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AMENABILITY AND THE STRUCTURE OF THE ALGEBRAS $A_p(G)$

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ABSTRACT. A number of characterizations are given of the class of amenable locally compact groups in terms of the ideal structure of the algebras $A_p(G)$. An almost connected group is amenable if and only if for some $1 < p < \infty$ and some closed ideal I of $A_p(G)$, I has a bounded approximate identity. Furthermore, G is amenable if and only if every derivation of $A_p(G)$ into a Banach $A_p(G)$-bimodule is continuous.

1. INTRODUCTION

Let G be a locally compact group. In [7] Eymard defined the Fourier algebra $A(G)$ of G to be the linear subspace of $C_0(G)$ consisting of all functions of the form $u(x) = (f * \tilde{g})^{\vee}(x)$, where $f, g \in L_2(G)$, $f^{\vee}(x) = f(x^{-1})$ and $\tilde{f}(x) = \overline{f(x^{-1})}$. The space $A(G)$ can be identified with a quotient space of the projective tensor product $L_2(G) \otimes_{\gamma} L_2(G)$. With respect to pointwise multiplication and the quotient norm, $A(G)$ is a commutative Banach algebra.

In [15], Herz introduced the L_p-versions of Eymard's algebra. He defined $A_p(G)$ to be the space of functions of the form $u(x) = \sum_{n=1}^{\infty} (f_i * \tilde{g}_i)^{\vee}$ where $f_i \in L_p(G)$, $g_i \in L_q(G)$, $1/p + 1/q = 1$, $1 < p < \infty$, and $\sum_{n=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$. Then

$$\|u\|_{A_p(G)} = \inf \left\{ \sum_{n=1}^{\infty} \|f_i\|_p \|g_i\|_q \,\middle|\, \sum_{n=1}^{\infty} (f_i * \overline{g}_i)^{\vee} \right\}$$

determines a norm on $A_p(G)$ with respect to which $A_p(G)$ is a Banach algebra. When $p = 2$, $A_p(G) = A(G)$.

We began a study of the structure of the closed ideals of $A(G)$ in [9 and 10]. The present investigation will extend many of the principal results of these earlier works to the $A_p(G)$ algebras. We will also prove a number of results which are new even for $p = 2$. In particular, we show that if G is almost connected, then G is amenable if and only if some closed ideal I in $A_p(G)$ has a bounded approximate identity for some $1 < p < \infty$. For the class of amenable

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groups we are able to improve considerably the main result of [6] which is concerned with invariant projections on $A_p(G)$-submodules of $\mathrm{PM}_p(G)$.

2. Preliminaries

Throughout this paper, G will denote a locally compact group with a fixed left Haar measure λ_G. For $1 < p < \infty$, $A_p(G)$ will denote the Banach algebra of continuous functions defined in §1.

Denote by $\mathrm{PF}_p(G)$ the closure of $L^1(G)$, considered as convolution operators on $L_p(G)$, with respect to the norm topology of $\mathscr{B}(L_p(G))$, the bounded linear operators on $L_p(G)$. The weak operator topology closure of $L^1(G)$ is denoted by $\mathrm{PM}_p(G)$. The spaces $\mathrm{PF}_p(G)$ and $\mathrm{PM}_p(G)$ are referred to as the p-pseudofunctions and the p-pseudomeasures respectively. It can be shown that $\mathrm{PM}_p(G) = A_p(G)^*$. When $p = 2$, $\mathrm{PM}_2(G)$ is usually denoted by $VN(G)$ while $\mathrm{PF}_2(G)$ is $C_r^*(G)$, the reduced group C^*-algebra of G (see [7]).

$B_p(G)$ is the multiplier algebra of $A_p(G)$, consisting of the continuous complex-valued functions v on G such that $vu \in A_p(G)$ for every $u \in A_p(G)$. Define a norm on B_p by

$$\|v\|_{B_p(G)} = \{\|uv\|_{A_p(G)} | u \in A_p(G) \text{ and } \|u\|_{A_p(G)} \leq 1\}.$$

Observe that if $v \in A_p(G)$, then $v \in B_p(G)$ and $\|v\|_{B_p(G)} \leq \|v\|_{A_p(G)}$. $B(G)$ denotes the Fourier-Stieltjes algebra of G. Then $B(G) = C^*(G)^*$ is the linear span of the continuous positive definite functions on G.

G is said to be amenable if there exists $m \in L_\infty(G)^*$ such that $m \geq 0$, $m(1_G) = 1$, and $m({}_xf) = m(f)$ for every $x \in G$, $f \in L_\infty(G)$. 1_A denotes the characteristic function of A and ${}_xf(y) = f(xy)$. Amenable groups include all abelian groups and all compact groups. The free group on two generators F_2 is nonamenable.

