

FLows IN FIBERS

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Dedicated to Professor Morisuke Hasumi on his 60th birthday

ABSTRACT. Let $H^\infty(\Delta)$ be the algebra of all bounded analytic functions on the open unit disc Δ , and let $\mathfrak{M}(H^\infty(\Delta))$ be the maximal ideal space of $H^\infty(\Delta)$. Using a flow, we represent a reasonable portion of a fiber in $\mathfrak{M}(H^\infty(\Delta))$. This indicates a relation between the corona theorem and the individual ergodic theorem. As an application, we answer a question of Forelli [3] by showing that there exists a minimal flow on which the induced uniform algebra is not a Dirichlet algebra. The proof rests on the fact that the closure of a nonhomeomorphic part in $\mathfrak{M}(H^\infty(\Delta))$ may contain a homeomorphic copy of $\mathfrak{M}(H^\infty(\Delta))$. Taking suitable factors, we may construct a lot of minimal flows which are not strictly ergodic.

1. INTRODUCTION

If $|\alpha| = 1$, then the fiber \mathfrak{M}_α of $\mathfrak{M}(H^\infty(\Delta))$ over α is defined to be

$$\mathfrak{M}_\alpha = \{\xi \in \mathfrak{M}(H^\infty(\Delta)); \xi(z) = \alpha\},$$

where z is the coordinate function. Then we have the decomposition

$$\mathfrak{M}(H^\infty(\Delta)) \setminus \Delta = \bigcup_{|\alpha|=1} \mathfrak{M}_\alpha.$$

Since the various fibers \mathfrak{M}_α are homeomorphic to one another, we restrict our attention to the fiber \mathfrak{M}_1 over 1 to look into the fringe $\mathfrak{M}(H^\infty(\Delta)) \setminus \Delta$.

Let Ω be a compact Hausdorff space on which the real line \mathbf{R} acts as a topological transformation group. This means that there is a one-parameter group $\{U_t\}_{t \in \mathbf{R}}$ of homeomorphisms of Ω onto itself such that the map $(\omega, t) \rightarrow U_t \omega$ is continuous on $\Omega \times \mathbf{R}$. The pair $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ is referred to as a (continuous) flow. For each ω in Ω and t in \mathbf{R} , the translate $U_t \omega$ of ω by t is denoted by $\omega + t$. We denote by $O(\omega)$ the orbit of ω , that is,

$$O(\omega) = \{\omega + t; t \in \mathbf{R}\}.$$

Then an ω in Ω is said to be *fixed* if $O(\omega)$ equals $\{\omega\}$. A flow $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ is called *minimal* if each orbit is dense in Ω , so there are no nontrivial closed invariant sets.

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Let $H^\infty(\mathbf{R}_+^2)$ denote the space of all bounded analytic functions in the upper half-plane \mathbf{R}_+^2 , and the space of their boundary-value functions is denoted by $H^\infty(\mathbf{R})$. They are identified each other by Fatou's theorem. Let $C(\Omega)$ be the space of all continuous complex-valued functions on Ω . A function ϕ in $C(\Omega)$ is *analytic* if the function $t \rightarrow \phi(\omega + t)$ lies in $H^\infty(\mathbf{R})$ for each ω in Ω . Let $A(\Omega)$ be the space of all analytic functions in $C(\Omega)$. Then $A(\Omega)$ is a uniformly closed subalgebra of $C(\Omega)$ containing constants. When there are no fixed points, $A(\Omega)$ separates the points on Ω , and Ω is the Choquet boundary of $A(\Omega)$ [13, Theorem I]. Throughout the paper, we shall always assume that $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ is an arbitrary flow without fixed points. Then $A(\Omega)$ becomes a uniform algebra on Ω , which is called the *induced uniform algebra* on $(\Omega, \{U_t\}_{t \in \mathbf{R}})$. Let $\mathcal{M}(A(\Omega))$ be the maximal ideal space of $A(\Omega)$. Although the structure is intractable, it has been investigated with the aid of ergodic theory (see, for instance, [12, 16 and 15]).

Let \mathbf{A} be a uniform algebra on a compact Hausdorff space \mathbf{Y} . Then \mathbf{A} is called a *Dirichlet algebra* on \mathbf{Y} if the real part $\operatorname{Re} \mathbf{A}$ of \mathbf{A} is uniformly dense in the space $C_{\mathbf{R}}(\mathbf{Y})$ of all real-valued functions in $C(\mathbf{Y})$. We say that \mathbf{A} is a *logmodular algebra* on \mathbf{Y} if

$$\log |\mathbf{A}^{-1}| = \{ \log |f|; f \in \mathbf{A}^{-1} \}$$

is uniformly dense in $C_{\mathbf{R}}(\mathbf{Y})$, where \mathbf{A}^{-1} denotes the set of all invertible elements of \mathbf{A} .

The purpose of this paper is to show how flows can be applied usefully to study the structure of $\mathcal{M}(H^\infty(\Delta))$. We first represent a portion of the fiber \mathfrak{M}_1 as a subset of the maximal ideal space of the induced uniform algebra on a flow. Then we point out that there appears a relation between the corona theorem and the individual ergodic theorem. Together with the characterization of analytic structure in $\mathcal{M}(H^\infty(\Delta))$, due to Hoffman, we second construct a minimal flow on which the induced uniform algebra is not a Dirichlet algebra. This provides a negative answer to a question posed by Forelli [3]:

When $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ is minimal, is $A(\Omega)$ a Dirichlet algebra on Ω ?

In the next section, we establish the notation and present some lemmas. Our representation of the fiber \mathfrak{M}_1 over 1 is given in §3 together with some related facts. After preparing some lemmas, we show in §4 the existence of a desired minimal flow. We close with some remarks in §5.

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2. PRELIMINARIES

We begin by collecting some notation and facts about a flow $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ which we shall use throughout. Let $M(\Omega)$ be the space of all bounded complex Borel regular measures on Ω . A measure in $M(\Omega)$ is said to be *quasi-invariant* if every translate of each null set is also a null set. A quasi-invariant measure is said to be *ergodic* if every invariant subset of Ω is either negligible or has negligible complement. Let μ be a positive quasi-invariant measure. A function ϕ in $L^\infty(\mu)$ is said to be *analytic* if the function $t \rightarrow \phi(\omega + t)$ lies in $H^\infty(\mathbf{R})$ for μ -a.e. ω in Ω .

Let us consider the representing measures for the induced uniform algebra $A(\Omega)$ on $(\Omega, \{U_t\}_{t \in \mathbf{R}})$. The Poisson kernel $P_z(t)$ for \mathbf{R}_+^2 is defined by $P_z(t) = s/\pi(s^2 + (t-u)^2)$, where $z = u + is$. For each $s > 0$ and for a point $\omega + u$ in the orbit $O(\omega)$ of ω , we define a homomorphism of $A(\Omega)$ by

$$(2.1) \quad \xi_z(\phi) = \phi * P_{is}(\omega + u) = \int_{-\infty}^{\infty} \phi(\omega + t) P_z(t) dt.$$

Then the product $O(\omega) \times (0, \infty)$ may be regarded as a subset of $\mathfrak{M}(A(\Omega))$ by (2.1). Since the representing measure for each point in $O(\omega) \times (0, \infty)$ is absolutely continuous to another one, $O(\omega) \times (0, \infty)$ makes a nontrivial Gleason part. On the other hand, there are representing measures of a different kind. Indeed, the Markov-Kakutani theorem shows that there is at least one invariant probability measure on Ω . Together with the Krein-Milman theorem, this assures the existence of an invariant, ergodic, probability measure m on Ω . Then m is a representing measure for $A(\Omega)$ [11, Theorem I]. It is known that every representing measure for $A(\Omega)$ is quasi-invariant if it is not a point mass [11, Theorem III].

Let $\beta\mathbf{Z}$ be the Stone-Čech compactification of the group \mathbf{Z} of integers, and let S_0 be the shift operator on \mathbf{Z} , which is defined by $S_0 n = n + 1$ for each n in \mathbf{Z} . It then follows from the Banach-Stone theorem [1, Chapter V, 8.8] that S_0 extends to a homeomorphism S of $\beta\mathbf{Z}$ onto itself. We denote by \mathbf{X} the quotient space obtained from $\beta\mathbf{Z} \times [0, 1]$ by identifying $(y, 1)$ with $(Sy, 0)$ for each y in $\beta\mathbf{Z}$. Then \mathbf{X} is a compact Hausdorff space. Let us define a flow $(\mathbf{X}, \{S_t\}_{t \in \mathbf{R}})$ by

$$(2.2) \quad S_t(y, s) = (S^{[t+s]}y, t + s - [t + s]),$$

where $[t]$ denotes the largest integer not exceeding t . We notice that the flow is far from a minimal one, since there exist closed invariant sets in profusion. Then there are many invariant, ergodic, probability measures on $M(\mathbf{X})$. We not only see that there are no fixed points in \mathbf{X} but also that there are no periodic orbits in \mathbf{X} . Then $A(\mathbf{X})$ is a uniform algebra on \mathbf{X} .

Let X_0 be the screw with pitch 1 and radius $1/2\pi$, that is, the quotient space obtained from $\mathbf{Z} \times [0, 1]$ by identifying each $(n, 1)$ with $(n + 1, 0)$. Since \mathbf{Z} is dense in $\beta\mathbf{Z}$, X_0 is also dense in \mathbf{X} . The map $t \rightarrow ([t], t - [t])$ is a homeomorphism of \mathbf{R} onto the subspace X_0 of \mathbf{X} . In what follows, we shall sometimes identify X_0 and $X_0 \times (0, \infty)$ with \mathbf{R} and \mathbf{R}_+^2 , respectively. Then \mathbf{R}_+^2 may be considered as a subset of $\mathfrak{M}(A(\mathbf{X}))$ by (2.1). Let $C_b(\mathbf{R})$ be the space of all bounded continuous functions on \mathbf{R} . Then each function in $C(\mathbf{X})$ determines a function in $C_b(\mathbf{R})$ by restricting it to X_0 .

Every function ϕ on \mathbf{X} has the automorphic extension $\phi^\#$ to $\beta\mathbf{Z} \times \mathbf{R}$ by

$$(2.3) \quad \phi^\#(y, t) = \phi(S^{[t]}y, t - [t]).$$

A function f on $\beta\mathbf{Z} \times \mathbf{R}$ is the automorphic extension of a function on \mathbf{X} if and only if $f(y, t) = f(Sy, t - 1)$ on $\beta\mathbf{Z} \times \mathbf{R}$.

Lemma 2.1. *Let ψ_0 be a function in $C_b(\mathbf{R})$. Then ψ_0 extends to a function ψ in $C(\mathbf{X})$ if and only if ψ_0 is uniformly continuous on \mathbf{R} .*

Proof. Suppose that ψ_0 is uniformly continuous on \mathbf{R} . For each s in $[0, 1)$, the sequence $\psi_0(n, s) = \psi_0(n + s)$, $n \in \mathbf{Z}$, lies in $l^\infty(\mathbf{Z})$. Then $\psi_0(n, s)$

extends to $\psi(y, s)$ in $C(\beta\mathbf{Z})$. Since \mathbf{X} is identified with $\beta\mathbf{Z} \times [0, 1)$, we may regard $\psi(y, s)$ as a function on \mathbf{X} . Let $\psi^\#(y, t)$ be the automorphic extension of $\psi(y, s)$ to $\beta\mathbf{Z} \times \mathbf{R}$. It follows from the uniform continuity of ψ_0 that the family $\{\psi^\#(y, t); y \in \beta\mathbf{Z}\}$ of the functions of t is equicontinuous on \mathbf{R} . This implies that $\psi^\#$ is continuous on $\beta\mathbf{Z} \times \mathbf{R}$. Thus ψ lies in $C(\mathbf{X})$.

The converse follows from a general fact. Indeed, it is an immediate consequence of the continuity of the map $(x, t) \rightarrow x + t$ on $\mathbf{X} \times \mathbf{R}$. \square

Let $C_{bu}(\mathbf{R})$ be the space of bounded uniformly continuous functions on \mathbf{R} . Then $C_{bu}(\mathbf{R})$ is a Banach algebra with the supremum norm, which is not separable. Lemma 2.1 shows that \mathbf{X} is the maximal ideal space of $C_{bu}(\mathbf{R})$ and that the flow $\{\mathbf{S}_t\}_{t \in \mathbf{R}}$ on \mathbf{X} is that which is induced by the natural translations of the functions in $C_{bu}(\mathbf{R})$. Such observation may provide an approach to almost periodic functions.

Lemma 2.2. *Let $A(\mathbf{X})$ be the induced uniform algebra on $(\mathbf{X}, \{\mathbf{S}_t\}_{t \in \mathbf{R}})$, and let ϕ_0 be a function in $H^\infty(\mathbf{R})$. If $r > 0$, then the convolution $\phi_0 * P_{ir}$, in the usual sense, extends to a function $\phi_r(x)$ in $A(\mathbf{X})$. Furthermore, the function $(x, r) \rightarrow \phi_r(x)$ is continuous on $\mathbf{X} \times (0, \infty)$.*

Proof. Since

$$|\phi_0 * P_{ir}(t-s) - \phi_0 * P_{ir}(t)| \leq \|\phi_0\|_\infty \cdot \|P_{s+ir} - P_{ir}\|_1,$$

$t \rightarrow \phi_0 * P_{ir}(t)$ is uniformly continuous on \mathbf{R} . By Lemma 2.1, $\phi_0 * P_{ir}$ extends to a $\phi_r(x)$ in $C(\mathbf{X})$. Let us consider the automorphic extension $\phi_r^\#(y, t)$ of ϕ_r to $\beta\mathbf{Z} \times \mathbf{R}$. Then $t \rightarrow \phi_r^\#(y, t)$ extends to \mathbf{R}_+^2 by its Poisson integral:

$$(2.4) \quad \phi_r^\#(y, z) = \phi_r^\#(y, t + is) = \phi_r^\# * P_{is}(y, t).$$

Since $\phi_0 * P_{ir}$ lies in $H^\infty(\mathbf{R})$, $z \rightarrow \phi_r^\#(n, z)$ is bounded analytic on \mathbf{R}_+^2 for each n in \mathbf{Z} . Then, since \mathbf{Z} is dense in $\beta\mathbf{Z}$, $z \rightarrow \phi_r^\#(y, z)$ lies also in $H^\infty(\mathbf{R}_+^2)$ for each y in $\beta\mathbf{Z}$. This shows that $t \rightarrow \phi_r(x + t)$ lies in $H^\infty(\mathbf{R})$. Thus ϕ_r is a function in $A(\mathbf{X})$.

