CHARACTERISTIC CYCLES OF DISCRETE SERIES
FOR R-RANK ONE GROUPS

JEN-TSEH CHANG

ABSTRACT. We determine the characteristic cycles of the discrete series representations for connected R-rank one linear groups. The computation is made through the moment maps; we determine their fibers and the cohomology in question case by case. The multiplicity of the discrete series, in terms of their Harish-Chandra parameters, is given by recursive formulae; for groups of type A and B closed formulae are obtained.

INTRODUCTION

In this note we determine explicitly the characteristic cycles for the discrete series representations of connected linear group of R-rank one. To give a brief account of our approach, we begin with recalling the general setting for these invariants (for details see [V]). Let G be a real linear semisimple Lie groups with Lie algebra $\mathfrak{g}_0$. Fix a maximal compact subgroup K. Write $K_{\mathbb{C}}$ and $\mathfrak{g}$ for the complexifications of K and $\mathfrak{g}_0$ respectively; we follow the same convention in denoting the complexification for other groups and algebras. Suppose M is a Harish-Chandra module for G, i.e., a $(U(\mathfrak{g}), K_{\mathbb{C}})$-module of finite length; here $U(\mathfrak{g})$ is the enveloping algebra for $\mathfrak{g}$. With respect to the usual degree filtration, $\operatorname{gr} U(\mathfrak{g})$ is just the symmetric algebra $S(\mathfrak{g})$; equivalently the algebra of regular functions $R(\mathfrak{g}^*)$ on $\mathfrak{g}^*$, the linear dual to $\mathfrak{g}$. To each good filtration on M, the graded object $\operatorname{gr} M$ is a finitely generated $S(\mathfrak{g}) = R(\mathfrak{g}^*)$-module. The characteristic cycle of M, denoted by $\operatorname{Ch}(M)$, is then the support with multiplicity of $\operatorname{gr} M$ on $\mathfrak{g}^*$. In particular this invariant captures both the Gelfand-Kirillov dimension and the Bernstein multiplicity of the underlying admissible representation. Also, by the nature of $(\mathfrak{g}, K_{\mathbb{C}})$-action, it is in fact a linear combination of closures of $K_{\mathbb{C}}$ orbits in the nilpotent cone of $\mathfrak{g}^*$, defined to be the part which is identified to the nilpotent cone in $\mathfrak{g}$ via the nondegenerate Killing form.

One way to approach these invariants is to use the localization theory of Harish-Chandra modules [B-B]. The module M can be localized to a sheaf of modules $\mathscr{M}$, equipped with an algebraic $K_{\mathbb{C}}$ action, over a certain twisted sheaf of differential operators on the flag variety X of $\mathfrak{g}$. With respect to a good filtration of $\mathscr{M}$, the graded object $\operatorname{gr}\mathscr{M}$ is then a coherent sheaf of $\mathscr{O}_{T^*X}$-modules on T^*X, the cotangent space of X. It is a simple observation that, in

case the support of $\mathcal{M}$ consists of only closed $K_{\mathbb{C}}$-orbits in X, $\mathrm{Ch}(M)$ is the support with multiplicity of the direct image sheaf of $\mathrm{gr}\,\mathcal{M}$ under the moment map on $\mathfrak{g}^*$. Moreover in this case, $\mathrm{gr}\,\mathcal{M}$ is supported entirely on the conormal bundles of closed $K_{\mathbb{C}}$ orbits in X. Note that when G has a compact Cartan subgroup, the discrete series Harish-Chandra modules correspond exactly to those sheaves supported on closed $K_{\mathbb{C}}$ orbits on X under the localization process. We remark that when $\mathcal{M}$ is not supported on a union of closed orbits, the above simple picture no longer holds; also the moment maps are generally very complicated. Nevertheless, the discrete series being the building blocks in classifying the Harish-Chandra modules, the understanding of the closed orbit cases would be useful in general. Our ultimate goal is to find an algorithm for computing the multiplicity for general discrete series in terms of the geometry of the flag variety.

In this note, we follow this line of approach and work out the details for connected linear groups of real rank one. In the spirit of our goal mentioned above, the main result of this note is Theorem 2.5 which says that in this special case, the relevant fixed-point variety of the moment maps in consideration can always be fibered over flag varieties with fibers also given by some flag varieties. In terms of explicit data, the main results are contained in Theorems A.7, B.5, C.6, and F.2 in which the characteristic variety and the multiplicity are determined; in the cases of A and B, closed formulae for the multiplicity are obtained. As a byproduct of this, our formulae for $SU(n, 1)$ coincide with those of King [K] up to a universal constant. King derived these formulae (i.e., the character polynomials) from the distribution character of the discrete series for $SU(n, 1)$. Also in the context of the Gelfand-Kirillov dimension, the case in FII (see Theorem F.2) provides a counterexample to Conjecture 5.1 in [K]; so a modification of the conjecture is needed. We will pursue this in a future publication.

As for the organization of this note, we collect the necessary ingredients from the localization theory [C] in §1. In §2, after recalling certain basic facts of general $\mathbb{R}$-rank one groups, we give a general form of our results (Theorem 2.5) and outline the procedure of the computations. In §§A, B, C, and F, we carry out the details case by case.

We would like to thank Jim Cogdell and Roger Zierau for helpful discussions, and the referee for valuable comments.

<div style="text-align:center">1</div>

Following the convention used in the introduction, we introduce some more notations. Write θ for the Cartan involution and $\mathfrak{p}$ for the corresponding (-1)-eigenspace in $\mathfrak{g}$. Fix an invariant Cartan-Killing form $\langle\ ,\ \rangle$, and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ in the following way: to each $f \in \mathfrak{g}$ the corresponding element $\xi_f \in \mathfrak{g}^*$ is given by, for all $g \in \mathfrak{g}$,

$$(1.1) \qquad\qquad \xi_f(g) = \langle f, g \rangle.$$

Set $N_{\mathfrak{k}}^*$ for the set of nilpotent elements in $(\mathfrak{g}/\mathfrak{k})^*$, i.e., the set consists of elements in $\mathfrak{g}^*$ which are identified to nilpotent elements in $\mathfrak{g}$.

We now recall the necessary setup of the localization theory (for details see [C]). As before, X denotes the flag variety for $\mathfrak{g}$ on which $G_{\mathbb{C}}$ acts. Each point in X represents a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, which in turn determines a

positive root system Φ^+. Our convention is such that the nilradical of $\mathfrak{b}$ is the span of the eigenspaces corresponding to the negative roots; thus $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$. With respect to a positive root system, as usual we write ρ for the half sum of all positive roots. If $\mathfrak{h}$ is θ-stable, write ρ_c for the half sum of all compact roots; and $\rho_n = \rho - \rho_c$. Also Φ_c (resp. Φ_n) for the set of compact (resp. noncompact) roots.

In this note we only consider the equal rank cases, i.e., G and K have the same rank. Suppose Z is a closed $K_\mathbb{C}$-orbit in X, in which we may choose a point represented by $\mathfrak{b}$ containing a Cartan subalgebra $\mathfrak{h}$ which is the complexification of a compact real Cartan subalgebra $\mathfrak{h}_0$; hence fixed pointwise by θ. Let $\lambda \in \mathfrak{h}^*$ be regular Φ^+-dominant integral such that $\lambda - \rho$ is the differential of a character ϕ on H. The set of data (Z, ϕ, Φ^+) then gives rise to a discrete series representation π_λ for the group G, with infinitesimal character determined by the Weyl group orbit of λ (see [C]); and all the discrete series representations arise in this fashion. The character on H tensored with the top exterior product of the normal bundle of Z in X has the differential $\lambda - \rho_c + \rho_n$. This character of H gives rise to a $K_\mathbb{C}$-homogeneous line bundle on Z, and we denote by $\mathscr{L}_{\lambda-\rho_c+\rho_n}$ its sheaf of local sections on Z; this is the lowest K-type sheaf for π_λ. We write Λ for the lattice of the differentials of all the characters on H, and Λ' for $\Lambda + \rho$; this is independent of ρ chosen. Note that since G is assumed to be connected, the character ϕ is determined by λ.

We now recall the definition of the moment map. Over each point in X, say x representing a Borel subalgebra $\mathfrak{b}_x$, the fiber $T_x^* X$ in $T^* X$ is given by $(\mathfrak{g}/\mathfrak{b}_x)^*$. The moment map $\gamma : T^* X \to \mathfrak{g}^*$ is then given by, on each fiber $T_x^* X$, the inclusion $(\mathfrak{g}/\mathfrak{b}_x)^* \hookrightarrow \mathfrak{g}^*$. Since Z is a closed $K_\mathbb{C}$-orbit, the conormal bundle $T_Z^* X$ is a closed irreducible subvariety of $T^* X$ and $\gamma(T_Z^* X) \subset N_{\mathfrak{k}}^*$. Since we will only be working with the restriction of γ on $T_Z^* X$ in this note, we will use γ for this restriction map from now on. When dealing with the image of the moment maps, it is generally more convenient to work in the Lie algebra rather than in its dual. Now $(\mathfrak{g}/\mathfrak{k})^* \simeq \mathfrak{p}$ and we have

$$(1.2) \qquad T_Z^* X \simeq K_\mathbb{C} \times_{K_\mathbb{C} \cap B} (\mathfrak{n}^- \cap \mathfrak{p}).$$

For $f \in \mathfrak{n}^- \cap \mathfrak{p} = \sum_{\alpha \in \Phi_n^+} \mathfrak{g}^{-\alpha}$, write $K_\mathbb{C}(f)$ for the centralizer of f in $K_\mathbb{C}$, and set

$$N_{K_\mathbb{C}}(f, \mathfrak{n}^- \cap \mathfrak{p}) := \{k \in K_\mathbb{C} : k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}\}.$$

Since the moment map is just the projection of the second factor in (1.2), noting that both projections from $T_Z^* X$ to Z and $N_{\mathfrak{k}}^*$ are $K_\mathbb{C}$-equivariant, we have through the second projection

$$(1.3) \qquad \gamma^{-1}(\xi_f) \simeq (N_{K_\mathbb{C}}(f, \mathfrak{n}^- \cap \mathfrak{p}))^{-1} \cdot z \subset Z.$$

Here z is the point in X represented by $\mathfrak{b}$. In light of this we consider $\gamma^{-1}(\nu)$ as a subvariety of Z from now on. For the following, notice that, γ being proper, $\gamma(T_Z^* X)$ is a closed irreducible subvariety in $N_{\mathfrak{k}}^*$; hence it is the closure of a single $K_\mathbb{C}$-orbit.

