A MAXIMALLY PATHOLOGICAL BROUWER HOMEOMORPHISM

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ABSTRACT. This paper constructs a Brouwer homeomorphism which does not act properly discontinuously on any nonempty invariant closed connected set in $\mathbb{R}^2$.

1. BACKGROUND

A Brouwer homeomorphism, h, is an orientation preserving homeomorphism of the plane which has no fixed point. Such homeomorphisms can be surprisingly more complex than simple translation, and, as the example presented here demonstrates, quite pathological. Before getting into the gory details of pathological Brouwer homeomorphisms, we will review a few classical results of Brouwer Theory, as presented in [B2] and [G].

Recall that a *translation arc* for $x \in \mathbb{R}^2$ is an arc $\alpha \subset \mathbb{R}^2$ joining x to $h(x)$ such that $\alpha \cap h(\alpha) = \{h(x)\}$. Furthermore, for any Brouwer homeomorphism, every point $x \in \mathbb{R}^2$ has a translation arc. Brouwer's Lemma says that for any Brouwer homeomorphism, h, and any translation arc, α, for h, $h^n(\alpha) \cap \alpha = \varnothing$ for $|n| > 1$. Thus, $A = \bigcup_{n=-\infty}^{\infty} h^n(\alpha) \subset \mathbb{R}^2$ is an embedded line, called a *translation line*. We will say that a homeomorphism, $h : \mathbb{R}^2 \to \mathbb{R}^2$, acts *properly discontinuously* on a set $B \subset \mathbb{R}^2$ if for every compact set $K \subset B$, the set $\{n \in \mathbb{Z} | h^n(K) \cap K \neq \varnothing\}$ is finite. Note that K need not be connected. Also, note that translation acts properly discontinuously on all sets.

In simple translation (say $(x, y) \to (x + 1, y)$), a translation line, A, is a proper subset of $\mathbb{R}^2$ with the following four nice properties:

(1) A is closed.
(2) A is connected.
(3) A is invariant.
(4) A is acted upon properly discontinuously.

In fact, any translation line which is properly embedded has these four properties. However, in the general case, A is not necessarily properly embedded, so A may not be closed.

Note that we can always find subsets for a Brouwer homeomorphism with any three of the above four properties. Any translation arc, α, contains no limit points of $A - \alpha$, other than the end points of α for sequences in $h(\alpha)$ and $h^{-1}(\alpha)$, and there is an ϵ-neighborhood of α which is "translated" along with

α (see e.g. [B1]). Thus A always has properties (2), (3), and (4). Furthermore, the closure of A always has properties (1), (2), and (3). A result related to Brouwer's lemma is: If h is a Brouwer homeomorphism, and $x \in \mathbb{R}^2$, then the orbit of x, $O(x) = \{h^n(x) : n \in \mathbb{Z}\}$ has no accumulation points in $\mathbb{R}^2$. Thus, the orbit of a point has properties (1), (3), and (4). A single point has properties (1), (2), and (4).

Main Result. *There is a Brouwer homeomorphism, $h : \mathbb{R}^2 \to \mathbb{R}^2$, which fails to act properly discontinuously on any proper closed connected invariant subset.*

By the facts presented above, no attribute can be eliminated from the preceding sentence. Such a Brouwer homeomorphism is maximally pathological in the sense that it acts unlike translation on all closed connected invariant sets. This answers the question raised by Brown, Slaminka, and Transue in [BST], where they present an example of a Brouwer homeomorphism which has no closed invariant line.

For illustrative purposes, we will first consider a relatively simple Brouwer homeomorphism, then complicate it to get a Brouwer homeomorphism with almost the properties we want, and finally complicate that Brouwer homeomorphism to get one which has all the desired properties. The construction of these examples, particularly the second, was inspired by [BST], although familiarity with that paper is not assumed here.

2. Simple example

The simple example, h_1, is a composition of two homeomorphisms, T and S_1, where T is unit left translation, and S_1 is described below.

S_1 preserves the leaves of the foliation of $\mathbb{R}^2$ by the lines $y = x + b$ where $b \in \mathbb{R}$. It fixes the integer points along the x-axis, and all points sufficiently far away from these fixed points will move 2 units to the right in the x-direction (over 2 and up 2 since S_1 preserves leaves). Between the integer points on the x-axis and the points which are sufficiently far away, we just "fill in" continuously (being sure not to introduce any more fixed points). (See Figure 1.)

Note that $h_1 = TS_1$ is essentially translation with some squeezing and stretching. However, h_1 is interesting in that the behavior of its orbits fall into four distinct classes. First is the unique orbit consisting of the integer points in the x-axis, which moves from right to left. Second are the orbits of points between the leaves which contain the integer points. These points primarily move from the lower left to the upper right, although they may take a slight detour right to left for a few iterations near the x-axis. Third are orbits of points on the bottom half of leaves which contain integer points. These points start in the lower left, move up and right until they near the integer points on the x-axis where they move out to the left, getting asymptotically close to the integer points on the x-axis. Finally, the orbits of points in the top half of the leaves containing the integer points start near the integer points on the x-axis to the right, move to the left, and then get away from the integer points on the x-axis, moving up and to the right.

While the opposing left-to-right and right-to-left movement appears interesting, it all "goes off to ∞" (i.e., any compact set containing right moving and left moving points is eventually mapped outside any compact set), and, as we noted above, we are left with what is basically translation. What we need is
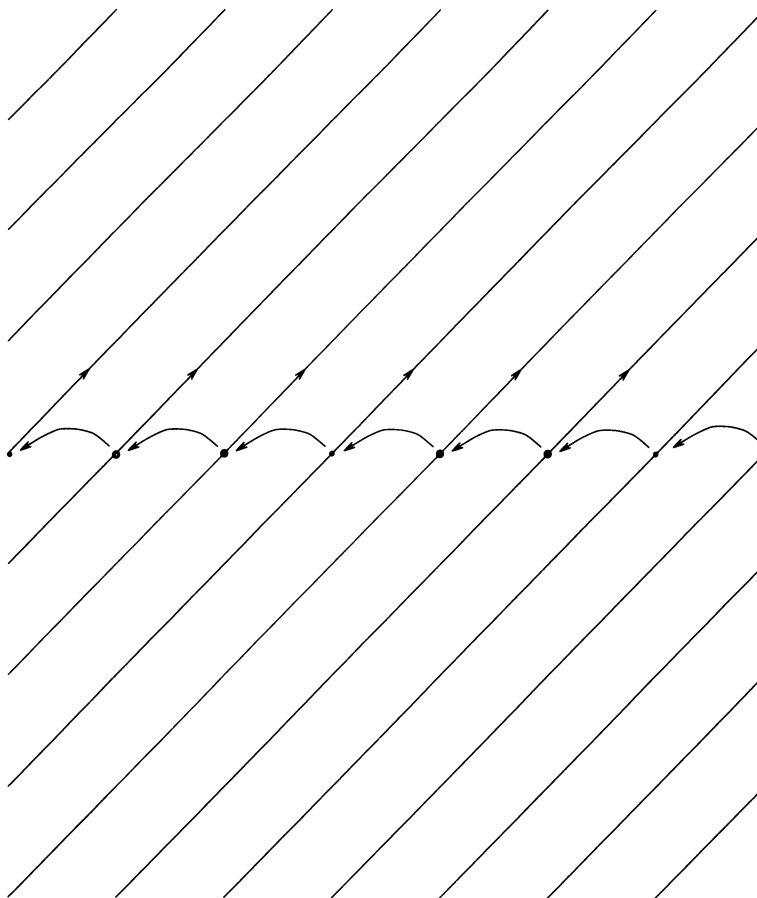
FIGURE 1

