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HARMONIC DIFFEOMORPHISMS OF THE HYPERBOLIC PLANE

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ABSTRACT. In this paper, we consider the Dirichlet problem at infinity for harmonic maps between the Poincaré model D of the hyperbolic plane $\mathbb{H}^2$, and solve this when given boundary data are C^4 immersions of $D(\infty)$, the boundary at infinity of D, to $D(\infty)$. Also, we present a construction of nonconformal harmonic diffeomorphisms of D, and give a complete description of the boundary behavior, including their first derivatives.

1. INTRODUCTION

Not much is known about nonconformal harmonic diffeomorphisms of D, the Poincaré model of the hyperbolic plane $\mathbb{H}^2$ of constant negative curvature. A class of these harmonic maps can be obtained by lifting to the universal covers the harmonic diffeomorphisms between closed Riemannian 2-manifolds of constant negative curvature with different conformal structures [13, 18]. Another class is given by the Gauss maps of entire spacelike nonumbilical constant mean curvature surfaces in the Minkowski 3-space $\mathbb{L}^3$, provided these surfaces are conformally equivalent to the unit disk and their lightlike sets coincide with the unit circle [8, 20]. In this respect, Choi and Treibergs [7] constructed an explicit one-parameter family of nonconformal harmonic diffeomorphisms of D, each of which is realized as the Gauss map of an entire spacelike constant mean curvature surface of revolution in $\mathbb{L}^3$, and describes completely their boundary behavior (see also Li and Tam [16]).

Motivated by these results we shall consider a Dirichlet problem, the *asymptotic Dirichlet problem*, for harmonic maps between D with C^0 boundary data from $D(\infty)$, the boundary at infinity of D, to $D(\infty)$. In this problem when constructing harmonic maps between D with unbounded image, a main technical difficulty arises in obtaining a priori growth estimates for them. However, we will prove in §3 that this difficulty can be resolved under a suitable assumption on the boundary data, and then this asymptotic Dirichlet problem can be solved. In our construction of harmonic maps between D, we first construct a suitable barrier map at the boundary $D(\infty)$ for each boundary map which is assumed to be a C^4 immersion between $D(\infty)$. Then a priori growth estimates will be established in Lemma 5 with the aid of their barrier maps. In the course of proof we also present a construction of nonconformal harmonic diffeomorphisms of D, and give a complete description of the boundary behavior,

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including their first derivatives. We would like to point out that using the heat flow method, a general theory for the existence of harmonic maps with bounded energy density between complete noncompact manifolds was developed by Li and Tam [15]. Applying this result to the Poincaré model D^m of the hyperbolic m-space $\mathbb{H}^m$, they also solved independently the asymptotic Dirichlet problem from D^m to D^n (for all $m, n \geq 2$) when given boundary data are C^3 maps of $D^m(\infty)$ to $D^n(\infty)$ with nonvanishing energy density. Recently, they [16] also obtained two fundamental results for uniqueness and regularity properties of proper harmonic maps from D^m to D^n. In particular, the above smoothness assumption on the boundary data can be relaxed from C^3 to $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ (for further developments see [21]).

As an application of our method, we shall construct, in §5, entire spacelike constant mean curvature surfaces M in $\mathbb{L}^3$, whose Gauss maps are harmonic diffeomorphisms of M to $\mathbb{H}^2$ with C^4 diffeomorphisms $\varphi: S^1$ $(\simeq M(\infty)) \to S^1$ $(\simeq \mathbb{H}^2(\infty))$ as prescribed boundary data. This is a converse of the result proved in Choi and Treibergs [7, 8].

Finally, we would like to mention some other related results for harmonic maps. Avilés, Choi, and Micallef [6] and the author [2] proved independently that the asymptotic Dirichlet problem from a strictly negatively curved Hadamard manifold M to a Hadamard manifold N can be solved uniquely for each C^0 boundary data from $M(\infty)$, the boundary at infinity of M, into N. On the other hand, nonexistence results for harmonic maps from Euclidean m-space $\mathbb{R}^m$ to a Hadamard manifold of negative sectional curvature bounded away from zero can be seen in Tachikawa and the author [4, 19].

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2. Preliminaries and statement of main results

Let M be a Hadamard n-manifold. The *boundary at infinity* or the *Eberlein-O'Neill boundary* $M(\infty)$ of M is defined to be the set of all asymptotic classes of geodesic rays in M; two rays γ_1 and γ_2 are *asymptotic* if $\mathrm{dist}_M(\gamma_1(t), \gamma_2(t))$ is bounded for all $t \geq 0$. One can define on $\overline{M} := M \cup M(\infty)$ a natural topology, the *cone topology* of $\overline{M}$, with respect to which the triple $(\overline{M}, M, M(\infty))$ is identified topologically with the Euclidean triple $(\overline{B}^n, B^n, S^{n-1})$, where $B^n = \{x \in \mathbb{R}^{n+1}; |x| < 1\}$ and $S^{n-1} = \partial B^n$ (see [9] for details). Note that there exists no natural (i.e., independent of each pole $o \in M$) smooth structure on $M(\infty)$ except when $M = \mathbb{H}^n$ (see [5]).

The *Poincaré model* D of the hyperbolic plane $\mathbb{H}^2$ is by definition the unit disk B in the complex plane $\mathbb{C}$ equipped with the metric

$$ds_D^2 = \frac{4\,dz\,d\overline{z}}{(1-|z|^2)^2} = \frac{4((dx^1)^2 + (dx^2)^2)}{(1-(x^1)^2-(x^2)^2)^2}, \tag{2.1}$$

where $z = x^1 + ix^2$ is the canonical global coordinate of $\mathbb{C}$. Throughout this paper, we always denote by $z = x^1 + ix^2$ the global coordinate on D as well as $\overline{B}$ (the closure of B in $\mathbb{C}$), and regard D as a Riemann surface by z. Also, we often regard $\overline{B}$ as a Riemannian 2-manifold with boundary equipped with flat Euclidean metric. It is easy to see that D is a Hadamard 2-manifold of constant negative curvature -1. The cone topology is defined also on $\overline{D} := D \cup D(\infty)$, and the triple $(\overline{D}, D, D(\infty))$ is identified topologically with the Euclidean triple $(\overline{B}, B, S^1)$. We may regard $D(\infty)$ as a C^∞ Riemannian 1-manifold by identifying $D(\infty) \simeq S^1$.

Let φ be a continuous map of S^1 to itself. We say that φ is *conformal* if φ is the restriction $f|_{S^1}$ of a holomorphic or antiholomorphic map f of $\overline{B}$ onto itself. By $\mathscr{D}(D), \mathscr{H}(\overline{D}), \mathscr{D}(D, B)$ and $\mathscr{H}(\overline{B}, \overline{D})$ we denote respectively the set of all diffeomorphisms of D, the set of all homeomorphisms of $\overline{D}$, the set of all diffeomorphisms of B onto D, and the set of all homeomorphisms of $\overline{B}$ onto $\overline{D}$.