A locally compact group is called a [SIN]-group if every neighborhood of e contains a compact neighborhood which is invariant under all inner automorphisms. For properties of [SIN]-groups see [22].

Set $\mathscr{A}$ be a commutative Banach algebra. $\Delta(\mathscr{A})$ will denote both the maximal ideal space of $\mathscr{A}$ and the multiplicative linear functions associated with these ideals. By means of the Gelfand transform, $\mathscr{A}$ can be realized as a subalgebra of $C_0(\Delta(\mathscr{A}))$. For an ideal I in $\mathscr{A}$, we define

$$Z(I) = \{x \in \Delta(\mathscr{A}) | u(x) = 0 \text{ for every } u \in I\}.$$

For $E \subset \Delta(\mathscr{A})$, define

$$I(E) = \{u \in \mathscr{A} | u(x) = 0 \text{ for every } x \in E\},$$
$$I_0(E) = \{u \in \mathscr{A} | \operatorname{supp} u \in \mathscr{F}(E)\},$$

where $\mathscr{F}(E) = \{K \subset \Delta(\mathscr{A}) | K \text{ is compact and } K \cap \overline{E} = \varnothing\}$. $I(E)$ and $I_0(E)$ are ideals in $\mathscr{A}$. $I(E)$ is closed. Moreover, if $Z(I) = E$, then $I_0(E) \subseteq I \subseteq I(E)$.

A closed subset E of $\Delta(\mathscr{A})$ is called a set of spectral synthesis, or simply an s-set if $I(E)$ is the only closed ideal I for which $Z(I) = E$.

Let G be a locally compact group. Let $A, B \subset G$ be closed. Let

$$\mathscr{S}_p(A, B) = \{u \in B_p(G) | u(A) \equiv 1, u(B) \equiv 0\},$$

$$s_p(A, B) = \begin{cases} \inf\{\|u\|_{B_p(G)} | u \in \mathscr{S}_p(A, B)\} & \text{if } \mathscr{S}_p(A, B) \neq \varnothing, \\ \infty & \text{if } \mathscr{S}_p(A, B) = 0, \end{cases}$$

$$\mathscr{F}(A) = \{K \subset G | K \text{ is compact}, K \cap A = \varnothing\},$$

$$s_p(A) = \sup\{s_p(A, K) | K \in \mathscr{F}(A)\},$$

$$I_p(A) = \{u \in A_p(G) | u(x) = 0 \text{ for every } x \in A\}.$$

A net $\{u_\alpha\}_{\alpha \in \mathscr{U}}$ in $A_p(G)$ is a bounded approximate identity in an ideal I if $u_\alpha \in I$, $\|u_\alpha\|_{A_p(G)} \leq M$ for all $\alpha \in \mathscr{U}$, and $\lim_\alpha \|uu_\alpha - u\|_{A_p(G)} = 0$ for every $u \in I$. Since left and right translations are isometric isomorphisms, the ideals $I(A)$, $I(xA)$, and $I(Ax)$ are all isometric ismorphic.

An ideal I is said to be idempotent if $I = I^2$, where $I^2 = \{\sum_{i=1}^n u_i v_i | u_i, v_i \in I\}$.

3. Amenability, bounded approximate identities and weak factorization

In this section we will extend and improve many of the results of [9] and [10] to the setting of the $A_p(G)$ algebras, $1 < p < \infty$. We will also consider weak factorization of ideals in $A_p(G)$.

Lemma 3.1. *Let G be an amenable locally compact group. Let $u \in B(G)$. Then $u \in B_p(G)$ for every $1 < p < \infty$. Furthermore, for each $I < p < \infty$, there exists a constant C_p independent of u such that $\|u\|_{B_p(G)} \leq C_p \|u\|_{B(G)}$.*

Proof. Since G is amenable, $B_p(G) = \mathrm{PF}_p(G)^*$. Let $K \subset G$ be compact. Since $u \in B(G)$, Cowling [3] has shown that $u_{|K} \in A_2(K) = \{v_{|K}; v \in A(G)\}$ and that

$$\|u_{|K}\|_{A_2(K)} = \inf\{\|v\|_{A(G)}; v_{|K} = u_{|K}\} \leq \|u\|_{B(G)}.$$