1.4. Proposition. *Under the above assumption, suppose that ν is a generic point in $\gamma(T_Z^* X)$, then as an algebraic cycle*

$$\mathrm{Ch}(\pi_\lambda) = \dim H^0(\gamma^{-1}(\nu), \mathscr{L}_{\lambda-\rho_c+\rho_n}|_{\gamma^{-1}(\nu)}) \cdot \gamma(T_Z^* X).$$

This follows from the above consideration and Corollary 2.9 in [C].

In the above context we consider the following parabolic subalgebra. Let S be the set of all simple compact roots and $\langle * \rangle$ be the set of roots generated by $*$. For such a set S, the corresponding parabolic subalgebra is defined as

$$(1.5) \qquad \mathfrak{q} = \mathfrak{l} + \mathfrak{u}^- \supset \mathfrak{b} \quad \text{with } \mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \langle S \rangle} \mathfrak{g}^\alpha .$$

The connected subgroups in $G_{\mathbb{C}}$ corresponding to $\mathfrak{q}, \mathfrak{l}, \ldots$ will be denoted by $Q, L, \ldots$ Let p be the natural map sending $\mathfrak{b}$ to $\mathfrak{q}$ from X to Y, the generalized flag variety of type S. Note that $L \subset K_{\mathbb{C}}$ by the definition. Write $z' = p(z)$ and $Z' = K_{\mathbb{C}} \cdot z' \subset Y$. Since $\mathfrak{u}^- \cap \mathfrak{p} = \mathfrak{n}^- \cap \mathfrak{p}$, we have a natural $K_{\mathbb{C}}$-equivariant projection

$$(1.6) \qquad p' : T_Z^* X \twoheadrightarrow T_{Z'}^* Y .$$

The fiber of p' coincides with that of p and is given by the full flag variety of the reductive group L. On $T_{Z'}^* Y$, we also have the notion of moment map and we will follow the same notation usage as in the full flag variety situation.

Finally we recall a useful normalization of a basis in $\mathfrak{g}$ due to Chevalley.

1.7. Chevalley normalization. *For each $\alpha \in \Phi$, a root vector X_α and $H_\alpha \in \sqrt{-1}\mathfrak{h}_0$ can be chosen such that for all $\alpha, \beta \in \Phi$:*

(1) $[X_\alpha, X_{-\alpha}] = H_\alpha$.

(2) $\beta(H_\alpha) = \langle \beta, \check{\alpha} \rangle$.

(3) $[X_\alpha, X_\beta] = 0$, *if* $\alpha + \beta \notin \Phi$ *and* $\alpha \neq -\beta$.

(4) $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$, *if* $\alpha + \beta \in \Phi$. *The $N_{\alpha,\beta}$ are nonzero rational constants such that $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$; also $N_{-\alpha,-\beta} = N_{-\beta,\alpha+\beta} = N_{\alpha+\beta,-\alpha}$.*

(5) $N_{\alpha,\beta} = \pm(p+1)$, *where p is the largest integer such that $\beta - p\alpha \in \Phi$.*

(6) $\sigma(X_\alpha) = -X_{-\alpha}$ *if σ is the complex conjugation with respect to the compact real form of $\mathfrak{g}$.*

(7) $\tau(X_\alpha) = \varepsilon_\alpha X_{-\alpha}$, *where $\varepsilon_\alpha = -1$ if α is compact, $\varepsilon_\alpha = 1$ if α is noncompact; τ is the conjugation on $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$.*

A basis with properties (1) to (6) always exists (as exhibited in VI-16, 17 of [S]). Note that $\tau = \theta \circ \sigma$. Applying θ to (6), (7) follows.

2

From now on, unless otherwise specified, G will be a connected simple linear group of real rank one which contains a compact Cartan subgroup. Suppose that $\mathfrak{g}$ is of rank n, then $\mathfrak{g}_0$ is one of the following real Lie algebras: $\mathfrak{su}(n, 1), \mathfrak{so}(2n, 1), \mathfrak{sp}(n-1, 1)$; and when $n = 4$, the exceptional $\mathfrak{f}_{4(-20)}$. As usual, these will be referred to in order as A, B, C, and F; also in the first three cases we consider, in order, $n \geq 1$, $n \geq 2$, and $n \geq 3$. We first recall some general facts about G.

2.1. Lemma. *The number of distinct closed $K_{\mathbb{C}}$-orbits in the flag variety X is given by*

(1) $n+1$, *if G is of type A.*

(2) 2, *if G is of type B.*

(3) n, *if G is of type C.*

(4) 3, *if G is of type F.*

For an arbitrary semisimple linear group G, the closed $K_{\mathbb{C}}$-orbits in X are parameterized by $W(\mathfrak{g})_\theta / W(K_{\mathbb{C}})$; where $W(\mathfrak{g})_\theta$ is the subgroup of θ-fixed elements in $W(\mathfrak{g})$. For the lemma, notice that $W(\mathfrak{g})_\theta = W(\mathfrak{g})$ for $\mathfrak{g}$ contains a compact Cartan subalgebra.

We now give a count for $K_{\mathbb{C}}$ orbits in $N_{\mathfrak{k}}^*$; note that $\{0\}$ is a $K_{\mathbb{C}}$-orbit.

2.2. Lemma. *The number and dimensions of nonzero $K_{\mathbb{C}}$-orbits in $N_{\mathfrak{k}}^*$ are given by*

   (1) 2 *of dimension* 1, *if G is of type A and* $n = 1$.
   (2) 3 *of dimensions* n, n, *and* $2n - 1$, *if G is of type A and* $n > 1$.
   (3) 1 *of dimension* $2n - 1$, *if G is of type B.*
   (4) 2 *of dimensions* $4n - 5$ *and* $2n - 1$, *if G is of type C.*
   (5) 2 *of dimensions* 15 *and* 11, *if G is of type F.*

The number of orbits in the lemma follows from Theorem 3.2 in Barbasch [B] and the Satake diagrams (see [He]). Except in the case (1) (i.e. $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$), the orbit with the largest dimension in each case is the regular dense orbit in $N_{\mathfrak{k}}^*$ and will be denoted by O_d; moreover its dimension is given by $\dim \mathfrak{p} - 1$ by Proposition 9 in Kostant-Rallis [K-R]. The dimensions of the other orbits will be clear from the calculations given later.

We now give an outline of the methods used in determining the fibers of the moment maps. Detailed computations will be given on a case by case basis. First recall that an S-triple $\{h, e, f\}$ (i.e., $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$) is called normal if $h \in \mathfrak{k}$ and $e, f \in \mathfrak{p}$ (see [K-R]). In every case, for a given closed $K_{\mathbb{C}}$-orbit Z, we shall exhibit a triple such that (see 1.7(7) for notations)

$$f \in \mathfrak{n}^- \cap \mathfrak{p}, \quad e = \tau(f), \quad h \in \sqrt{-1}\mathfrak{h}_0, \quad \text{and} \quad \gamma(T_Z^* X) = \overline{K_{\mathbb{C}} \cdot \xi_f}.$$

To such a triple, we define a parabolic subalgebra $\mathfrak{q}_1 = \mathfrak{l}_1 + \mathfrak{u}_1^-$ in $\mathfrak{k}$:

$$(2.3) \qquad \mathfrak{l}_1 = \text{the centralizer of } h \text{ in } \mathfrak{k}; \qquad \mathfrak{u}_1^- = \sum_{\{\alpha \in \Phi_c \,:\, \alpha(h) > 0\}} \mathfrak{g}^{-\alpha}.$$

This is the compact part of a θ-stable parabolic in $\mathfrak{g}$ obtained by the same defining procedure; for later use we call $\mathfrak{o}_1^-$ the nilradical of the bigger parabolic in $\mathfrak{g}$. Notice that from our assumption on G, $K_{\mathbb{C}}$ is connected reductive and therefore so is the Levi part L_1 of the parabolic Q_1.

2.4. Proposition. *$K_{\mathbb{C}}(f)$ is the semidirect product of the reductive group $K_{\mathbb{C}}(f) \cap L_1$ and the unipotent group $U_1^- \cap K_{\mathbb{C}}(f)$.*

This is a general fact; see Lemma 6.11 in Vogan [V]. Write $K_{\mathbb{C}}(f)_r$ for the reductive part $K_{\mathbb{C}}(f) \cap L_1$ of $K_{\mathbb{C}}(f)$. The main result on the fiber of the moment maps can be summarized in the following theorem. For notations see §1.

2.5 Theorem. *For G as above, suppose $\gamma(T_Z^* X) = \overline{K_{\mathbb{C}} \cdot \xi_f}$, then*

   (1) $U_1^- \subset K_{\mathbb{C}}(f)$ *and* $U_1^- \subset K_{\mathbb{C}} \cap Q$.
   (2) $N_{K_{\mathbb{C}}}(f, \mathfrak{n}^- \cap \mathfrak{p}) = (K_{\mathbb{C}} \cap Q) \cdot K_{\mathbb{C}}(f)$.
   (3) $K_{\mathbb{C}}(f)_r \cap Q$ *is a parabolic subgroup of $K_{\mathbb{C}}(f)_r$ with reductive part given by $K_{\mathbb{C}}(f)_r \cap L$ and $K_{\mathbb{C}}(f) \cdot z' \simeq K_{\mathbb{C}}(f)_r / K_{\mathbb{C}}(f)_r \cap Q$.*

(4) $\gamma^{-1}(\xi_f) = K_{\mathbf{C}}(f) \cdot (K_{\mathbf{C}} \cap Q) \cdot z = p^{-1}(K_{\mathbf{C}}(f) \cdot p(z))$.

(5) If $K_{\mathbf{C}} \cdot \xi_f \neq O_d$, then $Q_1 = K_{\mathbf{C}} \cap Q$ and $\gamma^{-1}(\xi_f) = p^{-1}(p(z))$.