Let $\Sigma_1 = (\Sigma_1, g)$ and $\Sigma_2 = (\Sigma_2, h)$ be oriented Riemannian 2-manifolds. Let (y^1, y^2) and (u^1, u^2) denote isothermal coordinates on Σ_1 and Σ_2 respectively, by which g and h are expressed locally as

$$g = \sigma(w)^2((dy^1)^2 + (dy^2)^2) = \sigma^2\, dw\, d\overline{w}, \qquad \sigma > 0,$$
$$h = \rho(u)^2((du^1)^2 + (du^2)^2) = \rho^2\, du\, d\overline{u}, \qquad \rho > 0,$$

where $w = y^1 + iy^2$ and $u = u^1 + iu^2$. A C^2 map $u \colon \Sigma_1 \to \Sigma_2$ is a *harmonic map* if $u(w)$, the representation of u in these coordinates, satisfies the following system of equations:

$$\tau(u)^j := \sigma^{-2} \sum_{\alpha=1}^{2} \left[\frac{\partial^2 u^j}{(\partial y^\alpha)^2} + \Gamma^j_{kl}(u) \frac{\partial u^k}{\partial y^\alpha} \frac{\partial u^l}{\partial y^\alpha} \right] = 0 \quad \text{for } j = 1, 2, \tag{2.2}$$

where Γ^j_{kl} denote the Christoffel symbols of the Levi-Civita connection of h. The system (2.2) is equivalent to the equation

$$\sigma^{-2} \left(\frac{\partial^2 u}{\partial w \partial \overline{w}} + \frac{2}{\rho} \frac{\partial \rho}{\partial u} \frac{\partial u}{\partial w} \frac{\partial u}{\partial \overline{w}} \right) = 0, \tag{2.3}$$

where
$$\frac{\partial u}{\partial w} = \frac{1}{2} \left(\frac{\partial u}{\partial y^1} - i \frac{\partial u}{\partial y^2} \right) \quad \text{and} \quad \frac{\partial u}{\partial \overline{w}} = \frac{1}{2} \left(\frac{\partial u}{\partial y^1} + i \frac{\partial u}{\partial y^2} \right).$$

It is immediate from (2.3) that the harmonicity of u does not depend on the choice of compatible metric of Σ_1, but depends only on its conformal structure, and u is harmonic if u is holomorphic or antiholomorphic.

Finally, we note that it follows from (2.1) and (2.3) that a C^2 map $u \colon D \to D$ is harmonic if and only if u satisfies the equation

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} + \frac{2\overline{u}}{1 - |u|^2} \frac{\partial u}{\partial z} \frac{\partial u}{\partial \overline{z}} = 0. \tag{2.4}$$

With these understood, our main results can be stated as follows.

Theorem 1. *Let $\varphi \colon D(\infty) \to D(\infty)$ be a C^4 immersion. Then there exists a harmonic map $u \in C^\infty(D, D) \cap C^0(\overline{D}, \overline{D})$ such that $u|_{D(\infty)} = \varphi$. Moreover u is holomorphic or antiholomorphic if and only if φ is conformal.*

Theorem 2. *Let φ be a C^4 diffeomorphism of $D(\infty)$. Then there exists a harmonic diffeomorphism $u \in \mathscr{D}(D) \cap \mathscr{H}(\overline{D})$ such that $u|_{D(\infty)} = \varphi$. Moreover u is conformal if and only if φ is conformal.*

Remark 1. If we choose, in Theorem 2, a nonconformal φ in particular, then we can obtain a nonconformal harmonic diffeomorphism u of D.

3. Proofs of Theorems 1 and 2

We first note the following lemma, which is an easy consequence of (2.3).

Lemma 1 (cf. [13]). *Let Σ_1, Σ_2, and Σ_3 be oriented Riemannian 2-manifolds. Suppose that $u: \Sigma_2 \to \Sigma_3$ is a harmonic map. If $v: \Sigma_1 \to \Sigma_2$ is a conformal map, then the composition $u \circ v: \Sigma_1 \to \Sigma_3$ is also a harmonic map.*

On account of Lemma 1, we can reduce the proofs of Theorems 1 and 2 to the following.

Theorem 1′. *Let $\varphi: S^1 \to D(\infty)$ be a C^4 immersion. Then there exists a harmonic map $u \in C^\infty(B, D) \cap C^0(\overline{B}, \overline{D})$ such that $u|_{S^1} = \varphi$. Moreover u is holomorphic or antiholomorphic if and only if φ is conformal.*

Theorem 2′. *Let φ be a C^4 diffeomorphism of S^1 onto $D(\infty)$. Then there exists a harmonic diffeomorphism $u \in \mathscr{D}(B, D) \cap \mathscr{H}(\overline{B}, \overline{D})$ such that $u|_{S^1} = \varphi$. Moreover u is conformal if and only if φ is conformal.*

Let $\varphi: S^1 \to S^1$ $(\simeq D(\infty))$ be the C^4 immersion in Theorem 1′. For each ρ $(0 < \rho < 1)$ let $u_\rho: B \to B_\rho := \{z \in D;\ \mathrm{dist}_D(0, z) < \log[(1+\rho)/(1-\rho)]\}$ be a harmonic map with $u_\rho|_{S^1} = \rho \cdot \varphi$ (cf. [13]). Associated with φ, we will first construct a *barrier map* (or an approximation) $\Phi: \overline{B} \to \overline{B}$ $(\simeq \overline{D})$ *at the boundary* S^1; Φ needs to satisfy the boundary conditions $\Phi|_{S^1} = \varphi$ and the inequality (3.11) in Lemma 4 below. Then from these properties and a priori gradient estimates for harmonic maps in [14], we will prove that a subsequence $\{u_{\rho_j}\}_{j\in\mathbb{N}}$ of $\{u_\rho\}_{0<\rho<1}$ converges to a harmonic map $u: B \to D$ such that $u|_{S^1} = \varphi$.

Now, we shall prove Theorem 1′ by a series of lemmas. Note that, without loss of generality, we may assume $\deg(\varphi) > 0$ throughout this section. Let $\Theta: \mathbb{R} \to \mathbb{R}$ denote a C^4 diffeomorphism which satisfies $\varphi(e^{i\theta}) = e^{i\Theta(\theta)}$.

Lemma 2. *Let $\mathscr{V}_1$ be a neighborhood of S^1 in $\mathbb{C}$. Define a C^2 map $\tilde{\varphi} = \tilde{\varphi}^1 + i\tilde{\varphi}^2: \mathscr{V}_1 \to \mathbb{C}$ by*

$$\tilde{\varphi}(z) = e^{i\Theta} + e^{i\Theta} \cdot \Theta' \cdot (r-1) + \tfrac{1}{2} e^{i\Theta}[-\Theta' + (\Theta')^2 - i\Theta''](r-1)^2 \tag{3.1}$$

for all $z = re^{i\theta} \in \mathscr{V}_1$, where prime denotes differentiation with respect to θ. Then the following hold.

$$\tilde{\varphi} = \varphi \quad on\ S^1, \tag{3.2}$$

$$1 - |\tilde{\varphi}|^2 = (1-r) \cdot \Theta' + (1-r)^2 f_0 \quad on\ \mathscr{V}_1, \tag{3.3}$$

$$\left|\frac{\partial \tilde{\varphi}}{\partial \overline{z}}\right| = (1-r)^2 f_1 \quad on\ \mathscr{V}_1, \tag{3.4}$$

$$\left|\frac{\partial^2 \tilde{\varphi}}{\partial z \partial \overline{z}}\right| = (1-r) f_2 \quad on\ \mathscr{V}_1, \tag{3.5}$$

where f_j $(0 \le j \le 2)$ *are bounded* C^0 *functions (depending only on the boundary data* φ*) on* $\mathscr{V}_1$. *Moreover, on a neighborhood* $\mathscr{V}_2$ $(\subset \mathscr{V}_1)$ *of* S^1, $\tilde{\varphi}|_{\mathscr{V}_2}: \mathscr{V}_2 \to \mathbb{C}$ *is a* C^2 *immersion.*

Proof. (3.2) and (3.3) follow directly from (3.1). Remark that Θ' has a positive infimum, since $\varphi: S^1 \to S^1$ is a C^4 immersion. Using the formulas

$$\frac{\partial}{\partial z} = \frac{1}{2}e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right),$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right),$$

$$\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\left[\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial \theta}\right)^2\right],$$

we have on S^1

$$\left|\frac{\partial \tilde{\varphi}}{\partial \overline{z}}\right| = 0, \quad \left|\frac{\partial}{\partial r}\left(\frac{\partial \tilde{\varphi}}{\partial \overline{z}}\right)\right| = 0, \quad \left|\frac{\partial^2 \tilde{\varphi}}{\partial z \partial \overline{z}}\right| = 0, \quad J(\tilde{\varphi}) = (\Theta')^2,$$

from which (3.4), (3.5) and the last assertion are immediate, where $J(\tilde{\varphi})$ stands for the Jacobian $|\partial\tilde{\varphi}/\partial z|^2 - |\partial\tilde{\varphi}/\partial\overline{z}|^2$ of $\tilde{\varphi}$.

