', s') to $(S^1 \times I \times I, S^1 \times I \times \partial I)$. If s' is connected, then there is an I-bundle $p: S' \to B$, where B is the Möbius band, such that $p|s': s' \to B$ is an I-bundle. □

Lemma 3.4. *Suppose that A'' is a component of $S' \cap A$. If A' is a disk, then A'' is a disk.*

Proof. To get a contradiction, suppose that A'' is an annulus. It follows by Lemma 3.3 that (S', s') is an S^1-pair. Then, by Lemma 3.3, S' is a solid torus and A' is a meridian disk for S'. Hence $A'' \cap A' \neq \varnothing$ because A'' is incompressible in S'. This is absurd since A is an embedded 2-manifold. □

Lemma 3.5. *If (S', s') is an S^1-pair component of (S, s) that is not an I-pair, then there is a Seifert fibering of S' in which both $A \cap S'$ and s' are unions of fibers.*

Proof. Suppose that $A \cap S'$ is not isotopic in $(S', \mathrm{cl}(\partial S' \backslash s'))$ to a union of fibers. Then by VI.34 of [J] there is a component A'' of $A \cap S'$ which meets every fiber in the given Seifert fibration of S'. In particular, A'' meets every fiber of s'. Therefore by the discussion immediately before Lemma 3.2 it follows that A'' is a disk. By Lemma 3.3(2), it follows that (S', s') can be given an I-pair structure and that is a contradiction. □

Lemma 3.6. *If (S', s') is an I-pair that is not an S^1-pair, then there is an I-pair structure on (S', s') such that $A \cap S'$ is a union of fibers.*

Proof. By Lemma 3.3, each component of $A \cap S'$ is a disk. By Lemma 3.2, no component of $A \cap S'$ is boundary parallel. If (S', s') is a product pair, then the result follows by Lemma 3.4 of [Wa]. Otherwise, we can ape the proof of Lemma 3.4 of [Wa]. □

Lemma 3.7. *(S_A, s_A) is a Seifert pair.*

Proof. Let A_δ be the union of components A^* of A such that each component of $A^* \cap \mathrm{Fr}(S; M)$ is an arc. Let $A_\alpha = A \backslash A_\delta$.

Now let (S_δ, s_δ) be the union of components of (S_A, s_A) that meet A_δ and let (S_α, s_α) be the union of components of (S_A, s_A) that meet A_α. It follows by Lemma 3.4 that $S_\alpha \cap S_\delta = \varnothing$. It follows from Lemmas 3.5 and 3.6 that (S_α, s_α) and (S_δ, s_δ) can be given an S^1-pair structure and an I-pair structure, respectively. It follows that (S_A, s_A) is a Seifert pair. □

Lemma 3.8. *s_A is isotopic in m separately into both F and G.*

Proof. This follows by Lemma 2.9. □

Let (T, t) be obtained from (S_A, s_A) by plugging, caulking, and filling so that (T, t) and $\mathrm{cl}(M \backslash T)$ has the fewest possible components with respect to this stipulation.

Lemma 3.9. *t is isotopic in m separately into F and G.*

Proof. This follows from Lemmas 3.8, 2.10, and 2.11. □

Lemma 3.10. *(T, t) is well-embedded in M, $\mathrm{Fr}(T; M)$ is essential in (M, m), and t is hard in m.*

Proof. First note that if r is a component of $\mathrm{Fr}(T; M)$, then either

(1) r is a component of $\mathrm{Fr}(S; M)$,

(2) $r = S^1 \times I$ and each component of $r \cap \mathrm{Fr}(S; M)$ is a disk which can be realized as $\beta \times I$, where β is a subarc of S^1, or

(3) each component of $r \cap \mathrm{Fr}(S; M)$ is an annulus that is hard in r.

Suppose that T' is the component of T which contains r and let $S' = S \cap T'$. Then (2) occurs only if each component of $A \cap S'$ is a disk which meets $\mathrm{Fr}(S'; M)$ in two arcs in its boundary and (3) occurs only if each component of $A \cap S'$ is an annulus.

We first show that $\mathrm{Fr}(T; M)$ is incompressible. To get a contradiction, let $D \subset M$ be a disk with $\partial D = D \cap \mathrm{Fr}(T; M)$ and let r be the component of $\mathrm{Fr}(T; M)$ that contains ∂D. Let (T', t') be the component of (T, t) that contains r. Since (S, s) is perfectly embedded in (M, m), it follows that r falls under either (2) or (3). So T' must contain at least one component of A.

We claim that $D \subset \mathrm{cl}(M \backslash T')$. Suppose $D' \subset T'$ and consider the following cases.

Suppose that (T', t') is an I-pair. In this case it can be argued that (T', t') is homeomorphic to $(D^2 \times I, D^2 \times \partial I)$. This is absurd because (T', t') must contain a component of A and the components of A are incompressible annuli and tori.

Now suppose that (T', t') is an S^1-pair. In this case it follows that T' is a solid torus and that D is a meridian disk for T'. Note that $t' = \varnothing$ because D must meet every fiber of t' and $D \cap t' = \varnothing$. Therefore $T' \cap A = \varnothing$ because $(D^2 \times S^1, \varnothing)$ contains no essential annuli or tori. We must conclude that $D \subset \mathrm{cl}(M \backslash T')$.

Let V be the component of $\mathrm{cl}(M \backslash T')$ that contains r and therefore D. We have two cases depending on whether r is an annulus or torus.

Suppose that r is an annulus. Let r' and r'' be the annuli into which ∂D splits r. Then $D \cup r'$ and $D \cup r''$ are disks in M whose boundaries are contained in m. Since m is incompressible in M, there are disks D' and D'' in m with $\partial D' = \partial(D \cup r')$ and $\partial D'' = \partial(D \cup r'')$. Let B' and B'' be the closed 3-cells in M bounded by $D \cup D' \cup r'$ and $D \cup D'' \cup r''$, respectively, by virtue of irreducibility. Note that $B' \cap B'' = D$, $B' \cap \partial M = D'$, $B'' \cap \partial M = D''$, and $V = B' \cup B''$. Therefore, $(V, D' \cup D'')$ is homeomorphic to $(D^2 \times I, D^2 \times \partial I)$.

We claim in this case that $(T' \cup V, t' \cup D' \cup D'')$ is an I-pair; this would contradict the minimality of $\#(\text{cl}(M \backslash T))$. It suffices to show that (T', t') is an I-pair.

We claim that r is covered by (2). Recall that (1) has already been ruled out. If (3) held, then D' would be a compressing disk for $\text{Fr}(S; M)$ in M.

Now observe that r is a component of $\text{Fr}(S_A; M)$. Let $S' = S \cap T'$. Since r is covered by (2), it follows that each component of $A \cap S'$ is a disk. Therefore by Lemmas 3.3(2) and 3.6, it follows that (T', t') is an I-pair. This contradicts the minimality of $\#(\text{cl}(M \backslash T))$.