Here $A \cdot B$ is the set $\{a \cdot b : a \in A, \ b \in B\}$ for subsets A, B in a group

Note that (4) follows from (2) and (3). It says that the relevant fixed-point variety $\gamma^{-1}(\xi_f)$ can be fibered over a generalized flag variety (of $K_{\mathbf{C}}(f)_\tau$ in Y) with fibers given by the flag varieties of certain smaller subgroups in $K_{\mathbf{C}}$. The theorem will be checked in each case and all items will be given explicitly; in fact, it will be clear that in all cases the fiber $\gamma^{-1}(\xi_f)$ is connected. The computation relies heavily on the following:

2.6. Bruhat decomposition. $K_{\mathbf{C}} = \coprod_w (K_{\mathbf{C}} \cap Q)wQ_1$: *a disjoint union with w ranging over a set of representatives of the double coset space $W(L)\backslash W(K_{\mathbf{C}})/W(L_1)$*.

Since $K_{\mathbf{C}} \cap Q$ normalizes $\mathfrak{n}^- \cap \mathfrak{p}$, we have, for $x \in K_{\mathbf{C}} \cap Q$ and $y \in Q_1$,

$$(2.7) \qquad xwy \cdot f \in \mathfrak{n}^- \cap \mathfrak{p} \Leftrightarrow y \cdot f \in \mathfrak{o}_1^- \cap w^{-1} \cdot \mathfrak{n}^- \cap \mathfrak{p}.$$

In each case, we use the explicit description of the double coset space to prove Theorem 2.5. Generally the situation in (5) is simpler; and it means that Q alone is sufficient to describe the fiber. For the dense orbit, the following fact is useful.

2.8. Dense orbit. *Under the $\mathbb{R}$-rank one assumption, $K_{\mathbf{C}} \cdot \xi_f = O_d$ if and only if $\mathrm{ad}(h)$ has only even eigenvalues on $\mathfrak{g}$.*

Note that the semisimple element $e - f$ is regular in the Lie algebra generated by $\{h, e, f\}$. Since the group G is assumed to be of real rank one, the above follows from Theorem 4 of Kostant-Rallis [K-R].

Theorem 2.5 enables us to compute the cohomology in 1.4. Recall $z' = p(z)$ and $Z' = K_{\mathbf{C}} \cdot z' \subset Y$, we have the following commutative diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ p\ } & Z' \\
\uparrow & & \uparrow \\
\gamma^{-1}(\xi_f) & \xrightarrow{\ p\ } & K_{\mathbf{C}}(f)_r \cdot z'.
\end{array}
$$

By Theorem 2.5, both horizontal maps are surjective and the diagram is a Cartesian square. We have by the base change, write $\mathscr{L}$ for $\mathscr{L}_{\lambda - \rho_c + \rho_n}$ for simplicity,

$$
\begin{aligned}
(2.9) \qquad H^0(\gamma^{-1}(\xi_f), \mathscr{L}|_{\gamma^{-1}(\xi_f)}) &= H^0(K_{\mathbf{C}}(f)_r \cdot z', \ p_*(\mathscr{L}|_{\gamma^{-1}(\xi_f)})) \\
&= H^0(K_{\mathbf{C}}(f)_r \cdot z', \ (p_*\mathscr{L})|_{K_{\mathbf{C}}(f)_r \cdot z'}).
\end{aligned}
$$

Since p is $K_{\mathbf{C}}$-equivariant, the fiber of $p_*\mathscr{L}$ at z' is just the representation of $K_{\mathbf{C}} \cap Q$ obtained by extending trivially the irreducible representation of L with the highest weight $\lambda - \rho_c + \rho_n$. Therefore $(p_*\mathscr{L})|_{K_{\mathbf{C}}(f)_r \cdot z'}$, is the sheaf of local sections of the $K_{\mathbf{C}}(f)_r$-homogeneous vector bundle on $K_{\mathbf{C}}(f)_r \cdot z'$ induced by the above representation of $K_{\mathbf{C}} \cap Q$ restricted to $K_{\mathbf{C}}(f)_r \cap Q$; which in view of 2.5(3) is trivial on the unipotent part of $K_{\mathbf{C}}(f)_r \cap Q$. Because of 2.5(3), we may choose a positive system for $(\mathfrak{k}(f)_r, \mathfrak{k}(f)_r \cap \mathfrak{h})$ by the restriction of $\Phi^+(\mathfrak{k}, \mathfrak{h})$. Now write $V^*(\mu)$ for the irreducible $*$-module with the highest weight μ (whenever this makes sense).

2.10. Lemma. (1) *If* $V^{K_{\mathbb{C}}(f)_r \cap L}(\mu)$ *occurs in* $V^L(\lambda - \rho_c + \rho_n)$ *when considered as representations for* $K_{\mathbb{C}}(f)_r \cap L$, *then* μ *is dominant with respect to the above positive root system for* $\mathfrak{k}(f)_r$.

(2) *Suppose that* $V^L(\lambda - \rho_c + \rho_n)$ *decomposes into* $\sum m_\mu \cdot V^{K_{\mathbb{C}}(f)_r \cap Q}(\mu)$. *Then the dimension of the cohomology in* 1.4 (*or* 2.6) *is given by* $\sum m_\mu \cdot \dim V^{K_{\mathbb{C}}(f)_r}(\mu)$.

Assuming (1), (2) follows from the Borel-Weil theorem for $K_{\mathbb{C}}(f)_r$. The first part will be clear from the case by case computations. Using the branching rules and the above lemma, we are able to compute the multiplicity explicitly; in cases A and B a closed formula in terms of λ is obtained.

We conclude this section with the following notational note. In $\mathfrak{gl}(n, \mathbb{C})$, we denote by E_{ij} the matrix unit with entry 1 where the ith row and the jth column meet, all other entries being 0. In the computations of the multiplicity, often we are more concerned with the dimensions of various representations. It is more convenient to express them in terms of the types of the groups and the highest weights involved. For this purpose, we write $V_l^A(\lambda)$ for the irreducible representation of the simple Lie algebra of type A_l with the highest weight λ; similarly for others.

In the following computations, we will constantly be referring back the notations, statements, and their numbering appearing in §2.

$$\text{A. } \mathfrak{g}_0 = \mathfrak{su}(n, 1); \ n \geq 1$$

When $n = 1$, $\mathfrak{g}_0$ is just the $\mathfrak{sl}(2, \mathbb{R})$ and the result on the characteristic cycles of its discrete series is well known (say in [C]). Therefore we assume that $n > 1$.

To fix coordinates for $\mathfrak{g}$ (also for comparing our results with those of King in [K]), we realize $\mathfrak{g}_0$ as the Lie algebra in $\mathfrak{gl}(n+1, \mathbb{R})$ of all matrices A with trace zero and $A^t I_{n,1} = -I_{n,1} \overline{A}$ where

$$I_{n,1} = \begin{pmatrix} -I_n & 0 \\ 0 & I_1 \end{pmatrix};$$

here I_k is the $k \times k$ identity matrix. We choose the compact Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{g}_0$ so that $\{h_j = E_{jj} - E_{j+1,j+1} : j = 1, \ldots, n\}$ is a basis for $\sqrt{-1}\mathfrak{h}_0$ over $\mathbb{R}$. Let ε_j be the linear form on the space of diagonal matrices in $\mathfrak{gl}(n+1, \mathbb{R})$ given by $\varepsilon_j(E_{kk}) = \delta_{jk}$. Each $\lambda \in \sqrt{-1}\mathfrak{h}_0^*$ can be expressed as $(\lambda_1, \ldots, \lambda_{n+1})$ with respect to the linear forms $\{\varepsilon_j\}_{j=1}^{n+1}$. The weight lattice Λ is a sublattice with finite index in

$$(A.1) \quad \left\{ (l_1, \ldots, l_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} l_j = 0 \text{ and } l_j - l_{j+1} \in \mathbb{Z} \text{ for } 1 \leq j \leq n \right\}.$$

The roots for the pair $(\mathfrak{g}, \mathfrak{h})$ are given by:

$$(A.2) \quad \begin{aligned} \Phi(\mathfrak{g}, \mathfrak{h}) &= \{\varepsilon_j - \varepsilon_k : 1 \leq j \neq k \leq n+1\}; \\ \Phi_n &= \{\pm(\varepsilon_j - \varepsilon_{n+1}) : 1 \leq j \leq n\}; \quad \Phi_c = \{\varepsilon_j - \varepsilon_k : 1 \leq j \neq k \leq n\}. \end{aligned}$$

We now fix a Weyl chamber $C \subset \sqrt{-1}\mathfrak{h}_0^*$ for the pair $(\mathfrak{k}, \mathfrak{h})$, noting that $\Phi_c = \Phi(\mathfrak{k}, \mathfrak{h})$, with the following positive roots:

$$(A.3) \quad \Phi^+(\mathfrak{k}, \mathfrak{h}) = \{\varepsilon_j - \varepsilon_k : 1 \leq j < k \leq n\}.$$

We have

(A.4) $\lambda = (\lambda_1, \ldots, \lambda_{n+1}) \in C \Leftrightarrow \lambda_1 \lambda_2 > \cdots > \lambda_n$.

Inside C there are $n+1$ Weyl chambers for the pair $(\mathfrak{g}, \mathfrak{h})$:

$$D_0 = \{s \in C: s_n > s_{n+1}\};$$
(A.5) $$D_i = \{s \in C: s_{n-i} > s_{n+1} > s_{n-i+1}\} \quad \text{for } 1 \leq i \leq n-1;$$
$$D_n = \{s \in C: s_{n+1} > s_1\}.$$

Write Φ_i^+ for the positive root system for the pair $(\mathfrak{g}, \mathfrak{h})$ which corresponds to D_i. It is quite useful to represent these positive systems by their colored Dynkin diagrams (black dots stand for noncompact simple roots, and the rest for compact simple roots):

And for $i = 1, \ldots, n-1$,

With β_j given by:

(A.6) $$\beta_j = \begin{cases} \varepsilon_j - \varepsilon_{j+1}, & \text{for } 1 \leq j \leq n-i-1, \\ \varepsilon_{n-i} - \varepsilon_{n+1} & \text{for } j = n-i, \\ \varepsilon_{n+1} - \varepsilon_{n-(i-1)}, & \text{for } j = n-(i-1), \\ \varepsilon_{j-1} - \varepsilon_j, & \text{for } n-(i-2) \leq j \leq n. \end{cases}$$

Each of these chambers gives rise to a θ-stable Borel subalgebra in $\mathfrak{g}$, and therefore a closed $K_{\mathbb{C}}$-orbit in X; call them $\mathfrak{b}_i$ and Z_i. The discrete series for G are parameterized by the regular elements in $\Lambda' \cap C$.