For ρ $(0 < \rho < 1)$ let B_ρ denote either $\{z \in B; |z| < \rho\}$ or $\{z \in D;\ \mathrm{dist}_D(0, z) < \log[(1+\rho)/(1-\rho)]\}$. From Lemma 2, we see that there exist constants ρ_1, ρ_2 $(0 < \rho_1, \rho_2 < 1)$ and a C^2 map $\Phi := \Phi^1 + i\Phi^2: \overline{B} \to \overline{B}$ such that

$$\overline{B} \setminus B_{\rho_1} \subset \mathscr{V}_2, \quad \Phi|_{\overline{B} \setminus B_{\rho_1}} = \tilde{\varphi}, \quad \Phi(\overline{B}_{\rho_1}) \subset \overline{B}_{\rho_2}. \tag{3.6}$$

Let Φ_ρ denote the C^2 map $\rho \cdot \Phi: \overline{B} \to \overline{B}_\rho$, and φ_ρ, the C^4 immersion $\rho \cdot \varphi: S^1 \to \partial \overline{B}_\rho$. By the existence and uniqueness theorem of harmonic maps [10, 11, 12], there exists a unique harmonic map $u_\rho \in C^\infty(B, B_\rho) \cap C^{3,\alpha}(\overline{B}, \overline{B}_\rho)$ (for any $\alpha \in (0, 1)$) such that $u_\rho|_{S^1} = \varphi_\rho$. Let $Z_\rho = \{z \in \overline{B}; u_\rho(z) = \Phi_\rho(z)\}$, the coincidence set of u_ρ and Φ_ρ. We shall next prove a priori growth estimates of u_ρ.

Lemma 3. *For each* ρ $(0 < \rho < 1)$

$$\Delta\{\mathrm{dist}_D(u_\rho(z), \Phi_\rho(z))\} \ge -C_1 \tag{3.7}$$

for all $z \in B \setminus Z_\rho$, *where* Δ *stands for the standard Laplacian* $4(\partial^2/\partial z\partial\overline{z})$ *on* $\mathbb{C}$ $(\simeq \mathbb{R}^2)$ *and* C_1 *is a positive constant depending only on the boundary data* φ.

Proof. Define $\Lambda: D \times D \to \mathbb{R}$ by $\Lambda(z_1, z_2) = \mathrm{dist}_D(z_1, z_2)$ for $z_1, z_2 \in D$. Using chain rule, we compute on $B \setminus Z_\rho$

$$\begin{aligned}
&\Delta[\mathrm{dist}_D(u_\rho(z), \Phi_\rho(z))] \\
&\quad = \sum_{\alpha=1}^{2}(\mathrm{Hess}(\Lambda))\left((u_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right) \oplus (\Phi_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right),\right. \\
&\qquad\qquad \left.(u_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right) \oplus (\Phi_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right)\right) \\
&\qquad + (d\Lambda)(\tau(u_\rho) \oplus \tau(\Phi_\rho)).
\end{aligned}$$

It is not hard to see by a similar argument in the proof of Lemma 3 in [12], that on $B \setminus Z_\rho$

$$(\mathrm{Hess}(\Lambda))\left((u_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right) \oplus (\Phi_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right), (u_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right) \oplus (\Phi_\rho)_*\left(\frac{\partial}{\partial x^\alpha}\right)\right) \geq 0$$

for $\alpha = 1, 2$. Since u_ρ is harmonic (i.e., $\tau(u_\rho) = 0$), we then have for $z \in B \setminus Z_\rho$

$$\begin{aligned}
&\Delta[\mathrm{dist}_D(u_\rho(z), \Phi_\rho(z))] \\
&\quad \geq ((d\Lambda)(u_\rho(z), \Phi_\rho(z)))\left(0 \oplus \sum_{j=1}^{2} \tau(\Phi_\rho)^j \frac{\partial}{\partial x^j}\right) \\
&\quad \geq -\left|\sum_{j=1}^{2} \tau(\Phi_\rho)^j \frac{\partial}{\partial x^j}\right|_D \geq -\frac{2}{1-|\Phi_\rho|^2}|\tau(\Phi_\rho)|,
\end{aligned}$$

where $|\cdot|_D$ is the norm with respect to ds_D^2 and $|\cdot|$ the Euclidean norm.

$\tau(\Phi_\rho)$ is given explicitly as

$$\begin{aligned}
|\tau(\Phi_\rho)| &= \left|\frac{\partial^2\Phi_\rho}{\partial z \partial \overline{z}} + \frac{2\overline{\Phi}_\rho}{1-|\Phi_\rho|^2}\frac{\partial \Phi_\rho}{\partial z}\frac{\partial \Phi_\rho}{\partial \overline{z}}\right| \\
&\leq \left|\frac{\partial^2\Phi_\rho}{\partial z \partial \overline{z}}\right| + \frac{2}{1-|\Phi_\rho|^2}|\overline{\Phi}_\rho| \cdot \left|\frac{\partial \Phi_\rho}{\partial z}\right| \cdot \left|\frac{\partial \Phi_\rho}{\partial \overline{z}}\right| \\
&\leq \rho\left|\frac{\partial^2\Phi}{\partial z \partial \overline{z}}\right| + \frac{2\rho^3}{1-\rho^2|\Phi|^2}|\overline{\Phi}| \cdot \left|\frac{\partial \Phi}{\partial z}\right| \cdot \left|\frac{\partial \Phi}{\partial \overline{z}}\right|.
\end{aligned}$$

From (3.4) and (3.5) we then obtain

$$|\tau(\Phi_\rho)| \leq C\left[(1-r) + \frac{(1-r)^2}{1-|\Phi|^2}\right]$$

and hence

$$\frac{2}{1-|\Phi_\rho|^2}|\tau(\Phi_\rho)| \leq C\left[\frac{1-r}{1-|\Phi|^2} + \left(\frac{1-r}{1-|\Phi|^2}\right)^2\right]. \tag{3.8}$$

From (3.3) we also obtain

$$1-|\Phi|^2 = (1-r)f_3, \tag{3.9}$$

f_3 being a bounded positive C^0 function on $\overline{B}$. It now follows from (3.8) and (3.9) that on $\overline{B}$

$$\frac{2}{1-|\Phi_\rho|^2}|\tau(\Phi_\rho)| \leq C_2, \tag{3.10}$$

where C_2 is a constant depending only on φ. Hence we obtain the estimate (3.7).

Lemma 4. *For each* ρ $(0 < \rho < 1)$

$$\mathrm{dist}_D(u_\rho(z), \Phi_\rho(z)) \leq \tfrac{1}{4}C_1 \quad \textit{for all } z \in \overline{B}. \tag{3.11}$$

Proof. From (3.7) we have

$$\Delta[\mathrm{dist}_D(u_\rho(z), \Phi_\rho(z)) + \tfrac{1}{4}C_1|z|^2] \geq 0 \quad \text{for all } z \in B\backslash Z_\rho.$$

Noting that $\Phi_\rho = \varphi_\rho = u_\rho$ on S^1, it follows from definition of Z_ρ and the maximum principle that

$$\mathrm{dist}_D(u_\rho(z), \Phi_\rho(z)) + \tfrac{1}{4}C_1|z|^2 \leq \tfrac{1}{4}C_1 \quad \text{for all } z \in \overline{B},$$

from which (3.11) follows.

Lemma 5. *Represent* Φ *and* u_ρ *as* $\Phi(z) = L(z)e^{i\cdot\arg(\Phi(z))}$ *and* $u_\rho(z) = R_\rho(z)e^{i\cdot\arg(u_\rho(z))}$ *for* $z \in \overline{B}$, *where* $L(z) = |\Phi(z)|$ *and* $R_\rho(z) = |u_\rho(z)|$. *Then for all* $z \in \overline{B}$ *and* ρ $(0 < \rho < 1)$ *we have*

$$e^{-C_1/4}(1 - \rho \cdot L(z)) \leq 1 - R_\rho(z) \leq e^{C_1/4}(1 - \rho \cdot L(z)), \tag{3.12}$$

(3.13)

$$\cos[\arg(u_\rho(z)) - \arg(\Phi(z))] \geq \left[\cosh^2\left(\log\frac{1+K_\rho(z)}{1-K_\rho(z)}\right) - \cosh\left(\frac{1}{4}C_1\right)\right] \Big/ \sinh^2\left(\log\frac{1+K_\rho(z)}{1-K_\rho(z)}\right),$$

where $K_\rho(z) = \min\{\rho \cdot L(z), R_\rho(z)\}$.