Since $\phi_0 * P_{ir} * P_{is} = \phi_0 * P_{i(r+s)}$ for $s > 0$, we see that $\phi_r^\#(y, t + is) = \phi_{r+s}^\#(y, t)$ by (2.4). Then the family $\{\phi_r^\#(y, t + is); y \in \beta\mathbf{Z}\}$ of functions of $t + is$ is equicontinuous on \mathbf{R}_+^2 . This implies that $(x, s) \rightarrow \phi_{r+s}(x)$ is continuous on $\mathbf{X} \times (0, \infty)$ for each $r > 0$. From this, the continuity of $(x, r) \rightarrow \phi_r(x)$ follows. \square

Proposition 2.3. *Let $A(\mathbf{X})$ be the induced uniform algebra on $(\mathbf{X}, \{\mathbf{S}_t\}_{t \in \mathbf{R}})$. Then $A(\mathbf{X})$ is a logmodular algebra on \mathbf{X} which is not a Dirichlet algebra.*

Proof. Suppose ϕ lies in $C_{\mathbf{R}}(\mathbf{X})$. For a given $\varepsilon > 0$, we choose an $r > 0$ such that $\|\phi - \phi * P_{ir}\|_\infty < \varepsilon$ by [2, Lemma 1]. On the other hand, the restriction ϕ_0 of ϕ to X_0 lies in $C_b(\mathbf{R})$. Let $H\phi_0$ be the Hilbert transform of ϕ_0 defined by

$$(2.5) \quad H\phi_0(t) = \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{|t-s|>\varepsilon} \phi_0(s) \left[\frac{1}{t-s} + \frac{s}{1+s^2} \right] ds.$$

Then $\psi_0 = \exp(\phi_0 + iH\phi_0)$ is an outer function in $H^\infty(\mathbf{R})$, which is invertible in $H^\infty(\mathbf{R})$. Observe that $\psi_0^{-1} * P_{ir}(t)$ is the inverse of $\psi_0 * P_{ir}(t)$ in $H^\infty(\mathbf{R})$. Since $\log|\psi_0| = \phi_0$, we see that

$$(2.6) \quad \log|\psi_0 * P_{ir}(t)| = \phi_0 * P_{ir}(t).$$

It follows from Lemma 2.2 that $\psi_0 * P_{ir}(t)$ and $\psi_0^{-1} * P_{ir}(t)$ extend to θ and θ^{-1} in $A(\mathbf{X})$. We see easily that $\log|\theta| = \phi * P_{ir}$ by (2.6). So $A(\mathbf{X})$ is a logmodular algebra on \mathbf{X} .

We now show that $A(\mathbf{X})$ is not a Dirichlet algebra. Let $u_0(t) = \frac{2}{\pi} \tan^{-1}(t)$. Since u_0 is uniformly continuous on \mathbf{R} , u_0 extends to a function u in $C_{\mathbf{R}}(\mathbf{X})$. However, we observe by (2.5) that

$$\text{dist}(u_0, \text{Re } H^\infty(\mathbf{R})) = \inf\{\|u_0 - \text{Re } \psi_0\|_\infty; \psi_0 \in H^\infty(\mathbf{R})\} = 1$$

(see [6, Chapter IV, Example 1.5 and Chapter V, Exercise 15]). Thus u cannot belong to the closure of $\text{Re } A(\mathbf{X})$ in $C_{\mathbf{R}}(\mathbf{X})$. \square

Let ν_0 be the measure on \mathbf{Z} defined by $\nu_0(\{n\}) = 1/(n^2 + 1)$, and let m_I be the restriction of Lebesgue measure to $I = [0, 1]$. By identifying \mathbf{R} with X_0 , $dt/(1 + t^2)$ and $\nu_0 \times m_I$ are mutually absolutely continuous. Since $l^\infty(\mathbf{Z}) = L^\infty(\nu_0)$, $L^\infty(\nu_0)$ is isometrically isomorphic to $C(\beta\mathbf{Z})$ and ν_0 extends to a normal measure ν in $M(\beta\mathbf{Z})$ (see [5, Chapter I, §9]). We then see that ν is quasi-invariant with respect to \mathbf{S} , to be exact, ν -null sets are preserved under the translation by \mathbf{S} . Therefore the measure $\nu \times m_I$ in $M(\mathbf{X})$ is a quasi-invariant measure on $(\mathbf{X}, \{\mathbf{S}_t\}_{t \in \mathbf{R}})$ by the definition (2.2).

Let $H^\infty(\mathbf{X})$ be the space of all analytic functions in $L^\infty(\nu \times m_I)$. Since $\nu \times m_I$ is concentrated on X_0 , that is, $\nu \times m_I(\mathbf{X} \setminus X_0) = 0$, $H^\infty(\mathbf{R})$ is isometrically isomorphic to $H^\infty(\mathbf{X})$ by

$$(2.7) \quad \Phi_0 \phi_0(n, s) = \phi_0(n + s), \quad \phi_0 \in H^\infty(\mathbf{R}),$$

for (n, s) in X_0 . Since the restriction of each function in $H^\infty(\mathbf{X})$ to X_0 determines a function in $H^\infty(\mathbf{R})$, Φ_0 maps $H^\infty(\mathbf{R})$ onto $H^\infty(\mathbf{X})$. Let $\mathfrak{M}(H^\infty(\mathbf{R}))$ and $\mathfrak{M}(H^\infty(\mathbf{X}))$ be the maximal ideal spaces of $H^\infty(\mathbf{R})$ and $H^\infty(\mathbf{X})$. Then the adjoint Φ_0^* of Φ_0 maps homeomorphically $\mathfrak{M}(H^\infty(\mathbf{X}))$ onto $\mathfrak{M}(H^\infty(\mathbf{R}))$. We shall modify Φ_0 in a little while.

The important feature of $H^\infty(\mathbf{X})$ is that each element has a useful Borel version.

Lemma 2.4. *Let ϕ be a function in $H^\infty(\mathbf{X})$. Then there is a Borel function $\tilde{\phi}$ on \mathbf{X} with the following properties:*

- (i) *The equation $\phi(x) = \tilde{\phi}(x)$ holds for $\nu \times m_I$ -a.e. x in \mathbf{X} .*
- (ii) *The function $x \rightarrow \tilde{\phi} * P_{ir}(x)$ lies in $A(\mathbf{X})$ for each $r > 0$. Furthermore, the function $(x, r) \rightarrow \tilde{\phi} * P_{ir}(x)$ is continuous on $\mathbf{X} \times (0, \infty)$.*
- (iii) *Let μ be a positive quasi-invariant measure on \mathbf{X} . Then we obtain*

$$(2.8) \quad \lim_{r \rightarrow +0} \int_{\mathbf{X}} \tilde{\phi} * P_{ir}(x) d\mu(x) = \int_{\mathbf{X}} \tilde{\phi}(x) d\mu(x).$$

Proof. We define the Borel function $\tilde{\phi}$ for a given ϕ in $H^\infty(\mathbf{X})$. Let ϕ_0 be the restriction of ϕ to X_0 . Then ϕ_0 is regarded as a function in $H^\infty(\mathbf{R})$. By Lemma 2.2, $\phi_0 * P_{ir}$ extends to a function $\phi_r(x)$ in $A(\mathbf{X})$, and $(x, r) \rightarrow \phi_r(x)$ is continuous on $\mathbf{X} \times (0, \infty)$. We set

$$(2.9) \quad f(y, z) = f(y, t + ir) = \phi_r^\#(y, t),$$

where $\phi_r^\#(y, t)$ is the automorphic extension of ϕ_r to $\beta\mathbf{Z} \times \mathbf{R}$. So f is continuous on $\beta\mathbf{Z} \times \mathbf{R}_+^2$, and $z \rightarrow f(y, z)$ lies in $H^\infty(\mathbf{R}_+^2)$ for each y in $\beta\mathbf{Z}$.

Let $\{r_n\}$ be a decreasing sequence that converges to 0. We then define the boundary-value function $f(y, t)$ by

$$f(y, t) = \limsup_{n \rightarrow \infty} \operatorname{Re} f(y, t + ir_n) + i \limsup_{n \rightarrow \infty} \operatorname{Im} f(y, t + ir_n).$$

Since $(y, t) \rightarrow f(y, t + ir_n)$ is continuous on $\beta\mathbf{Z} \times \mathbf{R}$, $f(y, t)$ is a Borel function on $\beta\mathbf{Z} \times \mathbf{R}$. On the other hand, it follows from Fatou's theorem that, for each y in $\beta\mathbf{Z}$,

$$(2.10) \quad f(y, t) = \lim_{r \rightarrow +0} f(y, t + ir)$$

for dt -a.e. t in \mathbf{R} . Since $f(y, t) = f(Sy, t - 1)$ by (2.9), there is a Borel function $\tilde{\phi}$ on \mathbf{X} whose automorphic extension is $f(y, t)$.

Since (2.10) shows that if n lies in \mathbf{Z} , then

$$\lim_{r \rightarrow +0} \phi_0 * P_{ir}(n + t) = \phi_0(n + t) = f(n, t)$$

for dt -a.e. t , we see that $\phi^\#(y, t) = f(y, t)$ for $d\nu \times dt$ -a.e. (y, t) . Thus (i) holds. We see easily that $\tilde{\phi} * P_{ir}(x) = \phi_r(x)$ on \mathbf{X} . Then (ii) follows from Lemma 2.2. To show (iii), we define the measure μ_1 in $M(\beta\mathbf{Z})$ by $\mu_1(E) = \mu(E \times [0, 1])$. Then μ_1 is quasi-invariant with respect to \mathbf{S} . It is easy to see that $\mu_1 \times m_I$ is a quasi-invariant measure on \mathbf{X} which is mutually absolutely continuous with respect to μ . Identifying \mathbf{X} with $\beta\mathbf{Z} \times [0, 1)$,

$$\lim_{r \rightarrow +0} \tilde{\phi} * P_{ir}(y, s) = \tilde{\phi}(y, s)$$

holds for m_I -a.e. s in $[0, 1)$ and for each y in $\beta\mathbf{Z}$. Thus (2.8) follows from Fubini's theorem and the bounded convergence theorem. \square

Let Φ_0 be the isometric isomorphism of $H^\infty(\mathbf{R})$ onto $H^\infty(\mathbf{X})$ by (2.7). For a given ϕ_0 in $H^\infty(\mathbf{R})$, we set $\phi = \Phi_0(\phi_0)$. Strictly speaking, ϕ is an equivalence class of functions modulo null functions. Then we assign the Borel function $\tilde{\phi}$ by Lemma 2.4 to $\Phi_0(\phi_0)$:

$$(2.11) \quad \Phi_0(\phi_0) = \tilde{\phi}, \quad \phi_0 \in H^\infty(\mathbf{R}).$$

From now on, Φ_0 will denote this modified isomorphism.

By virtue of this specialization, certain homomorphisms of $A(\mathbf{X})$ become the ones of $H^\infty(\mathbf{R})$. Since $H^\infty(\mathbf{R})$ and $H^\infty(\mathbf{X})$ are isometrically isomorphic via Φ_0 , it is a consequence of the following lemma, in which the uniqueness of extension depends on the corona theorem.

Lemma 2.5. *Suppose that ξ_0 lies in $\mathcal{M}(A(\mathbf{X})) \setminus \mathbf{X}$. Then ξ_0 extends uniquely to an element ξ of $\mathcal{M}(H^\infty(\mathbf{X}))$.*

Proof. Let μ be the representing measure for ξ_0 . Since μ is not a point mass, μ is quasi-invariant on \mathbf{X} . Using the Borel version by Lemma 2.4, we define

$$(2.12) \quad \xi(\phi) = \int_{\mathbf{X}} \tilde{\phi} d\mu, \quad \phi \in H^\infty(\mathbf{X}).$$

Since $(\phi\psi)^\sim = \tilde{\phi}\tilde{\psi}$ if ϕ and ψ lie in $H^\infty(\mathbf{X})$, it follows from (2.8) that ξ is a homomorphism of $H^\infty(\mathbf{X})$ which is an extension of ξ_0 .