some way to "bound" this behavior so it does not "escape." One way to do this is to "shove spikes" from the top and bottom into the foliation we used for S_1. This produces the second example, h_2, which fails to admit a nonempty closed compactly connected invariant subset upon which h_2 acts properly discontinuously.

3. DEFINITIONS OF SECOND AND THIRD EXAMPLES

When we say a set, M, is *compactly connected*, we mean that any two points in M are contained in a connected compact subset of M.

To define the second example, h_2, we need to make the concept of "shoving in spikes" precise. The spikes are the set $S = \{(x, y)|x \in \mathbb{Z}, y \leq -1$ or $y \geq 1\}$. We define a map $R : \mathbb{R}^2 \to (\mathbb{R}^2 - S)$ which preserves vertical lines by $R((x, y)) = (x, |2\operatorname{frac}(x) - 1|(y/|y|)(1 - 2^{-|y|}) + (1 - |2\operatorname{frac}(x) - 1|)y)$ if $y \neq 0$, and $R((x, y)) = (x, 0)$ if $y = 0$. R sends the foliation which we used for h_1 (Figure 1) to one which looks like Figure 2. Let g_λ be the leaf of this foliation which intersects the x-axis at $-\lambda$.

We can define h_2 by $h_2 = Rh_1R^{-1}$ on $\mathbb{R} - S$ and by $h_2((x, y)) = (x + 1, y)$
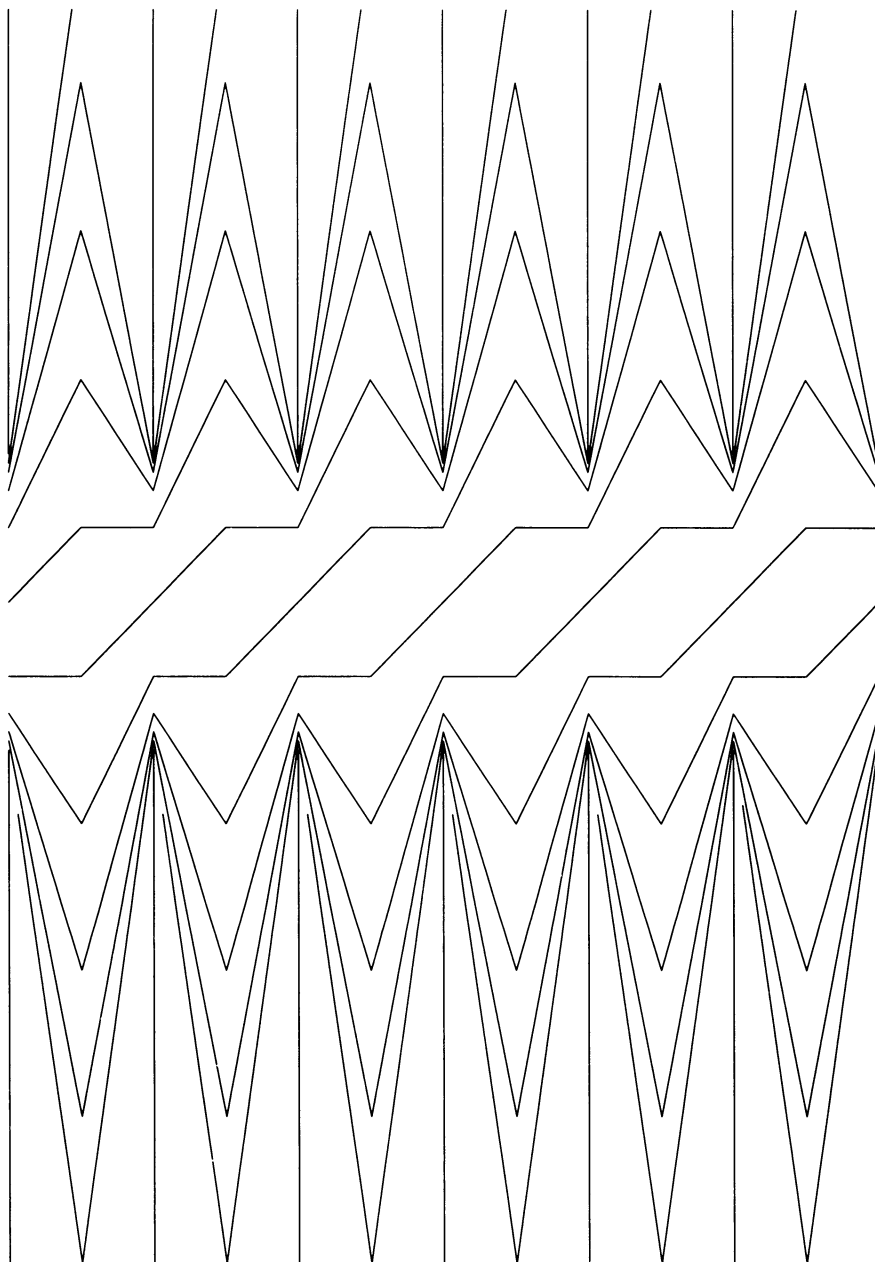
FIGURE 2

on S. As we shall see, h_2 has no proper closed compactly connected invariant subset of $\mathbb{R}^2$ which is acted upon properly discontinuously.

To complicate h_2 into the final example, we compose h_2 with a homeomorphism, D, made up of infinitely many nonoverlapping Dehn twists, i.e., $h_3 = Dh_2$. h_3 will not act properly discontinuously on any proper closed connected invariant set.