Proof. It follows from (3.11) that

$$\begin{aligned}\tfrac{1}{4}C_1 &\geq \mathrm{dist}_D(u_\rho, \Phi_\rho) \geq |\mathrm{dist}_D(0, u_\rho) - \mathrm{dist}_D(0, \Phi_\rho)| \\ &\geq \left|\log\frac{1+R_\rho}{1-R_\rho} - \log\frac{1+\rho\cdot L}{1-\rho\cdot L}\right|,\end{aligned}$$

which implies (3.12). On the other hand, by the law of cosines of geodesic triangles in D we get

$$\begin{aligned}&\cos[\arg(u_\rho) - \arg(\Phi)] \\ &\quad = [\cosh(\mathrm{dist}_D(0, u_\rho))\cosh(\mathrm{dist}_D(0, \Phi_\rho)) - \cosh(\mathrm{dist}_D(u_\rho, \Phi_\rho))] \\ &\qquad \times [\sinh(\mathrm{dist}_D(0, u_\rho))\sinh(\mathrm{dist}_D(0, \Phi_\rho))]^{-1} \\ &\quad \geq \left[\cosh^2\left(\log\frac{1+K_\rho}{1-K_\rho}\right) - \cosh(\mathrm{dist}_D(u_\rho, \Phi_\rho))\right] \Big/ \sinh^2\left(\log\frac{1+K_\rho}{1-K_\rho}\right),\end{aligned}$$

which, together with (3.11), then implies (3.13).

Lemma 6. *For each* k $(0 < k < 1)$, *there exists a constant* $l = l(k, \varphi)$ $(0 < l < 1)$ *such that*

$$u_\rho(\overline{B}_k) \subset \overline{B}_l \quad \textit{for all } \rho\ (0 < \rho < 1). \tag{3.14}$$

Proof. First it follows from (3.3) and (3.6) that there exists a constant l_0 $(0 < l_0 < 1)$ such that

$$\Phi_\rho(\overline{B}_k) \subset \Phi(\overline{B}_k) \subset \overline{B}_{l_0} \quad \text{for all } \rho. \tag{3.15}$$

Using (3.12) in (3.15) then yields

$$u_\rho(\overline{B}_k) \subset \overline{B}_l \quad \text{for all } \rho,$$

where $l = 1 - e^{-C_1/4}(1 - l_0)$.

For each k $(0 < k < 1)$, from (3.14) we have

$$u_\rho(\overline{B}_{(1+k)/2}) \subset \overline{B}_{l_1} \quad \text{for all } \rho\ (0 < \rho < 1), \tag{3.16}$$

where $l_1 = l_1(k, \varphi)$ $(0 < l_1 < 1)$. Then it follows from a priori gradient estimates for harmonic maps in [13, 14] that for all ρ $(0<\rho<1)$

$$\sup_{z\in B_k} \|du_\rho(z)\| \le C \sup_{z_1\in B_k} \frac{\mathrm{dist}_D(u_\rho(z_1), u_\rho(z_2))}{(1-k)/2}, \qquad z_2 \in B_{(1-k)/2}(z_1),$$

where $C = C(k, l_1)$ and $B_{(1-k)/2}(z_1) = \{z \in B; |z - z_1| < \frac{1}{2}(1-k)\}$. In consequence, it follows from (3.16) that

$$\sup_{z\in B_k} \|du_\rho(z)\| \le C\left(4\log\frac{1+l_1}{1-l_1}\right)(1-k)^{-1} \tag{3.17}$$

for all ρ $(0<\rho<1)$.

Let $\|u_\rho\|_{C^{2,\alpha}(B_k)}$ denote the $C^{2,\alpha}$-Hölder norm of $u_\rho \in C^\infty(B, D)$ in B_k. Then it also follows from (3.17) and a priori $C^{2,\alpha}$-estimates in [13, 14] that for each α $(0<\alpha<1)$

$$\|u_\rho\|_{C^{2,\alpha}(B_k)} \le C_3 \tag{3.18}$$

for all ρ $(0<\rho<1)$, where $C_3 = C_3(\alpha, k, l_1)$.

Once these are established, it follows from (3.14), (3.18) and the Ascoli-Arzelá theorem that there exist a subsequence $\{u_{\rho_j}\}_{j\in\mathbb{N}}$ of $\{u_\rho\}_{0<\rho<1}$ and a harmonic map $u \in C^2(B, D)$ such that on every compact subset of B, u_{ρ_j} converges to u in the C^2 topology. The harmonicity of u then implies $u \in C^\infty(B, D)$.

Put $u(z) = R(z)e^{i\cdot\arg(u(z))}$ for $z \in B$, where $R(z) = |u(z)|$. From (3.11)–(3.13) the following then holds for u.

Lemma 7. *For all $z \in B$ we have*

$$\mathrm{dist}_D(u(z), \Phi(z)) \le \tfrac{1}{4}C_1, \tag{3.19}$$

$$e^{-C_1/4}(1-L(z)) \le 1 - R(z) \le e^{C_1/4}(1-L(z)), \tag{3.20}$$

(3.21)

$$\cos[\arg(u(z)) - \arg(\Phi(z))] \ge \left[\cosh^2\left(\log\frac{1+K(z)}{1-K(z)}\right) - \cosh\left(\frac{1}{4}C_1\right)\right] \Big/ \sinh^2\left(\log\frac{1+K(z)}{1-K(z)}\right),$$

where $K(z) = \min\{L(z), R(z)\}$.

Lemma 8. *u extends to a continuous map of $\overline{B}$ onto $\overline{D}$ satisfying $u|_{S^1} = \varphi$.*

Proof. Take a point $z_0 \in S^1$ and a neighborhood T $(\subset \overline{D})$ of $\varphi(z_0) \in D(\infty)$. Let $W \in W(\varphi(z_0), \delta)$ be a neighborhood $\{z \in \overline{D}; |z - \varphi(z_0)| < \delta\}$ satisfying $W \subset T$ and

$$\mathrm{dist}_D(z_1, z_2) \ge \tfrac{1}{2}C_1 \tag{3.22}$$

for $z_1 \in W \cap D$ and $z_2 \in \partial T \cap D$. Then, from (3.2) and (3.6), there exists a positive constant $\varepsilon = \varepsilon(\varphi, \delta)$ such that

$$\Phi(\mathscr{V}(z_0, \varepsilon)) \subset W, \tag{3.23}$$

where $\mathscr{V}(z_0, \varepsilon) = \{z \in B; |z - z_0| < \varepsilon\}$. It then follows from (3.19), (3.22), and (3.23) that $u(\mathscr{V}(z_0, \varepsilon)) \subset T$. This implies that $\lim_{z\to z_0} u(z) = \varphi(z_0)$. Since $z_0 \in S^1$ is arbitrary, $u|_{S^1} = \varphi$.

Lemma 9. *u is holomorphic if and only if φ is conformal.*

Proof. We first assume that φ is conformal. Since $\deg(\varphi) > 0$ by assumption, there exists, by the argument principle, a holomorphic map f of $\overline{B}$ onto $\overline{B}$ ($\simeq \overline{D}$) such that $f|_{S^1} = \varphi$. It then follows from the uniqueness of u_ρ that $u_\rho = \rho \cdot f$ for all ρ $(0 < \rho < 1)$. Hence $u = f$.