The uniqueness of ξ follows from the corona theorem. Since $H^\infty(\mathbf{R})$ is isometrically isomorphic to $H^\infty(\mathbf{X})$, the corona theorem asserts that $X_0 \times (0, \infty)$

is dense in $\mathfrak{M}(H^\infty(\mathbf{X}))$ by (2.1). Observe that $t \rightarrow e^{it}$ is uniformly continuous on \mathbf{R} . Then Lemma 2.1 implies that e^{it} extends to a function $s_1(x)$ in $A(\mathbf{X})$ such that $|s_1(x)| = 1$ on \mathbf{X} . Since $t \rightarrow s_1(x+t)$ is periodic, we see that s_1 is not constant as a function in $L^1(\mu)$. Then there is an r with $0 < r < 1$ such that

$$(2.13) \quad |\xi_0(s_1)| = \left| \int_{\mathbf{X}} s_1 d\mu \right| \leq r.$$

Suppose ξ' in $\mathfrak{M}(H^\infty(\mathbf{X}))$ satisfies that $\xi' = \xi_0$ on $A(\mathbf{X})$. Since $A(\mathbf{X})$ is a logmodular algebra on \mathbf{X} , the representing measure μ for ξ_0 is unique. By the corona theorem and (2.13), there is a net $\{\xi_\alpha\}$ in $X_0 \times [-\log r, \infty)$ such that $\{\xi_\alpha\}$ converges to ξ' in $\mathfrak{M}(H^\infty(\mathbf{X}))$. By setting $\xi_\alpha = (x_\alpha, t_\alpha)$, (2.1) and (ii) of Lemma 2.4 show that

$$\int_{-\infty}^{\infty} \tilde{\phi} * P_{iu}(x_\alpha + t) P_{it_\alpha}(t) dt \rightarrow \int_{\mathbf{X}} \tilde{\phi} * P_{iu} d\mu$$

for each ϕ in $H^\infty(\mathbf{X})$ and for $u > 0$. Furthermore, we see that if $\{t_\alpha\}$ is bounded, then μ is the representing measure for $\xi_0 = (x, t)$ in $\mathbf{X} \times [-\log r, \infty)$. When $\{t_\alpha\}$ diverges, μ is an invariant measure on \mathbf{X} . In both cases, we obtain easily that

$$\int_{-\infty}^{\infty} \tilde{\phi}(x_\alpha + t) P_{it_\alpha}(t) dt \rightarrow \int_{\mathbf{X}} \tilde{\phi} d\mu,$$

by the condition $t_\alpha \geq -\log r$. This implies that $\{\xi_\alpha\}$ also converges to ξ in $\mathfrak{M}(H^\infty(\mathbf{X}))$. Hence $\xi' = \xi$, so the extension of ξ_0 is unique. \square

3. IN RELATION TO FIBERS

In this section, we represent a portion of the fiber \mathfrak{M}_1 of $\mathfrak{M}(H^\infty(\Delta))$ over 1 as a subset of the maximal ideal space $\mathfrak{M}(A(\mathbf{X}))$ of $A(\mathbf{X})$. This enable us to make clearer the structure of fibers and to give information about Gleason parts in them.

Let \mathbf{A} be a uniform algebra with the maximal ideal space $\mathfrak{M}(\mathbf{A})$. We usually identify each function in \mathbf{A} with its Gelfand transform on $\mathfrak{M}(\mathbf{A})$. A continuous map F of Δ into $\mathfrak{M}(\mathbf{A})$ is *analytic* if $f \circ F$ is analytic on Δ when f lies in \mathbf{A} . A one-to-one analytic map is said to be an *analytic disc*. With an analytic disc, we do not distinguish between the map F and its image $F(\Delta)$. Let $\mathfrak{M}(\mathbf{A})^\Delta$ be the set of all maps of Δ into $\mathfrak{M}(\mathbf{A})$. Then $\mathfrak{M}(\mathbf{A})^\Delta$ is a compact Hausdorff space in the product topology. If a net $\{F_\beta\}$ of analytic maps converges to F in $\mathfrak{M}(\mathbf{A})^\Delta$, then F is also analytic.

Observe that the function

$$(3.1) \quad w(z) = i \frac{1+z}{1-z}, \quad z \in \Delta,$$

maps conformally Δ onto \mathbf{R}_+^2 . Then w maps $\{-1, 0, 1\}$ to $\{0, i, \infty\}$. We next define the isometric isomorphism of $H^\infty(\Delta)$ onto $H^\infty(\mathbf{R}_+^2)$ by

$$(3.2) \quad \Phi_1(f) = f \circ w^{-1}, \quad f \in H^\infty(\Delta).$$

To investigate the fiber \mathfrak{M}_1 , we look into the behavior of functions in $H^\infty(\mathbf{R}_+^2)$ around at infinity.

Let $s(z)$ be the singular function in $H^\infty(\Delta)$ defined by

$$(3.3) \quad s(z) = \exp i w(z), \quad z \in \Delta.$$

Then $s(z)$ has a singularity at $z = 1$. Identifying s with its Gelfand transform on $\mathfrak{M}(H^\infty(\Delta))$, we set

$$(3.4) \quad H = \{\xi \in \mathfrak{M}(H^\infty(\Delta)); |s(\xi)| < 1\}.$$

Then H is an open set in $\mathfrak{M}(H^\infty(\Delta))$ that contains Δ . Observe that $H \setminus \Delta$ is contained in \mathfrak{M}_1 . More precisely, ξ in \mathfrak{M}_1 lies in $H \setminus \Delta$ if and only if ξ lies in the closure of an open disc inside Δ which is tangent to the unit circle at 1, that is, ξ is orocycular or nontangential (see [9, §6]).

By using the flow $(X, \{S_t\}_{t \in \mathbb{R}})$ in §2, the open set $\mathfrak{M}_1 \cap H$ in \mathfrak{M}_1 can be analyzed rather completely. Recall that we identify $X_0 \times (0, \infty)$ with \mathbb{R}_+^2 .

Theorem 3.1. *Let H be the open set in $\mathfrak{M}(H^\infty(\Delta))$ defined by (3.4). Then there is a homeomorphism π of H onto $\mathfrak{M}(A(X)) \setminus X$ with the following properties:*

(i) *Let π_1 denote the restriction of π to $\mathfrak{M}_1 \cap H$. Then π_1 maps homeomorphically $\mathfrak{M}_1 \cap H$ onto $\mathfrak{M}(A(X)) \setminus (X \cup \mathbb{R}_+^2)$.*

(ii) *The homeomorphism π preserves Gleason metrics. Furthermore, a continuous map F of Δ into H is analytic if and only if so is the map $\pi \circ F$ of Δ into $\mathfrak{M}(A(X)) \setminus X$.*

Proof. Let Φ_0 and Φ_1 be the isometric isomorphisms by (2.11) and (3.2), respectively. Setting $\Phi = \Phi_0 \circ \Phi_1$, we obtain an isometric isomorphism of $H^\infty(\Delta)$ onto $H^\infty(X)$ (see Figure 1). Let Φ^* denote the adjoint of Φ . Then Φ^* maps homeomorphically $\mathfrak{M}(H^\infty(X))$ onto $\mathfrak{M}(H^\infty(\Delta))$. We see also that Φ^* is an isometry of $H^\infty(X)^*$ onto $H^\infty(\Delta)^*$, where $H^\infty(X)^*$ and $H^\infty(\Delta)^*$ denote the dual spaces of $H^\infty(X)$ and $H^\infty(\Delta)$. So Φ^* maps each Gleason part of $\mathfrak{M}(H^\infty(X))$ to the one of $\mathfrak{M}(H^\infty(\Delta))$.

As in the proof of Lemma 2.5, $s_1(x)$ denotes the extension of $t \rightarrow e^{it}$ to a function in $A(X)$. We then set

$$H_1 = \{\xi \in \mathfrak{M}(H^\infty(X)); |s_1(\xi)| < 1\}.$$

Then H_1 is an open set in $\mathfrak{M}(H^\infty(X))$. Since $|s_1(\xi)| = 1$ on $\mathfrak{M}(H^\infty(X)) \setminus H_1$, H_1 is a union of Gleason parts. Since $\Phi_1 s(t) = e^{it}$ by (3.3), we see that $\Phi s = \Phi_0 \circ \Phi_1 s = s_1$. This implies that Φ^* maps homeomorphically H_1 onto H .

We next show that H_1 is homeomorphic to $\mathfrak{M}(A(X)) \setminus X$. When ξ lies in H_1 , we write $\sigma(\xi)$ for the restriction of ξ to $A(X)$. Then $\sigma(\xi)$ cannot lie in X , since $|s_1(\xi)| < 1$. Therefore σ maps continuously H_1 into $\mathfrak{M}(A(X)) \setminus X$. If ξ_0 lies in $\mathfrak{M}(A(X)) \setminus X$, then $|s_1(\xi_0)| < 1$, since the representing measure for ξ_0 is quasi-invariant. It follows from Lemma 2.5 that σ is a one-to-one map of H_1 onto $\mathfrak{M}(A(X)) \setminus X$.

To show the continuity of σ^{-1} , we take a neighborhood V of ξ in H_1 . Since H_1 is open in $\mathfrak{M}(H^\infty(X))$, we notice that $\mathfrak{M}(H^\infty(X)) \setminus V$ is compact in $\mathfrak{M}(H^\infty(X))$. For any η in $\mathfrak{M}(H^\infty(X)) \setminus V$, there is a function ϕ in $A(X)$ such that $\phi(\eta) = 1$ and $\phi(\xi) = 0$. Indeed, if η lies in $\mathfrak{M}(H^\infty(X)) \setminus H_1$, this follows from the fact that $|s_1| = 1$ only on $\mathfrak{M}(H^\infty(X)) \setminus H_1$. On the other hand, since σ is one-to-one, we see that $A(X)$ separates the points on H_1 . This shows the existence of such a function when η lies in $H_1 \setminus V$. Since $\mathfrak{M}(H^\infty(X)) \setminus V$

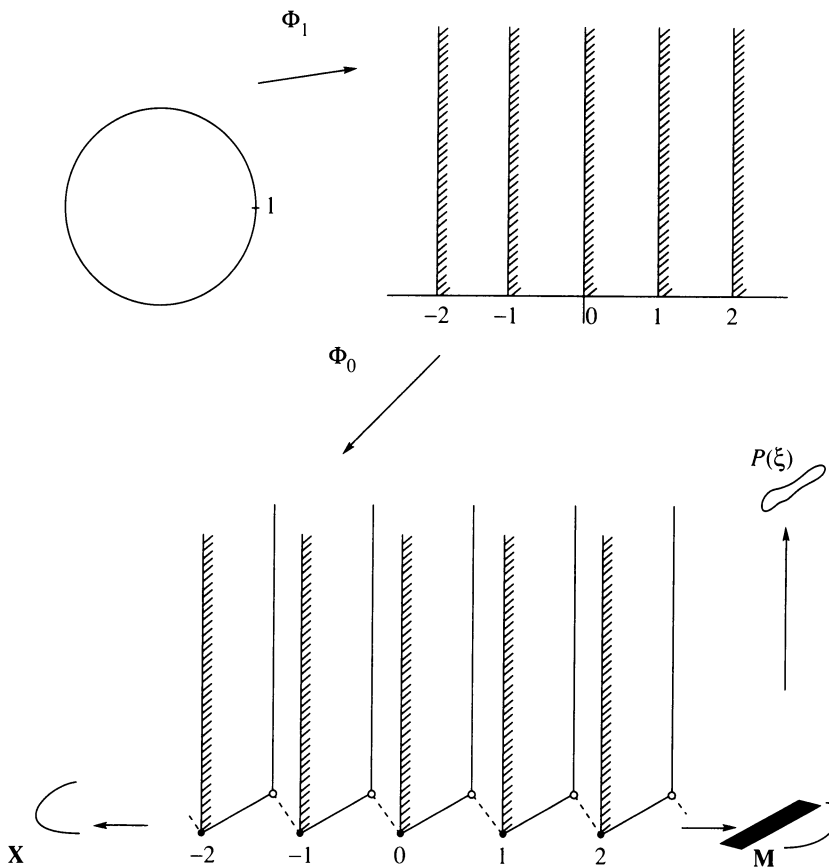


FIGURE 1

is compact, we choose such functions ϕ_1, \dots, ϕ_n in $A(\mathbf{X})$ so that the open subsets $\{|\phi_j| > 2/3\}$, $1 \leq j \leq n$, make a covering of $\mathfrak{M}(H^\infty(\mathbf{X})) \setminus V$. Define the neighborhood of $\sigma(\xi)$ in $\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}$ by

$$W = \{\eta_0 \in \mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}; |\phi_j(\eta_0)| < 1/3, 1 \leq j \leq n\}.$$

Then we see $\sigma^{-1}W$ is contained in V . Thus σ is a homeomorphism.

Let $\pi = \sigma \circ (\Phi^*)^{-1}$. Then π maps homeomorphically H onto $\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}$. It is easy to see that $\pi(\Delta) = \mathbf{R}_+^2$. Since $\mathfrak{M}_1 \cap H = H \setminus \Delta$, the restriction π_1 maps $\mathfrak{M}_1 \cap H$ onto $\mathfrak{M}(A(\mathbf{X})) \setminus (\mathbf{X} \cup \mathbf{R}_+^2)$. Thus (i) follows.

It follows from (iii) of Lemma 2.4 and Lemma 2.5 that

$$\|\xi - \eta\| = \|\sigma(\xi) - \sigma(\eta)\|$$

whenever ξ and η lie in H_1 . Since Φ^* is isometric, the map π preserves Gleason metrics. We notice that H and $\mathfrak{M}_1 \cap H$ are unions of Gleason parts.