Now suppose that r is a torus. Therefore r is covered by (3). It follows that ∂D cannot be isotoped in r into $r \backslash \text{Fr}(S; M)$. It is not difficult to argue, using the irreducibility of M, that V is a solid torus. (Note: The previous sentence is not the "standard mistake". There is something to be done.) Since r is a torus, it follows that (T', t') is an S^1-pair. The fact that $(T' \cup V, t')$ can be fibered as an S^1-pair follows because ∂D cannot be isotoped in r into $r \backslash \text{Fr}(S; M)$. Therefore $(T\cup, t)$ is a Seifert pair obtained by filling (T, t) in (M, m). This contradicts the minimality of $\#(\text{cl}(M \backslash T))$. Therefore, we may conclude that $\text{Fr}(T; M)$ is incompressible in M.

It follows immediately that (T, t) is well-embedded in (M, m), and hence that t is hard in m.

We now show that $\text{Fr}(T; M)$ is essential in (M, m). To get a contradiction, suppose that r is a component of $\text{Fr}(T; M)$ that is not essential in (M, m). Since r is incompressible, we may assume that there is a product $r \times I$ in M with $r \times 0 = r$ and $(\partial r \times I) \cup (r \times 1) \subset m$. Since (M, m) is good, it follows that r is not a torus. Therefore, r must be an annulus.

We may assume that $(r \times I) \cap \text{Fr}(T; M) = r$. So $(r \times I, (\partial r \times I) \cup (r \times 1))$ is a caulk for (T, t) in (M, m). Since (S, s) is perfectly embedded in (M, m), it follows that r is not covered by (1). Let (T', t') be the component of (T, t) which contains r.

Suppose that (T', t') is an S^1-pair. Then clearly $(T' \cup (r \times I), t' \cup (\partial r \times I) \cup (r \times 1))$ is an S^1-pair. This reduces $\#(\text{cl}(M \backslash T))$ and gives us a contradiction.

Now suppose that (T', t') is an I-pair that is not an S^1-pair. Then there is a disk $D \subset T'$ such that $D \cap (\partial T' \backslash t')$ is a pair of disjoint arcs α and β which span r; furthermore, we may assume that D is not parallel into $\partial T'$. Let D_α, $D_\beta \subset r \times I$ be the disks $\alpha \times I$ and $\beta \times I$, respectively. The disk $D \cup D_\alpha \cup D_\beta$ is a compressing disk for m in M. This is a contradiction. This ends the proof. □

Let (R, r) be a Seifert pair that is well-embedded in M with $\text{Fr}(R; M)$ essential in (M, m), r hard in m and isotopic separately in m into F and G, and (T, t) isotopic in (M, m) into (R, r) such that $(\#(R), \#(r))$ is minimal when taken in dictionary order. The existence of (R, r) follows from Lemma 3.10.

Lemma 3.11.

(1) (R, r) *is perfectly embedded in* (M, m).
(2) r *is irredundant in* m.

(3) (S, s) *and* A *are isotopic in* (M, m) *into* R.

Proof. To show that (R, r) is perfectly embedded in (M, m), we suppose that there is a component (R', r') of (R, r) that is isotopic in (M, m) into $(R\backslash R', r\backslash r')$. Then $(R\backslash R', r\backslash r')$ satisfies all of the demands put on (R, r) while $\#(R\backslash R') < \#(R)$. This is a contradiction. So (R, r) is perfectly embedded.

We now claim that r is irredundant in m. To get a contradiction, suppose there exists a reducing annulus B for r in m with respect to r'_1 and r'_2 such that r'_1 and r'_2 are both annuli. Let $J_i = B \cap r'_i$ for $i = 1, 2$. Let (R_i, r_i) be the component of (R, r) that contains r'_i for $i = 1, 2$. Note that (R_i, r_i) can be given the structure of an S^1-pair for each i. Let V be a regular neighborhood of B in M such that $V \cap R$ is a union of fibers of $R_1 \cup R_2$. Let $R_B = R \cup V$ and $r_B = r \cup B$. By Lemma 2.11, it follows that r_B is isotopic in m into r. Let (R^*, r^*) be obtained by caulking and filling (R_B, r_B) in (M, m). It follows as in the proof of Lemma 3.10 that (R^*, r^*) satisfies all of the demands put on (R, r) and $(\#(R^*), \#(r^*)) < (\#(R), \#(r))$, which is a contradiction. □

Suppose that (U, u) is a Seifert pair that is perfectly embedded in (M, m) such that u is hard and irredundant in m and isotopic in m separately into F and G. Suppose that if B is a round 2-manifold that is essential in (M, m) such that if ∂B is isotopic in m separately into F and G, then B is isotopic in (M, m) into (U, u). Then we say that (U, u) is an *engulfing Seifert pair for* (M, m) *controlled by* F *and* G. We write

$$(U, u) = \text{Eng}(M, m; F, G).$$

Theorem 3.12. *There exists an engulfing Seifert pair for* (M, m) *controlled by* F *and* G.

Proof. Suppose that there exists no engulfing Seifert pair for (M, m) controlled by F and G. Let $\mathbf{S}$ be the set of all Seifert pairs (R, r) that are perfectly embedded in (M, m) such that r is hard and strongly irredundant in m and isotopic in m separately into F and G.

Note that if $\mathbf{S} = \varnothing$, then M contains no round 2-manifold B that is essential in (M, m) such that ∂B is isotopic in m separately into F and G; by taking $\text{Eng}(M, m; F, G)$ to be the empty pair in this case, we are done.

Assume $\mathbf{S} \neq \varnothing$. If (S', s') and $(S'', s'') \in \mathbf{S}$, we say that $(S', s') \preceq (S'', s'')$ if there is an isotopy h_t of (M, m) such that $(h_1(S'), h_1(s')) \subset (S''\backslash \text{Fr}(S''; M), s''\backslash \partial s'')$. Suppose that $(S_1, s_1) \preceq (S_2, s_2) \preceq \cdots$ is a chain in $\mathbf{S}$. By Theorem V.2.1 of [JS], this chain is bounded above in $\mathbf{S}$. Therefore by Zorn's Lemma (see §3 of the introduction of [H], for example), $\mathbf{S}$ contains a maximal element, say (U, u).

Suppose that B is a round 2-manifold that is essential in (M, m) and is such that ∂B is isotopic in m separately in m into F and G. We claim that B is isotopic in (M, m) into (U, u). By Lemma 3.11, there is an $(R_B, r_B) \in \mathbf{S}$ such that (U, u) and B are isotopic in (M, m) into (R_B, r_B). Since (U, u) is maximal, it follows that (R_B, r_B) is isotopic in (M, m) into (U, u). Since B is isotopic in (M, m) into (R_B, r_B), we are done. □

4. Isotopies of pairs and triples

In this section, theorems are proved about isotoping round 2-manifolds in compact 3-manifold pairs and 3-triples. These two categories are virtually the same, but the triple notation is more convenient in the sequel.