Using these root systems, we now give a description of the nondense nilpotent orbits in $N_{\mathfrak{k}}^*$ (cf. 2.2(2)). With respect to Φ_0^+, the multiplicity of the simple noncompact root $\varepsilon_n - \varepsilon_{n+1}$ in any positive noncompact root is exactly one. Therefore in view of (A.2), the span of eigenvectors of positive noncompact roots is a $K_{\mathbb{C}}$-orbit in $N_{\mathfrak{k}}^*$; call it O_h. Denote by O_{ah} the orbit obtained in the same fashion using the positive system Φ_n^+. One checks that both of these orbits have dimension n; the subscripts allure to holomorphic and antiholomorphic (see remarks after A.7). Together with O_d, they exhaust all the $K_{\mathbb{C}}$-orbits in $N_{\mathfrak{k}}^*$ (cf. Lemma 2.2).

A.7. Theorem. *Suppose $\lambda \in \Lambda' \cap C$ is Φ_i^+-dominant, then $\mathrm{Ch}(\pi_\lambda) = c_i(\lambda) \cdot \overline{O_i}$, with c_i and O_i given by*:

(1) $$O_i = \begin{cases} O_d, & \text{if } i = 1, \ldots, n-1, \\ O_h, & \text{if } i = 0, \\ O_{ah}, & \text{if } i = n. \end{cases}$$

(2) *For* $i = 1, \ldots, n-1$,

$$c_i(\lambda) = \frac{(-1)^i}{1!2!\cdots(n-2)!} \begin{vmatrix} \lambda_1^{n-2} & \cdots & \lambda_{n-i}^{n-2} & \lambda_{n-i+1}^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-3} & \cdots & \lambda_{n-i}^{n-3} & \lambda_{n-i+1}^{n-3} & \cdots & \lambda_n^{n-3} \\ \vdots & & & & & \vdots \\ \lambda_1 & \cdots & \lambda_{n-i} & \lambda_{n-i+1} & \cdots & \lambda_n \\ 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{vmatrix}.$$

(3) *For* $i = 0$ *or* n,

$$c_i(\lambda) = \frac{1}{1!2!\cdots(n-2)!} \prod_{1 \le j < k \le n} (\lambda_j - \lambda_k).$$

Note that the above formulae for the multiplicity coincide with those in [K] up to a constant. In case $i = 0$ or n, the discrete series representations in question belong to the holomorphic discrete series. Moreover $Q_1 = K_{\mathbb{C}} \cap Q$, Theorem 2.5 and the above result then follow from Corollary 2.13 in [C], and the fact that $c_0(\lambda)$ is just the dimension of the irreducible representation of K with the highest weight $\lambda - \rho_c + \rho_{0,n}$; similar for c_n.

We now fix an $i \in \{1, \ldots, n-1\}$ and denote the simple roots in the Dynkin diagram for Φ_i^+ by $\beta_1, \ldots, \beta_n$ from left to right as listed in (A.6). In particular, the two noncompact simple roots are β_{n-i} and β_{n-i+1}.

Recall the normal basis listed in 1.7. Consider the normal triple

$$f = \sqrt{2}(X_{-(\varepsilon_1 - \varepsilon_{n+1})} + X_{-(\varepsilon_{n+1} - \varepsilon_n)}), \quad e = \tau(f), \quad h = [e, f] = 2H_{\varepsilon_1 - \varepsilon_n}.$$

Note that $f \in \mathfrak{n}^- \cap \mathfrak{p}$ and $\mathrm{ad}(h)$ has only even eigenvalues on $\mathfrak{g}$. By 2.8, we have $\gamma(T_Z^* X) = \overline{K_{\mathbb{C}} \cdot \xi_f} = \overline{O_d}$, and this gives A.7(1).

We proceed to the proof of 2.5 in this case. Notice that

$$\mathfrak{u}_1 \leftrightarrow \{\varepsilon_1 - \varepsilon_j, \varepsilon_j - \varepsilon_n : 2 \le j \le n-1\} \cup \{\varepsilon_1 - \varepsilon_n\};$$
$$\mathfrak{l}_1 \leftrightarrow \{\varepsilon_j - \varepsilon_k : 2 \le j \ne k \le n-1\}.$$

Here $\leftrightarrow$ means the space on the left is the span of the root-vectors with roots ranging in the set on the right; this convention will be used later also. Immediately we have $Q_1 \cdot f = \mathbb{C}f$ and $K_{\mathbb{C}}(f)$ is the semisimple part of Q_1. Now the representatives of the double coset space in 2.6 can be chosen from the set $\{\mathrm{id}, (\varepsilon_1, \varepsilon_n)\}$; the latter being the transposition. Therefore the condition in (2.7) is satisfied if and only if $w = \mathrm{id}$ and this gives Theorem 2.5.

To have a more concrete picture of the geometry involved, set $\mathfrak{l}_b \supset \mathfrak{h}$ to be the reductive subalgebra of $\mathfrak{k}$ containing all the eigenspaces with roots $\varepsilon_j - \varepsilon_k$ with $1 \le j \ne k \le n-i$; similarly $\mathfrak{l}_e$ for the one containing roots $\varepsilon_j - \varepsilon_k$ with $n-i+1 \le j \ne k \le n$; the subscripts allure to the positions of these parabolics in the Dynkin diagram, i.e., the beginning and the end. Note that $\mathfrak{l} = \mathfrak{l}_b \oplus \mathfrak{l}_e$, $[\mathfrak{l}_b, \mathfrak{l}_e] = 0$, and $L \cap Q_1$ is a parabolic in L_1 with Levi part $L_1 \cap (L_b \cdot L_e)$. We have

(A.8) $\qquad p^{-1}(z') \simeq (L_b/L_b \cap B) \times (L_e/L_e \cap B) \simeq \mathbb{P}^{n-i-1} \times \mathbb{P}^{i-1}$.

We are now ready to compute the multiplicity. The representation of L appearing in (2.9) is given by

$$H^0(L_b/L_b \cap B, \mathscr{L}|_{L_b/L_b \cap B}) \otimes H^0(L_e/L_e \cap B, \mathscr{L}|_{L_e/L_e \cap B}).$$

By 2.10, we need to examine the decomposition of the restriction of $V^L(\tilde{\lambda})$ on $L_1 \cap (L_b \cdot L_e)$. Recall that the positive system in question is Φ_i^+ in (A.5). Recall $\tilde{\lambda} = \lambda - \rho_c + \rho_n$, we have

(A.9)
$$\tilde{\lambda}_j = \begin{cases} \lambda_j + j - n/2, & \text{for } 1 \le j \le n - i, \\ \lambda_j + j - 1 - n/2, & \text{for } n - i + 1 \le j \le n, \\ \lambda_{n+1} + i - n/2, & \text{for } j = n + 1. \end{cases}$$

Note that the semisimple part of $L_1 \cap (L_b \cap L_e)$ is of type $A_{n-i-2} \times A_{i-2}$. Applying the branching rule to this case (Theorem 6.1 in [Bo]), we have

A.10. **Lemma.** *As $L_1 \cap (L_b \cdot L_e)$-modules,*

$$V^L(\tilde{\lambda}) = \sum V_{n-i-2, i-2}^{A \times A}(\lambda_2', \lambda_3', \dots, \lambda_{n-i}'; \lambda_{n-i+1}', \dots, \lambda_{n-1}');$$

with the parameters ranging over all possible λ_j' satisfying

$$\tilde{\lambda}_1 \ge \lambda_2' \ge \tilde{\lambda}_2 \ge \cdots \ge \lambda_{n-i}' \ge \tilde{\lambda}_{n-i} \quad \text{and} \quad \tilde{\lambda}_{n-i+1} \ge \lambda_{n-i+1}' \ge \cdots \ge \lambda_{n-1}' \ge \tilde{\lambda}_n.$$

In the above formula, the parameters before (resp. after) ";" indicate the weights on $L_b \cap L_1$ (resp. $L_e \cap L_1$).

Since λ is regular dominant, by (A.9), we have

(A.11)
$$\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \cdots \ge \tilde{\lambda}_{n-i} \ge \tilde{\lambda}_{n-i+1} \ge \cdots \ge \tilde{\lambda}_n.$$

In particular, all the parameters in Lemma A.10 are necessarily dominant for $K_C \cap L_1$. This verifies Lemma 2.10(1), and therefore 2.10(2) gives the multiplicity for π_λ. To get the closed form in A.7(2), we simplify notations first. Let $a_1 \ge \cdots \ge a_n$ be a sequence of numbers. Note that L_1 is of type A_{n-3} modulo the center.

A.12. **Definition.** (1) $[a_1, \dots, a_{n-1}] = \sum_{a_j \ge b_j \ge a_{j+1}} V_{n-3}^A(b_1, \dots, b_{n-2})$.

(2) $[a_1, \dots, a_k; a_{k+1}, \dots, a_n] = \sum V_{n-3}^A(b_1, \dots, b_{n-2})$, with the parameters ranging over the gaps of the sequence $a_1 \ge \cdots \ge a_k$ and separately $a_{k+1} \ge \cdots \ge a_n$.

By Lemma 2.10, the dimension of the cohomology in 1.4 is the dimension of the $[\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-i}; \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n]$. As virtual representation of L_1, we have

$$[\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-i}; \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n]$$
$$= [\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-i-1}, \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n]$$
$$- [\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-i-1}; \tilde{\lambda}_{n-i} - 1, \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n].$$

Inductively, it equals to
(A.13)
$$[\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-i-1}, \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n] - [\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-i-2}, \tilde{\lambda}_{n-i} - 1, \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n]$$
$$+ [\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-i-3}, \tilde{\lambda}_{n-i-1} - 1, \tilde{\lambda}_{n-i} - 1, \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n]$$
$$- \cdots + (-1)^{n-i-1}[\tilde{\lambda}_2 - 1, \tilde{\lambda}_3 - 1, \dots, \tilde{\lambda}_{n-i} - 1, \tilde{\lambda}_{n-i+1}, \dots, \tilde{\lambda}_n].$$

By the branching rule again, each term in the above formula is the restriction of the irreducible representation with the highest weight given by the indicated parameter of a simple group of type A_{n-1} to A_{n-2}. In particular the dimensions can be computed via the Weyl dimension formula for A_{n-1}.