Suppose that F is an analytic map of Δ into H . If ϕ lies in $A(\mathbf{X})$, then $\Phi^{-1}\phi(F(z))$ is analytic on Δ , since $\Phi^{-1}\phi$ lies in $H^\infty(\Delta)$. This implies that $\pi \circ F$ is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}$. Conversely, suppose that $\pi \circ F$ is analytic. Let ϕ be a function in $H^\infty(\mathbf{X})$. Then $\tilde{\phi}$ denotes the Borel version by Lemma 2.4. Since $\tilde{\phi} * P_{ir}$ is in $A(\mathbf{X})$ for $r > 0$, $\tilde{\phi} * P_{ir}(\pi \circ F(z))$ is analytic

on Δ . Then we see by (iii) of Lemma 2.4 that

$$\lim_{r \rightarrow +0} \tilde{\phi} * P_{ir}(\pi \circ F(z)) = \phi((\Phi^*)^{-1} \circ F(z)),$$

since the representing measure for $\pi \circ F(z)$ is quasi-invariant on \mathbf{X} . This shows that $\Phi^{-1}\phi(F(z))$ is analytic on Δ . Since Φ^{-1} maps $H^\infty(\mathbf{X})$ onto $H^\infty(\Delta)$, $F(z)$ is an analytic map of Δ into H . This completes the proof. \square

Let μ be an invariant, ergodic, probability measure on \mathbf{X} . Then μ is a representing measure for $A(\mathbf{X})$. As we mentioned earlier, there are many such measures. We identify here μ with the homomorphism of $A(\mathbf{X})$ by μ . By Theorem 3.1, μ determines a homomorphism of $H^\infty(\Delta)$ lying in \mathfrak{M}_1 . We claim that $\{\mu\}$ is a one-point part in $\mathfrak{M}(A(\mathbf{X}))$. Indeed, since $A(\mathbf{X})$ is a log-modular algebra by Proposition 2.3, each Gleason part is either a one-point part or an analytic disc. Let $H^2(\mu)$ be the closure of $A(\mathbf{X})$ in $L^2(\mu)$, and let $H_0^2(\mu)$ be the space of all functions in $H^2(\mu)$ which vanish at μ . Then the orthogonal complement of $H_0^2(\mu) \oplus \overline{H_0^2(\mu)}$ consists of constants by [11, Theorem I]. Recall that μ is represented as $\mu_0 \times m_I$, where μ_0 is an invariant measure on $\beta\mathbf{Z}$ with respect to \mathbf{S} . Suppose $\{\mu\}$ is not a one-point part. Then there is an inner function q in $H^2(\mu)$ such that $H_0^2(\mu) = qH^2(\mu)$ (see [5, Chapter V, §7]). Let $q^\#(y, t)$ be the automorphic extension of q to $\beta\mathbf{Z} \times \mathbf{R}$. Since the inner function $t \rightarrow q^\#(y, t)$ is not constant for μ_0 -a.e. y , we choose easily a non-negative function u in $L^\infty(d\mu_0 \times dt)$ satisfying $t \rightarrow q^\#(y, t)u(y, t)(1+t^2)^{-1}$ lies in $H^\infty(\mathbf{R})$. By setting $g(y, t) = u(y, t)(1+t^2)^{-1}$, a suitable choice of u shows that

$$\phi^\#(y, t) = \sum_{j=-\infty}^{\infty} g(\mathbf{S}^j y, t-j)$$

for some nonconstant function ϕ in $L^\infty(\mu)$. Then ϕ is orthogonal to $qH^2(\mu) \oplus \overline{qH^2(\mu)}$, thus we have a contradiction.

The following proposition is, of course, an immediate consequence of the corona theorem, so the point is the direct proof of it.

Proposition 3.2. *Let μ be an invariant, ergodic, probability measure on \mathbf{X} . Then the homomorphism in $\mathfrak{M}(A(\mathbf{X}))$ by μ is an accumulating point of \mathbf{R}_+^2 .*

Proof. For each ϕ in $A(\mathbf{X})$, the individual ergodic theorem shows that

$$(3.5) \quad \int_{\mathbf{X}} \phi d\mu = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \phi(x+t) dt$$

for μ -a.e. x in \mathbf{X} . It follows from Wiener's Tauberian theorem that the right side of (3.5) is equal to $\lim_{r \rightarrow \infty} \phi * P_{ir}(x)$ for each x such that (3.5) holds (see [14, Proof of Lemma 2.6]). By regarding $\mathbf{X} \times (0, \infty)$ as a subset of $\mathfrak{M}(A(\mathbf{X}))$, $\phi * P_{ir}(x)$ is the value of ϕ at the point (x, r) by (2.1). Let $\varepsilon > 0$, and let ϕ_1, \dots, ϕ_n lie in $A(\mathbf{X})$. Since the union of finite null sets is also null, we find a point (x, r) in $\mathbf{X} \times (0, \infty)$ which belongs to the neighborhood

$$W = \{\eta \in \mathfrak{M}(A(\mathbf{X})); |\phi_j(\eta) - \phi_j(\mu)| < \varepsilon, 1 \leq j \leq n\}$$

of μ in $\mathfrak{M}(A(\mathbf{X}))$. Since \mathbf{R}_+^2 is dense in $\mathbf{X} \times (0, \infty)$, there is a point in \mathbf{R}_+^2 lying in W . Thus μ is an accumulating point of \mathbf{R}_+^2 . \square

Let $\{\alpha_n\}$ be an interpolating sequence in Δ . We define

$$F_n(z) = \frac{z + \alpha_n}{1 + \overline{\alpha_n}z}, \quad z \in \Delta,$$

which is an analytic map of Δ into $\mathfrak{M}(H^\infty(\Delta))$, since Δ is regarded as a subset of $\mathfrak{M}(H^\infty(\Delta))$. If F is an accumulating point of $\{F_n\}$ in $\mathfrak{M}(H^\infty(\Delta))^\Delta$, then F is an analytic disc. Conversely, all analytic structure in $\mathfrak{M}(H^\infty(\Delta)) \setminus \Delta$ comes about in this manner. This is the characterization of nontrivial parts in $\mathfrak{M}(H^\infty(\Delta))$ due to Hoffman (see [9] or [6, Chapter X]). We define similarly

$$L_n(z) = \frac{w_n - \overline{w_n}z}{1 - z}, \quad z \in \Delta,$$

for an interpolating sequence $\{w_n\}$ in \mathbf{R}_+^2 . Then L_n is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{X}))$ and lies in $[\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}]^\Delta$. Suppose that an accumulating point L of $\{L_n\}$ in $\mathfrak{M}(A(\mathbf{X}))^\Delta$ lies in $[\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}]^\Delta$. It then follows from Hoffman's theorem and Theorem 3.1 that L is an analytic disc in $\mathfrak{M}(A(\mathbf{X}))$. This fact will be used in §4.

Recall that an interpolating Blaschke product $B(z)$ on \mathbf{R}_+^2 with zeros $\{w_n = t_n + is_n\}$ is *thin* if $B(z)$ satisfies that

$$\lim_{n \rightarrow \infty} 2s_n |B'(w_n)| = 1.$$

Suppose that $\{s_n\}$ is bounded away from zero. Let $\{L_n\}$ and L be as in above. Then L is an analytic disc which is homeomorphic to Δ and L^{-1} is the product cB of B by a unimodular constant c (see [6, Chapter X, Exercise 8]). From this fact, we obtain the following:

Proposition 3.3. *There is a nontrivial part P in $\mathfrak{M}(A(\mathbf{X}))$ satisfying the following properties:*

- (i) *Let L be the analytic map of Δ onto P . Then L is a homeomorphism.*
- (ii) *Let μ be the representing measure for $L(0)$. Then μ is invariant on \mathbf{X} . Furthermore, there is a measurable function $K(z, x)$ on $\Delta \times \mathbf{X}$ such that $x \rightarrow K(z, x)$ is invariant on \mathbf{X} and*

$$\phi(L(z)) = \int_{\mathbf{X}} \phi(x) K(z, x) d\mu(x), \quad \phi \in A(\mathbf{X}).$$

Proof. Let us consider the case that $\{s_n\}$ diverges, and let $P = L(\Delta)$. Then we see that the representing measure for $L(z)$ is invariant for each z in Δ . Since $A(\mathbf{X})$ is logmodular, each representing measure for $L(z)$ is mutually absolutely continuous to μ . Let $\theta(x)$ be the invariant function by the right side of (3.5) with cB in place of ϕ . It is easy to see that the function

$$K(z, x) = \frac{1}{2\pi} \frac{1 - |z|^2}{|\theta(x) - z|^2}$$

satisfies the desired property. \square

4. ON A QUESTION OF FORELLI

Let \mathcal{C} be the class of all closed, nonempty, invariant subsets in a flow $(\Omega, \{U_t\}_{t \in \mathbf{R}})$. A set in \mathcal{C} is *minimal* in $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ if it is minimal with respect to the inclusion relation of \mathcal{C} . Then Zorn's lemma guarantees the existence of a minimal set. We notice that $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ is a minimal flow if and only if Ω is minimal by itself. A minimal flow is said to be *strictly ergodic* if there is exactly one invariant probability measure.

As a generalization of Wermer's maximality theorem [8, Chapter 6], Forelli [4] showed that if $(\Omega, \{U_t\}_{t \in \mathbb{R}})$ is minimal, then the induced uniform algebra $A(\Omega)$ is a maximal algebra. In connection with this result, he has asked the question stated in Introduction (see [12, §6] for a nice account of related topics). In response to this question, Muhly [11, Theorem II] gave a sufficient condition: If $(\Omega, \{U_t\}_{t \in \mathbb{R}})$ is strictly ergodic, then $A(\Omega)$ is a Dirichlet algebra on Ω . Thus, under the condition of strictly ergodicity, many interesting results have been obtained.

Our objective in this section is to give a negative answer to the question:

Theorem 4.1. *There exists a minimal flow on which the induced uniform algebra is not a Dirichlet algebra.*

Let $(X, \{S_t\}_{t \in \mathbb{R}})$ be the flow in §2, and let M be a minimal set in it. We denote the restriction of $\{S_t\}_{t \in \mathbb{R}}$ to M by $\{T_t\}_{t \in \mathbb{R}}$. Then $(M, \{T_t\}_{t \in \mathbb{R}})$ is a minimal flow (see Figure 1 for the situation). We shall show the induced uniform algebra $A(M)$ is not a Dirichlet algebra on M .

The outline of our proof runs as follows. As usual, $M \times (0, \infty)$ is regarded as a subset of the maximal ideal space $\mathfrak{M}(A(M))$ of $A(M)$. For a given x_0 in M , we choose a suitable divergent sequence $\{u_n\}$ such that an accumulating point ξ of $\{(x_0, u_n)\}$ lies in a nontrivial part $P(\xi)$ that is homeomorphic to Δ . This follows from Hoffman's theorem and Theorem 3.1. Furthermore, by virtue of Marshall's theorem [10], we see that the closure $\overline{P(\xi)}$ of $P(\xi)$ is a homeomorphic replica of $\mathfrak{M}(H^\infty(\Delta))$ and that the restriction $A|_{\overline{P(\xi)}}$ of $A(M)$ to $\overline{P(\xi)}$ is isometrically isomorphic to $H^\infty(\Delta)$. Let Γ_0 be the Šilov boundary of $A|_{\overline{P(\xi)}}$. Since $H^\infty(\Delta)$ is not a Dirichlet algebra on its Šilov boundary, a real measure on Γ_0 is orthogonal to $A|_{\overline{P(\xi)}}$. Although Γ_0 lies off M , the Riesz representation theorem assures the existence of a real invariant measure on M which is orthogonal to $A(M)$. Consequently, $A(M)$ cannot be a Dirichlet algebra.

We begin by constructing artificial Blaschke products. Let N be a positive integer, and fix $\gamma = t + iu$ in \mathbb{R}_+^2 . We then set

$$(4.1) \quad \zeta_j = Nj + \gamma, \quad j \in \mathbb{Z}.$$

Then the points $\{\zeta_j\}$ are periodically distributed on the horizontal line $\{\operatorname{Im} z = u\}$ and form an interpolating sequence in \mathbb{R}_+^2 . We denote by $B(N, \gamma)(z)$ the Blaschke product with zeros $\{\zeta_j\}$ (see [6, Chapter II, (2.3)]). It is useful to represent $B(n, \gamma)(z)$ by another form. Observe that $z \rightarrow e^{2\pi iz/N}$ is an inner function in $H^\infty(\mathbb{R}_+^2)$ with period N . Then we have that

$$(4.2) \quad B(N, \gamma)(z) = c \frac{e^{2\pi iz/N} - e^{2\pi i\gamma/N}}{1 - e^{2\pi iz/N} \overline{e^{2\pi i\gamma/N}}},$$

where c is a unimodular constant. In particular, if $u > 1$, then we write

$$(4.3) \quad z_j = Nj + iu, \quad j \in \mathbb{Z}.$$

Since z_j and \bar{z}_{-j} are axial symmetric with respect to the imaginary axis, $B(N, iu)$ has a simple form

$$(4.4) \quad B(N, iu)(z) = - \prod_{j=-\infty}^{\infty} \frac{z - z_j}{z - \bar{z}_j} = - \frac{z - iu}{z + iu} \prod_{j=1}^{\infty} \frac{(z - iu)^2 - N^2 j^2}{(z + iu)^2 - N^2 j^2}.$$

Setting $\gamma = t + iu$, we see that $B(N, \gamma)(z) = cB(N, iu)(z - t)$ for a unimodular constant c .

The pseudo-hyperbolic distances ρ_1 and ρ on Δ and \mathbf{R}_+^2 are defined by

$$(4.5) \quad \begin{cases} \rho_1(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right| & \text{on } \Delta, \text{ and} \\ \rho(z, w) = \left| \frac{z - w}{z - \overline{w}} \right| & \text{on } \mathbf{R}_+^2, \end{cases}$$

respectively. Let L be a linear fractional map of Δ onto \mathbf{R}_+^2 . Then we see that $\rho_1(z, w) = \rho(L(z), L(w))$ on $\Delta \times \Delta$. Let $0 < \lambda < 1$, and set

$$U(w, \lambda) = \{z \in \mathbf{R}_+^2; \rho(z, w) < \lambda\}$$

for $w = t + is$ in \mathbf{R}_+^2 . Then an easy calculation shows that $U(w, \lambda)$ is an open disc with center $(t, s(1 + \lambda^2)/(1 - \lambda^2))$ and radius $2s\lambda/(1 - \lambda^2)$. It follows from [6, Chapter I, Lemma 1.4] that if ζ lies in $U(w, \lambda)$, then $U(\zeta, \lambda)$ is contained in $U(w, 2\lambda/(1 + \lambda^2))$.

Lemma 4.2. *Let $K > 0$, and let $0 < \delta < \varepsilon < 1$. Then there are a positive integer $N = N(K, \varepsilon, \delta)$ and a real number $u = u(K, \varepsilon, \delta)$, $u > 1$, satisfying the following properties:*

(i) *Let $\{z_j\}$ be the sequence by (4.3) with these N and u , and let*

$$U_j = \{z; \rho(z, z_j) < 2(1 - \varepsilon)/(1 + (1 - \varepsilon)^2)\}$$

for each j in \mathbf{Z} . Then the discs $\{U_j\}$ are pairwise disjoint, and each U_j is contained in the half-plane $\{\operatorname{Im} z > K\}$ (see Figure 2 on the next page).