We will construct D as follows: Let d_0 be a Dehn twist which acts on the topological annulus, A_0, which surrounds an ϵ-neighborhood ($\epsilon \ll 1$) of
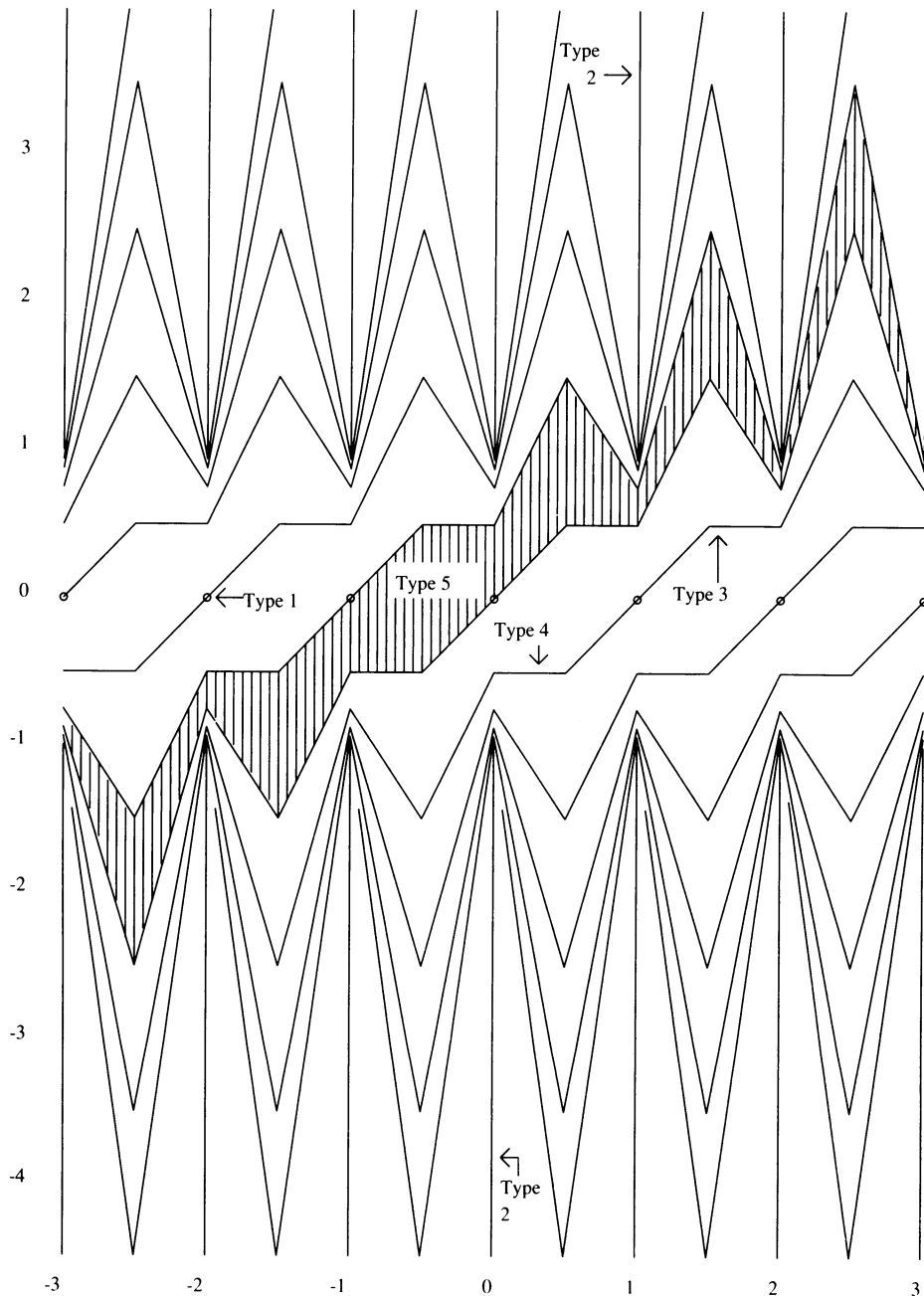
FIGURE 3

the segment $[-2, 2]$ on g_0; A_0 is contained in a 2ϵ-neighborhood of this segment, and has "thickness" $\delta < \epsilon$. Let A_i and d_i, $i \in \mathbb{Z}$, be the right and left translates of A_0 and d_0 such that $A_i \cap g_i \neq \varnothing$. D is defined to be the infinite composition of the d_i's, i.e., $D|_{A_i} = d_i$ and D is the identity elsewhere.

To get a better picture of the dynamics of h_2 (and consequently the dynamics of h_3), look at the following regions, which we call fundamental (after [A]) (see Figure 3):

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(1) Each integer point on the x-axis.

(2) Each spike $\{(x, y) : x = n, y \geq 1\}$ or $\{(x, y) : x = n, y \leq -1\}$ where $n \in \mathbb{Z}$.

(3) The right half of each integer leaf of the foliation, g_n, $n \in \mathbb{Z}$, with $x > -n$.

(4) The left half of each integer leaf of the foliation, g_n, $n \in \mathbb{Z}$, with $x < -n$.