Until after the proof of Lemma 4.4, we will take (M, m) to be a good 3-manifold pair and (Q, q) to be a Seifert pair that is well-embedded in (M, m).

Suppose that (Q', q') is an I-pair component of (Q, q) that is not also an S^1-pair. Suppose that A is an annulus in M with $\partial A \subset m$ which is isotopic in (M, m) to a component of $\mathrm{Fr}(Q'; M)$. Then we say that A is *bewildered in* (M, m) *with respect to* (Q, q). If A is bewildered in (M, m) with respect to (Q, q) and there is no S^1-pair component of (Q, q) into which A is isotopic in (M, m), then we say that A is *lost in* (M, m) *with respect to* (Q, q).

If R is a round 2-manifold that is properly embedded in M with $\partial R \subset m$ such that $(R, \partial R) \subset (Q, q)$ and every bewildered component of R is contained in an S^1-pair component of (Q, q), then we say that (R, r) is *safe in* (M, m) *with respect to* (Q, q).

We say that (Q, q) is *guided in* (M, m) if there is no annulus that is lost in (M, m) with respect to (Q, q). Here again, the terms "guided" and "bewildered" also mean something analogous one dimension lower.

We say that (Q, q) is *irredundant in* (M, m) if q is irredundant in m.

Let F' be the union of components of $\mathrm{Fr}(Q; M)$ that are lost in (M, m) with respect to (Q, q). Let F be a 2-manifold in $M\backslash Q$ that is parallel in (M, m) to F'. Let Q_F be a regular neighborhood of F in $M\backslash Q$ and let $q_F = Q_F \cap m$.

Lemma 4.1. *$(Q \cup Q_F, q \cup q_F)$ is guided in (M, m).*

Proof. This follows because each component of (Q_F, q_F) is an S^1-pair. □

Lemma 4.2. *Suppose that A is a round 2-manifold that is essential in (M, m) and contained in (Q, q). If (Q, q) is guided in (M, m), then there is an isotopy h_t of (M, m) such that $h_1(A)$ is safe in (M, m) with respect to (Q, q).*

Proof. Choose A within its (M, m) isotopy class so that $A \subset Q\backslash \mathrm{Fr}(Q; M)$ and the non-S^1-pair components of (Q, q) contain the fewest components of A.

Let A' be a bewildered component of A. We claim that A' is contained in an S^1-pair component of (Q, q). Let (Q', q') be the component of (Q, q) that contains A'. To get a contradiction, suppose that (Q', q') is not an S^1-pair. Since (Q, q) is guided in (M, m), there is an S^1-pair component (Q'', q'') of (Q, q) such that A' is isotopic in (M, m) to a 2-manifold $A'' \subset (Q'', q'')$. By Proposition 5.4 of [Wa], there is a pair $(A' \times I, \partial A' \times I) \subset (M, m)$ with $A' \times 0 = A'$ and $A' \times 1 = A''$. By choice of A', we may assume that $(A' \times I) \cap A = A'$. We may use $A' \times I$ to perform an isotopy of (M, m) that leaves $A\backslash A'$ fixed and reduces the number of bewildered components of A which are contained in non-S^1-pair components of (Q, q). This is our sought-after contradiction. □

Lemma 4.3. *If (Q, q) is guided in (M, m), then q is guided in m.*

Proof. Let q^* be the union of nonannulus components of q. Let J be a component of ∂q^* and let (Q', q') be the component of (Q, q) which contains

J. Then (Q', q') is an I-pair. Let A_J be the component of $\mathrm{Fr}(Q'; M)$ that contains J. Since (Q, q) is guided in (M, m), there is an S^1-pair component (Q'', q'') of (Q, q) such that A_J is isotopic in (M, m) into (Q'', q''). Then J is isotopic in m into q''. Since (M, m) is good and (Q'', q'') is an S^1-pair, every component of q'' is an annulus. Therefore q is guided in m. □

Let us now make the demand that (Q, q) be perfectly embedded in (M, m). Let (Q^*, q^*) be a Seifert pair that is guided in (M, m) and is obtained from $(Q \cup Q_F, q \cup q_F)$ by perhaps adding regular neighborhoods of annulus components of $\mathrm{cl}(m \setminus (q \cup q_F))$ and caulking and filling in (M, m) so that $(\#(q^*), \#(M \setminus Q^*))$ is minimal in dictionary order. We refer to (Q^*, q^*) as a *guide for* (Q, q) *in* (M, m).

Lemma 4.4.

(1) $(Q, q) \subset (Q^*, q^*)$,
(2) (Q^*, q^*) *is guided and irredundant in* (M, m), *and*
(3) $\mathrm{Fr}(Q^*; M)$ *is essential in* (M, m).

Proof. It is easy to see that $(Q, q) \subset (Q^*, q^*)$ and that (Q^*, q^*) is guided in (M, m).

To get a contradiction, suppose that q^* is not irredundant in m. Then there is a reducing annulus A for q^* in m with respect to two annulus components of q^*. Therefore each component of ∂A is contained in an S^1-pair component of (Q^*, q^*). Let V be a regular neighborhood of A in M so that $V \cap Q^*$ is a union of fibers of the S^1-pairs which contain ∂A. Then $(Q^* \cup V, q^* \cup A)$ is a Seifert pair and is guided in (M, m) but $\#(q^* \cup A) < \#(q^*)$. That $\mathrm{Fr}(Q^*; M)$ is essential in (M, m) follows as in Lemma 3.10. □

The most natural statement of Lemma 4.5 requires the use of the language of triples. For the remainder of the section, we will assume that (M, m, m') is a good 3-triple and that $(Q, q, q') \subset (M, m, m')$ is a Seifert triple.

We say that (Q, q, q') is *guided* (*irredundant*) *in* (M, m, m') if $(Q, q \cup q')$ is guided (irredundant) in $(M, m \cup m')$.

Suppose that (Q, q, q') is guided, irredundant, and well-embedded in (M, m, m') and that $\mathrm{Fr}(Q; M)$ is essential in (M, m, m'). Suppose that $(R, r, r') \subset (M, m, m')$ is a round triple such that R is essential in (M, m, m'). Let h_t be an isotopy of (M, m, m') such that $h_1(R)$ is safe in $(M, m' \cup m)$ with respect to $(Q, q \cup q')$. Suppose that g_t is an isotopy of m' such that $g_1(r')$ is safe in m' with respect to q'.

Lemma 4.5. *There is an isotopy* f_t^* *of* (M, m, m') *such that* $f_1^*(R) \subset Q$ *and* $f_t^*|m = h_t|m$ *and* $f_t^*|m' = g_t$ *for each* $t \in I$.