(ii) *Let γ in \mathbf{R}_+^2 satisfy that $\rho(\gamma, iu) < 1 - \varepsilon$, and let $\{\zeta_j\}$ be the sequence by (4.1). If $B(N, \gamma)(z)$ denotes the Blaschke product with zeros $\{\zeta_j\}$, then*

$$(4.6) \quad |B(N, \gamma)(z)| \geq 1 - \delta \quad \text{on } \{0 \leq \operatorname{Im} z \leq K\},$$

and

$$(4.7) \quad \|B(N, \gamma)(z) - \rho(z, \zeta_j)\| < \varepsilon \quad \text{on } U_j.$$

(iii) *To each $\delta' > 0$, there correspond $a = a(\gamma)$, $1 - \delta < a < 1$, and $K', K' > K$, such that*

$$(4.8) \quad \sup\{|B(N, \gamma)(z) - a|; \operatorname{Im} z > K'\} < \delta',$$

that is, the left side of (4.8) converges to 0 as $K' \rightarrow \infty$. Furthermore, for each square $Q = [t_0, t_0 + l] \times [0, l]$, $l > K'$, the inequality

$$(4.9) \quad \sum_{\zeta_j \in Q} \operatorname{Im} \zeta_j \leq \varepsilon l$$

holds.

Proof. By setting $\lambda = 2(1 - \varepsilon)/(1 + (1 - \varepsilon)^2)$, U_j is the disc with center $(Nj, u(1 + \lambda^2)/(1 - \lambda^2))$ and radius $2u\lambda/(1 - \lambda^2)$. Thus the discs $\{U_j\}$ have the property (i) whenever N and u satisfy that

$$(4.10) \quad u > K(1 + \lambda)/(1 - \lambda) \quad \text{and} \quad N > 4u\lambda/(1 - \lambda^2).$$

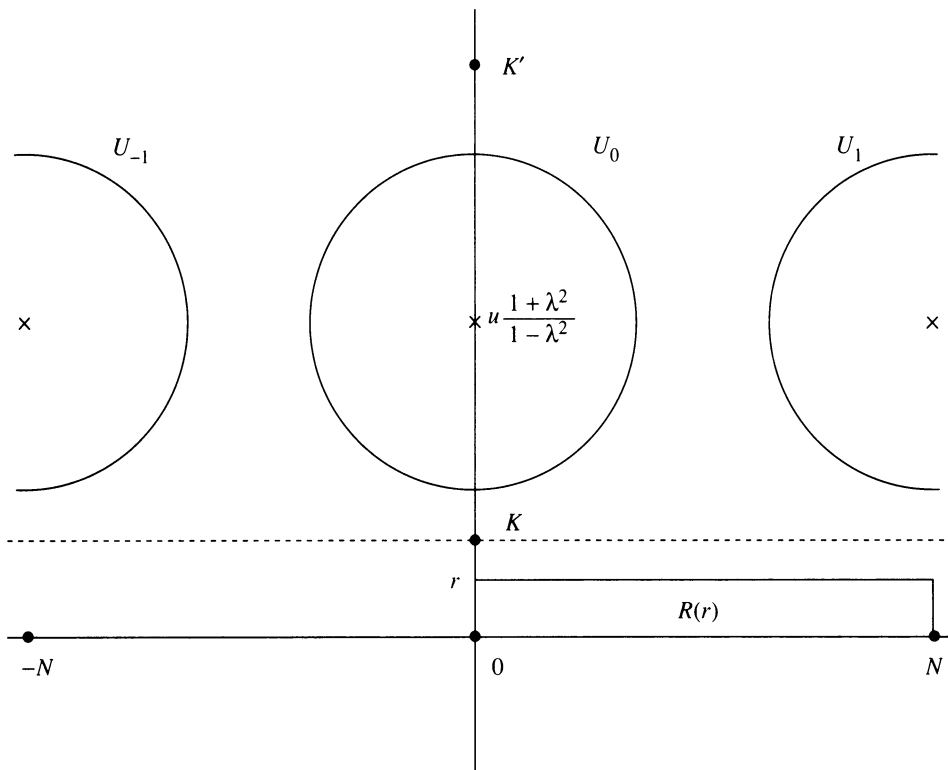


FIGURE 2

Let (N, u) be a pair with the property (i). We notice that u and N/u may be chosen as large as desired. Define that

$$W_0 = \{\gamma; \rho(\gamma, iu) \leq 1 - \varepsilon\}.$$

Then W_0 is a compact disc contained in U_0 . Let $R(r)$, $0 \leq r \leq K$, be the rectangle $[0, N] \times [0, r]$ (see Figure 2). We claim that $(z, \gamma) \rightarrow |B(N, \gamma)(z)|$ is uniformly continuous on $R(K) \times W_0$. Indeed, we find easily a constant $c > 0$ such that

$$(4.11) \quad 1 - \rho(z, Nj + \gamma)^2 < c/j^2, \quad j \neq 0,$$

on $R(K) \times W_0$. Then the product $|B(N, \gamma)(z)| = \prod_{j=-\infty}^{\infty} \rho(z, Nj + \gamma)$ converges uniformly on $R(K) \times W_0$. Since each $(z, \gamma) \rightarrow \rho(z, Nj + \gamma)$ is continuous on $R(K) \times W_0$, the uniform continuity of $|B(N, \gamma)(z)|$ follows from the compactness of $R(K) \times W_0$. Consequently, since $|B(N, \gamma)(z)| = 1$ on $R(0) \times W_0$, there is an r , $0 < r < K$, such that $|B(N, \gamma)(z)| > 1 - \delta$ on $R(r) \times W_0$. Since $\rho(z, Nj + \gamma) = \rho(z, \zeta_j) = \rho(kz, k\zeta_j)$ we see that

$$|B(N, \gamma)(z)| = \prod_{j=-\infty}^{\infty} \rho(kz, k\zeta_j) = |B(kN, k\gamma)(kz)|$$

for each positive integer k . Then $|B(kN, k\gamma)(z)| > 1 - \delta$ on $[0, kN] \times [0, kr]$. Therefore, if we replace N and u by larger ones, we make (N, u) satisfy $|B(N, \gamma)(z)| > 1 - \delta$ on $R(K) \times W_0$. Since $|B(N, \gamma)(z)|$ is periodic with period N , (4.6) follows.

We see similarly that (4.11) holds on $U_0 \times W_0$ with a suitable constant $c > 0$. Since the partial product $\prod_{j=-m}^m \rho(z, Nj + \gamma)$ converges uniformly to $|B(N, \gamma)(z)|$ on $U_0 \times W_0$, there is an m such that

$$\left| \prod_{|j| \geq m+1} \rho(z, Nj + \gamma) - 1 \right| < \varepsilon$$

on $U_0 \times W_0$. Replacing N with one larger than mN , we may assume

$$\|B(N, \gamma)(z) - \rho(z, \gamma)\| < \varepsilon$$

on $U_0 \times W_0$. Then (4.7) follows from the periodicity of $|B(N, \gamma)(z)|$. Thus we obtain the property (ii).

We write $\gamma = t + is$. Since $\rho(\gamma, iu) < 1 - \varepsilon$, we have easily the inequality,

$$u \frac{1 - (1 - \varepsilon)}{1 + (1 - \varepsilon)} < s < u \frac{1 + (1 - \varepsilon)}{1 - (1 - \varepsilon)}.$$

Since N/u is chosen as large as desired, we may assume that $e^{-2\pi s/N} > 1 - \delta$. Define $a(\gamma) = e^{-2\pi s/N}$. By (4.2), we see that (4.8) holds for some $K' > K$.

If we make N/u and K' sufficiently large, (4.9) holds for each square $Q = [t_0, t_0 + l] \times [0, l]$, $l > K'$, so the proof is complete. \square

Let $\{\varepsilon_n\}$ be a decreasing sequence such that $0 < \varepsilon_n < 1$ and $\sum \varepsilon_n < \infty$. We then choose a sequence $\{\delta_n\}$, $0 < \delta_n < 1$, such that

$$(4.12) \quad \prod_{k=n}^{\infty} (1 - \delta_k) > 1 - \varepsilon_n, \quad n \geq 1.$$

For a given $K_n > 0$, let N_n, u_n, γ_n , and a_n be as in Lemma 4.2 with respect to the above ε_n and δ_n . We also set $z_{nj} = N_n j + iu_n$ and $\zeta_{nj} = N_n j + \gamma_n$, $j \in \mathbf{Z}$.

By induction, we obtain immediately the following:

Lemma 4.3. *There is an increasing sequence $\{K_n\}$ with the following properties:*

(i) *Let $n \geq 2$. Then*

$$(4.13) \quad \left| \prod_{k=m}^{n-1} |B(N_k, \gamma_k)(z - t_k)| - \prod_{k=m}^{n-1} a_k \right| < \varepsilon_n$$

holds on $\{\operatorname{Im} z > K_n\}$, where $0 \leq t_k \leq N_k - 1$ and $m = 1, 2, \dots, n-1$.

(ii) *Let $U_{nj} = \{z; \rho(z, z_{nj}) < 2(1 - \varepsilon_n)/(1 + (1 - \varepsilon_n)^2)\}$. Then each U_{nj} , $j \in \mathbf{Z}$, is contained in $\{K_n < \operatorname{Im} z < K_{n+1}\}$.*

(iii) *If $Q = [t_0, t_0 + l] \times [0, l]$, $l > K_{n+1}$, then the inequality*

$$\sum_{\zeta_{nj} \in Q} \operatorname{Im} \zeta_{nj} \leq \varepsilon_n l, \quad n \geq 1,$$

holds.

Recall our notation \mathbf{M} for a minimal set in $(\mathbf{X}, \{\mathbf{S}_t\}_{t \in \mathbf{R}})$. Identifying \mathbf{X} with $\beta\mathbf{Z} \times [0, 1)$, we fix an x_0 in \mathbf{M} of the form $(y_0, 0)$ in $\beta\mathbf{Z} \times [0, 1)$. For each $n \geq 1$, we set

$$Z_n(k) = \{N_n j + k; j \in \mathbf{Z}\},$$

where $k = 0, 1, \dots, N_n - 1$. Since \mathbf{Z} is the disjoint union of $Z_n(k)$, there is a unique k_n such that the closure of $Z_n(k_n)$ in $\beta\mathbf{Z}$ contains the above point y_0 . We then define

$$(4.14) \quad w_{nj} = z_{nj} + k_n = (N_n j + k_n) + iu_n.$$

Then $\{w_{nj}; j \in \mathbf{Z}\}$ is an interpolating sequence for each $n \geq 1$. We notice that $B(N_n, k_n + iu_n)(z)$ is the Blaschke product with zeros $\{w_{nj}; j \in \mathbf{Z}\}$. Recall that $\mathbf{X} \times [0, \infty)$ is embedded in $\mathfrak{M}(A(\mathbf{X}))$ by (2.1). By identifying $X_0 \times [0, \infty)$ with \mathbf{R}_+^2 , the point $(x_0, u_n) = ((y_0, 0), u_n)$ lies in the closure of $\{w_{nj}; j \in \mathbf{Z}\}$ in $\mathfrak{M}(A(\mathbf{X}))$. We notice that (x_0, u_n) lies also in $\mathfrak{M}(A(\mathbf{M}))$, since $\mathbf{M} \times [0, \infty)$ is a subset of $\mathfrak{M}(A(\mathbf{M}))$. It will be shown that the accumulating point ξ of $\{(x_0, u_n); n \geq 1\}$ in $\mathfrak{M}(A(\mathbf{M}))$ lies in a nontrivial part $P(\xi)$ that is homeomorphic to Δ .

Let us show that $\{w_{nj}; n \geq 1, j \in \mathbf{Z}\}$ is an interpolating sequence in \mathbf{R}_+^2 . These points are clearly separated by (ii) of Lemma 4.3. Since $\sum \varepsilon_n < \infty$, it follows from (iii) of Lemma 4.3 that

$$(4.15) \quad \sum_{w_{nj} \in Q} \operatorname{Im} w_{nj} \leq \left(1 + \sum \varepsilon_n\right) l$$

for each square $Q = [t_0, t_0 + l] \times [0, l]$. Therefore, by [6, Chapter VII, Theorem 1.1], $\{w_{nj}; n \geq 1, j \in \mathbf{Z}\}$ is an interpolating sequence.