Let $i: A(G) \to A_p(G)$ be the canonical injection. Since i is continuous [15], there exists a constant C_p such that $\|v\|_{A_p(G) \leq C_p} \|v\|_{A(G)}$ for every $v \in A(G)$. Let $v_0 \in A(G)$ be such that $v_{0|K} = u_{|K}$. Then $i(v_0)_{|K} = u_{|K}$, so $u_{|K} \in A_p(K)$. Moreover,

$$\begin{aligned} \|u_{|K}\|_{A_p(K)} &\leq \inf\{\|i(v)\|_{A_p(G)}; v \in A(G), v_{|K} = u_{|K}\} \\ &\leq \inf\{C_p \|v\|_{A_p(G)}; v \in A(G), v_{|K} = u_{|K}\} \\ &\leq C_p \|u\|_{B(G)}. \end{aligned}$$

By the converse of Cowling's result, $u \in B_p(G)$ and $\|u\|_{B_p(G)} \leq C_p \|u\|_{B(G)}$. □

Proposition 3.2. *Let G be an amenable locally compact group. Let $A \subset G$ be closed. If $s_2(A) < \infty$, then $s_p(A) < \infty$ for every $1 < p < \infty$.*

Proof. It follows from Lemma 3.1 that if $K \in \mathscr{F}(A)$ and $u \in \mathscr{S}_2(A, K)$, then $u \in \mathscr{S}_p(A, K)$ and $\|u\|_{B_p(G) \leq C_p} \|u\|_{B(G)}$. Hence, $s_p(A) \leq C_p s_2(A)$. □

Corollary 3.3. *Let G be an amenable locally compact group. Let H be a closed subgroup of G which is either* (i) *open,* (ii) *compact,* (iii) *normal. Then $s_p(H) < \infty$. Furthermore, if G is a* [SIN]*-group and H is any closed subgroup, then $s_p(H) < \infty$.*

Proof. If H satisfies (i), (ii), or (iii) above, then by [10, Lemma 3.6], $s_2(H) = 1$. Proposition 3.2 implies that $s_p(H) < \infty$.

If G is a [SIN]-group, then $s_2(H) = 1$ for every closed subgroup H of G [11, Proposition 3.10]. Again, by Proposition 3.2, $s_p(H) < \infty$. □

A straightforward modification of the proof of [10, Proposition 3.2] establishes the next proposition.

Proposition 3.4. *Let G be an amenable locally compact group. Let A be a closed set of spectral synthesis for $A_p(G)$. If $s_p(G) < \infty$, then $I_p(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ which satisfies*

(i) $\|u_\alpha\|_{A_p(G)} \leq 2 + s_p(A)$ *for every* $\alpha \in \mathscr{U}$,

(ii) $u_\alpha \in A_p(G) \cap C_{00}(G)$ *for every* $\alpha \in \mathscr{U}$,

(iii) *if* $K \in \mathscr{F}(A)$, *there exists a sequence* $\{u_{K_n}\} \subseteq \{u_\alpha\}_{\alpha\in\mathscr{U}}$ *such that* $\|vu_{K_n} - v\|_{A_p(G)} \leq 1/n$ *for every* $v \in A_p(G)$ *with* $\operatorname{supp} v \subseteq K$.

Corollary 3.5. *Let G be an amenable locally compact group. Let H be a closed subgroup of G which is* (i) *open,* (ii) *compact, or* (iii) *normal. Then $I_p(H)$ has a bounded approximate identity for every $1 < p < \infty$ and hence $I_p(H)$ is idempotent. If G is a* [SIN]*-group, then $I_p(H)$ has a bounded approximate identity, and hence is idempotent for every closed subgroup H of G and every $1 < p < \infty$.*

Proof. Since G is amenable, every closed subgroup is a set of spectral synthesis for each $A_p(G)$, $1 < p < \infty$ [14]. The result follows immediately from Corollary 3.5 and from Cohen's Factorization Theorem [17, Theorem 32.22]. □

Definition 3.6. Let $\mathscr{R}(G)$ denote the ring of subsets of G generated by the open left-cosets of G. Define $\mathscr{R}_c(G) = \{A \subset G; A \text{ is closed}, A \in \mathscr{R}(G_d)\}$ where G_d is the group G together with the discrete topology. The sets $A \in \mathscr{R}_c(G)$ can be characterized by following an argument due to Gilbert [12] originally presented for abelian groups. We have $A \in \mathscr{R}_c(G)$ if and only if $A = \bigcup_{i=1}^n x_i(H_i \backslash \Delta_i)$ where H_i is a closed subgroup of G, $\Delta_i \in \mathscr{R}(H_i)$, and $x_i \in G$ for every $1 \leq i \leq n$ [11, Lemma 3.5].