Note that $\tilde{\lambda}_j - 1 = \lambda_j + j - 1 - n/2$ for $1 \leq j \leq n - i$. By the formula on p. 214 of [Z], each dimension is given by the Vandermonde determinant of a subset of $\{\lambda_j - 1 + n/2\}_{j=1}^n$, hence that of $\{\lambda_j\}_{j=1}^n$. In summary, we have $0!1! \cdots (n-2)! \cdot \dim H^0(\gamma^{-1}(\xi_f), \mathscr{L}|_{\gamma^{-1}(\xi_f)})$ is given by

$$\begin{vmatrix} \lambda_1^{n-2} & \cdots & \lambda_{n-i-1}^{n-2} & \lambda_{n-i+1}^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-3} & \cdots & \lambda_{n-i-1}^{n-3} & \lambda_{n-i+1}^{n-3} & \cdots & \lambda_n^{n-3} \\ \vdots & & & & & \vdots \\ \lambda_1 & \cdots & \lambda_{n-i-1} & \lambda_{n-i+1} & \cdots & \lambda_n \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$- \begin{vmatrix} \lambda_1^{n-2} & \cdots & \lambda_{n-i-2}^{n-2} & \lambda_{n-i}^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-3} & \cdots & \lambda_{n-i-2}^{n-3} & \lambda_{n-i}^{n-3} & \cdots & \lambda_n^{n-3} \\ \vdots & & & & & \vdots \\ \lambda_1 & \cdots & \lambda_{n-i-2} & \lambda_{n-i} & \cdots & \lambda_n \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{vmatrix} + \cdots.$$

By the row expansion formula for determinants, this then proves the formula for c_i in Theorem A.7(2).

$$\text{B. } \mathfrak{g}_0 = \mathfrak{so}(2n, 1); \; n \geq 2$$

We realize $\mathfrak{so}(2n, 1)$ as the subalgebra in $\mathfrak{gl}(2n+1, \mathbb{R})$ of all traceless matrices A satisfying $A^t I_{2n,1} = -I_{2n,1}A$. We choose a compact Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{g}_0$ so that it has a basis given by $\{h_i = E_{2i-1,2i} - E_{2i,2i-1}\}_{i=1}^n$. Let ε_j be the linear form on $\sqrt{-1}\mathfrak{h}_0$ given by $\varepsilon_j(\sqrt{-1}h_k) = \delta_{jk}$. The roots are given by

(B.1)
$$\Phi = \{\pm\varepsilon_i \ (1 \leq i \leq n), \ \pm\varepsilon_i \pm \varepsilon_j \ (1 \leq i \neq j \leq n)\},$$
$$\Phi_c = \{\text{long roots}\}, \quad \Phi_n = \{\text{short roots}\}.$$

We now fix a positive system for $\Phi_c = \Phi(\mathfrak{k}, \mathfrak{h})$ with the positive roots given by

(B.2)
$$\{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\varepsilon_{n-1} + \varepsilon_n\}.$$

Note that $\mathfrak{k} \simeq \mathfrak{so}(2n, \mathbb{C})$. The corresponding Weyl chamber $C \subset \sqrt{-1}\mathfrak{h}_0^*$ is given by

(B.3)
$$\lambda = (\lambda_1, \ldots, \lambda_n) \in C \Leftrightarrow \lambda_1 > \cdots > \lambda_{n-1} > |\lambda_n|.$$

Here the coordinates are assigned with respect to the basis $\{\varepsilon_j\}_{j=1}^n$. There are two Weyl chambers for the pair $(\mathfrak{g}, \mathfrak{h})$:

(B.4)
$$D = \{s \in C : s_n > 0\}, \qquad \overline{D} = \{s \in C : s_n < 0\}.$$

The colored Dynkin diagram for Φ^+ determined by D is

Note that $\overline{D} = s_{\varepsilon_n}D$, it has the same colored Dynkin diagram. As before, consider $\Lambda' = \Lambda + \rho$, and write $\tilde{\lambda} = \lambda - \rho_c + \rho_n$.

B.5. **Theorem.** *Suppose that* $\lambda \in \Lambda' \cap C$, *then* $\text{Ch}(\pi_\lambda) = c(\lambda) \cdot \overline{O_d}$; *where*

$$c(\lambda) = \tfrac{1}{2} \dim V^B_{n-1}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}).$$

Equivalently, using Weyl's dimension formula,

$$c(\lambda) = \frac{1}{2} \cdot \frac{\prod_{1 \le j < k \le n-1}(\lambda_j - \lambda_k)(\lambda_j + \lambda_k) \prod_{1 \le j \le n-1} \lambda_j}{\prod_{1 \le j < k \le n-1}(k - j)(2n - 1 - (k + j)) \prod_{1 \le j \le n-1}(n - \tfrac{1}{2} - j)}.$$

In particular it is homogeneous in λ. Reflecting by the simple root ε_n, it suffices to show this for $\lambda \in D$. Since there is only one $K_\mathbb{C}$-orbit in $N^*_{\mathfrak{k}}$, only the multiplicity needs to be verified.

We consider the triple

(B.6) $f = X_{-\varepsilon_1}, \quad e = X_{\varepsilon_1}, \quad \text{and} \quad h = [e, f] = H_{\varepsilon_1}.$

The corresponding parabolic is then

$$\mathfrak{u}_1 \leftrightarrow \{\varepsilon_1 \pm \varepsilon_j : j > 1\},$$
$$\mathfrak{l}_1 \leftrightarrow \{\varepsilon_i \ (2 \le i \le n), \pm\varepsilon_i \pm \varepsilon_j \ (2 \le i < j \le n)\}.$$

Notice that $Q_1 \cdot f = \mathbb{C}f$ and $K_\mathbb{C}(f)$ is the semisimple part of $\dot{Q}_1$. Also the double coset space in 2.6 can be represented by $\{\text{id}, \sigma_{\varepsilon_1}, (\varepsilon_1, \varepsilon_n)\}$; the middle one is the sign change on ε_1. Immediately the condition in (2.7) holds only if $w = \text{id}$. This proves Theorem 2.5 in this case.

To compute the multiplicity, note that L is of type A_{n-1}, and $L \cap L_1$ is of type A_{n-2}; modulo the center in both cases. By the branching rule each constituent of $V^L(\tilde{\lambda})$ (the representation of L appearing in (2.9)), when viewed as an $L \cap L_1$-module, has a highest weight μ satisfying $\tilde{\lambda}_1 \ge \mu_1 \ge \cdots \ge \tilde{\lambda}_{n-1} \ge \mu_{n-1} \ge \tilde{\lambda}_n$. Now $\tilde{\lambda}_j = \lambda_j - (n - j) + \tfrac{1}{2}$, for all j; in particular $\tilde{\lambda}_n > 0$. This then verifies 2.10(1). Thus

$$c(\lambda) = \sum_{\lambda_1 \ge \mu_1 \ge \cdots \ge \mu_{n-1} \ge \tilde{\lambda}_n} \dim V^D_{n-1}(\mu)$$

$$= \frac{1}{2} \sum_{\tilde{\lambda}_1 \ge \mu_1 \ge \cdots \ge \tilde{\lambda}_{n-1} \ge |\mu_{n-1}|} \dim V^D_{n-1}(\mu)$$

$$= \frac{1}{2} \dim V^B_{n-1}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}).$$

The second equality follows from the conjugation between

$$V^D_{n-1}(\mu_1, \dots, \mu_{n-2}, \mu_{n-1}) \quad \text{and} \quad V^D_{n-1}(\mu_1, \dots, \mu_{n-2}, -\mu_{n-1});$$

the third is the branching rule (Theorem 12.1 in [Bo]). This completes the proof of Theorem B.5.

$$\text{C. } \mathfrak{g}_0 = \mathfrak{sp}(n - 1, 1); \ n \ge 3$$

Recall that $\mathfrak{sp}(n, \mathbb{C})$ is the algebra of matrices A in $\mathfrak{gl}(2n, \mathbb{C})$ satisfying $A^t J_n = -J_n A$ with

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We realize $\mathfrak{sp}(n-1,1)$ as the subalgebra of matrices A in $\mathfrak{sp}(n,\mathbb{C})$ satisfying $A^t K_{n-1,1} = -K_{n-1,1}A$ with

$$K_{n-1,1} = \begin{pmatrix} -I_{n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I_{n-1} \end{pmatrix}.$$

We choose the compact Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{g}_0$ so that $\sqrt{-1}\mathfrak{h}_0$ has a basis $\{h_i\}_{i=1}^n$ with h_i given by $E_{ii} - E_{n+i,n+i}$ for $1 \le i \le n$. Let ε_j be the linear form on $\sqrt{-1}\mathfrak{h}_0$ given by $\varepsilon_j(h_k) = \delta_{jk}$. Then we have

$$\Phi = \{\pm 2\varepsilon_i \ (1 \le i \le n), \pm\varepsilon_i \pm \varepsilon_j \ (1 \le i < j \le n)\},$$

(C.1) $$\Phi_n = \{\pm\varepsilon_j \pm \varepsilon_n \ (1 \le j \le n-1)\},$$

$$\Phi_c = \{\pm 2\varepsilon_i \ (1 \le i \le n-1), \pm\varepsilon_i \pm \varepsilon_j \ (1 \le i < j \le n-1)\} \cap \{\pm 2\varepsilon_n\}.$$

Note that $\mathfrak{k} = \mathfrak{sp}(n-1,\mathbb{C}) \oplus \mathfrak{sp}(1,\mathbb{C})$. Fix a positive system for $\Phi_c = \Phi(\mathfrak{k},\mathfrak{h})$ with the following simple roots

(C.2) $$\{\varepsilon_j - \varepsilon_{j+1} \ (1 \le j \le n-2), 2\varepsilon_{n-1}\} \cup \{2\varepsilon_n\}.$$

The corresponding Weyl chamber is given by

(C.3) $$\lambda = (\lambda_1, \dots, \lambda_n) \in C \Leftrightarrow \lambda_1 > \cdots > \lambda_{n-1} > 0 \quad \text{and} \quad \lambda_n > 0.$$

There are n Weyl chambers for the pair $(\mathfrak{g},\mathfrak{h})$:

$$D_1 = \{s \in C : s_{n-1} > s_n\};$$

(C.4) $$D_i = \{s \in C : s_{n-i} > s_n > s_{n-i+1}\} \quad \text{for } 2 \le i \le n-1;$$

$$D_n = \{s \in C : s_n > s_1\}.$$

The corresponding colored Dynkin diagrams are

and for $i = 2, \dots, n-1$,

with β_j given by

(C5) $$\beta_j = \begin{cases} \varepsilon_j - \varepsilon_{j+1} & \text{for } 1 \le j \le n-i-1, \\ \varepsilon_{n-i} - \varepsilon_n, & \text{for } j = n-i, \\ \varepsilon_n - \varepsilon_{n-(i-1)}, & \text{for } j = n-(i-1), \\ \varepsilon_{j-1} - \varepsilon_j, & \text{for } n-(i-2) \le j \le n-1, \\ 2\varepsilon_{n-1}, & \text{for } j = n. \end{cases}$$

Note that Φ_1^+ and Φ_n^+ are the Borel-de Siebenthal chambers. There are two closed $K_\mathbb{C}$-orbits in $N_\mathfrak{k}^*$ (cf. 2.2); call the smaller one O_s.