Let $B_m(z)$ be the Blaschke product with zeros $\{w_{nj}; n \geq m, j \in \mathbf{Z}\}$. Since

$$(4.16) \quad B_m(z) = \prod_{n=m}^{\infty} B(N_n, k_n + iu_n)(z),$$

$B_m(z)$ has the periodic zeros $\{w_{nj}; j \in \mathbf{Z}\}$ on each horizontal line $\{\operatorname{Im} z = u_n\}$, $n \geq m$, (see Figure 3 for the distribution of zeros of $B_1(z)$). We see

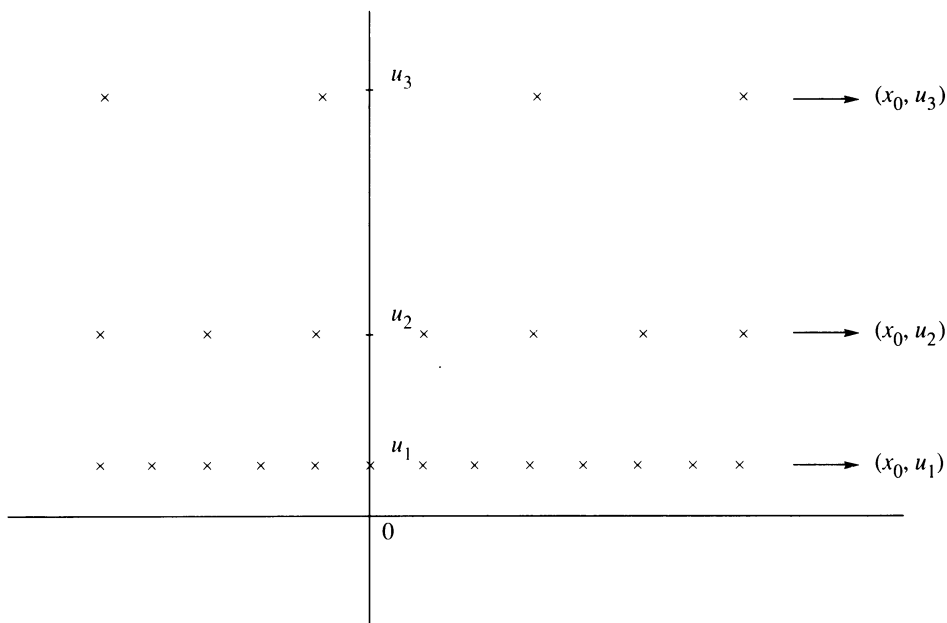


FIGURE 3

by (4.16) and [6, Chapter II, Theorem 6.1] that $B_m(z)$ extends analytically to $\{\operatorname{Im} z \geq -K_m\}$. It follows from (4.12) and (ii) of Lemma 4.2 that $|B_m(z)| \leq (1 - \varepsilon_m)^{-1}$ on $\{\operatorname{Im} z \geq -K_m\}$. Let $0 < s < K_m$, and define the bounded function F on \mathbf{R} by $F(t) = B_m(t - is)$. Since $B_m(t) = F * P_{is}(t)$, we see that $t \rightarrow B_m(t)$ is uniformly continuous on \mathbf{R} . Hence Lemma 3.1 implies that $B_m(t)$ extends to a function $B_m(x)$ in $A(\mathbf{X})$. Recall that $O(x_0) \times (0, \infty)$ is a nontrivial part in $\mathfrak{M}(A(\mathbf{X}))$. However, since x_0 is in the minimal set \mathbf{M} , $O(x_0) \times (0, \infty)$ is not homeomorphic to \mathbf{R}_+^2 . On the other hand, $O(x_0) \times (0, \infty)$ contains every (x_0, u_n) , which is an accumulating point of $\{w_{nj}; j \in \mathbf{Z}\}$.

Let $\{\varepsilon_n\}$, $\{\gamma_n\}$, and $\{U_{nj}\}$ be as in Lemma 4.3, and let $\{w_{nj}\}$ be the sequence defined by (4.14). We then set

$$\begin{aligned} V_{nj} &= \{z; \rho(z, w_{nj}) < 2(1 - \varepsilon_n)/(1 + (1 - \varepsilon_n)^2)\} \\ &= U_{nj} + k_n. \end{aligned}$$

Observe that $B(N_n, \gamma_n)(z - k_n)$ is (a unimodular constant multiple of) the interpolating Blaschke product with zeros $\{\zeta_{nj} + k_n; j \in \mathbf{Z}\}$, where $\zeta_{nj} = N_n j + \gamma_n$. By (iii) of Lemma 4.3, (4.15) holds with $\zeta_{nj} + k_n$ in place of w_{nj} . Then $\{\zeta_{nj} + k_n; n \geq 1, j \in \mathbf{Z}\}$ is also an interpolating sequence. Let $C_m(z)$ be the Blaschke product with zeros $\{\zeta_{nj} + k_n; n \geq m, j \in \mathbf{Z}\}$. So we may write

$$(4.17) \quad C_m(z) = \prod_{n=m}^{\infty} B(N_n, k_n + \gamma_n)(z)$$

(compare with (4.16)).

Let $\{\delta_n\}$, $\{K_n\}$, and $\{a_n\}$ be also as in Lemma 4.3. Then we see that $|B(N_n, k_n + \gamma_n)(z)| > 1 - \delta_n$ on $\{0 < \operatorname{Im} z < K_n\}$ and $1 - \delta_n < a_n < 1$ by Lemma 4.2. It follows from (4.12) that the product

$$(4.18) \quad a^{(m)} = \prod_{n=m}^{\infty} a_n$$

converges and satisfies that $1 - \varepsilon_m < a^{(m)} < 1$.

Lemma 4.4. *Let V_{nj} and $C_m(z)$ be as in above. Then*

$$\sup\{|C_m(z)| - a^{(m)}\rho(z, \zeta_{nj} + k_n)|; z \in V_{nj}\}$$

converges to 0 as $n \rightarrow \infty$.

Proof. We set, at our convenience, that

$$E_n(z) = |B(N_n, k_n + \gamma_n)(z)|,$$

and

$$F_n(z) = a^{(n+1)} E_n(z) \prod_{k=m}^{n-1} E_k(z).$$

Then $|C_m(z)| = \prod_{k=m}^{\infty} E_k(z)$ by (4.17). Since V_{nj} is contained in $\{K_n < \operatorname{Im} z < K_{n+1}\}$ by (ii) of Lemma 4.3, it follows from (4.6) that

$$\prod_{k=n+1}^{\infty} (1 - \delta_n) < \prod_{k=n+1}^{\infty} E_k(z) < 1, \quad z \in V_{nj}.$$

Observe that

$$|F_n(z) - a^{(n+1)}\rho(z, \zeta_{nj} + k_n) \prod_{k=m}^{n-1} E_k(z)| \leq |E_n(z) - \rho(z, \zeta_{nj} + k_n)|.$$

Then we see by (4.7), (4.13), and (4.18) that

$$\begin{aligned} & ||C_m(z) - a_n^{-1}a^{(m)}\rho(z, \zeta_{nj} + k_n)| \\ & \leq ||C_m(z) - F_n(z)| + \left| F_n(z) - a^{(n+1)}\rho(z, \zeta_{nj} + k_n) \prod_{k=m}^{n-1} a_k \right| \\ & \leq \left| |C_{n+1}(z)| - a^{(n+1)} \right| + |E_n(z) - \rho(z, \zeta_{nj} + k_n)| + \left| \prod_{k=m}^{n-1} E_k(z) - \prod_{k=m}^{n-1} a_k \right| \\ & < \left(1 - \prod_{k=n+1}^{\infty} (1 - \delta_k) \right) + \varepsilon_n + \varepsilon_n \\ & < 3\varepsilon_n \end{aligned}$$

for each z in V_{nj} . Since a_n^{-1} tends to 1 as $n \rightarrow \infty$, the conclusion follows. \square

We denote by ξ_n the homomorphism (x_0, u_n) in $\mathfrak{M}(A(\mathbf{X}))$. To each $n \geq 1$, there corresponds a subnet $\{w_{n\beta}\}$ of $\{w_{nj}; j \in \mathbf{Z}\}$ converging to ξ_n in $\mathfrak{M}(A(\mathbf{X}))$. Let ξ be an accumulating point of $\{\xi_n\}$. Then ξ is also an accumulating point of the interpolating sequence $\{w_{nj}; n \geq 1, j \in \mathbf{Z}\}$. Therefore, by Hoffman's theorem and Theorem 3.1, ξ belongs to a nontrivial part $P(\xi)$. Precisely, let L_{nj} be the linear fractional map of Δ onto \mathbf{R}_+^2 by

$$(4.19) \quad L_{nj}(z) = \frac{w_{nj} - \overline{w_{nj}}z}{1 - z}, \quad z \in \Delta.$$

Then L_{nj} is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{X}))$. Taking a finer subnet $\{L_{n\beta}\}$ of $\{L_{nj}\}$, if necessary, we may assume that

$$(4.20) \quad \lim_{\beta} L_{n\beta} = L_n, \quad n \geq 1,$$

in $\mathfrak{M}(A(\mathbf{X}))^\Delta$, where L_n is in $[\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}]^\Delta$ and satisfies that $L_n(0) = \xi_n$. Let us determine the explicit form of L_n . For simplicity, we write $x + t + is$ for $(x + t, s)$ in $\mathbf{X} \times [0, \infty)$. It then follows from (4.19) that

$$(4.21) \quad L_n(z) = x_0 + \frac{i u_n(1 + z)}{1 - z}, \quad z \in \Delta,$$

which maps Δ onto the nontrivial part $O(x_0) \times (0, \infty)$. We notice that $L_{n\beta}$ and L_n lie in $[\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}]^\Delta$, so the convergence in (4.20) is the same one in $\mathfrak{M}(H^\infty(\mathbf{X}))^\Delta$ by Theorem 3.1. As we mentioned earlier, L_n is not a homeomorphism, since x_0 lies in the minimal set \mathbf{M} . We take again a subnet $\{L_\alpha\}$ of $\{L_n\}$ such that

$$(4.22) \quad \lim_{\alpha} L_\alpha = L_0$$

for a map L_0 in $\mathfrak{M}(A(\mathbf{X}))^\Delta$ with $L_0(0) = \xi$. Since $\lim_{\alpha} u_\alpha = \infty$, we see by (4.21) that L_0 is also in $[\mathfrak{M}(A(\mathbf{X})) \setminus \mathbf{X}]^\Delta$. It then follows from Theorem 3.1 and Hoffman's theorem that L_0 is an analytic map and the range $L_0(\Delta)$ is the

nontrivial part $P(\xi)$ containing ξ . We recall that the meaning of (4.22) is that if z lies in Δ , then

$$(4.23) \quad \lim_{\alpha} \phi \circ L_{\alpha}(z) = \phi \circ L_0(z)$$

for each ϕ in $A(\mathbf{X})$.

We next show that L_0 may be considered as an analytic map of Δ into $\mathfrak{M}(A(\mathbf{M}))$. Let $E_{\mathbf{X}}$ be the closure of $\mathbf{M} \times [0, \infty)$ in $\mathfrak{M}(A(\mathbf{X}))$. Since $L_{\alpha}(z)$, $z \in \Delta$, is in $\mathbf{M} \times [0, \infty)$ by (4.21) and $\lim_{\alpha} u_{\alpha} = \infty$, we see that $L_0(z)$ lies in $E_{\mathbf{X}} \setminus \mathbf{M} \times [0, \infty)$. Then the whole $P(\xi)$ is contained in $E_{\mathbf{X}} \setminus \mathbf{M} \times [0, \infty)$. On the other hand, $\mathbf{M} \times [0, \infty)$ is also considered as a subset of $\mathfrak{M}(A(\mathbf{M}))$. We denote similarly by $E_{\mathbf{M}}$ the closure of $\mathbf{M} \times [0, \infty)$ in $\mathfrak{M}(A(\mathbf{M}))$. We have to examine the relation between $E_{\mathbf{X}}$ and $E_{\mathbf{M}}$.

Let $A|_{\mathbf{M}}$ be the space of the restrictions to \mathbf{M} of functions in $A(\mathbf{X})$. Then $A|_{\mathbf{M}}$ is a subalgebra of $A(\mathbf{M})$ containing constants. So there is a natural restriction map σ_1 of $E_{\mathbf{M}}$ to $E_{\mathbf{X}}$ which is the identity map on $\mathbf{M} \times [0, \infty)$. Let us show that σ_1 is a homeomorphism of $E_{\mathbf{M}}$ onto $E_{\mathbf{X}}$. Since $A(\mathbf{X})$ is a logmodular algebra, σ_1 is a one-to-one continuous map. If η lies in $E_{\mathbf{X}}$, then there is a net $\{\eta_{\beta}\}$ in $\mathbf{M} \times [0, \infty)$ converging to η in $E_{\mathbf{X}}$. Since $E_{\mathbf{M}}$ is compact, we may assume that $\{\eta_{\beta}\}$ converges to η_1 in $E_{\mathbf{M}}$. We then have that $\sigma_1(\eta_1) = \eta$. Since $E_{\mathbf{M}}$ is compact, $E_{\mathbf{M}}$ and $E_{\mathbf{X}}$ are homeomorphic via the map σ_1 .

We see that the closure of $A|_{\mathbf{M}}$ is a logmodular algebra on \mathbf{M} , then so is $A(\mathbf{M})$. If $\sigma_1(\eta_1) = \eta$, then η_1 and η have the same representing measures. Identifying L_0 with $\sigma_1^{-1} \circ L_0$, we see that (4.23) holds for each ϕ in $A(\mathbf{M})$. This implies that L_0 is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{M}))$.

Since the representing measures for the points in $P(\xi)$ are mutually absolutely continuous, $P(\xi)$ is also a nontrivial part with respect to $A(\mathbf{M})$ by [5, Chapter VI, §1.2].

Let μ be the representing measure for a point η in $E_{\mathbf{M}} \setminus \mathbf{M} \times [0, \infty)$. We show that μ is invariant on $(\mathbf{M}, \{T_t\}_{t \in \mathbf{R}})$, although the similar fact has been used tacitly in the proofs of Lemma 2.5 and (ii) of Proposition 3.3. Indeed, there is a net (x_{β}, u_{β}) in $\mathbf{M} \times [0, \infty)$ converging to η , where $\lim_{\beta} u_{\beta} = \infty$. Let μ_{β} be the representing measure for (x_{β}, u_{β}) . Then $\{\mu_{\beta}\}$ converges to μ in the weak- $*$ topology on $M(\mathbf{M})$. Since

$$|P_{iu}(t-s) - P_{iu}(t)| \leq \frac{s}{u} (1 + \frac{s}{u}) P_{iu}(t-s),$$

we see that

$$\int_{\mathbf{M}} \phi(x+s) d\mu(x) = \int_{\mathbf{M}} \phi(x) d\mu(x), \quad \phi \in C(\mathbf{M}),$$

for each s in \mathbf{R} . Then μ is invariant on \mathbf{M} . In particular, the representing measure for each point in $P(\xi)$ is invariant on \mathbf{M} .