Proof. Let $m' \times [-1, 2]$ be a regular neighborhood of m' in M with $m' = m' \times -1$ such that $Q \cap (m' \times [-1, 2]) = q' \times [-1, 2]$ and $R \cap (m' \times [-1, 2]) = r' \times [-1, 2]$. Let $U = m' \times [-1, 1]$ and let $\bar{M} = \mathrm{cl}(M \setminus U)$. Define $\mu: M \to \bar{M}$ by $\mu(x, s) = (x, \frac{1}{3}(s + 4))$ for $(x, s) \in m' \times [-1, 2]$ and let $\mu(x) = x$ elsewhere. It follows that μ is a homeomorphism. Observe that $\mu(Q) = Q \cap \bar{M}$ and $\mu(R) = R \cap \bar{M}$.

We define an isotopy $f_t : M \to M$ as follows. Let $t \in I$ be given. Let $f_t(x) = \mu h_t \mu^{-1}(x)$ for every $x \in \bar{M}$, let $f_t(x, s) = (h_{st}(x), s)$ for every $(x, s) \in m' \times [0, 1]$, and $f_t(x, s) = (g_{|s|t}(x), s)$ for every $(x, s) \in m' \times [-1, 0]$.

Since $\mu(R) = R \cap \bar{M}$ and $\mu(Q) = Q \cap \bar{M}$, it follows that $f_1(R \cap \bar{M}) \subset Q \cap \bar{M}$. It is not necessarily true that $f_1(R) \subset Q$. We claim that this can be remedied by composing with an isotopy that is fixed off $m' \times (-1, 1)$.

Let J be a component of r', q'_J the component of q' which contains $f_1(J)$, A the component of R which contains J, and $A' = A \cap U = J \times [-1, 1]$. Note that $J = J \times -1$ and $J' = J \times 1$. Let $\eta : m' \times [-1, 1] \to m'$ be the projection map.

We claim that $f_1(\partial A) \subset q'_J \times \{-1, 1\}$. It suffices to show that $\eta f_1(J') \subset q'_J$. Let q'_0 be the component of q' such that $\eta f_1(J') \subset q'_0$. To get a contradiction, suppose that $q'_0 \neq q'_J$. Since q' is guided and irredundant in m', it follows that exactly one of q'_J and q'_0 is an annulus. Since $f_1(J) = g_1(J)$ and $g_1(r')$ is safe in m' with respect to q', it follows that q'_J is an annulus; therefore q'_0 is not an annulus. However, because $\eta f_1(J')$ is isotopic in m' into $m' \backslash q'_0$, it can be shown that A is bewildered in $(M, m \cup m')$ with respect to $(Q, q \cup q')$. This is because one can lift the parallelism between $\eta f_1(J') = h_1(J)$ and a component of $\partial q'_0$ to a parallelism between $h_1(A)$ and a component of the frontier of the I-triple component of (Q, q, q') that contains both q'_0 and $h_1(A)$.

Since $h_1(R)$ is safe in $(M, m \cup m')$ with respect to $(Q, q \cup q')$, it follows that $h_1(A)$ is contained in an S^1-pair component of $(Q, q \cup q')$. Since $f_1(J') = \mu h_1 \mu^{-1}(J) \subset h_1(A)$, it follows that $f_1(J')$ is contained in an S^1-pair component of $(Q, q \cup q')$. Consequently, q'_0 is an annulus. This is a contradiction. So it follows that $\eta f_1(J') \subset q'_J$. Therefore $f_1(\partial A') \subset q'_J \times \{-1, 1\}$.

Since $Q \cap U = q' \times [-1, 1]$, we may now perform an isotopy of (M, m, m') fixed off $m' \times (-1, 1)$ that pushes $f_1(R) \cap U$ into $Q \cap U$. Let f_t^* be f_t followed by this isotopy. □

5. Stack isotopy lemmas

In this section, we will prove a number of results about isotopies in the categories $\mathscr{S}$ and $\mathscr{A}$ and how they are related to one another. The main result is Theorem 5.5, which is a version of our Main Theorem in the category $\mathscr{A}$.

Lemma 5.1. *Let (A, α) and $(B, \beta) \in \mathscr{A}$.*

(1) *If $f: A \to B$ is an $\mathscr{A}$-map, then $\mathscr{Q}f: \mathscr{Q}(A) \to \mathscr{Q}(B)$ is a proper map.*
(2) *If $H_t: A \to A$ is an $\mathscr{A}$-isotopy, then $\mathscr{Q}(H_t): \mathscr{Q}(A) \to \mathscr{Q}(A)$ is an isotopy of $(\mathscr{Q}(A), p(a_0))$, where $p: \bigcup_{n=0}^{\infty} A_n \to \mathscr{Q}(A)$ is the quotient map.*

Proof. This is left as an exercise. □

For the remainder of this section, we will take (M, μ) to be an admissible good 3-stack.

Suppose that S is a Seifert stack that is a substack of M such that (S_n, s_n, s'_n) is guided (irredundant) in (M_n, m_n, m'_n) for each $n \geq 0$. Then we say that S is *guided* (*irredundant*) *in* M.

Lemma 5.2. *Suppose that S is a Seifert stack that is an admissible substack of (M, μ). Suppose that S is guided and irredundant in M and that* Fr(S) *is essential in M. Let R be a round stack that is an admissible substack of (M, μ)*

and essential in M. If R is $\mathscr{S}$-isotopic in M into S, then R is $\mathscr{A}$-isotopic in M into S.

Proof. Let $H_t\colon M \to M$ be an $\mathscr{S}$-isotopy such that $(h_n)_t(R_n) \subset S_n$ for every $n \geq 0$. By Lemma 4.2, we may assume that $(h_n)_1(R_n)$ is safe in $(M_n, m_n \cup m'_n)$ with respect to $(S_n, s_n \cup s'_n)$ for $n \geq 0$.

By Lemma 4.5 there is an isotopy $(h^*_n)_t$ of (M_n, m_n, m'_n) such that $(h^*_n)_t|m_n = (h_n)_t|m_n$ and $(h^*_n)_t|m' = \mu_n^{-1}((h_{n+1})_t|m_{n+1})\mu_n$ and $(h^*_n)_t(R_n) \subset S_n$ for each $n \geq 0$. Let $H^*_t = \{(h^*_n)_t\}$. It is easy to check that H^*_t is an $\mathscr{A}$-isotopy. □

The remainder of the section is devoted to the proof of Theorem 5.5. For $n \geq 0$, let $e(s, t)$ be a guide for $\operatorname{Eng}(M_n, m_n \cup m'_n; s, t)$ in $(M_n, m_n \cup m'_n)$, where $s \cup t \subset m_n \cup m'_n$.

We define the substack S_0 of M as follows. For $n \geq 0$ let

$$(S_{0,n}, s''_{0,n}) = e(m_n \cup m'_n, m_n \cup m'_n).$$

Put $s_{0,n} = s''_{0,n} \cap m_n$ and $s'_{0,n} = s''_{0,n} \cap m'_n$. Let $S_0 = \{(S_{0,n}, s_{0,n}, s'_{0,n})\}$.