(5) Each region between two successive integer leaves of the foliation, g_n and g_{n+1}, $n \in \mathbb{Z}$.
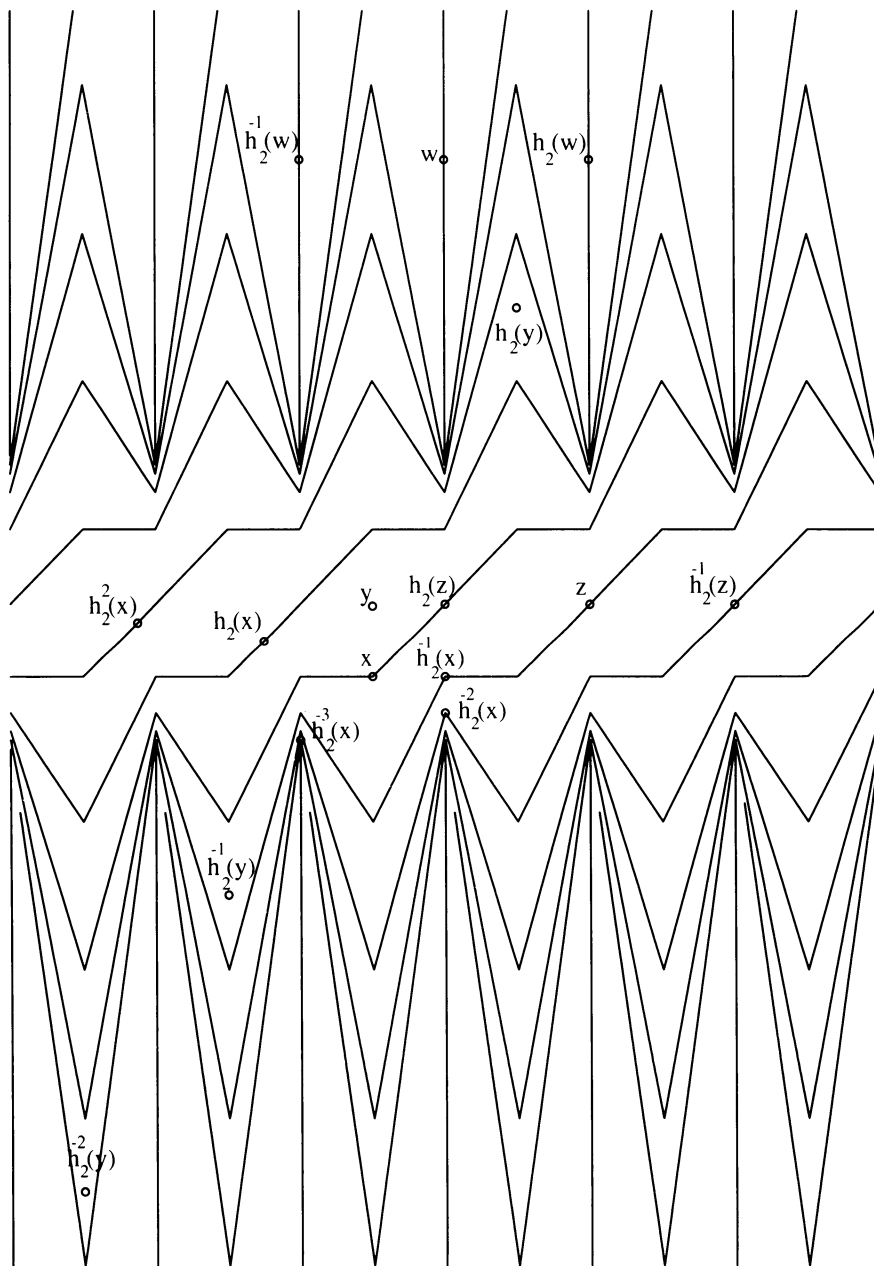

FIGURE 4

Now, examine the images of points in these regions under successive forward and backward iterations of h_2 (see Figure 4). Note that no region is setwise fixed, and each is taken to another of the same type. Points that make up a region of type 1 move from right to left under forward iterations, and the union of all such points, which we will call P, is setwise fixed and closed. Points in regions of type 2 move from left to right, and the union of the spikes in the top half-plane, which we will call M, is setwise fixed and closed, as is the union of the spikes in the bottom half-plane, which we will call N. Points in regions of type 3 eventually move to the right under both forward and backward iterations of h_2. Under forward iterations, these points converge to the set M, i.e., given an x in a region of type 3, for all $\epsilon > 0$, there exists $n \in \mathbb{Z}$ such that $(h_2)^m(x)$ is within the ϵ-neighborhood of $M \cup \{(x,y)|y \geq 1/\epsilon\}$ for all $m > n$. Under backward iterations, these points converge to the set P. Similarly, points in regions of type 4 move to the left under both positive and negative iterations of h_2. Under positive iterations, these points converge to the set P. Under negative iterations, these points converge to the set N. Points in regions of type 5 eventually move from left to right under forward iterations of h_2. Under positive iterations, such a point will converge to the set M, and under negative iterations, it will converge to the set N.

The infinitely many Dehn twists we composed h_2 with to get h_3 cause there to be infinitely many different types of fundamental regions for h_3. As a result, rather than doing a detailed analysis of each type of fundamental region, we will simply note that, away from the very thin Dehn twists which make up D, h_3 behaves the same as h_2.

4. Key argument

We will say that a set, I, has *x-divergent points* under a Brouwer homeomorphism, h, if there are points $w, z \in I$ such that either the x-coordinate of $h^n(w)$ goes to $+\infty$ and the x-coordinate of $h^n(z)$ goes to $-\infty$ as $n \to +\infty$ or the x-coordinate of $h^n(w)$ goes to $+\infty$ and the x-coordinate of $h^n(z)$ goes to $-\infty$ as $n \to -\infty$. The key idea in showing that h_2 and h_3 have the stated properties is contained in the following two claims:

Claim 1. If I is a compact connected set in $\mathbb{R}^2 - S$, and I has x-divergent points under h_2 (or h_3), then $h_2^n(I)$ (or $h_3^n(I)$) intersects, for infinitely many n, any vertical segment connecting a top spike to a bottom spike, i.e., $V_m = (x,y) : x = m, -1 < y < 1$, where $m \in \mathbb{Z}$.