Lemma 4.5. *Let ξ_n denote the homomorphism (x_0, u_n) defined above, and let ξ be an accumulating point of $\{\xi_n\}$ in $\mathfrak{M}(A(\mathbf{M}))$. Then ξ lies in a nontrivial part $P(\xi)$ in $\mathfrak{M}(A(\mathbf{M}))$ satisfying the following properties:*

(i) *Let $\overline{P(\xi)}$ be the closure of $P(\xi)$ in $\mathfrak{M}(A(\mathbf{M}))$. Then each point in $\overline{P(\xi)}$ has a unique representing measure which is invariant on \mathbf{M} .*

(ii) The part $P(\xi)$ is homeomorphic to Δ . Precisely, the analytic map L_0 by (4.22) has its continuous inverse.

Proof. We have shown that ξ lies in a nontrivial part $P(\xi)$, in which every representing measure is invariant on \mathbf{M} . Since the set of all invariant probability measures on \mathbf{M} is weak- $*$ compact in $M(\mathbf{M})$, each point in $\overline{P(\xi)}$ has also an invariant representing measure. Thus (i) follows.

Let L_{nj} , L_n , and L_0 be as in (4.19), (4.21), and (4.22), respectively. Let $B_m(z)$, $m \geq 1$, be the Blaschke product defined by (4.16). Since $B_m(z)$ extends to a function $B_m(x)$ in $A(\mathbf{X})$, we may regard $B_m(x)$ as a function in $A(\mathbf{M})$ by identifying with its restriction to \mathbf{M} . Since

$$\rho(L_{nj}(z), L_{nj}(0)) = \rho_1(z, 0) = |z|, \quad z \in \Delta,$$

by (4.5), Lemma 4.4 implies that

$$\sup\{||B_m \circ L_{nj}(z)| - a^{(m)}|z||; |z| \leq 1 - \varepsilon_n\}$$

converges to 0 as $n \rightarrow \infty$, where $a^{(m)}$ is the constant by (4.18). By (4.20) and (4.22), we obtain that $|B_m \circ L_0(z)| = a^{(m)}|z|$ on Δ . It then follows from Schwarz's lemma that $(c_m B_m) \circ L_0(z) = z$ for a constant c_m with $|c_m| = (a^{(m)})^{-1}$. Since B_m is continuous on $\mathfrak{M}(A(\mathbf{M}))$, L_0 has the continuous inverse. \square

To extend (ii) of Lemma 4.5 (see Lemma 4.8), we need the following:

Lemma 4.6. Let L_0 be the analytic map of Δ into $\mathfrak{M}(A(\mathbf{X}))$ by (4.22), and let $b(z)$ be an arbitrary Blaschke product on Δ with zeros $\{\alpha_k\}$, that is,

$$b(z) = \prod_{k=1}^{\infty} \frac{\bar{\alpha}_k}{|\alpha_k|} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z}$$

(we consider $\bar{\alpha}_k/|\alpha_k| = -1$, if $\alpha_k = 0$). If $0 < \varepsilon < 1$, then there is a Blaschke product $B(z)$ on \mathbf{R}_+^2 such that $B(z)$ extends to a function $B(x)$ in $A(\mathbf{X})$ satisfying that

$$(4.25) \quad cB \circ L_0(z) = b(z), \quad z \in \Delta,$$

for some constant c with $|c| < 1 + \varepsilon$.

Proof. We have already done the hard work back in Lemma 4.4. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be as in (4.12), and let $\{r_k\}$ be a sequence such that $0 < r_k < 1$ and $\prod r_k > (1 + \varepsilon)^{-1}$. We assume that $|\alpha_k| \leq |\alpha_{k+1}|$ for all $k \geq 1$.

Let w_{nj} and L_{nj} be as in (4.14) and (4.19), respectively. Then we see by (4.5) that $\rho_1(z, w) = \rho(L_{nj}(z), L_{nj}(w))$ on $\Delta \times \Delta$. Since $L_{nj}(0) = w_{nj}$, to each $k \geq 1$, there corresponds $m_1(k)$ such that

$$\rho(w_{nj}, L_{nj}(\alpha_k)) = \rho_1(0, \alpha_k) = |\alpha_k| < 1 - \varepsilon_n$$

for all $n \geq m_1(k)$. Observe that $L_{nj}(\alpha_k) = N_n j + L_{n0}(\alpha_k)$ by (4.19). By (iii) of Lemma 4.3, $\{L_{nj}(\alpha_k); n \geq m_1(k), j \in \mathbf{Z}\}$ is an interpolating sequence in \mathbf{R}_+^2 . Making $m_1(k)$ large, we may assume that

$$(4.26) \quad \sum_{n=m_1(k)}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\operatorname{Im} L_{nj}(\alpha_k)}{1 + |L_{nj}(\alpha_k)|^2} < \left(\frac{1}{2}\right)^k.$$

Let $m(k) \geq m_1(k)$. We denote by $C_{m(k)}(z)$ the Blaschke product on \mathbf{R}_+^2 with zeros $\{L_{nj}(\alpha_k); n \geq m(k), j \in \mathbf{Z}\}$. Then $C_{m(k)}(z)$ is the Blaschke product in (4.17) with $L_{n0}(\alpha_k)$ in place of γ_n . Let $a^{(m(k))}$ be the constant by (4.18) with respect to $C_{m(k)}(z)$. If we make $m(k)$ sufficiently large, then $a^{(m(k))}$ satisfies that $r_k < a^{(m(k))} < 1$. By Lemma 4.4, we see that

$$\sup\{|C_{m(k)}(z)| - a^{(m(k))}\rho(z, L_{nj}(\alpha_k))\}; z \in V_{nj}\}$$

converges to 0 as $n \rightarrow \infty$. By induction, we choose an increasing sequence $\{m(k)\}$, $m(k) \geq m_1(k)$, such that

$$\sup\left\{\left|\prod_{i=1}^k C_{m(i)}(z) - \prod_{i=1}^k a^{(m(i))}\rho(z, L_{nj}(\alpha_i))\right|; z \in V_{nj}\right\} < \varepsilon_k$$

for each n with $m(k+1) \geq n > m(k)$, and that

$$|C_{m(k)}(z)| \geq \prod_{i=m(k)}^{\infty} (1 - \delta_i) > 1 - \delta_k$$

on $\{0 \leq \operatorname{Im} z < K_{m(k)}\}$ by (4.6) and (4.17). We then set

$$B(z) = \prod_{i=1}^{\infty} C_{m(i)}(z) \quad \text{and} \quad a = \prod_{i=1}^{\infty} a^{(m(i))}.$$

If $m(k+1) \geq n > m(k)$, then we have by (4.12) that

$$\left| |B(z)| - \left| \prod_{i=1}^k C_{m(i)}(z) \right| \right| \leq \left| \left| \prod_{i=k+1}^{\infty} C_{m(i)}(z) \right| - 1 \right| < 1 - \prod_{i=k+1}^{\infty} (1 - \delta_i) < \varepsilon_k$$

on V_{nj} . This implies that

$$\sup\left\{\left| |B(z)| - \prod_{i=1}^k a^{(m(i))}\rho(z, L_{nj}(\alpha_i)) \right|; z \in V_{nj}\right\} < 2\varepsilon_k.$$

Let $b_k(z)$ be the partial product of $b(z)$:

$$b_k(z) = \prod_{i=1}^k \frac{\bar{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z}.$$

Then, whenever $m(k+1) \geq n > m(k)$, we obtain that

$$\sup\left\{\left| |B \circ L_{nj}(z)| - \prod_{i=1}^k a^{(m(i))}|b_k(z)| \right|; |z| \leq 1 - \varepsilon_n \right\} < 2\varepsilon_k.$$

Consequently, it follows from (4.20) and (4.22) that

$$|B \circ L_0(z)| = a|b(z)|, \quad z \in \Delta.$$

Since $a > (1 + \varepsilon)^{-1}$, the maximum principle shows that (4.25) holds for a constant c with $|c| = a^{-1}$. \square

Recall that we identify ϕ in $A(\mathbf{M})$ with its Gelfand transform on $\mathfrak{M}(A(\mathbf{M}))$. Then $\phi|_{P(\xi)}$ denotes the restriction of ϕ to $P(\xi)$. Let $A|_{P(\xi)}$ denote the algebra

of all $\phi|_{P(\xi)}$ for ϕ in $A(\mathbf{M})$. Then the analytic map L_0 in (4.22) induces naturally the isometric isomorphism Ψ of $A|_{P(\xi)}$ to $H^\infty(\Delta)$ by

$$(4.28) \quad \Psi(\phi|_{P(\xi)})(z) = \phi \circ L_0(z), \quad z \in \Delta.$$

If $b(z)$ is a Blaschke product, let us agree to also call $cb(z)$ a Blaschke product when c is a unimodular constant. Together with Bernard's trick, Marshall has shown that the unit ball of $H^\infty(\Delta)$ is the norm closed convex hull of the set of Blaschke products (see [10] or [6, Chapter V, Corollary 2.6]).

Lemma 4.7. *Let f be a function in $H^\infty(\Delta)$ with $\|f\|_\infty = 1$, and let $\varepsilon > 0$. Then there is a function ψ in $A(\mathbf{M})$ with $\|\psi\|_\infty < 1 + \varepsilon$ such that $\psi \circ L_0(z) = f(z)$ on Δ . Consequently, the isomorphism Ψ by (4.28) maps $A|_{P(\xi)}$ onto $H^\infty(\Delta)$.*

Proof. It follows from Marshall's theorem that there is a sequence $\{q_n\}$ of convex combinations of Blaschke products on Δ such that

$$f = q_0 + \frac{\varepsilon}{3} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n q_n.$$

On the other hand, Lemma 4.6 shows that we may choose a convex combination Q_n of Blaschke products on \mathbf{R}_+^2 which extends to a function in $A(\mathbf{X})$ such that $(1 + \varepsilon/3)Q_n \circ L_0 = q_n$, $n \geq 0$. Since $\|Q_n\|_\infty \leq 1$, the series

$$\psi(x) = \left(1 + \frac{\varepsilon}{3}\right) \left(Q_0(x) + \frac{\varepsilon}{3} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n Q_n(x)\right)$$

converges uniformly on \mathbf{X} , and belongs to $A(\mathbf{X})$. Of course, the restriction of ψ to \mathbf{M} lies in $A(\mathbf{M})$. It is easy to see that $\|\psi\|_\infty < 1 + \varepsilon$ and $\psi \circ L_0(z) = f(z)$ on Δ . \square

Let $A|_{\overline{P(\xi)}}$ similarly be the algebra of all restrictions to the closure $\overline{P(\xi)}$ of functions in $A(\mathbf{M})$, and let $\widehat{H}^\infty(\Delta)$ be the uniform algebra on $\mathfrak{M}(H^\infty(\Delta))$ of all Gelfand transforms of functions in $H^\infty(\Delta)$.

The following lemma is an easy consequence of the corona theorem and Lemma 4.7, so the proof is omitted.

Lemma 4.8. *Let Γ and Γ_0 be the Šilov boundaries of $\widehat{H}^\infty(\Delta)$ and $A|_{\overline{P(\xi)}}$, respectively. Let L_0 be the analytic map of Δ onto $P(\xi)$ by (4.22). Then L_0 extends to a homeomorphism of $\mathfrak{M}(H^\infty(\Delta))$ onto $\overline{P(\xi)}$, which maps Γ onto Γ_0 .*

Observe that Γ_0 is contained in $\overline{P(\xi)} \setminus P(\xi)$. We denote by λ_ω the representing measure for ω in Γ_0 . Then λ_ω is invariant on the flow $(\mathbf{M}, \{T_t\}_{t \in \mathbf{R}})$ by (i) of Lemma 4.5. For each ϕ in $C(\mathbf{M})$, we set

$$\tau\phi(\omega) = \int_{\mathbf{M}} \phi(x) d\lambda_\omega(x), \quad \omega \in \Gamma_0.$$

Then the map $\omega \rightarrow \tau\phi(\omega)$ is continuous on Γ_0 . Indeed, by identifying each ω in Γ_0 with its unique representing measure λ_ω in $M(\mathbf{M})$, Γ_0 becomes a subset of $M(\mathbf{M}) = C(\mathbf{M})^*$, the dual space of $C(\mathbf{M})$. The weak- $*$ closure of $\{\lambda_\omega; \omega \in \Gamma_0\}$ consists of invariant probability measures which are multiplicative on $A(\mathbf{M})$. Since Γ_0 is closed in $\mathfrak{M}(A(\mathbf{M}))$ and since representing measures

are unique, the weak- $*$ closure of $\{\lambda_\omega; \omega \in \Gamma_0\}$ has to be itself. Then the map $\lambda_\omega \rightarrow \omega$ is continuous from one compact space to another, so the continuity of $\omega \rightarrow \tau\phi(\omega)$ follows. We observe easily that τ is a positive linear operator of $C(\mathbf{M})$ into $C(\Gamma_0)$ with $\|\tau\| = 1$.

Let ν be a positive measure in $M(\Gamma_0)$. We then define the linear functional F on $C(\mathbf{M})$ by

$$F(\phi) = \int_{\Gamma_0} \tau\phi(\omega) d\nu(\omega), \quad \phi \in C(\mathbf{M}).$$

We see that $|F(\phi)| \leq \|\nu\| \cdot \|\phi\|_\infty$ and $F(T_t\phi) = F(\phi)$, where $T_t\phi(x) = \phi(x+t)$. We also see that F is positive and $F(1) = \nu(\Gamma_0) = \|\nu\|$. Then the Riesz representation theorem assures the existence of a positive invariant measure μ in $M(\mathbf{M})$ such that

$$(4.29) \quad F(\phi) = \int_{\mathbf{M}} \phi(x) d\mu(x), \quad \phi \in C(\mathbf{M}).$$

It is clear that μ is uniquely determined and $\|\mu\| = \|\nu\|$. However, ν may not be determined uniquely by μ , which causes a difficulty.