Theorem 3.7. *Let G be an amenable* [SIN]*-group. Let $A \in \mathscr{R}_c(G)$. Then $I_p(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ such that*

(i) $u_\alpha \in A_p(G) \cap C_{00}(G)$,

(ii) *if* $K \in \mathscr{F}(A)$, *then there exists a sequence* $\{u_{K_n}\} \subseteq \{u_\alpha\}_{\alpha\in\mathscr{U}}$ *such that if* $v \in I(A)$ *and* $\operatorname{supp} v \subseteq K$, *then*

$$\|u_{K_n}v - v\|_{A_p(G)} \leq 1/n.$$

In particular, A is an s-set for $A_p(G)$.

Proof. That A is an s-set follows immediately if we establish the existence of a bounded approximate identity satisfying (i).

It follows from Corollary 3.5 that $I_p(H)$ has a bounded approximate identity for any closed subgroup H of G.

Let $A = \bigcup_{i=1}^n x_i(H_i \backslash \Delta_i)$ where $\Delta_i \in \mathscr{R}(H_i)$. By Host's idempotent theorem, $1_{\Delta_i} \in B(Hi)$. By [4, Theorem 2], 1_{Δ_i} extends to a $u \in \mathscr{S}_2(\Delta_i, H_i \backslash \Delta_i)$. Lemma 3.1 shows that $u \in \mathscr{S}_p(\Delta_i, H_i \backslash \Delta_i)$.

As G is amenable, $A_p(G)$ has a bounded approximate identity [24, p. 96]. Arguing as in [11, Lemma 3.9 and Theorem 3.11], we see that $I_p(H_i)$ has a bounded approximate identity and therefore so must $I_p(H_i \backslash \Delta_i)$. It is then clear that $I_p(x_i(H_i \backslash \Delta_i))$ has a bounded approximate identity. By again following

[11, Theorem 3.11], we see that $I_p(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathscr{U}}$.

That $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ can be chosen to satisfy (i) and (ii) follows from a careful examination of the proofs of [10, Proposition 3.2; 11, Lemma 3.9, and Theorem 3.11]. □

Theorem 3.8. *Let G be an amenable group. Let X be a weak*-closed $A_p(G)$-submodule of* $\mathrm{PM}_p(G)$. *Then the following are equivalent:*

(i) *X is invariantly complemented,*
(ii) *${}^{\perp}X$ has a bounded approximate identity.*

Furthermore, if G is any locally compact group for which ${}^{\perp}X$ has a bounded approximate identity whenever X is a weak-closed invariantly complemented submodule of* $\mathrm{PM}_p(G)$, *then G is amenable.*

Proof. As G is amenable, $A_p(G)$ has a bounded approximate identity. The first part follows from [10, Proposition 6.4].

Let $X = \{0\}$. Then X is invariantly complemented and $A_p(G) = {}^{\perp}X$ has a bounded approximate identity if and only if G is amenable so the last statement follows. □

Corollary 3.9. *Let G be an amenable locally compact group. Let H be a closed subgroup of G which is* (i) *open,* (ii) *compact or* (iii) *normal. Then $I_p(H)^{\perp}$ is invariantly complemented. Furthermore, if G is also a* [SIN]*-group and $A \in \mathscr{R}_c(G)$, then $I_p(A)^{\perp}$ is invariantly complemented.*

Corollary 3.9 extends for the class of amenable groups a recent result of Derighetti [6, Théorème 2] who shows that $I_p(H)^{\perp}$ is invariantly complemented whenever H is a closed normal subgroup and G is an arbitrary locally compact group. In the case $p = 2$, Derighetti's result is due to Lau and Losert [21, Theorem 2].

Theorem 3.10. *Let G be an abelian locally compact group. Then for every $1 < p < \infty$, $A_p(G)$ is an amenable Banach algebra.*

Proof. Let $\widehat{G}$ denote the dual group of G. Then since $\widehat{G}$ is amenable, $L^1(\widehat{G})$ is amenable [19, Theorem 2.5]. Hence $A(G)$ is an amenable Banach algebra. For every $1 < p < \infty$, $A(G)$ embeds continuously into $A_p(G)$ as a dense subalgebra. Therefore, $A_p(G)$ is amenable [19, Proposition 5.3]. □

Corollary 3.11. *Let G be an abelian locally compact group. Let X be a weak*-closed $A_p(G)$-submodule of* $\mathrm{PM}_p(G)$. *Then X is complemented if and only if X is invariantly complemented.*

Proof. Assume that X is complemented. Then since $A_p(G)$ is amenable, ${}^{\perp}X$ has a bounded approximate identity [5, Theorem 3.7]. It follows from Theorem 3.8 that X is invariantly complemented.