C.6. Theorem. *Suppose $\lambda \in \Lambda' \cap C$ is Φ_i^+-dominant, then $\mathrm{Ch}(\pi_\lambda) = c_i(\lambda)\overline{O}_i$ where*

    (1) $O_n = O_s$ and $O_i = O_d$ if $i \neq n$.

    (2) $c_n(\lambda) = \dim V_{n-1}^C(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-1})$.

    (3) *For $i \neq n$, $c_i(\lambda)$ is given by the recursive formula in C.17.*

Consider the case Φ_n^+ first. We introduce the triple

$$f = X_{-(\varepsilon_n - \varepsilon_1)}, \quad e = \tau(f), \quad h = [e, f] = H_{\varepsilon_n - \varepsilon_1}.$$

C.7. Proposition. *Assumptions as above, $\gamma(T_Z^*X) = \overline{O}_s$, $O_s = K_\mathbb{C} \cdot \xi_f$, and $\dim O_s = 2n - 1$.*

As a corollary, C.6(2) is now established.

Proof. In this case, $K_\mathbb{C} \cap Q$ (in (2.3)) is the entire factor $\mathfrak{sp}(n-1, \mathbb{C})$. Checking the reflections about $2\varepsilon_n$, we have $N_{K_\mathbb{C}}(f, \mathfrak{n}^- \cap \mathfrak{p}) = K_\mathbb{C} \cap Q$. On the other hand,

$$\dim \gamma(T_Z^*X) \leq \dim Y = n^2 - (n-1)^2 = 2n - 1.$$

Note that $\dim O_d = 4n - 5 > 2n - 1$ for $n \geq 3$. This gives the proposition. $\square$

From now on, let $\Phi^+ = \Phi_i^+$ for a fixed $i \in \{1, \ldots, n-1\}$. Consider

$$f = \sqrt{2}(X_{-(\varepsilon_1 - \varepsilon_n)} + X_{-(\varepsilon_n + \varepsilon_{n-1})}), \quad e = \tau(f), \quad h = [e, f] = 2H_{\varepsilon_1 + \varepsilon_{n-1}}.$$

Note that $f \in \mathfrak{n}^- \cap \mathfrak{p}$ and by 2.8, $O_d = K_\mathbb{C} \cdot \xi_f$. Also

(C.8) $\mathfrak{u}_1 \leftrightarrow \{\varepsilon_1 - \varepsilon_j, -\varepsilon_j + \varepsilon_{n-1} \ (2 \leq j \leq n-2); \ \varepsilon_1 + \varepsilon_j \ (1 \leq j \leq n-1)\}$.

Now the semisimple part $\mathfrak{l}_1'$ of $\mathfrak{l}_1$ has three simple factors:

(C.9)
$$\mathfrak{l}_{1,1} \leftrightarrow \{\varepsilon_j \pm \varepsilon_k \ (2 \leq j, k \leq n-2)\},$$
$$\mathfrak{l}_{1,2} \leftrightarrow \{\pm(\varepsilon_1 - \varepsilon_{n-1})\}, \quad \mathfrak{l}_{1,3} \leftrightarrow \{\pm 2\varepsilon_n\}.$$

We now proceed to the proof of 2.5. This case is slightly more complicated than A and B.

C.10. Claim. $N_{K_\mathbb{C}}(f) \subset (K_\mathbb{C} \cap Q) \cdot Q_1$.

It follows from (C.8) and (C.9) that

(C.11)
$$U_1^- \cdot f = f, \quad L_{1,1} \cdot f = \mathbb{C}f,$$
$$L_{1,2}L_{1,3} \subset \mathbb{C}f \cup \mathbb{C}\{f \pm X_{\varepsilon_n - \varepsilon_{n-1}} \pm X_{-\varepsilon_1 - \varepsilon_n}\}.$$

On the other hand, the representatives of the double coset in 2.6 can be chosen from

(C.12) $\{\mathrm{id}, (\varepsilon_1, \varepsilon_j) \ (j > n - i + 1)\} \cdot \{\mathrm{id}, (\varepsilon_{n-i+1}, \varepsilon_k) \ (k \leq n - i)\} \cdot \{\mathrm{id}, \sigma_{\varepsilon_1}\}$.

In case $i = 1$, the first set consists of the id only. Comparing (C.11) and (C.12), one sees that the condition in (2.7) holds only if $w = \mathrm{id}$. This gives the claim.

To finish the proof of 2.5, it remains to check that if $q_1 \in Q_1$ and $q_1 \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}$, then modulo a factor in $K_\mathbb{C} \cap Q$ on the left, $q_1 \in K_\mathbb{C}(f)$. In view of (C.11), we may assume that $q_1 \in L_{1,2} \cdot L_{1,3}$. We then proceed as before using (C.11) and the Bruhat decomposition of $L_{1,2} \cdot L_{1,3}$ with respect to the standard Borel subgroup $L_{1,2} \cdot L_{1,3} \cap Q$. We omit the details. In particular, one checks

that the centralizer of f is the $\mathfrak{sl}(2, \mathbb{C})$ generated by the triple, under a proper normalization,

(C.13) $X_{\varepsilon_1 - \varepsilon_{n-1}} + X_{2\varepsilon_n}, \quad X_{\varepsilon_{n-1} - \varepsilon_1} + X_{-2\varepsilon_n}, \quad H_{\varepsilon_1 - \varepsilon_{n-1}} + H_{2\varepsilon_n}.$

Putting the above together, we have established 2.5 in this case and

C.14. Corollary. $K_{\mathbb{C}}(f)_r$ *is connected and its simple factors are of type* C_{n-3} *and* A_1.

To examine the fiber $\gamma^{-1}(\xi_f)$ more closely, we proceed in the same manner as in the case of $\mathfrak{su}(n, 1)$. Let $\mathfrak{l}_b \supset \mathfrak{h}$ be the reductive subalgebra of $\mathfrak{k}$ containing all the eigenspaces corresponding to roots $\varepsilon_j - \varepsilon_k$ with $1 \leq j \neq k \leq n - i$; similarly $\mathfrak{l}_e$ the one containing roots $\pm\varepsilon_j \pm \varepsilon_k$ with $n - i + 1 \leq j, k \leq n - 1$ (if $i = 1$, the latter is just $\pm 2\varepsilon_n$). Then $\mathfrak{l} = \mathfrak{l}_b + \mathfrak{l}_e$, $[\mathfrak{l}_b, \mathfrak{l}_e] = 0$, and they are of type A_{n-i-1} and C_{i-1} respectively for $i > 1$; when $i = 1$, they are A_{n-2} and A_1. The fiber $p^{-1}(z')$ of p is the product of the flag varieties for $\mathfrak{l}_b$ and $\mathfrak{l}_e$. Now $K_{\mathbb{C}}(f)_r \cap L$ is a parabolic in $K_{\mathbb{C}}(f)_r$ with Levi part $K_{\mathbb{C}}(f)_r \cap (L_b \cdot L_e)$ whose semisimple part is of type $A_{n-i-2} \times C_{i-2}$ for $i > 1$; when $i = 1$ it is given by $A_{n-4} \times A_1$ in view of the triple before C.14.

To compute the multiplicity, recall $\tilde{\lambda} = \lambda - \rho_c + \rho_n$ and the positive system in question is $\Phi^+ = \Phi_i^+$, we have for $i > 1$,

(C.15) $\tilde{\lambda}_j = \begin{cases} \lambda_j - (n - 1 - j), & \text{for } 1 \leq j \leq n - i, \\ \lambda_j - (n - j), & \text{for } n - i + 1 \leq j \leq n - 1, \\ \lambda_n + (i - 2), & \text{for } j = n. \end{cases}$

When $i = 1$, we ignore the second line for the formulae for $\tilde{\lambda}$. Applying the branching rules (p. 381 in [Z] for type C), we have

C.16. Lemma. *As* $K_{\mathbb{C}}(f)_r \cap (L_b \cdot L_e)$ *modules,*
  (1) *for* $i = 1$,

$$V^L(\tilde{\lambda}) = \sum V_{n-4, 1}^{A \times A}(\lambda_2', \ldots, \lambda_{n-2}'; \tilde{\lambda}_n),$$

with the parameters ranging over all λ_j' *satisfying*

$q_1 \geq \lambda_2' \geq q_2 \geq \cdots \geq \lambda_{n-2}' \geq q_{n-2}, \quad$ *for all* $\tilde{\lambda}_1 \geq q_1 \geq \tilde{\lambda}_2 \geq \cdots \geq q_{n-2} \geq \tilde{\lambda}_{n-1}.$

  (2) *for* $i > 1$,

$$V^L(\tilde{\lambda}) = \sum V_{n-i-2, i-2}^{A \times C}(\lambda_2', \lambda_3', \ldots, \lambda_{n-i}'; \lambda_{n-i+1}', \ldots, \lambda_{n-2}'),$$

with the parameters ranging over all possible λ_j' *satisfying*

$$\tilde{\lambda}_1 \geq \lambda_2' \geq \tilde{\lambda}_2 \geq \cdots \geq \lambda_{n-i}' \geq \tilde{\lambda}_{n-i}$$

and

$$q_{n-i+1} \geq \lambda_{n-i+1}' \geq \cdots \geq q_{n-2} \geq \lambda_{n-2}' \geq q_{n-1}$$

for all q_j *satisfying*

$$\tilde{\lambda}_{n-i+1} \geq q_{n-i+1} \geq \cdots \geq \tilde{\lambda}_{n-1} \geq q_{n-1} \geq 0.$$