Let $k \geq 1$ be given. Suppose that substacks $S_1, \dots, S_k$ of M have been defined. Define S_{k+1} as follows. Let

$$(S_{k+1,0}, s''_{k+1,0}) = e(s_{k,0} \cup s'_{k,0}, \mu_0^{-1}(s_{k,1})),$$

and for $n \geq 1$ let

$$(S_{k+1,n}, s''_{k+1,n}) = e(s_{k,n} \cup s'_{k,n}, \mu_{n-1}(s'_{k,n-1}) \cup \mu_n^{-1}(s_{k,n+1})).$$

Let $s_{k+1,n} = s''_{k+1,n} \cap m_n$ and $s'_{k+1,n} = s''_{k+1,n} \cap m'_n$.

Lemma 5.3. *Suppose that R is a round stack that is an admissible substack of (M, μ). If R is essential in M, then R is $\mathscr{S}$-isotopic in M into S_k for each $k \geq 0$.*

Proof. We proceed by induction on k.

Say $k = 0$. Let $n \geq 0$ be given. It follows by construction of S_0 that there is an isotopy of (M_n, m_n, m'_n) that takes (R_n, r_n, r'_n) into $(S_{0,n}, s_{0,n}, s'_{0,n})$. Therefore there is an $\mathscr{S}$-isotopy of M that takes R into S_0.

Suppose that $k \geq 0$, and suppose that R is $\mathscr{S}$-isotopic in M into S_k. Let G be the image of R at the end of this $\mathscr{S}$-isotopy. Then $(G_n, g_n, g'_n) \subset (S_{k,n}, s_{k,n}, s'_{k,n})$ for each $n \geq 0$.

Since R is an admissible substack of (M, μ), it follows that $\mu_n(g'_n)$ is isotopic in m_{n+1} to $g_{n+1} \subset s_{k,n+1}$ for $n \geq 0$. It follows that (G_n, g_n, g'_n) is isotopic in (M_n, m_n, m'_n) into $(S_{k+1,n}, s_{k+1,n}, s'_{k+1,n})$ by construction of S_{k+1}. Therefore G is $\mathscr{S}$-isotopic in M into S_{k+1}. The proof follows immediately. □

Let $n \geq 0$ be given. Let F and G be compact 2-manifolds in $m_n \cup m'_n$. We say that $F \preceq G$ if F is isotopic in $m_n \cup m'_n$ into $G \backslash \partial G$. Let $(F)^*$ denote the strongly irredundant associate of F in $m_n \cup m'_n$.

Note that for $k \geq 0$, we have $s''_{k+1,n} \preceq s''_{k,n}$ and therefore $(s''_{k+1,n})^* \preceq (s''_{k,n})^*$. By Corollary 2.6 it follows that there is an integer $\Xi(n)$ such that $(s''_{\Xi(n),n})^*$ is isotopic in $m_n \cup m'_n$ to $(s''_{k,n})^*$ for $k \geq \Xi(n)$.

For $n \geq 0$, let $\Gamma(n) = \max\{\Xi(k) | 0 \leq k \leq n\}$.

Lemma 5.4. *Let* $n \geq 0$ *be given. Then* $(\mu_n(s'_{\Gamma(n),n}))^*$ *is isotopic in* m_{n+1} *to* $(s_{\Gamma(n),n+1})^*$.

Proof. The proof follows by Corollary 2.7. □

For $n \geq 0$, let $(T_n, t''_n) = (S_{\Gamma(n),n}, s''_{\Gamma(n),n})$. Let $t_n = t''_n \cap m_n$ and $t'_n = t''_n \cap m'_n$. By Lemmas 4.4 and 4.3, it follows that t'_n and t_{n+1} are guided and irredundant in m'_n and m_{n+1}, respectively. Therefore by Lemma 5.4 and Lemma 2.13, it follows that $\mu(t'_n)$ is isotopic in m_{n+1} to t_{n+1}. By an isotopy of M_n fixed on a neighborhood of m_{n+1} in M_{n+1}, we may assume that $\mu(t'_n) = t_{n+1}$ for each $n \geq 0$. Let $T = \{(T_n, t_n, t'_n)\}$.

Theorem 5.5. *If* R *is a round stack which is an admissible substack of* (M, μ) *and is essential in* M, *then* R *is* $\mathscr{A}$*-isotopic in* M *into* T. *Furthermore,* T *is a Seifert stack which is an admissible substack of* (M, μ) *and* $\mathrm{Fr}(T)$ *is essential in* M.

Proof. That T is a Seifert stack, an admissible substack of (M, μ), and $\mathrm{Fr}(T)$ is essential in M follows by construction.

Suppose that R is a round stack that is an admissible substack of (M, μ) and is essential in M. It follows by Lemma 5.3 that R is $\mathscr{S}$-isotopic in M into S_k for each $k \geq 0$. Therefore, (R_n, r_n, r'_n) is isotopic in (M_n, m_n, m'_n) into $(S_{k,n}, s_{k,n}, s'_{k,n})$ for every $k \geq 0$ and $n \geq 0$. Therefore (R_n, r_n, r'_n) is isotopic in (M_n, m_n, m'_n) into (T_n, t_n, t'_n) for every $n \geq 0$, i.e., R is $\mathscr{S}$-isotopic in M into T. Since T is guided and irredundant in M, it follows by Lemma 5.2 that R is $\mathscr{A}$-isotopic in (M, μ) into T. □

6. Seifert stacks and round stacks

The two main results of this section are Theorems 6.6 and 6.7.

Theorem 6.6 states that if (S, σ) is an admissible Seifert stack and each component of $\mathscr{Q}(S)$ is noncompact, then the associated noncompact 3-manifold pair is a Seifert pair. The assumption of the noncompactness of $\mathscr{Q}(S)$ is necessary. However, this presents no problem in the sequel.

Theorem 6.7 states that the image of an admissible, essential round substack under the quotient map is strongly essential.

In the rest of the section, let (S, σ) be an admissible Seifert stack such that no component of $s_n \cup s'_n$ is a torus for $n \geq 0$. Let $\Sigma = \mathscr{Q}(S)$ and let $p\colon \bigcup_{n=0}^{\infty} S_n \to \Sigma$ be the quotient map. Put $\Phi = p(s_0)$. Suppose that (S, σ) is such that Σ is noncompact and connected.

Note that if (T, t'') is a Seifert pair, and one component of t'' is an annulus, then (T, t'') may be fibered as an S^1-pair. Consequently, if (T, t, t') is a component of (S_n, s_n, s'_n) for some n and one component of $t \cup t'$ is an annulus, then every component of $t \cup t'$ is an annulus.

For $n \geq 0$, let $\mathscr{V}_n$ be the set of components of $p(S_n)$ and let $\mathscr{E}_n$ be the set of components of $p(s'_n)$ (which is equal to $p(s_{n+1})$). Let $\mathscr{V} = \bigcup_{n=0}^{\infty} \mathscr{V}_n$ and $\mathscr{E} = \bigcup_{n=0}^{\infty} \mathscr{E}_n$.