Next, we assume that u is holomorphic. Since $u|_{S^1} = \varphi$, u has only finitely many zeros $z_1, \dots, z_n$ in B. By the Poisson-Jensen formula (cf. [1]), we then obtain

$$\log\left|u(z)\prod_{j=1}^{n}\left(\frac{1-\overline{z}_j z}{z-z_j}\right)\right| = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\log|u(e^{i\theta})|\,d\theta,$$

which implies

$$u(z) = e^{i\alpha}\prod_{j=1}^{n}\left(\frac{z-z_j}{1-\overline{z}_j z}\right) \quad \text{for } z \in B, \tag{3.24}$$

where α is a real constant, since $u|_{S^1} = \varphi$. It follows from (3.24) that u extends to a holomorphic map of $\overline{B}$ onto $\overline{B}$ ($\simeq D$) with $u|_{S^1} = \varphi$. Hence φ is conformal. We note that $\deg(\varphi) = n$. This completes the proof of Theorem 1′.

Remark 2. When φ is not conformal, it follows from Lemma 12.2 in [13] and Lemma 9 that the zeros of $|\partial u/\partial z|$ or $|\partial u/\partial \overline{z}|$ are isolated in B.

Proof of Theorem 2′. Let u be a harmonic map constructed in the proof of Theorem 1′. To prove Theorem 2′, it suffices to show $u \in \mathscr{D}(B, D)$ when $\deg(\varphi) = \pm 1$. We may assume, without loss of generality, that $\deg(\varphi) = 1$.

On account of the existence theorem of harmonic diffeomorphisms [13, 18], we have

$$u_\rho \in \mathscr{D}(B, B_\rho) \cap C^{3,\alpha}(\overline{B}, \overline{B}_\rho) \quad \text{for all } \rho \ (0 < \rho < 1),$$

where $0 < \alpha < 1$. Combining (3.3) and (3.6) with (3.12), it is verified that for each k $(0 < \rho_1 < k < 1)$, there exist constants l, l' and ρ_3 $(0 < \rho_3 < 1$, $0 < l < l' < 1)$ such that

$$\overline{B}_l \subset u_\rho(\overline{B}_{(1+k)/2}) \subset \overline{B}_{l'} \quad \text{for all } \rho \geq \rho_3. \tag{3.25}$$

It then follows from (3.25) together with Theorem 7.1 in [13] that

$$(J(u_\rho))(z) \geq ((1 - |u_\rho(z)|^2)/2)^2\delta^{-1} > 0 \tag{3.26}$$

for all $\rho \geq \rho_3$ and $z \in B_k$, where $\delta = \delta(k, l, l') > 0$. Applying (3.26) to the subsequence $\{u_{\rho_j}\}_{j\in\mathbb{N}}$ and letting $\rho_j \to 1$, we know that u satisfies

$$(J(u))(z) \geq ((1 - |u(z)|^2)/2)^2\delta^{-1} > 0 \quad \text{for all } z \in B_k,$$

and then from (3.3) and (3.20)

$$(J(u))(z) > 0 \quad \text{for all } z \in B, \tag{3.27}$$

which implies that u is a local diffeomorphism of B to D. Hence $u \in \mathscr{D}(B, D)$, completing the proof of Theorem 2′.

Remark 3. Since

$$J(u) = |\partial u/\partial z|^2 - |\partial u/\partial \overline{z}|^2, \tag{3.28}$$

it follows from (3.27) and (3.28) that

$$|\partial u/\partial z| > 0 \text{ (resp. } |\partial u/\partial \overline{z}| > 0) \quad \text{in } B, \tag{3.29}$$

if $\deg(\varphi) = 1$ (resp. $\deg(\varphi) = -1$).

4. The boundary behavior of $|du|$ and $J(u)$

In this section, we shall investigate the boundary behavior of $|du|$ and $|J(u)|$, which will be of use in §5.

Proposition 1. *Let u be a harmonic map constructed in Theorem 1′. Regard u as a map $u \in C^\infty(B, B) \cap C^0(\overline{B}, \overline{B})$. Then u is a Lipschitz map of $\overline{B}$ onto itself, i.e., there exists a positive constant C_4 depending only on φ such that*

$$\left| \sum_{j,\alpha=1}^{2} \left(\frac{\partial u^j}{\partial x^\alpha}\right)^2 (z) \right| \le C_4 \quad \textit{for } z \in B. \tag{4.1}$$

Proof. We first regard u and Φ as maps $u, \Phi \in C^\infty(D, D) \cap C^0(\overline{D}, \overline{D})$. Let σ be a positive constant and fix it. Take a point z_0 in D satisfying

$$\log \frac{1 + |z_0|}{1 - |z_0|} \ge 4\sigma. \tag{4.2}$$

Let $\widetilde{B}_\sigma(z_0)$ denote the open geodesic ball $\{z \in D;\ \mathrm{dist}_D(z, z_0) < \sigma\}$.

We shall estimate $\widetilde{B}_\sigma(z_0)$ from the outside by a truncated cone. Let $r_1 = r_1(|z_0|)$, $r_2 = r_2(|z_0|)$, and $\xi = \xi(z_0)$ $(0 < \xi < \pi/2)$ be constants satisfying

$$0 < r_1 < |z_0| < r_2 < 1,$$

$$\sigma = \log\left(\frac{1 - r_1}{1 - |z_0|} \cdot \frac{1 + |z_0|}{1 + r_1}\right) = \log\left(\frac{1 - |z_0|}{1 - r_2} \cdot \frac{1 + r_2}{1 + |z_0|}\right). \tag{4.3}$$

$$\cosh(2\sigma) = \cosh^2\left(\log \frac{1 + |z_0|}{1 - |z_0|}\right) - \cos\xi \, \sinh^2\left(\log \frac{1 + |z_0|}{1 - |z_0|}\right). \tag{4.4}$$

Then it is immediate from (4.2)–(4.4) that the truncated cone

$$\mathscr{T} = \mathscr{T}_0(z_0, r_1, r_2, \xi) = \{z \in D;\ r_1 < |z| < r_2,\ \sphericalangle_0(z, z_0) < \xi\}$$

satisfies

$$\widetilde{B}_\sigma(z_0) \subset \mathscr{T}, \tag{4.5}$$

where $\sphericalangle_0(z, z_0)$ denotes the angle between the vectors $\overrightarrow{0z}$ and $\overrightarrow{0z_0}$ in B.

Next, we shall estimate $\Phi(\mathscr{T})$ from the outside by a geodesic ball. It follows from (3.3) and (3.6) that there exist a bounded positive C^0 function h of $\overline{B}$ and a constant r_0 $(0 < \rho_1 < r_0 < 1)$ such that

$$1 - |\Phi| = (1 - r)h \quad \text{on } \overline{B}, \tag{4.6}$$

$$|\partial(\arg(\Phi))/\partial\theta| \le C_5 \quad \text{on } \overline{B} \setminus B_{r_0}, \tag{4.7}$$

where $C_5 = C_5(\varphi)$. It is an immediate consequence of (4.3) and (4.6) that we have the estimate

$$\begin{aligned} &|\mathrm{dist}_D(\Phi(z), 0) - \mathrm{dist}_D(\Phi(z_0), 0)| \\ &\quad = \left|\log\left(\frac{1-|\Phi(z)|}{1-|\Phi(z_0)|}\cdot\frac{1+|\Phi(z_0)|}{1+|\Phi(z)|}\right)\right| \\ &\quad \le \left|\log\frac{1-|z|}{1-|z_0|}\right| + \log(2A_0) \\ &\quad \le \sigma + \log(2A_0) \quad \text{for all } z \in \mathscr{T}, \end{aligned} \tag{4.8}$$

where $A_0 = \sup_{\overline{B}} h / \inf_{\overline{B}} h$. Also, form (4.3), (4.4), and (4.7), there exists a constant r_3 $(r_0 < r_3 < 1)$ such that

$$\begin{aligned} &0 < C_5\xi < \pi/2, \\ &0 < 1-\cos(C_5\xi) < 2(C_5)^2(1-\cos\xi) \quad \text{for } r_1 = r_1(|z_0|) \ge r_3. \end{aligned} \tag{4.9}$$