Proof. Since h_2 and h_3 restricted to $\mathbb{R}^2 - S$ are homeomorphisms of $\mathbb{R}^2 - S$ to $\mathbb{R}^2 - S$, $h_2^n(I) \subset \mathbb{R}^2 - S$ and $h_3^2(I) \subset \mathbb{R}^2 - S$. Furthermore, since h_2 and h_3 are continuous and I is connected, $h_2(I)$ and $h_3(I)$ are connected. Each V_m separates $\mathbb{R}^2 - S$. Since I contains x-divergent points, $h_2^n(I)$ (or $h_3^n(I)$) must contain a point on either side of each V_m for some n, and thus, since the V_m separate $\mathbb{R}^2 - S$, intersect that V_m. Without loss, assume that I contains x-divergent points as $n \to +\infty$. Then if $h_2^n(I)$ (or $h_3^n(I)$) has a "left moving" point, z, to the left of some V_m and a "right moving" point to the right of the same V_m, then $h_2^n(I)$ (or $h_3^n(I)$) must intersect that V_m for all $n > N$ (see Figure 5). Thus we have infinitely many images of $h_2^n(I)$ (or $h_3^n(I)$) intersecting V_m. □
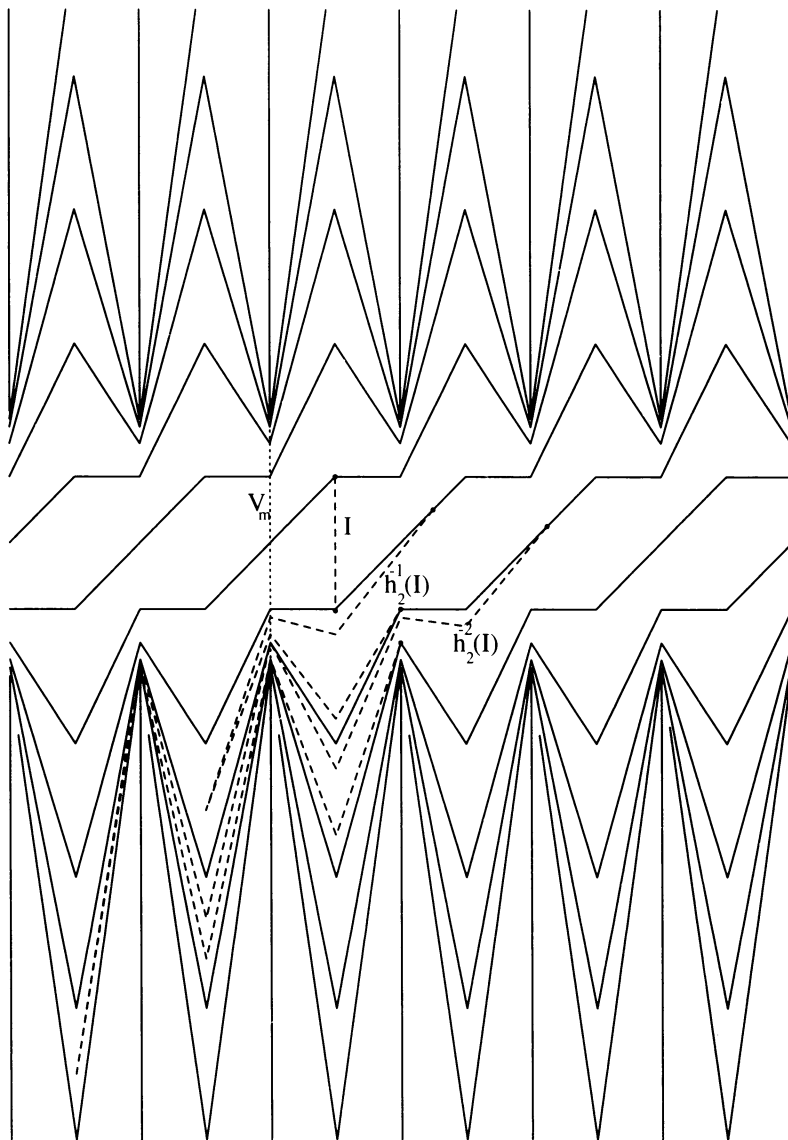
FIGURE 5

Claim 2. If a compact connected I with x-divergent points is contained in a closed invariant (under h_2 (or h_3)) subset, J, then h_2 (or h_3) cannot act properly discontinuously on J

Proof. Since J is invariant, and $I \subset J$, all the images of I are in J. If we take a compact set, $K \subset \mathbb{R}^2$ such that some $V_m \subset K$, then $K \cap J$ is compact (since J is closed). So $I \cup (K \cap J)$ is a compact subset of J. Since $h_2^n(I) \cap K \neq \varnothing$ for infinitely many n, and $h_2^n(I) \cap K \subset J$, $h_2^n(I \cup (K \cap J)) \cap (I \cup (K \cap J)) \neq \varnothing$ for infinitely many n. Thus h_2 does not act properly discontinuously on J. The argument is the same for h_3. $\square$

5. The compactly connected case

It is now left to show that all closed compactly connected invariant subsets under h_2 and all closed connected invariant subsets under h_3 contain a compact connected set, I, with x-divergent points. We will first consider the case where our closed invariant set, J, is compactly connected.

Note that while h_2 preserves the foliation by g_λ's shown in Figure 2 and translates the leaves, h_3 does not preserve this foliation. h_3 does, however, preserve the leaves outside a 2ϵ-neighborhood of the leaves with integer subscripts (the integer leaves), and translates them. For, example, both h_2 and h_3 take $g_{1/2}$ to $g_{3/2}$. We will call the leaves, $g_{n+1/2}$, half way between two successive integer leaves, g_n and g_{n+1}, half-integer leaves.

Claim 3. Any connected set, C, in $\mathbb{R}^2$ which contains points on both sides of a properly embedded strip, B (e.g. $\{(x, y) \in \mathbb{R}^2 | 0 \le x \le 1\}$), must contain a connected subset $C' \subset C \cap B$ which contains a point in C on a side of B and a point in the interior of B.

Proof. Consider the connected components of $C \cap B$ and $C \backslash (\text{interior } B)$. These components form a set of connected subsets $\{C_\alpha\}$ such that $\bigcup C_\alpha = C$. Note that each C_α must contain a point in one of the sides of B. Suppose that no C_α contains a point in the interior of B. Then C is disjoint since we assumed it contained points on both sides of B. We can let C' be any C_α which contains a point in the interior of B. $\square$

Claim 4. Any nonempty connected invariant set, J, under h_2 (or h_3) must contain a point on every half-integer leaf (in fact, every leaf) of the foliation by g_λ's.

Proof. If J contains a point on a half-integer leaf, $g_{n+1/2}$, then the image under h_2 of that point is on $g_{n+3/2}$ and the preimage is on $g_{n-1/2}$, so iteration gives us a point on every half-integer leaf. If J contains a point between two half-integer leaves, the images of that point must also be in J. But the images of that point must be between different pairs of half-integer leaves, so since there is some half-integer leaf between any two images of such a point and the leaf divides the plane into two parts, J must have a point on a half-integer leaf. If J contains a point in a region of type 2, again, the images of that point, which are in different regions of type 2, must also be in J. But each region of type 2 is disconnected from all other regions of type 2, so J must contain a point outside of a region of type 2. Thus J always has a point on each half-integer leaves. Since any other leaf separates half-integer leaves, J must contain a point on every leaf. $\square$

Claim 5. If J is a compactly connected closed invariant set under h_2 (or h_3), then there is a $I \subset J$ such that I is compact, connected, and contains x-divergent points.