We say that $e \in \mathscr{E}$ is incident on $V \in \mathscr{V}$ iff $e \cap V \neq \varnothing$, i.e., $e \subset \partial V$. Let $\mathscr{G}$ be the graph with vertex set $\mathscr{V}$ and edge set $\mathscr{E}$ with this incidence relation. Since Σ is connected, it follows that $\mathscr{G}$ is connected. It follows that $\mathscr{G}$ is infinite because Σ is noncompact.

Lemma 6.1. *Any two edges of $\mathscr{G}$ are homeomorphic.*

Proof. Let $e, e' \in \mathscr{E}$. There is a path $(e_1, \dots, e_r)$ with $e = e_1$ and $e' = e_r$ because $\mathscr{G}$ is connected.

We wish to show that e is homeomorphic to e'. If $r = 1$, then we are done. Suppose that $r > 1$ and that e_i is homeomorphic to e for $1 \leq i < r$. Let V be the vertex of $\mathscr{G}$ on which e_{r-1} and e_r are both incident.

Let n be the integer such that V is a component of $p(S_n)$ and put $v'' = V \cap p(s_n \cup s'_n)$. Then e_{r-1} and e_r are both components of v''. If (V, v'') is an I-pair, then e_{r-1} is homeomorphic to e_r. On the other hand, if (V, v'') is an S^1-pair, then each component of v'' is an annulus. This ends the proof. □

Lemma 6.2. *If each edge of $\mathscr{G}$ is an annulus, then (Σ, Φ) is an S^1-pair.*

Proof. It follows that each component of $(S_n, s_n \cup s'_n)$ can be fibered as an S^1-pair for each $n \geq 0$. Since each component of $s_n \cup s'_n$ is an annulus, we may assume that $\sigma: s'_n \to s_{n+1}$ is fiber-preserving for $n \geq 0$. Therefore, (Σ, Φ) can be given the structure of an S^1-pair. □

Lemma 6.3. *If no edge of $\mathscr{G}$ is an annulus, then every vertex of $\mathscr{G}$ is of degree at most* 2.

Proof. This follows because if (T, t'') is an I-pair, then t'' has at most two components. □

Let us now assume that no edge of $\mathscr{G}$ is an annulus. Since $\mathscr{G}$ is infinite, it follows that at most one vertex of $\mathscr{G}$ has degree 1 with the rest having degree 2.

Let V_0 be a vertex of $\mathscr{G}$ chosen as follows. If $\mathscr{G}$ has a vertex of degree 1, let V_0 be that vertex; otherwise let V_0 be any vertex. Let Σ' be a component of $\text{cl}(\Sigma \backslash V_0)$ and let $\Phi' = \Sigma' \cap V_0$. Let $\mathscr{V}' = \{V \in \mathscr{V} | V \subset \Sigma'\}$ and let $\mathscr{E}' = \{e \in \mathscr{E} | e \subset \Sigma' \backslash \Phi'\}$. Let $\mathscr{G}'$ be the subgraph of $\mathscr{G}$ determined by $(\mathscr{V}', \mathscr{E}')$. Then exactly one vertex of $\mathscr{G}'$ has degree 1, call it V_1. Every other vertex of $\mathscr{G}'$ has degree 2. Therefore we may index the vertices of $\mathscr{G}'$ as V_k for $k \geq 1$ so that V_k is adjacent to V_{k+1} in $\mathscr{G}'$.

Lemma 6.4. (Σ', Φ') *is homeomorphic to* $(\Phi' \times [0, \infty), \Phi' \times 0)$.

Proof. Let $v_1 = V'_1 \cap (\Phi' \cup V_2)$. For $n \geq 2$, let $v_n = V_n \cap (V_{n-1} \cup V_{n+1})$. Then (V_n, v_n) is a product I-pair for $n \geq 1$. Therefore, there is a homeomorphism of pairs $h_n: (V_n, v_n) \to (\Phi' \times [n-1, n], \Phi' \times \partial [n-1, n])$ for $n \geq 1$.. Consequently, one may construct a homeomorphism $h: (\Sigma', \Phi') \to (\Phi' \times [0, \infty), \Phi' \times 0)$. □

Let $v_0 = V_0 \cap \text{cl}(\Sigma \backslash V_0)$.

Lemma 6.5. (Σ, Φ) *is homeomorphic to* $(V_0 \backslash v_0, \Phi)$.

Proof. Let $v_0 \times I$ be a regular neighborhood of v_0 in V_0 such that $v_0 \times 1 = v_0$. Let $V_0^* = V_0 \backslash (v_0 \times (0, 1])$. There is a homeomorphism of pairs $f: (V_0, v_0) \to (V_0^*, v_0 \times 0)$ which is fixed on Φ. By Lemma 6.4, f may be extended to a homeomorphism $\bar{f}: (\Sigma, \Phi) \to (V_0 \backslash v_0, \Phi)$. □

Theorem 6.6. (Σ, Φ) *is a Seifert pair.*

Proof. Combine Lemmas 6.2 and 6.5. □

For the remainder of the section, let (M, μ) be a good admissible stack. Suppose that R is a round stack that is an admissible substack of (M, μ) and is essential in M. Let $W = \mathscr{Q}(M)$ and let $p\colon \bigcup_{n=0}^{\infty} M_n \to W$ be the quotient map. For $n \geq 0$, put $N_n = p(M_n)$, $w_n = p(m_n)$, and $V_n = \bigcup_{k=0}^{n} N_k$. Note that $\{V_n\}$ is an exhaustion of W. Let $F_k = p(R_k)$ for $k \geq 0$ and let $F^* = \bigcup_{k=0}^{\infty} F_k$. Since R is essential in M, it follows that F_l is essential in $(N_l, w_l \cup w_{l+1})$ for $l \geq 0$.

Theorem 6.7. *F^* is strongly essential in (W, w_0).*

Proof. Suppose that the statement is not true. There is a component F of F^* which is either not essential in (W, w_0) or is not trapped in (W, w_0) by some compact subset of W.

First suppose that F is not essential in (W, w_0). Hence, there is a parallelism (Q, q) in (W, w_0) from F to a 2-manifold in w_0. Since Q is proper in W and w_0 is compact, it follows that F is compact.

Let $k > 0$ be given. Suppose that T is a component of $w_k \cap Q$. Then $\partial T \subset F$. It is not difficult to show that T is incompressible in Q. It follows that T is parallel in (Q, F) to a 2-manifold in F. Let (U, u) be the parallelism in (Q, F) from T to the 2-manifold in F. We may choose T so that $U \cap (\bigcup_{n=0}^{\infty} w_n) = T$. Now u is in a component of F_l for some l. So $(U, u) \subset (N_l, w_l \cup w_{l+1})$. This contradicts that F_l is essential in $(N_l, w_l \cup w_{l+1})$.