Since λ is regular dominant, by (C.4) and (C.15) we have $\tilde{\lambda}_{n-i} > \tilde{\lambda}_{n-i+1}$ and $\tilde{\lambda}_n \geq 0$. This says that all the parameters appearing in the above sum are

618   JEN-TSEH CHANG

dominant for the factor of type C_3 in $K_{\mathbb{C}}(f)_r$; also dominant for the factor of type A_1 when $i = 1$. As for the other factor when $i > 1$, notice that by (C.13), $(\mathfrak{l}_{1,2} + \mathfrak{l}_{1,3}) \cap \mathfrak{l}$ is just $\mathbb{C}(H_{\varepsilon_1 - \varepsilon_{n-1}} + H_{\varepsilon_n})$. In particular, the restriction to $(L_{1,2} \cdot L_{1,3}) \cap L$ of $V^L(\tilde{\lambda})$ is the direct sum of all weight spaces. Let ν be an arbitrary weight for $V^L(\tilde{\lambda})$, then $\langle \nu, \varepsilon_1 \rangle \geq \tilde{\lambda}_{n-i}$ and $\langle \nu, \varepsilon_{n-1} \rangle \leq \tilde{\lambda}_{n-i+1}$ (say by looking into the weight polygons of types A_{n-i-1} and C_{i-1}). Putting together, we have in view of (C.1) and (C.15),

$$\langle \nu, H_{\varepsilon_1 - \varepsilon_{n-1}} + H_{2\varepsilon_n} \rangle = \langle \nu, \varepsilon_1 - \varepsilon_{n-1} + \varepsilon_n \rangle \geq \tilde{\lambda}_{n-i} - \tilde{\lambda}_{n-i+1} + \tilde{\lambda}_n > 0.$$

This verifies Lemma 2.10(1).

C.17. Proposition. (1) *With λ'_j given in C.16(1),*

$$c_1(\lambda) = \sum_{\lambda'} \dim V^C_{n-3}(\lambda'_2, \ldots, \lambda'_{n-2}).$$

(2) *For $i > 1$,*

$$c_i(\lambda) = \left(\sum_{\lambda'} \dim V^C_{n-3}(\lambda'_2, \ldots, \lambda'_{n-2}) \right) \cdot \left(\sum_{\nu} \dim V^A_1(\langle \nu, \varepsilon_1 - \varepsilon_{n-1} + \varepsilon_n \rangle) \right),$$

with λ'_j specified by C.16(2) and ν ranging over all the weights (counted with multiplicity) of $V^L(\tilde{\lambda})$.

The highest weight for each term appearing in (2) for A_1 is given by the parameter times the dual fundamental weight. We remark that using the Freudenthal's formula and the tricks used in the case of $\mathfrak{su}(n, 1)$, one could simplify the above formulae.

For $i = 1$, note that the triple in (C.13) are contained in the intersection $K_{\mathbb{C}}(f)_r \cap Q$. Thus only the factor of type C_{n-3} contributes in the multiplicity. That gives C.17(1). When $i > 1$, the base $K_{\mathbb{C}}(f)_r \cdot z'$ is given by the product of $L_{1,1}/(L_{1,1} \cap (L_b \cdot L_e))$ and $L_{1,2} \cdot L_{1,3}/(L_{1,2} \cdot L_{1,3}) \cap Q)$. The first part gives the terms for C_3, and the second part follows from the discussion preceding C.17.
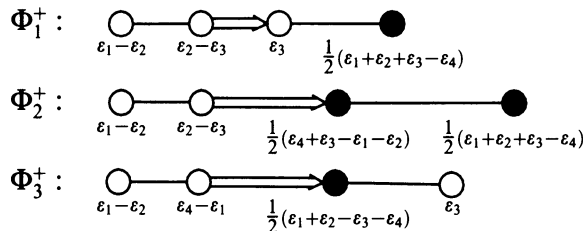
F. $\mathfrak{g}_0 = \mathfrak{f}_{4(-20)}$

We now consider the case when G is of type FII; in this case G is unique. Without giving an explicit realization, we fix a compact Cartan subalgebra $\mathfrak{h}_0$; therefore a maximal compact $\mathfrak{k}_0 \simeq \mathfrak{so}(9)$ is also fixed. Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ be an orthonormal basis, over $\mathbb{R}$, for $\sqrt{-1}\mathfrak{h}_0^*$ so that a positive root system Φ_1^+ can be chosen with simple roots given by

$$\Phi_1^+ \leftrightarrow \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3, \tfrac{1}{2}(\varepsilon_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)\}.$$

Moreover we may assume that this positive system satisfies the Borel–de Siebenthal property. Therefore the noncompact simple root must be either $\varepsilon_1 - \varepsilon_2$ or $\tfrac{1}{2}(\varepsilon_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$ (cf. p. 478 in [He]). Since there are exactly three $K_{\mathbb{C}}$-conjugation classes of positive root system for $\mathfrak{g}$ (i.e., three closed $K_{\mathbb{C}}$-orbits in X, cf. 2.2), one checks that the noncompact simple root must by $\tfrac{1}{2}(\varepsilon_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$ (otherwise by reflecting about $\varepsilon_1 - \varepsilon_2$, there will be too many

conjugacy classes). Reflecting about this noncompact simple root, the three closed $K_{\mathbb{C}}$-orbits are represented by the following colored Dynkin diagrams:

$$\Phi_1^+: \quad \underset{\varepsilon_1-\varepsilon_2}{\circ}\!\!-\!\!\underset{\varepsilon_2-\varepsilon_3}{\circ}\!\!\Rightarrow\!\!\underset{\varepsilon_3}{\circ}\qquad\underset{\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)}{\bullet}$$

$$\Phi_2^+: \quad \underset{\varepsilon_1-\varepsilon_2}{\circ}\!\!-\!\!\underset{\varepsilon_2-\varepsilon_3}{\circ}\!\!\Rightarrow\!\!\underset{\frac{1}{2}(\varepsilon_4+\varepsilon_3-\varepsilon_1-\varepsilon_2)}{\bullet}\qquad\underset{\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)}{\bullet}$$

$$\Phi_3^+: \quad \underset{\varepsilon_1-\varepsilon_2}{\circ}\!\!-\!\!\underset{\varepsilon_4-\varepsilon_1}{\circ}\!\!\Rightarrow\!\!\underset{\frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)}{\bullet}\qquad\underset{\varepsilon_3}{\circ}$$

The common $\Phi^+(\mathfrak{k},\mathfrak{h})=\Phi_c^+$ has simple roots $\{\varepsilon_4-\varepsilon_1,\varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_3,\ \varepsilon_3\}$. This determines a Weyl chamber C in $\sqrt{-1}\mathfrak{h}_0$:

(F.1) $\qquad\qquad \lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\in C \Leftrightarrow \lambda_4>\lambda_1>\lambda_2>\lambda_3.$

Also the set of noncompact roots is given by

$$\Phi_n=\{\tfrac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)\}.$$

As before $\Lambda'=\Lambda+\rho$.

F.2. Theorem. *Suppose that $\lambda\in\Lambda'\cap C$ is Φ_i^+-dominant, then $\mathrm{Ch}(\pi_\lambda)=c_i(\lambda)\cdot\overline{O_d}$ where*

(1) $c_1(\lambda)=\dim V_3^B(\tilde\lambda_1,\tilde\lambda_2,\tilde\lambda_3).$
(2) $c_i(\lambda)$ *is given in* F.10 *for $i=2,3$.*

In particular, all three family of discrete series have the same characteristic variety. This gives a counterexample to Conjecture 5.1 in [K].

For an easy reference, we label half of the noncompact roots as

$$\begin{aligned}
\beta_1&=\tfrac{1}{2}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4), & \alpha_1&=\tfrac{1}{2}(-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4),\\
\beta_2&=\tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4), & \alpha_2&=\tfrac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4),\\
\beta_3&=\tfrac{1}{2}(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4), & \gamma_1&=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4),\\
\beta_4&=\tfrac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4), & \gamma_2&=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4).
\end{aligned}$$

The span of the eigenspaces with roots above will be denoted by $\mathscr{S}_+$; also $\mathscr{S}_-=\tau(\mathscr{S}_+)$.

To exhibit the normal triple, note that $\beta_i\in\Phi_j^+$ for all $1\le i\le 4$ and $1\le j\le 3$. Consider

(F.3) $\qquad f=\sqrt{2}(X_{-\beta_2}+X_{-\beta_3}),\quad e=\tau(f),\quad h=[e,f]=2H_{\varepsilon_4}.$

By 2.8 again, $O_d=K_{\mathbb{C}}\cdot\xi_f$; this gives the first part of F.2. O_d being even, by (the proof of) Lemma 4 in [K-R], the moment map

(F.4) $\qquad\qquad K_{\mathbb{C}}\times_{Q_1}\mathscr{S}_-\to\overline{O_d}$ is birational $K_{\mathbb{C}}$-equivariant.

Note that in the case of Φ_1^+, $Q_1=K_{\mathbb{C}}\cap Q$ and F.2(1) is obtained. Associated to this triple,

(F.5) $\qquad\qquad \mathfrak{u}_1\leftrightarrow\{\varepsilon_4,\varepsilon_4\pm\varepsilon_i\ (1\le i\le 3)\}.$

Immediately one sees that $U_1^- \cdot f = f$. The reductive part $\mathfrak{l}_1$ has center $\mathbb{C}H_{\varepsilon_4}$ and

(F.6) $\mathfrak{l}_1' \leftrightarrow \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3\}$.

Now $\mathscr{S}_-$ is the radical of noncompact roots having a positive projection on H_{ε_4} (i.e., $\mathscr{S}_- = \mathfrak{p} \cap \mathfrak{o}_-$ in 2–3); therefore $L_1' \simeq \mathrm{Spin}(7, \mathbb{C})$ acts on $\mathscr{S}_-$ (also on $\mathscr{S}_+$). Examining the weights, this is just the spin representation. In particular, we may endow a structure of split octonians on $\mathscr{S}_-$, with an L_1'-invariant standard inner product; the corresponding norm will be denoted by $\| \ \|$.