The case $p = 2$ in Corollary 3.11 is again due to Lau and Losert [21]. It is worthwhile to note that Theorem 3.10 and Corollary 3.11 will hold whenever G is such that $A(G)$ is amenable. There is reason to believe that this is the case precisely when G is amenable.

For $A \subseteq G$, let $\operatorname{bdy} A$ and $\operatorname{int} A$ denote the boundary of A and the interior of A respectively. □

Proposition 3.12. *Let $A \subset G$ be closed. Assume that $I_p(A)$ factorizes weakly. Then $\lambda(\operatorname{bdy} A) = 0$ and $\operatorname{int}(A)$ is a clopen subset of G.*

Proof. $I_p(A)$ is a weakly selfadjoint subalgebra of $A_p(G)$. Hence by [8, Theorem 1.3], there exists an $0 < M < \infty$ such that for every $K \in \mathscr{F}(A)$, there exists a $u_K \in I_p(A)$ with $\|u_K\|_{A_p(G)} \le M$, $u_K(x) \ge 1$ for every $x \in K$ and $u_K \ge 0$. Let $\mathscr{F}(A)$ be directed by inclusion. Then $\{u_K\}_{K\in\mathscr{F}(A)}$ is a bounded net in $W_p(G) = \mathrm{PF}_p(G)^*$. Hence we can assume that $\{u_K\}_{K\in\mathscr{F}(A)}$ converges in the weak*-topology to some $u \in W_p(G)$ with $\|u\|_{W_p(G)} \le M$ (otherwise choose a convergent subnet). It is easy to see that $u(x) \ge 0$, $u(x) \ge 1$ on $G\backslash A$ and that $u(x) = 0$ for every $x \in \operatorname{int}(A)$. Hence $u(x) \ge 1$ on $\operatorname{bdy}(A)$. Since u is continuous, $\operatorname{int}(A)$ is clopen.

Assume that $\lambda(\operatorname{bdy} A) > 0$. Then we can find a subset V of $\operatorname{bdy} A$ with $0 < \lambda(V) < \infty$. Let $f = 1_V$. Then $f \in L_1(G) \subseteq \mathrm{PF}_p(G)$ and

$$0 < \lambda(V) \le \int_G u(x) f(x)\,dx = \langle u, f\rangle = \lim_k \langle u_K, f\rangle = 0$$

since $u_K(x) = 0$ for every $x \in \operatorname{bdy} A$. As this is impossible, $\lambda(\operatorname{bdy} A) = 0$. □

Proposition 3.13. *Let $A \subset G$ be closed. Suppose that I is a closed ideal of $A_p(G)$ with $Z(I) = A$. If I has a bounded approximate identity, then $\lambda(\operatorname{bdy} A) = 0$ and $\operatorname{int}(A)$ is clopen. Moreover, $1_{G\backslash \operatorname{int} A} \in W_p(G) = \mathrm{PF}_p(G)^*$.*

Proof. Let $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ be a bounded approximate identity in I. We may assume that $\mathrm{w}^* - \lim_\alpha u_\alpha = u$ for some $u \in W_p(G)$.

Let $v \in I$. Let $f \in L_1(G)$ and $\varepsilon > 0$. Then

$$\langle uv, f\rangle = \langle u, vf\rangle = \lim_\alpha \langle u_\alpha, vf\rangle = \lim_\alpha \langle u_\alpha v, f\rangle.$$

Let

$$(*) = |\langle uv, f\rangle - \langle v, f\rangle| \le |\langle uv, f\rangle - \langle u_\alpha v, f\rangle| + |\langle u_\alpha v, f\rangle - \langle v, f\rangle|.$$

Choose α such that $|\langle uv, f\rangle - \langle u_\alpha v, f\rangle| < \varepsilon$ and $\|u_\alpha v - v\|_{A_p(G)} < \varepsilon$. Then $(*) \le \varepsilon + \varepsilon\|f\|_{\mathrm{PF}_p(G)}$. Since ε is arbitrary and f is fixed, it follows that $\langle uv, f\rangle = \langle v, f\rangle$ for every $f \in L_1(G)$. Hence $uv = v$. It follows also that $u(x) = 1$ on $G\backslash A$ and that $u(x) = 0$ on $\operatorname{int} A$. Therefore $u = 1_{G\backslash \operatorname{int} A}$. □

Proposition 3.14. *Let I be a closed ideal in $A_p(G)$ with $Z(I) = A$. If either I is weakly selfadjoint or I has a bounded approximate identity, then either $\lambda(A) > 0$ or G is amenable.*

Proof. Assume that $\lambda(A) = 0$. Let $K \subset G$ be compact. Let $y \in C_{00}^+(G)$, with $\operatorname{supp}\varphi \subseteq K$. Let $\varepsilon, \varepsilon_1 > 0$. We can find an open neighborhood V_{ε_1} of A such that $\lambda(V)\|\varphi\|_\infty < \varepsilon_1$.