Now suppose that F is not trapped in (W, w_0) by any compact subset of W. Then $\partial F = \varnothing$ because otherwise the compact w_0 would trap F in (W, w_0). Therefore F is either an open annulus or a torus. Let k be the least integer such that $F \cap V_k \neq \varnothing$. There is a proper map $h\colon F \times I \to W$ such that $h(x, 0) = x$ and $h(F \times 1) \cap V_k = \varnothing$.

Let T be a component of $h^{-1}(w_k)$. By the methods of Lemma 6.5 of [He], we may assume that T is incompressible in $F \times I$. Hence there is a parallelism (Q, q) in $(F \times I, F \times 0)$ between T and a 2-manifold in $F \times 0$. We may choose T so that $Q \cap (\bigcup_{n=0}^{\infty} h^{-1}(w_n)) = T$.

Note that $h(q) \subset F_l$ for some l by our choice of T. So $h|Q\colon Q \to N_l$. It follows from Proposition 5.4 of [Wa] that F_l is not essential in $(N_l, w_l \cup w_{l+1})$. This is a contradiction. □

7. The Main Theorem

In this section, the Main Theorem is proved. Preliminary to this we prove a number of lemmas which relate round manifolds which are strongly essential in very good 3-manifold pairs to round stacks which are essential in good stacks.

For the remainder of this section, we will insist that (W, w) be a very good 3-manifold pair. It follows, after standard arguments, that W is irreducible and end-irreducible. We fix R to be a round 2-manifold that is strongly essential in (W, w).

Let K be a compact, connected 3-submanifold of W with $w \subset K$. Let T be the union of the components of R that are trapped by K.

Let S be a compact 2-manifold in T such that $K \cap T \subset S \backslash \mathrm{Fr}(S; T)$ and such that $S \cap T'$ is connected for every component T' of T.

Let M be a compact, connected 3-submanifold of W such that $K \cup S \subset M \backslash \mathrm{Fr}(M; W)$ and such that $(M, w \cup \mathrm{Fr}(M; W))$ is good.

If $C \subset W$, then we say that an isotopy h_t of (W, w) is *frugal rel* C whenever h_t has compact support and is fixed on C. For an isotopy h_t of (W, w), define $c(h_t) = \#(h_1(R) \cap \mathrm{Fr}(M; W))$.

Choose h_t to be an isotopy of (W, w) that is frugal rel $K \cup S$ such that $c(h_t) \leq c(g_t)$ for any isotopy g_t of (W, w) that is frugal rel $K \cup S$.

Suppose that R' is the closure of a component of $h_1(R) \backslash \mathrm{Fr}(M; W)$.

Lemma 7.1. *The* 2*-manifold* R' *is not parallel in* $(M, \mathrm{Fr}(M; W))$ *or* $(\mathrm{cl}(W \backslash M), \mathrm{Fr}(M; W))$ *to a* 2*-manifold in* $\mathrm{Fr}(M; W)$.

Proof. To get a contradiction, suppose that there is a parallelism (Q, q) in either $(M, \mathrm{Fr}(M; W))$ or $(\mathrm{cl}(W \backslash M), \mathrm{Fr}(M; W))$ between R' and a 2-manifold in $\mathrm{Fr}(M; W)$. Since no component of $\mathrm{Fr}(M; W)$ is a torus, it follows that $\partial R' \neq \varnothing$.

We claim that $R' \cap (K \cup S) = \varnothing$. Assume the contrary to get a contradiction. Then $Q \subset M$ because $K \cup S \subset M \backslash \mathrm{Fr}(M; W)$. It now follows that R' contains a component S' of $S(= h_1(S))$ because $S \subset M \backslash \mathrm{Fr}(M; W)$ and $\partial R' \subset \mathrm{Fr}(M; W)$. Let T' be the component of $h_1(T)$ which contains S'. Since $K \cap T' \subset S'$, pushing along Q isotops T' into $W \backslash K$. This contradicts the fact that K traps T' in (W, w). Therefore $R' \cap (K \cup S) = \varnothing$.

We now claim that $Q \cap (K \cup S) = \varnothing$. Again assume the contrary. Since $R' \cap (K \cup S) = \varnothing$, it follows that Q must contain a component of $K \cup S$. Consequently, $K \cup S \subset Q$ because K is connected and every component of S meets K. By pushing along Q, $h_1(T)$ can be isotoped into $W \backslash K$. This is a contradiction of the fact that K traps every component of T.

Since $Q \cap (K \cup S) = \varnothing$, there is an isotopy g_t of (W, w) that is frugal rel $K \cup S$ such that $c(g_t) < c(h_t)$. This is a contradiction, which ends the proof. □

Observe that every component of $R \cap \mathrm{Fr}(M; W)$ is a simple closed curve because $\partial R \subset M \backslash \mathrm{Fr}(M; W)$.

Lemma 7.2. *Every component of* $h_1(R) \cap \mathrm{Fr}(M; W)$ *is noncontractible in both* $\mathrm{Fr}(M; W)$ *and* $h_1(R)$.

Proof. It is not difficult to show that $\mathrm{cl}(W \backslash M)$ is irreducible. The conclusion follows by standard arguments and Lemma 7.1. □

Lemma 7.3.

(1) $h_1(R) \cap M$ *is a compact round* 2*-manifold that is essential in* $(M, w \cup \mathrm{Fr}(M; W))$.

(2) $h_1(R) \cap \mathrm{cl}(W \backslash M)$ *is a round* 2*-manifold that is strongly essential in* $(\mathrm{cl}(W \backslash M), \mathrm{Fr}(M; W))$.

Proof. By Lemma 7.2, every component of $h_1(R) \cap \mathrm{Fr}(M; W)$ is noncontractible in R. Since R is a round 2-manifold, it follows that if $\mathbf{R}$ is a component of $h_1(R)$, then any two components of $\mathbf{R} \cap \mathrm{Fr}(M; W)$ are parallel in $\mathbf{R}$. So $h_1(R) \cap M$ and $h_1(R) \cap \mathrm{cl}(W \backslash M)$ are round and incompressible in M and $\mathrm{cl}(W \backslash M)$, respectively.

Part (1) now follows by Lemma 7.1.

By Lemma 7.1, it follows that

$$h_1(R) \cap \mathrm{cl}(W \backslash M)$$

is essential in $(\mathrm{cl}(W \backslash M), \mathrm{Fr}(M; W))$.

Now suppose that $\mathbf{R}$ is a component of $h_1(R)$. If $\mathbf{R}$ meets M, then $\mathrm{Fr}(M; W)$ traps $\mathrm{cl}(\mathbf{R}\backslash M)$ in $(\mathrm{cl}(W\backslash M), \mathrm{Fr}(M; W))$. Suppose that $\mathbf{R}\cap M = \varnothing$. Then there exists a compact $L \subset W$ which traps $\mathbf{R}$ in (W, w). Note that $\mathrm{cl}(L\backslash M)$ traps $\mathbf{R}$ in $(\mathrm{cl}(W\backslash M), \mathrm{Fr}(M; W))$. □

Let (E, e) be an irreducible 3-manifold pair and let F be a 2-sided 2-manifold that is properly embedded and incompressible in E such that $\partial F \cap e = \varnothing$. Let A be a compact round 2-manifold that is essential in (E, e). Let g_t be an isotopy of (E, e) fixed on e such that $\#(g_1(A)\cap F) \leq \#(f_1(A)\cap F)$ for any isotopy f_t of (E, e) that is fixed on e.