Note that if $|z_0| \ge (\sqrt{2}-1)/(\sqrt{2}+1)$, then

$$\sinh\left(\log\left(\frac{1+|z_0|}{1-|z_0|}t\right)\right) \le 2t\ \sinh\left(\log\frac{1+|z_0|}{1-|z_0|}\right) \quad \text{for } t > 0. \tag{4.10}$$

Put now $r_4 = \max\{r_3, (\sqrt{2}-1)/(\sqrt{2}+1)\}$. Then, when $r_1 = r_1(|z_0|) \ge r_4$, it follows from (4.4), (4.6), (4.7), (4.9), and (4.10) that we have for $z \in \mathscr{T}$

$$\begin{aligned} &\cosh(\mathrm{dist}_D(\Phi(z_0), |\Phi(z_0)|e^{i\cdot\arg(\Phi(z))})) \\ &\quad = \cosh^2\left(\log\frac{1+|\Phi(z_0)|}{1-|\Phi(z_0)|}\right) \\ &\qquad - \cos[\arg(\Phi(z)) - \arg(\Phi(z_0))]\sinh^2\left(\log\frac{1+|\Phi(z_0)|}{1-|\Phi(z_0)|}\right) \\ &\quad = 1 + [1-\cos\{\arg(\Phi(z)) - \arg(\Phi(z_0))\}]\sinh^2\left(\log\frac{1+|\Phi(z_0)|}{1-|\Phi(z_0)|}\right) \\ &\quad \le 1 + (1-\cos(C_5\xi))\sinh^2\left(\log\left(\frac{1+|z_0|}{1-|z_0|}\cdot\frac{2}{A_1}\right)\right) \\ &\quad \le 1 + 2(C_5)^2(1-\cos\xi)\left(\frac{4}{A_1}\right)^2\sinh^2\left(\log\frac{1+|z_0|}{1-|z_0|}\right) \\ &\quad \le C_6\left[\cosh^2\left(\log\frac{1+|z_0|}{1-|z_0|}\right) - \cos\xi\ \sinh^2\left(\log\frac{1+|z_0|}{1-|z_0|}\right)\right] \\ &\quad = C_6\cosh(2\sigma) \end{aligned}$$

and hence

$$\mathrm{dist}_D(\Phi(z_0), |\Phi(z_0)|e^{i\cdot\arg(\Phi(z))}) \le \log(2C_6\cosh(2\sigma)), \tag{4.11}$$

where $A_1 = \inf_{\overline{B}} h$ (≤ 1) and $C_6 = \max\{1, 32(C_5/A_1)^2\}$. On the other hand, when $r_1 = r_1(|z_0|) < r_4$ or $\log[(1+|z_0|)/(1-|z_0|)] < 4\sigma$, we get from (4.3) and (4.6)

$$\Phi(\mathscr{T}) \subset B_{r_5}, \tag{4.12}$$

where $r_5 = r_5(\sigma, r_4, h)$ $(0 < r_5 < 1)$. By putting

$$\lambda = \max\left\{\sigma + \log(2A_0) + \log(2C_6\ \cosh(2\sigma)),\ 2\ \log\frac{1+r_5}{1-r_5}\right\},$$

it then follows from (4.8), (4.11) and (4.12) that

$$\Phi(\mathscr{T}) \subset \widetilde{B}_\lambda(\Phi(z_0)). \tag{4.13}$$

We now obtain from (3.19), (4.5), and (4.13) that

$$u(\widetilde{B}_\sigma(z_0)) \subset \widetilde{B}_{\lambda+C_1/2}(u(z_0)). \tag{4.14}$$

It is verified from (4.14) and a priori gradient estimates for harmonic maps (Theorem 6.1 in [13]) that

$$\frac{1-|z_0|^2}{1-|u(z_0)|^2}\left|\sum_{j,\alpha=1}^{2}\left(\frac{\partial u^j}{\partial x^\alpha}\right)^2(z_0)\right|^{1/2} \le \frac{C_7(\lambda+\frac{1}{2}C_1)}{\sigma}, \tag{4.15}$$

where $C_7 = C_7(\sigma, \lambda + \frac{1}{2}C_1)$. We note that from (3.20) and (4.6)

$$1-|u(z_0)|^2 \le 2A_2 e^{C_1/4}(1-|z_0|^2), \tag{4.16}$$

where $A_2 = \sup_{\overline{B}} h \ (\ge 1)$. In consequence, it follows from (4.15) and (4.16) that

$$\left|\sum_{j,\alpha=1}^{2}\left(\frac{\partial u^j}{\partial x^\alpha}\right)^2(z_0)\right|^{1/2} \le \frac{2A_2C_7e^{C_1/4}(\lambda+\frac{1}{2}C_1)}{\sigma}.$$

Since z_0 is an arbitrary point in D, u is a Lipschitz map of $\overline{B}$ onto itself. This completes the proof.

Proposition 2. *Let u be a harmonic diffeomorphism, constructed in Theorem 2′, with $\deg(\varphi) = 1$. Regard u as a map $u \in C^\infty(B, B) \cap C^0(\overline{B}, \overline{B})$. Then there exists a positive constant $\delta_1 = \delta_1(\varphi)$ such that*

$$J(u) \ge \delta_1^{-1} > 0 \quad \textit{in } B. \tag{4.17}$$

Remark 4. Combined with (3.28), (4.17) implies

$$\left|\frac{\partial u}{\partial z}\right|^2 \ge \delta_1^{-1} > 0 \quad \text{in } B. \tag{4.18}$$

Lemma 10. *Let $\overline{\widetilde{B}_{l_j}}(z_j) = \{z \in D;\ \mathrm{dist}_D(z, z_j) \le l_j\}$ for $j = 1, 2$. Suppose that a harmonic diffeomorphism $v: \overline{\widetilde{B}_{l_1}}(z_1) \to \overline{\widetilde{B}_{l_2}}(z_2)$ satisfies*

$$\mathrm{Vol}(v(\widetilde{B}_s(z_1))) \ge \mu > 0 \quad \textit{for some } s\ (0 < s < l_1).$$

Then there exists a positive constant δ such that

$$\|J(v)(z)\| \ge \delta^{-1} \quad \textit{for } z \in \widetilde{B}_l(z_1), \tag{4.19}$$

where $0 < l < l_1$, $\delta = \delta(l_1, l_2, s, \mu, l)$, and $\|J(v)(z)\|$ stands for the normalized Jacobian of $v(z)$ by the metric ds_D^2.

Proof. Note that, since the isometry group of D acts transitively on D, there exist a bounded positive C^0 function $f: [0, 1] \to \mathbb{R}$ and a conformal diffeomorphism $w = w(z_1, l_1): \overline{B}\ (\subset \mathbb{C}) \to \overline{\widetilde{B}_{l_1}}(z_1)$ such that

$$\|J(w)(z)\| = f(|z|) \quad \text{for } z \in \overline{B}, \tag{4.20}$$

where f is defined independently of z_1. Then, by applying Theorem 7.1 in [13] to the harmonic diffeomorphism $v \circ w: \overline{B} \to \overline{\widetilde{B}_{l_2}(z_2)}$, we obtain

$$\|J(v \circ w)\| \geq \delta_2^{-1} > 0 \quad \text{on } B_k, \tag{4.21}$$

where $0 < k < 1$ and $\delta_2 = \delta_2(l_2, s, \mu, k, f)$. Then, combining (4.21) with (4.20) yields (4.19).