In either of the above cases, we can find a $u \in I$ with $\|u\|_{A_p(G)} < M < \infty$ (M independent of K) which is such that $\inf\{R_e u(x); x \in K\backslash V\} \ge 1 - \varepsilon$. Then

$$|\langle u, \varphi\rangle| \le \|L_\varphi\|_{CV_{p'}} \|u\|_{A_p(G)} \le M\|L_\varphi\|_{CV_{p'}}$$

where $\|L_\varphi\|_{CV_{p'}}$ is the norm of φ as a convolution operator on $L_{p'}(G)$.

But

$$R_e\langle u, \varphi\rangle = \int_G R_e u(x)\varphi(x)\,dx \ge (1-\varepsilon)\|\varphi\|_1 - \varepsilon_1.$$

Therefore $\|\varphi\|_1 \le M\|L_\varphi\|_{CV_{p'}}$.

As K was arbitrary, $\|\psi\|_1 \leq M\|L_\psi\|_{CV_{p'}}$ for every $\psi \in C_{00}^+(G)$. Given $\psi \in C_{00}^+(G)$, we have

$$\|\psi\|_1^n = \|\psi^{*n}\|_1 \leq M\|L_{\psi^{*n}}\|_{CV_{p'}} \leq M\|L_\psi\|_{CV_{p'}}.$$

Hence $\|\psi\|_1 = \|L_\psi\|_{CV_{p'}}$ for every $\psi \in C_{00}^+(G)$. This implies that G is amenable. □

In [11], we established a connection between the existence of bounded approximate identities in $A(G)$ and the existence of either "large" amenable subgroups or open amenable subgroups. It was conjectured that if there is a closed ideal in $A(G)$ with a bounded approximate identity, then G must have an open amenable subgroup. We have further evidence to support this conjecture and more.

Theorem 3.15. *Let G be a connected locally compact group. Let I be a nonzero closed ideal in $A_p(G)$, $1 < p < \infty$, which is such that either I has a bounded approximate identity or I is weakly selfadjoint and weakly factorizes. Then G is amenable.*

Proof. Let $Z(I) = A$. If G is nonamenable, then by Proposition 3.14 $\lambda(A) > 0$. It follows from Propositions 3.12 and 3.13 that $\mathrm{int}(A)$ is a nonempty clopen set. Since G is connected, $A = G$. But then $I = \{0\}$ which is impossible. Hence G is amenable. □

Corollary 3.16. *Let G be an almost connected locally compact group. Let I be a nonzero closed ideal in $A_p(G)$, $1 < p < \infty$, with a bounded approximate identity, then G is amenable.*

Proof. Let $A = Z(I)$. Since I is nonzero, $A \neq G$. Therefore, by translating if necessary, we can assume that there exist $x_0 \in G_0 \backslash A$, where G_0 is the connected component of G. Let $I_{G_0} = \{u_{|G_0}\,;\, u \in I\}^-$ (the closure of $\{u_{|G_0}\,;\, u \in I\}$ in $A_p(G)$). Since $A_p(G)_{|G_0} = A_p(G_0)$ [16, Theorem 1], I_{G_0} is a closed ideal in $A_p(G_0)$. Let $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ be a bounded approximate identity in I. As $\|u_{|G_0}\|_{A_p(G_0)} \leq \|u\|_{A_p(G)}$ for every $u \in A_p(G)$, $\{u_{|G_0}\}_{\alpha\in\mathscr{U}}$ is a bounded approximate identity for the closed ideal I_{G_0}, which is nonzero since $x_0 \in G_0\backslash A$. By Theorem 3.15, G_0 is amenable. But G is almost connected, so G/G_0 is compact and hence amenable. Therefore G is also amenable [24, Proposition 13.4]. □

4. Cofinite ideals in $A_p(G)$

Definition 4.1. An ideal I in $A_p(G)$ is called cofinite if the dimension of $A_p(G)/I$ is finite. The codimension of I is $\dim A_p(G)/I$.

We can proceed as in [10, §5] to obtain the following characterization of amenable groups which extends [10, Corollary 5.6, Lemma 5.7, and Theorem 5.8].