Proof. By Claim 4, there are points in J on successive half-integer leaves, $g_{n-1/2}$ and $g_{n+1/2}$. Since J is compactly connected, we can let I be a compact connected set containing points in $g_{n-1/2}$ and $g_{n+1/2}$. On the leaves $g_{n-1/2}$ and $g_{n+1/2}$, h_2 (and h_3) increases the x-coordinate of every point by 1 under each positive iteration (and therefore decreases the x-coordinate of every point

in these sets by 1 under each negative iteration). Thus, for $w \in g_{n-1/2}$ or $w \in g_{n+1/2}$, the x coordinate of $h_i^m(w)$ goes to $+\infty$ as $m \to +\infty$ for $i = 2, 3$, and the x-coordinate of $h_i^m(w)$ goes to $-\infty$ as $m \to -\infty$ for $i = 2, 3$. So, we need to show that there is some other point, $z \in I$ such that the x-coordinate of $h_i^m(z)$ goes to $-\infty$ as $m \to +\infty$ or the x-coordinate of $h_i^m(z)$ goes to $+\infty$ as $m \to -\infty$ for $i = 2, 3$.

First we will consider h_2. Note that since g_n separates $g_{n-1/2}$ and $g_{n+1/2}$, I must contain a point on g_n. Under h_2, g_n is composed of points of type 1, type 3, and type 4. As we noted in §3, if z is of type 1 or type 4, the x-coordinate of $h_2^m(z)$ goes to $-\infty$ as $m \to +\infty$, and if z is of type 1 and type 3, the x-coordinate of $h_2^m(z)$ goes to $+\infty$ as $m \to -\infty$. Thus I contains x-divergent points under h_2.

Now, we consider what happens when J is invariant under h_3. The idea is the same as in the h_2 case: there is some set between $g_{n-1/2}$ and $g_{n+1/2}$ which separates them and consists of points which are x-divergent from the points in $g_{n-1/2}$ and $g_{n+1/2}$ under h_3. The set is, however, more complicated than g_n.

Let B_n be the bounded component of $\mathbb{R}^2 - A_n$. Consider $B_n \cap g_n$. We can divide this set into three parts, Bg_n^1, the point on the x-axis (which is of type 1 under h_2), Bg_n^3, the points above the x-axis (which were of type 3 under h_2), and Bg_n^4, the points below the x-axis (which were of type 4 under h_2). Now, one can easily see that $h_3^i(Bg_n^1 \cup Bg_n^3) \subset B_{n+i} \cap g_{n+i}$ for $i < 0$ and $h_3^i(Bg_n^1 \cup Bg_n^4) \subset B_{n+i} \cap g_{n+i}$ for $i > 0$, thus these points are x-divergent from those in $g_{n+1/2}$ and $g_{n-1/2}$. However, it is not only these points, but the points in $B_k \cap g_k$ for $k \in \mathbb{Z}$ and all the images of these points, $h_3^j(B_k \cap g_k)$ for $j \in \mathbb{Z}$, which are x-divergent from $g_{n+1/2}$ and $g_{n-1/2}$. In particular, the set $\bigcup_{i=-\infty}^{+\infty} h_3^{-i}(B_{n+i} \cap g_{n+i})$ separates $g_{n+1/2}$ and $g_{n-1/2}$ and consists of points which are x-divergent from the points in these two leaves. Thus, any compact connected set which contains points in both $g_{n+1/2}$ and $g_{n-1/2}$ must contain x-divergent points under h_3. $\square$

Thus Claim 2 and Claim 5 directly imply the following:

Initial Result. h_2 *and* h_3 *do not act properly discontinuously on any nonempty closed compactly connected invariant subset of* $\mathbb{R}^2$.

Note that invariance under h_2 and h_3 are not equivalent.

6. Connected but not compactly connected sets

Now, let us consider what connected but not compactly connected sets look like. The example to keep in mind is the fish bone set (see Figure 6), $F = A \cup B \cup C \cup D \cup E$, where

(1) $A = \{$ segments from $(0, n)$ to $(1 - 1/n, 0)$ for $n \in \mathbb{Z}$, $n \geq 2\}$,
(2) $B = \{$ segments from $(0, n)$ to $(1/n - 1, 0)$ for $n \in \mathbb{Z}$, $n \geq 2\}$,
(3) $C = \{(0, y) | y \in \mathbb{R}, y \geq 0\}$,
(4) $D = \{(1, y) | y \in \mathbb{R}, y \geq 0\}$,
(5) $E = \{(-1, y) | y \in \mathbb{R}, y \geq 0\}$.

We have the following claim about the structure of closed connected but not compactly connected sets:
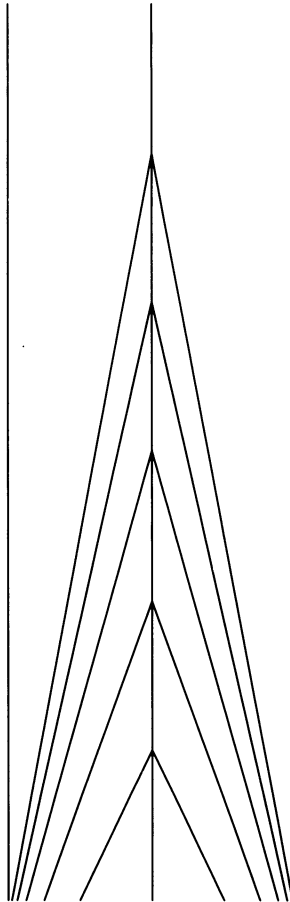
FIGURE 6

Claim 6. Closed connected but not compactly connected sets are composed of disjoint unbounded maximal compactly connected subsets, called *ray sets*.

Proof. Clearly, if J is a connected but not compactly connected set, we may separate J into compactly connected components (points are compactly connected). Any two maximal compactly connected components must be disjoint because if X and Y are two such sets and $z \in X \cap Y \neq \varnothing$ then for any $x \in X$ and $y \in Y$, there are compact subsets $X' \subset X$ and $Y' \subset Y$ which connect x to z and y to z. So $X' \cup Y'$ is a compact set which connects x to y.

Let J_0 be a maximal compactly connected component of a closed connected but not compactly connected set, J, and suppose that J_0 is bounded. If $\{x_i\}$ is a sequence in J_0 with accumulation point x_0, then, since J is closed, $x_0 \in J$, and, since J_0 is bounded and compactly connected and $J_0 \cup x_0$ must be connected, J_0 must contain x_0. Thus J_0 is closed.

Now, since J is connected and $J \neq J_0$, there must be a sequence $\{y_i\}$ not in J_0 which converges to a point in J_0. We will divide the ways this sequence can converge into three cases.