F.7. Lemma. $L_1 \cdot f = O_d \cap \mathscr{S}_- = \{x \in \mathscr{S}_- : \|x\| \neq 0\}$ *and the isotropy subgroup at* f *is isomorphic to the exceptional group* G_2.

Proof. The first equality follows from (F.4). The second part follows from the structure of the spin representation of $\mathrm{Spin}(7, \mathbb{C}) : L_1'$ acts transitively on the unit sphere on $\mathscr{S}_-$ with isotropy subgroup G_2 (see Theorem 11.40 in [Ha]). $\square$

Before entering the proof of 2.5, we compute first $K_{\mathbb{C}}(f)_r \cap Q$. We will use some elementary facts about these exceptional Lie algebras. From the above lemma, the inclusion $K_{\mathbb{C}}(f)_r \subset L_1'$ is the same as the inclusion $G_2 \subset \mathrm{Spin}(7, \mathbb{C})$ by viewing G_2 as the automorphism group of the Cayley algebra $\mathscr{S}_-$ with the neutral element given by f. By the Triality Theorem (see Theorems 11.19, 11.40, and 11.42 in [Ha]), this inclusion can also be realized as G_2 acting on the seven-dimensional pure Cayley numbers in $\mathscr{S}_-$; and $\mathrm{Spin}(7, \mathbb{C})$ acts on which via the standard vector representation. In which context, an explicit realization of Cartan subalgebras and their roots are given in [R-S]; and we will use their results freely. To identify our situation to those in [R-S], note that

(F.8) $x \in \sqrt{-1}\mathfrak{h}_0 \cap \mathfrak{k}(f) \Leftrightarrow -x_1 + x_2 + x_3 = x_4 = 0$.

Here the coordinates on $\mathfrak{h}_0$ are dual to the orthonormal basis $\{\varepsilon_j\}_{j=1}^4$, i.e., $x_i = \varepsilon_i(x)$. Identifying t_1, t_2, t_3 in §1 of [R-S] with $\varepsilon_2, \varepsilon_3, -\varepsilon_1$ above respectively, we may then use their descriptions.

F.9. Proposition. *The simple roots for* $K_{\mathbb{C}}(f)_r \simeq G_2$ *are given by* $\varepsilon_2 - \varepsilon_3$ *and* ε_3 *with the Cartan subalgebra described in* (F.8). *Moreover* $K_{\mathbb{C}}(f)_r \cap Q$ *is the maximal parabolic subgroups in* $K_{\mathbb{C}}(f)_r \simeq G_2$ *corresponding to (i.e., containing the eigenspaces of roots in) the sets* $\{\varepsilon_2 - \varepsilon_3\}$ *for* Φ_2^+, $\{\varepsilon_3\}$ *for* Φ_3^+.

Proof. By formulas on p. 10 of [R-S], the root vectors for G_2 are

$$Y_{\pm(\varepsilon_2-\varepsilon_3)} = X_{\pm(\varepsilon_2-\varepsilon_3)}, \qquad Y_{\pm\varepsilon_3} = X_{\pm(\varepsilon_1-\varepsilon_2)} + X_{\pm\varepsilon_3} .$$

Recalling the definitions of Φ_2^+ and Φ_3^+, in view of Proposition 1.4, the proposition follows. $\square$

We now proceed to the proof of 2.5. The representatives for the double coset space in 2.6 can be chosen from

$$\Phi_2^+ : \mathrm{id}, \sigma_{\varepsilon_4}, (\varepsilon_4, \varepsilon_3), \sigma_{\varepsilon_4}(\varepsilon_4, \varepsilon_3);$$
$$\Phi_3^+ : \mathrm{id}, \sigma_{\varepsilon_4}, (\varepsilon_4, \varepsilon_3).$$

To examine the condition in (2.7), note that its right-hand sides are

$$(\Phi_2^+) \quad w \cdot \mathscr{S}_- \cap \mathfrak{n}^- \cap \mathfrak{p} \leftrightarrow \begin{cases} \{-\alpha_2\}, & \text{if } w = \sigma_{\varepsilon_4}, \\ \{-\beta_3, -\beta_4, \alpha_2, -\gamma_2\}, & \text{if } w = (\varepsilon_4, \varepsilon_3), \\ \{-\beta_1, -\beta_2, -\gamma_2\}, & \text{if } w = \sigma_{\varepsilon_4}(\varepsilon_4, \varepsilon_3), \end{cases}$$

$$(\Phi_3^+) \quad w \cdot \mathscr{S}_- \cap \mathfrak{n}^- \cap \mathfrak{p} \leftrightarrow \begin{cases} \{\alpha_1, \alpha_2\}, & \text{if } w = \sigma_{\varepsilon_4}, \\ \{-\beta_3, -\beta_4, \alpha_2, -\gamma_2\}, & \text{if } w = (\varepsilon_4, \varepsilon_3). \end{cases}$$

Notice that the L_1'-invariant product on $\mathscr{S}_-$ pairs the root-vectors with opposite weights (for the algebra $\mathfrak{l}_1'$). One checks immediately that $w \cdot \mathscr{S}_- \cap \mathfrak{n}^- \cap \mathfrak{p}$ above are all isotropic with respect to the invariant inner product. Therefore by Lemma F.7, the condition in (2.7) holds only if $w = \mathrm{id}$. This plus the analysis in F.7 and F.9 proves Theorem 2.5 in this case.

To compute the multiplicity, by a direct computation,

$$\begin{aligned} \tilde{\lambda}_1 &= \lambda_1 - 2, & \tilde{\lambda}_1 &= \lambda_1 - 3/2, \\ \tilde{\lambda}_2 &= \lambda_2 - 1, & \tilde{\lambda}_2 &= \lambda_2 - 1/2, \\ \tilde{\lambda}_3 &= \lambda_3, & \tilde{\lambda}_3 &= \lambda_3 - 1/2, \\ \tilde{\lambda}_4 &= \lambda_4 - 2, & \tilde{\lambda}_4 &= \lambda_4 - 5/2. \end{aligned}$$

The left-hand side is for Φ_2^+; the other for Φ_3^+. In view of (F.8), it is convenient to express weights for $K_{\mathbb{C}}(f)_r \simeq G_2$ by $\mu = (\mu_1, \mu_2, \mu_3) = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3$ (restricted to the plane $-x_1 + x_2 + x_3 = 0$). Also (μ_1, μ_2, μ_3) is equivalent to $(\mu_1 - a, \mu_2 + a, \mu_3 + a)$ for any constant a, and it is dominant iff $\mu_2 \geq \mu_3 \geq 0$.

F.10. Proposition.

(1) $c_2(\lambda) = \sum_{\tilde{\lambda}_1 \geq \lambda_2' \geq \tilde{\lambda}_2 \geq \lambda_3' \geq \tilde{\lambda}_3} \dim V_2^G((\lambda_2' + \lambda_3'), \lambda_2', \lambda_3')$.

(2) $c_3(\lambda) = \sum_{\tilde{\lambda}_4 \geq \lambda_1' \geq \tilde{\lambda}_1 \geq \lambda_2' \geq \tilde{\lambda}_2} \dim V_2^G(\lambda_1', \lambda_2', \lambda_3)$.

Proof. For Φ_2^+, modulo the center, $K_{\mathbb{C}} \cap Q$ is of type A_2 with simple roots $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_2 - \varepsilon_3$; and the intersection $K_{\mathbb{C}}(f)_r \cap L$ is of type A_1 determined by the simple root $\varepsilon_2 - \varepsilon_3$. The representation occurring in the cohomology in (2.9) is $V_2^A(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$. The branching rule from A_2 to A_1 gives rise to all the parameters appearing in the summand above. One checks immediately that they are dominant in G_2 from (F.1) and the form for $\tilde{\lambda}$ above; so 2.10(1) is verified. This gives the first part.

As for Φ_3^+, $K_{\mathbb{C}} \cap Q$ is of type $A_2 \times A_1$ with simple roots $\{\varepsilon_4 - \varepsilon_1, \varepsilon_1 - \varepsilon_2\}$, and $\{\varepsilon_3\}$ modulo the center. Notice that the intersection $K_{\mathbb{C}}(f)_r \cap L$ in this case, modulo the center, is generated by the vectors $X_{(\varepsilon_1 - \varepsilon_2)} + X_{\varepsilon_3}$ and $X_{-(\varepsilon_1 - \varepsilon_2)} + X_{-\varepsilon_3}$; with the corresponding element $H_{\varepsilon_1 - \varepsilon_2} + H_{\varepsilon_3}$ in $\mathfrak{h}$. The representation of the cohomology in (2.9) is $V_{2,1}^{A \times A}(\tilde{\lambda}_4, \tilde{\lambda}_1, \tilde{\lambda}_2; \tilde{\lambda}_3)$. The branching rule from A_2 to A_1 (corresponding to $\varepsilon_1 - \varepsilon_2$) gives rise to the parameters in the summand above. Again they are dominant in G_2; therefore 2.10(1) and the second part are verified. $\square$

REFERENCES

[B]   D. Barbasch, *Fourier inversions for unipotent orbital integrals*, Trans. Amer. Math. Soc. **249** (1979), 51–83.

[B-B]   A. Beilinson and J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Ser. I Math. **292** (1981), 15–18.

[Bo]   H. Boerner, *Representations of groups*, North-Holland, Amsterdam, 1970.

[C]   J.-T. Chang, *Characteristic cycles of holomorphic discrete series*, Trans. Amer. Math. Soc. **334** (1992), 213–227.

[Ha]   F. Reese Harvey, *Spinors*, Rice Univ., 1987.

[He]   S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.

[K.]   D. King, *Character polynomials of discrete series representations*, Non Commutative Harmonic Analysis and Lie Groups, Lecture Notes in Math., vol. 880, Springer-Verlag, New York, 1981.

[K-R]   B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[R-S]   S. Rallis and G. Schiffman, *Theta correspondence associated to G_2*, Amer. J. Math. (to appear).

[S]   J.-P. Serre, *Algèbres de Lie semi-simples complexes*, Benjamin, New York, 1966.

[V]   D. Vogan, Jr., *Associated varieties and unipotent representations,* preprint, 1990.

[Z]   D. P. Želobenko, *Compact Lie groups and their representations*, Transl. Math. Monographs, vol. 40, Amer. Math. Soc., Providence, RI, 1973.

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078-0613

E-mail address: `changj@math.okstate.edu`