Theorem 4.2. *Let G be a locally compact group. Then the following are equivalent:*

(a) *G is amenable.*
(b) *For every $1 < p < \infty$ and every cofinite ideal $I \subset A_p(G)$, $I = I(A)$ for some finite set $A = \{x_1, \ldots, x_n\} \subset G$ with $n = \mathrm{codim}\, I$.*

(c) *For every* $1 < p < \infty$ *and every cofinite ideal* $I \subset A_p(G)$, I *has a bounded approximate identity.*

(d) *For some* $1 < p < \infty$ *and some closed cofinite ideal* $I \subset A_p(G)$, $I^2 = I$.

Lemma 4.3. *Let* G *be an amenable locally compact group. Let* I *be a closed in* $A_p(G)$ *with infinite codimension. Then there exist sequences* $\{u_n\}$, $\{v_n\}$ *in* $A_p(G)$ *such that* $u_n v_1 \cdots v_{n-1} \notin I$ *but* $u_n v_1 \cdots v_n \in I$.

The proof of this lemma is identical to that of the case $p = 2$ (see [9, Lemma 2]).

Definition 4.4. Let $\mathscr{A}$ be a Banach algebra and let X be a Banach $\mathscr{A}$-bimodule. A derivation $D: \mathscr{A} \to X$ is a linear map which satisfies $D(uv) = u \cdot D(v) + D(u) \cdot v$ for every $u, v \in \mathscr{A}$. Let X be a left Banach $\mathscr{A}$-module. A linear operator $T: \mathscr{A} \to X$ is said to be of class $\mathscr{T}$ if for every $u, v \in \mathscr{A}$

$$T(uv) = u \cdot T(v) + L(u, v)$$

where $L(\cdot, \cdot)$ is a bilinear operator from $\mathscr{A} \times \mathscr{A}$ to X for which $v \mapsto L(u, v)$ is continuous for each $u \in \mathscr{A}$.

Proceeding as in [9, Theorem 1] and [10, Theorem 5.9] we have the following

Theorem 4.5. *Let* G *be a locally compact group. Then the following are equivalent:*

(i) G *is amenable.*

(ii) *For every* $1 < p < \infty$, *every homomorphism from* $A_p(G)$ *with finite-dimensional range is continuous.*

(iii) *For every* $1 < p < \infty$, *every derivation of* $A_p(G)$ *into a finite-dimensional commutative Banach* $A_p(G)$*-bimodule is continuous.*

(iv) *For every* $1 < p < \infty$, *every derivation of* $A_p(G)$ *into a Banach* $A_p(G)$*-bimodule is continuous.*

Theorem 4.6. *Let* G *be a compact group. Let* $1 < p < \infty$. *If* $S: A_p(G) \to X$ *is of class* $\mathscr{T}$, *then* S *is continuous.*

Proof. First assume that G is separable. Then $A_p(G)$ is a separable Banach algebra with identity. Since $A_p(G)$ is a normal algebra, every prime ideal is contained in a unique maximal ideal (see [1, p. 97]). Let J be a closed prime ideal. Then since $Z(J) = \{x_0\}$ for some $x_0 \in G$, $J = I(\{x_0\})$. Hence J is a maximal ideal. Moreover, if $I = I(\{x\})$ is a maximal ideal, then $I^2 = I$. It follows from [1, Theorem 4.2] that every operator of class $\mathscr{F}$ is bounded.

Now assume that G is an arbitrary compact group and that $T: A_p(G) \to X$ is class $\mathscr{F}$. Assume that T is discontinuous. Let $\{u_n\}$ be a sequence in $A_p(G)$ such that $u_n \to 0$ while $Tu_n \not\to 0$. For each n we can find a compact normal subgroup K_n of G such that G/K_n is separable and u_n is constant on cosets of K_n. Let $K = \bigcap_{n=1}^{\infty} K_n$. Then K is compact and normal. Furthermore each u_n is constant on cosets of K and G/K is separable.