Proof of Proposition 2. First, keeping the proof of Proposition 1 in mind, we shall show that there exists a positive constant σ such that

$$u(\widetilde{B}_{\sigma/2}(z_0)) \supset \widetilde{B}_1(u(z_0)) \quad \text{for all } z_0 \in D. \tag{4.22}$$

We note that, since $\varphi: S^1 \to S^1$ is a C^4 diffeomorphism, there exist positive constants r_6 $(0 < r_3 < r_6 < 1)$ and C_8

$$0 < C_8 < \left|\frac{\partial(\arg(\Phi))}{\partial\theta}\right| \quad \text{on } \overline{B} \setminus B_{r_6}. \tag{4.23}$$

Choose a positive constant σ large enough so that

$$\frac{1}{4}\sigma + \min\left\{\log\frac{1}{4A_0}, \log\frac{C_9}{2}\right\} \geq 1 + \frac{1}{2}C_1, \tag{4.24}$$

where $C_9 = \min\{1, (C_8)^2/32(A_2)^2\}$. Let z_0 be a point in D satisfying

$$\log[(1 + |z_0|)/(1 - |z_0|)] \geq \sigma. \tag{4.25}$$

Let $k_1 = k_1(|z_0|)$, $k_2 = k_2(|z_0|)$, and $\eta = \eta(z_0)$ $(0 < \eta < \pi/2)$ be constants satisfying

$$\begin{gathered} 0 < k_1 < |z_0| < k_2 < 1, \\ \frac{1}{4}\sigma = \log\left(\frac{1 - k_1}{1 - |z_0|} \cdot \frac{1 + |z_0|}{1 + k_1}\right) = \log\left(\frac{1 - |z_0|}{1 - k_2} \cdot \frac{1 + k_2}{1 + |z_0|}\right), \end{gathered} \tag{4.26}$$

$$\cosh\left(\frac{1}{4}\sigma\right) = \cosh^2\left(\log\frac{1 + |z_0|}{1 - |z_0|}\right) - \cos\eta \, \sinh^2\left(\log\frac{1 + |z_0|}{1 - |z_0|}\right). \tag{4.27}$$

Then, it follows from (4.25)–(4.27) that the truncated cone

$$\mathscr{T} = \mathscr{T}_0(z_0, k_1, k_2, \eta) = \{z \in D;\ k_1 < |z| < k_2, \sphericalangle_0(z, z_0) < \eta\}$$

satisfies

$$\mathscr{T} \subset \widetilde{B}_{\sigma/2}(z_0). \tag{4.28}$$

Let $W_1 = \{z \in \overline{\mathscr{T}}; |z| = k_1\}$, $W_2 = \{z \in \overline{\mathscr{T}}; |z| = k_2\}$, and $W_3 = \{z \in \overline{\mathscr{T}}; \sphericalangle_0(z, z_0) = \eta\}$, respectively. Put $\partial\mathscr{T} = W_1 \cup W_2 \cup W_3$. It follows from (4.24) and (4.26) that for $z \in W_1$

$$\begin{aligned} &\operatorname{dist}_D(\Phi(z_0), 0) - \operatorname{dist}_D(\Phi(z), 0) \\ &\quad = \log\left(\frac{1 - |\Phi(z)|}{1 - |\Phi(z_0)|} \cdot \frac{1 + |\Phi(z_0)|}{1 + |\Phi(z)|}\right) \geq \log\left(\frac{(1 - k_1)h(z)}{(1 - |z_0|)h(z_0)} \cdot \frac{1}{2}\right) \\ &\quad \geq \log\left(\frac{1 - k_1}{1 - |z_0|} \cdot \frac{1 + |z_0|}{1 + k_1} \cdot \frac{1}{4A_0}\right) \\ &\quad = \frac{1}{4}\sigma + \log\left(\frac{1}{4A_0}\right) \geq 1 + \frac{1}{2}C_1 > 0. \end{aligned} \tag{4.29}$$

Similarly, for $z \in W_2$

$$\operatorname{dist}_D(\Phi(z), 0) - \operatorname{dist}_D(\Phi(z_0), 0) \geq 1 + \tfrac{1}{2}C_1 > 0. \tag{4.30}$$

It is immediate from (4.23), (4.26), and (4.27) that there exists a constant k_3 $(0 < r_6 < k_3 < 1)$ such that for $k_1 = k_1(|z_0|) \geq k_3$

$$0 < C_8\eta < \pi/2, \qquad 1 - \cos(C_8\eta) \geq \tfrac{1}{2}(C_8)^2(1 - \cos\eta). \tag{4.31}$$

Note that if $|z_0| \geq (\sqrt{2} - (2A_2)^{-1})/(\sqrt{2} + (2A_2)^{-1})$, then

$$\frac{1 + |z_0|}{1 - |z_0|} \cdot \frac{1}{2A_2} > 1, \tag{4.32}$$

$$\sinh\left(\log\left(\frac{1 + |z_0|}{1 - |z_0|} \cdot \frac{1}{2A_2}\right)\right) \geq \frac{1}{4A_2} \sinh\left(\log\frac{1 + |z_0|}{1 - |z_0|}\right) > 0. \tag{4.33}$$

Put $k_4 = \max\{k_3, (\sqrt{2}-(2A_2)^{-1})/(\sqrt{2}+(2A_2)^{-1})\}$. Then, when $k_1 = k_1(|z_0|) \geq k_4$, it is verified from (4.6), (4.23), (4.27), and (4.31)–(4.33) that for $z \in W_3$

$$\begin{aligned}
&\cosh(\operatorname{dist}_D(\Phi(z_0), |\Phi(z_0)|e^{i\cdot\arg(\Phi(z))})) \\
&= \cosh^2\left(\log\frac{1 + |\Phi(z_0)|}{1 - |\Phi(z_0)|}\right) - \cos[\arg(\Phi(z)) - \arg(\Phi(z_0))] \\
&\quad \cdot \sinh^2\left(\log\frac{1 + |\Phi(z_0)|}{1 - |\Phi(z_0)|}\right) \\
&= 1 + [1 - \cos\{\arg(\Phi(z)) - \arg(\Phi(z_0))\}] \\
&\quad \cdot \sinh^2\left(\log\frac{1 + |\Phi(z_0)|}{1 - |\Phi(z_0)|}\right) \\
&\geq 1 + (1 - \cos(C_8\eta))\sinh^2\left(\log\left(\frac{1 + |z_0|}{1 - |z_0|} \cdot \frac{1}{2A_2}\right)\right) \\
&\geq 1 + \frac{1}{2}(C_8)^2(1 - \cos\eta)\left(\frac{1}{4A_2}\right)^2 \sinh^2\left(\log\frac{1 + |z_0|}{1 - |z_0|}\right) \\
&\geq C_9\left[\cosh^2\left(\log\frac{1 + |z_0|}{1 - |z_0|}\right) - \cos\eta\sinh^2\left(\log\frac{1 + |z_0|}{1 - |z_0|}\right)\right] \\
&\geq C_9\cosh\left(\frac{1}{4}\sigma\right) \geq \frac{1}{2}C_9e^{\sigma/4}
\end{aligned}$$

and then from (4.24)

$$\operatorname{dist}_D(\Phi(z_0), |\Phi(z_0)|e^{i\cdot\arg(\Phi(z))}) \geq \frac{1}{4}\sigma + \log\frac{C_9}{2} \geq 1 + \frac{1}{2}C_1. \tag{4.34}$$

When $\log[(1 + |z_0|)/(1 - |z_0|)] \geq \sigma$ and $k_1 = k_1(|z_0|) \geq k_4$, we get from (4.29), (4.30), and (4.34) that

$$\operatorname{dist}_D(\Phi(z_0), \Phi(\partial\mathscr{T})) \geq 1 + \tfrac{1}{2}C_1. \tag{4.35}$$

Combining (4.35) with (3.19) then yields

$$\operatorname{dist}_D(u(z_0), u(\partial\mathscr{T})) \geq 1.$$

It then follows from (4.28) and $u \in \mathscr{D}(D)$ that

$$u(\widetilde{B}_{\sigma/2}(z_0)) \supset u(\mathscr{T}) \supset \widetilde{B}_1(u(z_0)). \tag{4.36}$$

When $\log[(1+|z_0|)/(1-|z_0|)] < \sigma$ or $k_1 = k_1(|z_0|, \sigma) < k_4$, we can choose σ satisfying (4.22) by (4.6) and (4.26). Together with (4.36), this yields (4.22).