We next extend τ to a positive linear operator of $L^1(\mu)$ to $L^1(\nu)$ with $\|\tau\| = 1$. Let ϕ be a function in $C_{\mathbf{R}}(\mathbf{M})$. We write $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are respectively the positive and negative parts of ϕ . Since $|\tau\phi_1 - \tau\phi_2| \leq \tau\phi_1 + \tau\phi_2$ and $|\phi| = \phi_1 + \phi_2$, we have that $\|\tau\phi\|_1 \leq \|\phi\|_1$. From this fact, the assertion follows easily. We denote also by τ this extended operator of $L^1(\mu)$ to $L^1(\nu)$.

Let τ^* be the adjoint of the extended operator τ . Then τ^* is a positive linear operator of $L^\infty(\nu)$ to $L^\infty(\mu)$, and its range consists of invariant functions. When ϕ is in $L^1(\mu)$, we put

$$(4.30) \quad \begin{cases} m_T\phi(x) = \frac{1}{2T} \int_{-T}^T \phi(x+t) dt, \text{ and} \\ m\phi(x) = \lim_{T \rightarrow \infty} m_T\phi(x). \end{cases}$$

Then the mean ergodic theorem shows that $m\phi$ lies in $L^1(\mu)$ and $\tau(m\phi) = \tau\phi$. We see also that $m_T\phi$ lies in $C(\mathbf{M})$ if ϕ is in $C(\mathbf{M})$.

Let $b(z)$ be a Blaschke product on Δ . By Lemma 4.6, we may choose a decreasing sequence $\{r_n\}$ converging to 1 and a sequence $\{B_n\}$ of Blaschke products on \mathbf{R}_+^2 such that

$$r_n B_n \circ L_0(\eta) = b(\eta), \quad \eta \in \mathfrak{M}(H^\infty(\Delta)),$$

where L_0 is identified with its extension to $\mathfrak{M}(H^\infty(\Delta))$ by Lemma 4.8. This yields that $\tau(r_n B_n)(\omega) = b \circ L_0^{-1}(\omega)$ on Γ_0 . Since $b \circ L_0^{-1}$ lies in $L^\infty(\nu)$, $\tau^*(b \circ L_0^{-1})(x)$ is an invariant function in $L^\infty(\mu)$. We claim that

$$\tau \circ \tau^*(b \circ L_0^{-1})(\omega) = b \circ L_0^{-1}(\omega), \quad \omega \in \Gamma_0.$$

Indeed, if we set $\phi_n = m(r_n B_n)$ by (4.30), then ϕ_n is an invariant function in $L^\infty(\mu)$ with $\|\phi_n\|_\infty \leq r_n$ and satisfies that $\tau\phi_n = b \circ L_0^{-1}$ on Γ_0 . So we have that

$$\|\nu\| = \int_{\Gamma_0} |b \circ L_0^{-1}|^2 d\nu = \int_{\Gamma_0} (\tau\bar{\phi}_n)(b \circ L_0^{-1}) d\nu = \int_{\mathbf{M}} \bar{\phi}_n \tau^*(b \circ L_0^{-1}) d\mu.$$

Since $\|\tau^*(b \circ L_0^{-1})\|_\infty \leq 1$ and $\|\mu\| = \|\nu\|$, this shows that

$$\begin{aligned} \|\phi_n - \tau^*(b \circ L_0^{-1})\|_2^2 &= \|\phi_n\|_2^2 - 2 \operatorname{Re} \int_{\mathbf{M}} \bar{\phi}_n \tau^*(b \circ L_0^{-1}) d\mu + \|\tau^*(b \circ L_0^{-1})\|_2^2 \\ &\leq r_n^2 \|\mu\| - 2\|\mu\| + \|\mu\| = (r_n^2 - 1)\|\mu\|. \end{aligned}$$

Since $\|\phi_n - \tau^*(b \circ L_0^{-1})\|_1 \leq (r_n^2 - 1)^{1/2} \|\mu\|$ and $\tau\phi_n = b \circ L_0^{-1}$, we see that $\tau \circ \tau^*(b \circ L_0^{-1}) = b \circ L_0^{-1}$. Thus we obtain the following:

Lemma 4.9. *Let L_0 , τ , and τ^* be as in above. If b_1 and b_2 are Blaschke products on Δ , then we have the equation*

$$\int_{\Gamma_0} (b_1 \circ L_0^{-1})(\bar{b}_2 \circ L_0^{-1}) d\nu = \int_{\mathbf{M}} \tau^*(b_1 \circ L_0^{-1}) \tau^*(\bar{b}_2 \circ L_0^{-1}) d\mu.$$

Setting $b_1 = b_2 = b$ in Lemma 4.9, we see that $\tau^*(b \circ L_0^{-1})$ is a unimodular function.

Under these observations, we may show that $A(\mathbf{M})$ is not a Dirichlet algebra on \mathbf{M} . The following proof is due to the referee, which improves our former one so much.

Proof of Theorem 4.1. It suffices to show that there is a nonzero real measure in $M(\mathbf{M})$ which is orthogonal to $A(\mathbf{M})$. It is well known that $\widehat{H}^\infty(\Delta)$ is not a Dirichlet algebra on its Šilov boundary Γ (see, for example, [8, Chapter 10]). By Lemma 4.7, we see that $A|_{\overline{P(\xi)}}$ is not a Dirichlet algebra on Γ_0 . Consequently, we may find a nonzero real measure ν in $M(\Gamma_0)$ which is orthogonal to $A|_{\overline{P(\xi)}}$. By the Jordan decomposition theorem, ν is represented as $\nu = \nu_1 - \nu_2$, where ν_1 and ν_2 are positive and negative variations of ν . We may assume that both ν_1 and ν_2 are probability measures on Γ_0 . It follows from (4.29) that there are invariant probability measures μ_i , $i = 1, 2$, such that

$$\int_{\Gamma_0} \tau \phi d\nu_i = \int_{\mathbf{M}} \phi d\mu_i, \quad \phi \in C(\mathbf{M}).$$

Let τ_i denote the extension of τ to $L^1(\mu_i)$, as described in above. The adjoint of τ_i is similarly denoted by τ_i^* . Let $0 < \varepsilon < \frac{1}{3}$. Since ν_1 and ν_2 are mutually singular, we may choose a unimodular continuous function θ on Γ_0 such that

$$\left| \int_{\Gamma_0} \theta d(\nu_1 - \nu_2) \right| \geq \|\nu_1\| + \|\nu_2\| - \varepsilon = 2 - \varepsilon.$$

Let $L^\infty(\mathbf{T})$ be the space of bounded (Lebesgue) measurable functions on the unit circle \mathbf{T} . Since Γ is the maximal ideal space of $L^\infty(\mathbf{T})$, $\theta \circ L_0$ is regarded as a unimodular function in $L^\infty(\mathbf{T})$. Then the Douglas-Rudin theorem and Frostman's theorem show that there are Blaschke products b_1 and b_2 on Δ such that $\|\theta \circ L_0 - b_1 \bar{b}_2\|_\infty < \frac{1}{2}\varepsilon$ (see [6, Chapter 5, Theorem 2.1]). This implies that

$$\left| \int_{\Gamma_0} (b_1 \circ L_0^{-1})(\bar{b}_2 \circ L_0^{-1}) d(\nu_1 - \nu_2) \right| \geq 2 - 2\varepsilon.$$

By Lemma 4.9, we see that

$$\left| \int_{\mathbf{M}} \tau_1^*(b_1 \circ L_0^{-1}) \tau_1^*(\bar{b}_2 \circ L_0^{-1}) d\mu_1 - \int_{\mathbf{M}} \tau_2^*(b_1 \circ L_0^{-1}) \tau_2^*(\bar{b}_2 \circ L_0^{-1}) d\mu_2 \right| \geq 2 - 2\varepsilon.$$

There are two sequences of Blaschke products $\{B_n^{(1)}\}$ and $\{B_n^{(2)}\}$ on \mathbf{R}_+^2 and two decreasing sequences $\{r_n^{(1)}\}$ and $\{r_n^{(2)}\}$ converging to 1 such that each $B_n^{(j)}$, $j = 1, 2$, extends to be continuous on \mathbf{X} and $r_n^{(j)} B_n^{(j)} \circ L_0 = b_j$ on $\mathfrak{M}(H^\infty(\Delta))$. Let m_T and m_i be the operators by (4.30) with respect to $L^1(\mu_i)$. Recall that, for ϕ in $C(\mathbf{M})$, $m_T(\phi)$ lies in $C(\mathbf{M})$ and $\|m_T(\phi) - m_i(\phi)\|_2$ converges to 0 as $T \rightarrow \infty$. On the other hand, from the paragraph preceding Lemma 4.9, we see that

$$\lim_{n \rightarrow \infty} \int_{\mathbf{M}} \left| m_i(r_n^{(j)} B_n^{(j)}) - \tau_i^*(b_j \circ L_0^{-1}) \right|^2 d\mu_i = 0$$

for $i, j = 1, 2$. Hence there exists a k such that

$$\left| \int_{\mathbf{M}} m_1(r_k^{(1)} B_k^{(1)}) m_1(r_k^{(2)} \overline{B_k^{(2)}}) d\mu_1 - \int_{\mathbf{M}} m_2(r_k^{(1)} B_k^{(1)}) m_2(r_k^{(2)} \overline{B_k^{(2)}}) d\mu_2 \right| \geq 2 - 3\varepsilon.$$

Therefore there is a $T > 0$ such that

$$\left| \int_{\mathbf{M}} m_T(r_k^{(1)} B_k^{(1)}) m_T(r_k^{(2)} \overline{B_k^{(2)}}) d(\mu_1 - \mu_2) \right| \geq 2 - 4\varepsilon > \frac{2}{3}.$$

Since $m_T(r_k^{(j)} B_k^{(j)})$ is in $C(\mathbf{M})$, this shows that $\mu_1 \neq \mu_2$. Since $\nu = \nu_1 - \nu_2$ is orthogonal to $A|_{\overline{P(\xi)}}$, we obtain a nonzero real measure $\mu = \mu_1 - \mu_2$ in $M(\mathbf{M})$ which is orthogonal to $A(\mathbf{M})$. This completes the proof. \square

5. REMARKS

Let $(\mathbf{M}, \{T_t\}_{t \in \mathbf{R}})$, $A(\mathbf{M})$, and $P(\xi)$ be as in §4, and let μ be the representing measure for ξ . Then μ is an invariant probability measure in $M(\mathbf{M})$ which is not ergodic. We denote by $H^\infty(\mu)$ the weak- $*$ closure of $A(\mathbf{M})$ in $L^\infty(\mu)$. Then $H^\infty(\mu)$ is a weak- $*$ Dirichlet algebra in $L^\infty(\mu)$, since $A(\mathbf{M})$ is a logmodular algebra.

(i) Recall that, since $(\mathbf{M}, \{T_t\}_{t \in \mathbf{R}})$ is minimal, $A(\mathbf{M})$ is a maximal subalgebra of $C(\mathbf{M})$. By contrast to this fact, it follows from [11, Corollary 3.1] that $H^\infty(\mu)$ is not a weak- $*$ maximal algebra in $L^\infty(\mu)$.

(ii) For each ϕ in $L^1(\mu)$, $m\phi$ denotes the invariant function by (4.30). We now set

$$\mathcal{H} = \{m\phi; \phi \in A(\mathbf{M})\}.$$

We then see that \mathcal{H} is isometrically isomorphic to $H^\infty(\Delta)$. Of course, $m\phi$ cannot be continuous on \mathbf{M} except for constants.

(iii) Let $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ be a minimal flow. Recall that if $(\Omega, \{U_t\}_{t \in \mathbf{R}})$ is strictly ergodic, then $A(\Omega)$ is a Dirichlet algebra on Ω . Then it is natural to ask whether strict ergodicity is necessary when $A(\Omega)$ is a Dirichlet algebra. We may, however, construct a minimal flow $(\mathbf{Y}, \{V_t\}_{t \in \mathbf{R}})$ not being strictly ergodic, on which $A(\mathbf{Y})$ is a Dirichlet algebra. Indeed, let $B_1(z)$ be the Blaschke product in (4.16) with $m = 1$. Since $B_1(z)$ extends to be continuous on \mathbf{X} , it determines a unimodular function $B_1(x)$ in $A(\mathbf{M})$. Let \mathfrak{B} be the closed subalgebra of $C(\mathbf{M})$ generated by $\{T_t B_1, \overline{T_s B_1}; t, s \in \mathbf{R}\}$. Taking a suitable factor of $(\mathbf{M}, \{T_t\}_{t \in \mathbf{R}})$, we have a minimal flow $(\mathbf{Y}, \{V_t\}_{t \in \mathbf{R}})$ such that \mathfrak{B} is isometrically isomorphic to $C(\mathbf{Y})$. We may consider that $A(\mathbf{Y})$ contains the uniform algebra generated by $\{T_t B_1; t \in \mathbf{R}\}$ and all the constant functions. Then it is not difficult to show that $(\mathbf{Y}, \{V_t\}_{t \in \mathbf{R}})$ is the desired one. We mention that since $C(\mathbf{Y})$ is separable, \mathbf{Y} is metrizable.

Incidentally, the referee has pointed out that there can be a minimal flow $(\Omega, \{U_t\}_{t \in \mathbb{R}})$ where Ω is metrizable, and yet $A(\Omega)$ is not a Dirichlet algebra. In this situation, Choquet's theorem works well. So we would expect some information about the invariant measures representing homomorphisms of $A(\Omega)$.

Note added in proof. One application of the flow $(X, \{S_t\}_{t \in \mathbb{R}})$ should be noted: Let μ be an invariant, ergodic, probability measure in $M(X)$, and let $H^2(\mu)$ and $H_0^2(\mu)$ be as in the paragraph preceding Proposition 3.2. Then the invariant subspace $H_0^2(\mu)$ is generated by one of its elements. This answers an old question in the setting of weak- $*$ Dirichlet algebras (see [7, Chapter 5, §4]). For the proof, see our subsequent note *Single generator problem*.

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