There exists an isometric isomorphism ρ from $A_p(G/K)$ onto the subspace of $A_p(G)$ consisting of functions which are constant on cosets of K [16, Proposition 6]. X becomes a left Banach $A_p(G/K)$-module with respect to the module action as defined by $\tilde{u} \circ x = \rho(\tilde{u})x$ for every $\tilde{u} \in A_p(G/K)$. Similarly $\widetilde{T}: A_p(G/K) \to X$ defined by $\widetilde{T}(\tilde{u}) = T(\rho(\tilde{u}))$ for every $\tilde{u} \in A_p(G/K)$ is of class $\mathcal{F}$. If we choose $\tilde{u}_n \in A_p(G/K)$ such that $\rho(\tilde{u}_n) = u_n$, then $\tilde{u}_n \to 0$, but $\widetilde{T}(\tilde{u}_n) \nrightarrow 0$. Hence $\widetilde{T}$ is discontinuous. By the above argument, this is impossible. □

Remark. The class $\mathcal{T}$ was introduced by Bade and Curtis [1] in order to handle both homomorphisms and derivations simultaneously in their investigation of automatically continuous linear functions. They also considered the linear map T from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule which satisfies Leibniz rule of order n. That is, there exist operators T_j and $\widetilde{T}_j$, $j = 1, \dots, n-1$, such that T_j and $\widetilde{T}_j$ satisfy a Leibniz rule of order j and

$$T(ab) = aT(b) + \sum_{j=1}^{n-1} T_j(a)\widetilde{T}_{n-j}(b) + T(a)b.$$

In particular, if $T_j = \widetilde{T}_j$, then $\{T_1, \dots, T_n\}$ is called a higher derivation of rank n. It is a simple induction argument to show that if S satisfies a Leibniz rule of order n then S is bounded if every $T: \mathcal{A} \to X$ of class $\mathcal{T}$ is bounded.

Bade and Curtis also observe that if X is a separable Banach space with an ordered Schauder basis $\{x_i\}$ and if $\mathcal{A}$ is an algebra such that every operator that satisfies a Leibniz rule of order n is continuous, then every homomorphism ρ from $\mathcal{A}$ into $\mathcal{B}(X)$ for which $\rho(u)$ is upper triangular for all $u \in \mathcal{A}$ is continuous (see [1, pp. 99–100]).

Theorem 4.7. *Let G be a compact group. Let $1 < p < \infty$. Let X be a Banach $A_p(G)$-bimodule. If $T: A_p(G) \to X$ is a linear map which satisfies a Liebniz rule of order n, then T is continuous. In particular, if $\{T_1, \dots, T_n\}$ is a derivation of rank n on $A_p(G)$, then each T_i is continuous.*

Corollary 4.8. *Let G be a compact group. Let $1 < p < \infty$. Let X be a Banach space with an ordered Schauder basis. If $\rho: A_p(G) \to \mathcal{B}(X)$ is a homomorphism for which each $\rho(u)$, $u \in A_p(G)$, can be represented by an upper triangular matrix, then ρ is continuous.*

In [26], Warner and Whitley proved that if X is a closed subspace in $L^1(\mathbb{R}) \cong A(\mathbb{R})$ of codimension n for which every $f \in X$ belongs to at least n distinct maximal ideals, then X is an ideal. In particular, $X \cong I(\{x_1, \dots, x_n\})$ for some subset $\{x_1, \dots, x_n\}$ of $\mathbb{R}$. This result is related to a classic theorem of Gleason-Kahane-Zelazko for subspaces of codimension 1 in unitary Banach algebras (see [13, 20]). Warner and Whitley ask for which locally compact abelian groups does $L^1(G)$ and hence $A(G)$ have this property. Recent work of Chen and Cohen [2] has led to the solution of this question. Using a recent result of Rao, we may prove the following.

Proposition 4.9. *Let G be a locally compact group. Let $1 < p < \infty$. Assume that g is σ-compact and separable. Then $A_p(G)$ has property $(*)$:*

$(*)$ If X is a closed subspace of $A_p(G)$ of codimension n which is such that every $u \in X$ vanishes at least n distinct points in G, then X is an ideal in $A_p(G)$. In particular, $X = I(\{x_1, \dots, x_n\})$ for some $\{x_1, \dots, x_n\} \subset G$.

Moreover, if $A_p(G)$ has property $(*)$, then G is σ-compact and separable.

Proof. The algebras $A_p(G)$ are all semisimple, selfadjoint regular Banach algebras. If G is separable, then $\{x\}$ is a G_δ-set for each $x \in G$. Finally, since G

is σ-compact, the statement follows from [25, Theorem 2.3] and the fact that finite subsets of G are s-sets for $A_p(G)$.

Conversely, if G is not σ-compact, then since $A_p(G) \subset C_p(G)$, every function $u \in A_p(G)$ vanishes at infinitely many points of G. Hence $(*)$ cannot hold.

If G is not separable and $u(x_0) = 0$ for some $u \in A_p(G)$ and $x_0 \in G$, then there exists a compact normal subgroup K of G, $K \neq \{e\}$, such that $u(x_0k) = 0$ for every $k \in K$. Hence $A_p(G)$ is not strongly separating and thus $(*)$ fails (see [25, p. 242]). □

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