Now, owing to (4.14) and (4.22) we can apply Lemma 10 to $u \in \mathscr{D}(D)$. Then, there exists a positive constant $\delta_3 = \delta_3(\sigma, \lambda + \frac{1}{2}C_1)$ such that

$$\|J(u)\| \geq \delta_3^{-1} > 0 \quad \text{in } \widetilde{B}_{\sigma/2}(z_0).$$

Since δ_3 is independent of z_0, we obtain

$$\left(\frac{1-|z|^2}{1-|u(z)|^2}\right)^2 (J(u))(z) \geq \delta_3^{-1} > 0 \quad \text{for } z \in D. \tag{4.37}$$

Combining (4.37) with (3.20) and (4.6) then implies

$$\begin{aligned}(J(u))(z) &\geq \left(\frac{1-|u(z)|^2}{1-|z|^2}\right)^2 \delta_3^{-1} \geq \left(\frac{e^{-C_1/4}(1-|\Phi|)}{1-|z|^2}\right)^2 \delta_3^{-1} \\ &\geq \left(\frac{e^{-C_1/4}h(z)(1-|z|)}{1-|z|^2}\right)^2 \delta_3^{-1} \\ &\geq \left(\frac{A_1 e^{-C_1/4}}{2}\right)^2 \delta_3^{-1} \quad \text{for } z \in D\ (\simeq B).\end{aligned}$$

This completes the proof of Proposition 2.

5. Entire spacelike constant mean curvature surfaces in $\mathbb{L}^3$

In this section, applying Theorem 2′ together with Proposition 1 and Remark 4 to the representation formula for spacelike surfaces in $\mathbb{L}^3$ [3], we shall construct entire spacelike constant mean curvature surfaces M in $\mathbb{L}^3$, whose Gauss maps are harmonic diffeomorphisms of M to $\mathbb{H}^2$ with C^4 diffeomorphisms $\varphi: S^1\ (\simeq M(\infty)) \to S^1\ \ (\simeq \mathbb{H}^2(\infty))$ as prescribed boundary data.

We first review briefly relevant facts on Gauss maps of spacelike surfaces in $\mathbb{L}^3$. The Minkowski 3-space $\mathbb{L}^3$ is $\mathbb{R}^3$ equipped with Lorentzian metric $ds^2 = dx^2 + dy^2 - dz^2$. The *upper hyperboloid* $\mathbb{H}^2 = \{(x, y, z) \in \mathbb{L}^3;\ x^2 + y^2 - z^2 = -1,\ z > 0\}$ of future-directed unit timelike vectors in $\mathbb{L}^3$ is the hyperbolic plane with respect to the induced metric. The map $\mathscr{P}$ of $\mathbb{H}^2$ onto D defined by

$$\mathscr{P}: (x, y, z) \to \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

is an isometry. An immersed surface M in $\mathbb{L}^3$ whose induced metric is Riemannian is called *spacelike.* The future-directed unit normal vectors of M defines the *Gauss map* $G: M \to \mathbb{H}^2$. We shall also call the composition $\mathscr{P} \circ G: M \to D$ the Gauss map of M. The Gauss map of M is harmonic if and only if the mean curvature of M is constant [17].

With these understood, we prove

Theorem 3. *Let φ be a C^4 diffeomorphism of S^1 to $D(\infty)$ $(\simeq S^1)$ with* $\deg(\varphi) = 1$. *Then there exist a harmonic diffeomorphism $u \in \mathscr{D}(B, D) \cap \mathscr{H}(\overline{B}, \overline{D})$ and an entire spacelike embedding $X: B \to \mathbb{L}^3$ with the following properties:*

(1) $u|_{S^1} = \varphi$.

(2) *$M := X(B)$ has constant mean curvature 1, the Gauss map of M is given by u, and the Gaussian curvature K of M satisfies $-1 \leq K \leq -a^2$, where $0 < a = a(\varphi) \leq 1$.*

(3) *$X = (X^1, X^2, X^3)$ is given explicitly as*

$$X^1(z) = 2\,\mathrm{Re} \int^z \frac{1+\overline{u}^2}{(1-|u|^2)^2} \frac{\partial u}{\partial z}\, dz + c_1,$$

$$X^2(z) = 2\,\mathrm{Re} \int^z (-i) \frac{1-\overline{u}^2}{(1-|u|^2)^2} \frac{\partial u}{\partial z}\, dz + c_2,$$

$$X^3(z) = 2\,\mathrm{Re} \int^z \frac{2\overline{u}}{(1-|u|^2)^2} \frac{\partial u}{\partial z}\, dz + c_3$$

for $z \in B$, where $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ and the integral being taken along an arbitrary path from a fixed point to the point $z \in B$.

(4) *If φ is not conformal, then the umbilical points of M are isolated.*

To prove Theorem 3, we prepare the following proposition.

Proposition 3 ([3]). *Let M be a simply connected Riemann surface, $H\colon M \to D$ be a nonvanishing real C^∞ function on M, and $\Psi\colon M \to D$ be a nowhere antiholomorphic C^∞ map of M to D. Suppose H and Ψ satisfy the following differential equation:*

$$H\left(\frac{\partial^2 \Psi}{\partial w \partial \overline{w}} + \frac{2\overline{\Psi}}{1-|\Psi|^2} \frac{\partial \Psi}{\partial w} \frac{\partial \Psi}{\partial \overline{w}}\right) = \frac{\partial H}{\partial \overline{w}} \frac{\partial \Psi}{\partial w}, \tag{5.1}$$

where w is a complex coordinate on M compatible with its complex structure. Then there exists a spacelike immersion $X\colon M \to \mathbb{L}^3$ with the following properties:

(1) *The mean curvature of M is H, and the Gauss map of M is given by Ψ.*

(2) *The induced metric g on M and the Gaussian curvature K of M are given by*

$$g = \left(\frac{2}{H(1-|\Psi|^2)} \left|\frac{\partial \Psi}{\partial w}\right|\right)^2 |dw|^2, \tag{5.2}$$

$$K = H^2 \left(\left|\frac{\partial \Psi}{\partial \overline{w}} \Big/ \frac{\partial \Psi}{\partial w}\right|^2 - 1\right). \tag{5.3}$$

(3) *$X = (X^1, X^2, X^3)$ is given explicitly as*

$$X^1(w) = 2\,\mathrm{Re} \int^w \frac{1}{H} \frac{1+\overline{\Psi}^2}{(1-|\Psi|^2)^2} \frac{\partial \Psi}{\partial w}\, dw + c_1,$$

$$X^2(w) = 2\,\mathrm{Re} \int^w \frac{-i}{H} \frac{1-\overline{\Psi}^2}{(1-|\Psi|^2)^2} \frac{\partial \Psi}{\partial w}\, dw + c_2,$$

$$X^3(w) = 2\,\mathrm{Re} \int^w \frac{2}{H} \frac{\overline{\Psi}}{(1-|\Psi|^2)^2} \frac{\partial \Psi}{\partial w}\, dw + c_3,$$

for $w \in M$, where $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ and the integral being taken along an arbitrary path from a fixed point to the point w.

(4) $(\partial\Psi/\partial\overline{w})(w_0) = 0$ *at a point* $w_0 \in M$ *if and only if* w_0 *is an umbilical point of* M.

Proof of Theorem 3. Let u be a harmonic diffeomorphism with $u|_{S^1} = \varphi$, which is constructed in Theorem $2'$. In Proposition 3, take $M = B$, $H = 1$, and $\Psi = u$. It follows from (2.4), (3.29), and Remark 2 that Ψ is nowhere antiholomorphic, H and Ψ satisfy (5.1), and that if φ is not conformal, then the zeros of $\partial\Psi/\partial\overline{w}$ are isolated in M. Combining (3.28), (4.1), and (4.17) with (5.3) also yields $-1 \leq K \leq -a^2$. Hence assertions (1)–(4) hold.

It remains to prove that M is entire in $\mathbb{L}^3$. Note that a complete spacelike surface in $\mathbb{L}^3$ is entire. Thus it suffices to show that M is complete. But, substituting (3.20), (4.1), and (4.18) in (5.2), it is not hard to see that $M = (B, g)$ is quasi-isometric to $D = (B, ds_D^2)$, and that M is complete. This completes the proof.

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