Case 1. Suppose $\{y_i\}$ is contained in some other maximal compactly connected component, J_1 (see Figure 7A). Let K be the set of points, x, such that there are sequences in both J_0 and J_1 which converge to x. If any connected

component, K', of K is compact, then $J_0 \cup K'$ and $J_1 \cup K'$ are compactly connected and so $J_0 \cup J_1 \cup K'$ ($= J_0 \cup J_1$) is compactly connected. But this contradicts J_0 maximal. If no component of K is bounded, we get a contradiction of J_0 bounded, since J_0 is closed.

Case 2. We may assume that each point of the sequence $\{y_i\}$ is on a different compactly connected component of J. Suppose that infinitely many y_i's are in unbounded compactly connected components. We may then assume that each y_i is in an unbounded J_i (see Figure 7B). Let us look at an ϵ-neighborhood and a 2ϵ-neighborhood of J_0. Let L_i be the connected component of $J_i \cap 2\epsilon$-neighborhood of J_0 which contains y_i. Now the L_i are compact connected sets which are all contained in a compact set, so they must converge to a compact connected set, L_0. Since we know the L_i contain a sequence which converges to a point in J_0, L_0 must be in J_0. However, by a modification of Claim 3 for the annulus, all L_i must contain points in the 2ϵ-neighborhood minus the ϵ-neighborhood, and therefore so must the convergence set, giving us a contradiction.

Case 3. We may assume that for every sequence, $\{y_i\}_\alpha$, each point of that sequence is on a different bounded compactly connected component. Consider all sequences $\{y_i\}_\alpha$ with each point on a different component, J_{i_α}, which converge to a point in J_0. We will say J_{i_α} is *closer* to J_0 than J_{k_β} if there is an ϵ-neighborhood of J_0, $N_\epsilon(J_0)$, for some ϵ such that $J_{i_\alpha} \cap N_\epsilon(J_0) \neq \varnothing$ and $J_{k_\beta} \cap N_\epsilon(J_0) = \varnothing$. If neither J_{i_α} nor J_{k_β} is closer to J_0 than the other, we will say J_{i_α} is as close as J_{k_β} to J_0. We need the following subclaim to complete this case.

Subclaim. For every $\epsilon > 0$, there is a J_{I_α} such that $J_{I_\alpha} \subset N_\epsilon(J_0)$ and if J_{i_β} is either closer to J_0 than J_{I_α} or as close as J_{I_α} to J_0, then $J_{i_\beta} \subset N_\epsilon(J_0)$.

Proof of subclaim. Suppose there is no such J_{I_α}. Then there is a sequence of J_{i_β}'s which contains a sequence of points with an accumulation point in J_0 and a sequence with an accumulation point outside of $N_\epsilon(J_0)$. By passing to convergent subsequences and applying reasoning similar to that in case 2, this gives a contradiction. □

Pick an ϵ and let J_{I_α} be a component as in the above claim (see Figure 7C). Let $L = J_0 \cup J_{I_\alpha} \cup \{$ all J_{i_β} closer to J_0 than $J_{I_\alpha}\}$. Note that $L \subset N_\epsilon(J_0)$ and that there is a $\delta < \epsilon$ such that $J \cap N_\delta(J_0) \subset L$.

L is closed because each J_{i_α} is closed and all the J_{i_β} components are compact and connected and all contained in an ϵ-neighborhood of J_0, so any sequence of J_{i_β}'s which converges must converge to a compact connected set, K, in $N_\epsilon(J_0)$. Now, since J is closed, $K \subset J$. Furthermore, since every J_{i_β} contains a point closer than or as close as J_{I_α} to J_0, there must be a sequence with an accumulation point as close as or closer than J_{I_α}. But this accumulation point must be in K, so $K \subset L$. Thus L is compact.

Now in a compact set, the compactly connected components are the connected components, so J_0, J_{I_α}, and the J_{i_β}'s are connected components of L. Connected components have arbitrarily small closed and open neighborhoods around them, so, in particular, there must be such a neighborhood around J_0 in L. But $N_\delta(J_0) \cap J \subset L$, so there is also such a neighborhood around J_0 in J. But then J_0 is disconnected from the rest of J, contradicting the connectedness of J.
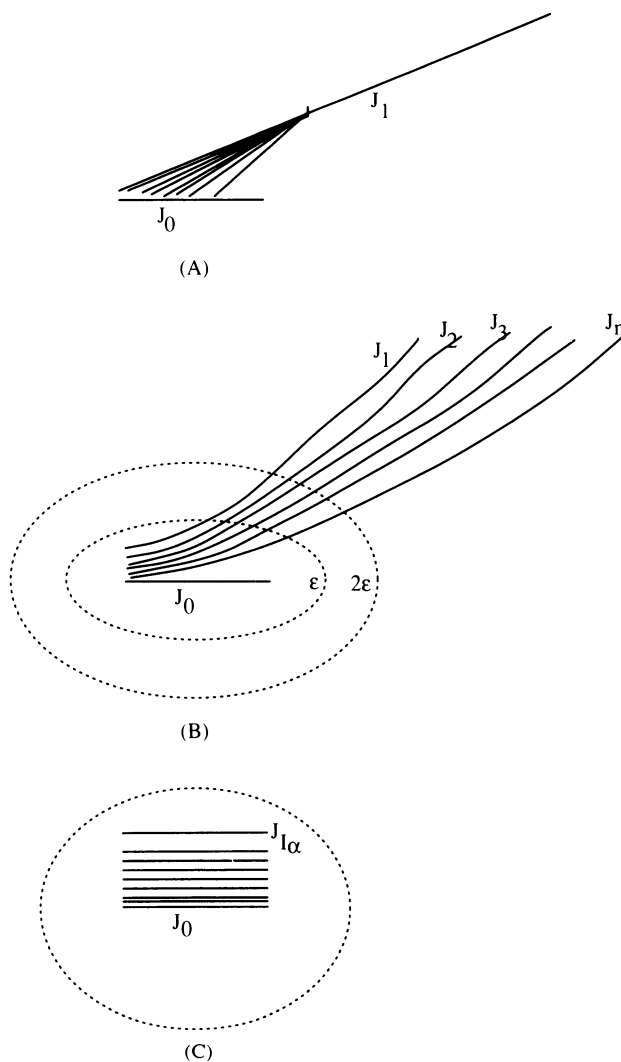
FIGURE 7

Thus there can be no bounded compactly connected components in a closed connected but not compactly connected set. □

7. PROOF OF MAIN RESULT

Now that the structure of connected not compactly connected sets has been established, we are ready to prove the main result:

Main Result. h_3 *does not act properly discontinuously on any nonempty closed invariant subset, J, of $\mathbb{R}^2$.*