Let E' be the manifold obtained by splitting E along F. Let A' be a component of $A\cap E'$. Let F' be the preimage of F by the quotient map $E' \to E$.

Lemma 7.4. *A' is essential in $(E', e\cup F')$.*

Proof. This follows by the essentiality of A in (E, e) and the minimality of $\#(G_1(A)\cap F)$. □

Let V be a very good exhaustion for W such that V_n is connected for every $n \geq 0$.

Let $n \geq 0$ be given. Let T_n be the union of the components of R that are trapped by V_n. Let S_n be a compact 2-submanifold of T_n such that $V_n \cap T_n \subset S_n \setminus \mathrm{Fr}(S_n; T_n)$ and S_n meets each component of T_n in a connected set. Let $\nu(n) > n$ be an integer such that $V_n \cup S_n \subset V_{\nu(n)}\backslash\mathrm{Fr}(V_{\nu(n)}; W)$. We may assume that $\nu(k) > \nu(l)$ if $k > l$.

Let h_t^n be an isotopy of (W, w) that is frugal rel $V_n \cup S_n$ such that $\#(h_1^n(R)\cap \mathrm{Fr}(V_{\nu(n)}; W))$ is minimal for all isotopies of (W, w) that are frugal rel $V_n \cup S_n$.

We define a function σ from the set of nonnegative integers to itself as follows. Let $\sigma(0)$ be chosen so that $\sigma(0) > \nu(0)$ and $V_{\sigma(0)}\backslash\mathrm{Fr}(V_{\sigma(0)}; W)$ contains the support of h_t^0. Let $k \geq 1$ be given. Let $\sigma(k) > \nu(\sigma(k-1))$ so that $V_{\sigma(k)}\backslash\mathrm{Fr}(V_{\sigma(k)}; W)$ contains the support of $h_t^{\sigma(k-1)}$.

Define $h_t\colon W \to W$ by $h_t|V_{\sigma(0)} = h_t^0|V_{\sigma(0)}$ and $h_t|\mathrm{cl}(V_{\sigma(k)}\backslash V_{\sigma(k-1)}) = h_t^{\sigma(k-1)}|\mathrm{cl}(V_{\sigma(k)}\backslash V_{\sigma(k-1)})$ for every $k \geq 1$. It follows that h_t is an isotopy of (W, w).

For $k \geq 0$, let $\xi(k) = \nu(\sigma(k))$. Let $E_0 = V_{\xi(0)}$ and let $e_0 = w\cup\mathrm{Fr}(V_{\xi(0)}; W)$. For $k \geq 1$, let $E_k = \mathrm{cl}(V_{\xi(k)}\backslash V_{\xi(k-1)})$ and let $e_k = \mathrm{Fr}(E_k; W)$. Let

$$F_0 = \bigcup_{0<n<\xi(0)} \mathrm{Fr}(V_n; W).$$

For $k \geq 1$, let

$$F_k = \bigcup_{\xi(k-1)<n<\xi(k)} \mathrm{Fr}(V_n; W).$$

For $k \geq 0$, let $A_k = h_1(R)\cap E_k$.

For $k \geq 0$, let g_t^k be an isotopy of (E_k, e_k) fixed on e_k such that $\#(g_1^k(A_k)\cap F_k)$ is minimal Let $g_t\colon W \to W$ be defined by $g_t|E_l = g_t^l|E_l$ for $l \geq 0$. Then g_t is an isotopy of (W, w). Let $Q = g_1 h_1(R)$. Let $M_0 = V_0$, let $m_0 = w$, and let $m_0' = \mathrm{Fr}(V_0; W)$. For $k \geq 1$, let $M_k = \mathrm{cl}(V_k\backslash V_{k-1})$, $m_k = \mathrm{Fr}(V_{k-1}; W)$, and let $m_k' = \mathrm{Fr}(V_k; W)$. For $k \geq 0$, define $\mu_k\colon m_k' \to m_{k+1}$ to be the identity map. Put $M = \{(M_n, m_n, m'_n)\}$ and $\mu = \{\mu_n\}$. Then (M, μ) is a good admissible stack. For $k \geq 0$, let $X_k = Q \cap M_k$, $x_k = Q \cap m_k$, and $x'_k = Q \cap m'_k$. Let $X = \{(X_k, x_k, x'_k)\}$.

Lemma 7.5. *X is a round stack that is an admissible substack of (M, μ) and is essential in M.*

Proof. This follows by Lemmas 7.3 and 7.4. □

Theorem 7.6 (The Main Theorem). *There is a Seifert pair $(\Sigma, \Phi) \subset (W, w)$ such that Σ is proper in W, $\mathrm{Fr}(\Sigma; W)$ is strongly essential in (W, w), and if R is a round 2-manifold that is strongly essential in (W, w) then R is isotopic in (W, w) into (Σ, Φ).*

Proof. By Theorem 5.5, there is a Seifert stack U that is an admissible substack of (M, μ) such that X is $\mathscr{A}$-isotopic in (M, μ) into U. Note that $W = \mathscr{Q}(M)$ and let $p\colon \bigcup_{n=0}^{\infty} M_n \to W$ be the quotient map. Note that $w = p(m_0)$ and $Q = p(\bigcup_{n=0}^{\infty} X_n)$. Put $\Pi = p(\bigcup_{n=0}^{\infty} U)$ and $\Theta = p(u_0)$. It follows by Lemma 5.1 that Q (and therefore R) is isotopic in (W, w) into (Π, Θ).

It follows by Theorem 6.7 that $\mathrm{Fr}(\Pi; W)$ is strongly essential in (W, w) By Theorem 6.6, it follows that every noncompact component of (Π, Θ) is a Seifert pair. We construct (Σ, Φ) from (Π, Θ) as follows. Let Π' be the union of the compact components of Π and put $\Theta' = \Pi \cap \Theta$.

If (π, θ) is a component of (Π', Θ'), let $(\sigma_\pi, \phi_\theta) = \mathrm{Eng}(\pi, \theta; \theta, \theta)$. Then $(\sigma_\pi, \phi_\theta)$ is a Seifert pair that is perfectly embedded in (π, θ) into which any essential round 2-manifold in (π, θ) can be isotoped by an isotopy of (π, θ). It follows that $\mathrm{Fr}(\sigma_\pi; W)$ is strongly essential in (W, w). Let $\Sigma' = \bigcup\{\sigma_\pi \mid \pi$ is a component of $\Pi'\}$. Let $\Sigma = (\Pi \backslash \Pi') \cup \Sigma'$ and $\Phi = \Sigma \cap w$. Then (Σ, Φ) is the Seifert pair promised by the theorem. □

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