Proof. As before, Let B_i be the bounded component of $\mathbb{R}^2 - A_i$, and let X_i be the integer point on the x-axis contained in B_i. Initially, suppose J contains a point, z, inside some B_j on the integer leaf, g_j. Now, either J is compactly connected or z is in some ray set $J_i \subset J$ which is compactly connected and
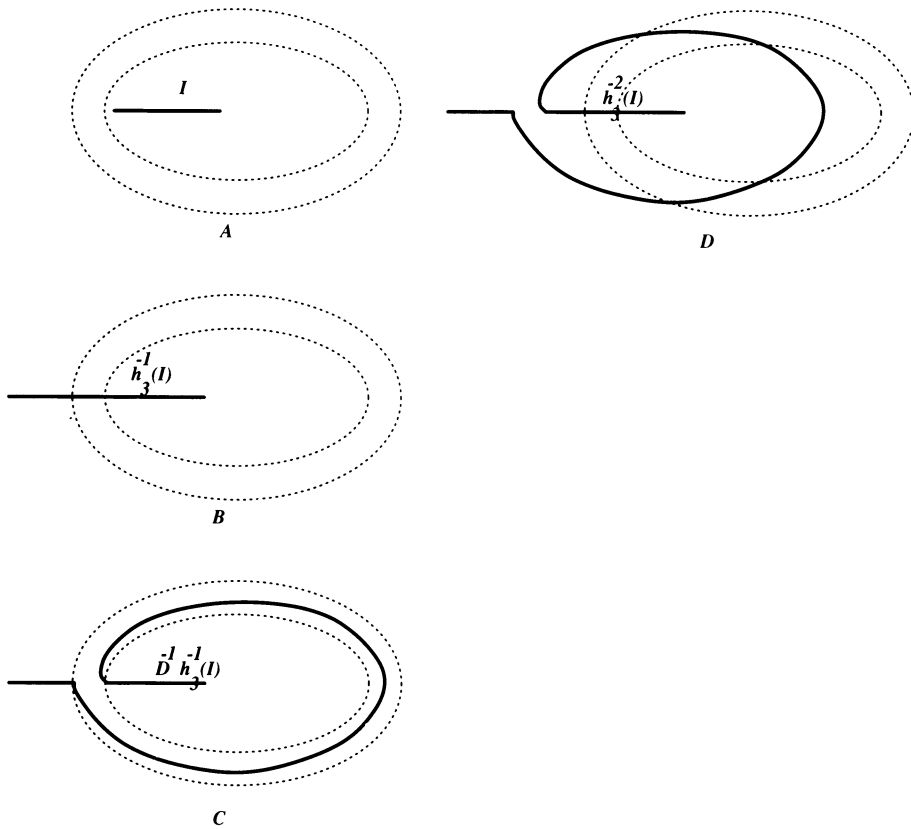
FIGURE 8

unbounded. Either way, z must be compactly connected to other points in B_j, i.e., there is a compact connected subset, I, of J in B_j which contains z and other points in B_j. The proof breaks into two cases, depending on whether or not $I \subset g_j$.

Case 1. Suppose there is a compactly connected $I \subset J \cap B_j$ containing z and a point not in g_j. Without loss, assume that z is on the bottom half of g_j and consider forward iterations of h_3. By Claim 3, we may assume that I is within a $1/2\epsilon$-neighborhood of g_j. $h_3^n(z)$ goes to the left (has decreasing x-coordinate) as $n \to +\infty$, while the points, w, not on g_j will slowly move across the B_k's with $w - X_i$ increasing until $h_3^n(I)$ intersects the boundary of some B_{j+n} within $1/2\epsilon$ of g_{j+n} on the right side of B_{j+n}. Now this section of the boundary of B_{j+n} is mapped out of $B_{j+n+1} \cup A_{j+n+1}$ and to the right under h_3, and all further images of I will contain w's further to the right, with x-coordinates $\to +\infty$ as $n \to +\infty$. So I contains x-divergent points, and eventually gets stretched to the left and right under h_3.

Case 2. Suppose we have a $z \in B_j \cap g_j$, but there is no compactly connected set in B_j for any j connecting a z to a point in $B_j - g_j$. Then, since J is compactly connected or $z \in J_i$, where J_i is some ray set, J must contain a "segment" I on g_j which connects z to the boundary of B_j, ∂B_j (otherwise

we would have a compactly connected component). Since we are assuming that there is no I as in case 1, we may choose I and z so that $h_3^n(z) \in B_{j+n}$ for $-1 \leq n \leq 1$. Without loss, say that I connects z to $\partial B_j \cap g_j$ on the bottom half of g_j (see Figure 8A) and look at negative iterations of h_3. Note that $h_3^{-1} = h_2^{-1}D^{-1}$. $h_3^{-1}(I)$ is contained in g_{j-1} and crosses A_{j-1} (see Figure 8B). $D^{-1}h_3^{-1}(I) \subset g_{j-1} \cup A_{j-1}$ and it runs from z to ∂B_{j-1} along g_{j-1}, wraps once around A_{j-1}, intersecting g_{j-1} on its top half, and leaves A_{j-1} along the bottom half of g_{j-1} (see Figure 8C). Now, the point at which $D^{-1}h_3^{-1}(I)$ intersects the top half of g_{j-1} is mapped to a point in $B_{j-2} \cap g_{j-2}$ on the top half of g_{j-2} by h_2^{-1} and the points in $D^{-1}h_3^{-1}(I)$ near this intersection not on G_{j-1} are mapped to points nearby inside B_{j-2}. Thus $h_3^{-2}(I)$ contains a compact connected set containing a point in $B_{j-2} \cap g_{j-2}$ and a point in $B_{j-2} - g_{j-2}$ (see Figure 8D), which contradicts our assumptions and reduces us to the previous case.

Thus, if h_3 is to act properly discontinuously on J, we must have $J \cap (B_j \cap g_j) = \varnothing$ for all j, so suppose now that this is the case. Now, since J is invariant, we must have $J \cap (\bigcup_{i=-\infty}^{+\infty} \bigcup_{j=-\infty}^{+\infty} h_3^i(B_j \cap g_j)) = \varnothing$. But $\bigcup_{i=-\infty}^{+\infty} h_3^{-i}(B_i \cap g_i)$ divides the plane into two unbounded regions, and, more importantly, separates leaves, g_λ (for $|\lambda - \text{int}(\lambda)| > 2\epsilon$), from their images. Thus, if J is a connected closed invariant set, it must contain a point in this set. Thus h_3 does not act properly discontinuously on any nonempty closed connected invariant subset. $\square$

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