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ON REALIZATION OF BJÖRNER'S 'CONTINUOUS PARTITION LATTICE' BY MEASURABLE PARTITIONS

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ABSTRACT. Björner [1] showed how a construction by von Neumann of examples of *continuous geometries* can be adapted to construct a continuous analogue of finite partition lattices. Björner's construction realizes the continuous partition lattice abstractly, as a completion of a direct limit of finite lattices. Here we give an alternative construction realizing a continuous partition lattice concretely as a lattice of measurable partitions. This new lattice contains the Björner lattice and shares its key properties. Furthermore its automorphism group is the full automorphism group (mod 0) of the unit interval with Lebesgue measure, whereas, as we show, the Björner lattice possesses only a proper subgroup of these automorphisms.

1. INTRODUCTION

Consider the Boolean lattice of subsets of a finite set or the modular lattice of subspaces of a finite-dimensional vector space. The basic invariant attached to an element in one of these lattices is its rank, which defines the cardinality of a subset or the dimension of a subspace. Continuous analogues of subset and subspace lattices exist, the discrete-valued rank function being supplanted by a continuously increasing function taking arbitrary real values and representing the probability of an event, for instance, or a kind of continuous dimension.

It is natural to ask for a similar continuous analogue of the semimodular lattice of partitions of a finite set. In this paper we construct such a thing out of appropriately chosen measurable partitions of the unit interval. What results is a new continuous partition lattice which resembles, and in fact contains, one constructed in a purely abstract fashion by Björner [1].

The intention here is to fulfill a sort of historical mandate whereby each abstract continuous analogue is found to have a concrete counterpart once the finite set underlying the discrete example of rank n is replaced by the real unit interval X. There is a theme in the history: a naive definition suggests itself but fails, then is saved by judicious sharpening. The prototypical situation is the impossibility of defining Lebesgue measure for all subsets of X. Measurability is our salvation; the continuous Boolean algebra is the lattice of measurable subsets of X (mod 0). For vector spaces, we turn to the closed subspaces in the Hilbert space $L^2(X)$, but alas their lattice is not even modular. We are saved

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this time by von Neumann; the projections in a "factor of type II_1" define a modular sublattice and it is a continuous geometry [4, 6].

As for partitions, there is a fully developed theory [3] of measurable partitions of X, but as we ought to expect by now, the lattice of these is far too general to carry a rank function with the properties we desire in a continuous partition lattice. Of course there is once again a saving grace, namely to consider only what we shall presently define to be *continuous partitions*.

Let us now indicate in some detail the general plan of the paper.

After measure-theoretic preliminaries we define continuous partitions in §2. We order them by refinement and define the rank function r, then establish that we have constructed a complete lattice Π and that r is strictly increasing. Next we bring out the structural characteristics of upper and lower intervals in Π, which are analogous to those of finite partition lattices. Aided by this information we prove that the rank function is semimodular. We also single out the *modular elements* and discuss their connection with the theorem on semimodularity.

In §3 we consider the metric induced by the rank function and show that Π is a complete metric space. From this we deduce that the rank function is an isometry from any maximal chain in Π to the interval $[0, 1]$.

In §4 we observe that the preceding results allow us to embed the Björner lattice Π_∞ in our lattice Π. To get more information we characterize explicitly the image of Π_∞, which is a complete sublattice of Π. Using the modular elements we show that the automorphisms of Π are exactly those induced by automorphisms (mod 0) of the underlying Lebesgue space, and that only a proper subgroup of these restrict to automorphisms of Π_∞. This is a negative answer to a conjecture made by Björner, and it shows that at least in this respect the new lattice Π has somewhat nicer properties than Björner's Π_∞.

2. The lattice Π and its rank function

For measure-theoretic foundations we follow Rohlin [3]. All measure spaces will have finite total measure, equal to 1 unless otherwise specified.

Let $(B; 0, 1, \cap, \cup, ^c)$ be a complete Boolean algebra with a countably additive strictly increasing measure μ. Equivalently, B is the σ-algebra of measurable sets (mod 0) in some measure space. If $x\Delta y = (x \cap y^c) \cup (y \cap x^c)$ denotes the symmetric difference, then B is a complete metric space in the metric $d(x, y) = \mu(x\Delta y) = \mu(x \cup y) - \mu(x \cap y)$. B is separable as a metric space iff it is countably generated as a complete lattice, or equivalently as a σ-algebra. Such a B is called a *Lebesgue lattice*. Up to measure-preserving isomorphism, there is a unique Lebesgue lattice without atoms; the general Lebesgue lattice is then a direct sum of this with countably many atoms.

In principle, everything in this paper can be constructed beginning with an abstract nonatomic Lebesgue lattice B. However, for the sake of both convenience and intuition it will be useful to regard B as the lattice of measurable sets (mod 0) in a measure space X. In this context, the appropriate requirement on X is that it be a *Lebesgue space*, which for our purposes can be defined as a space isomorphic to a real interval with Lebesgue measure, plus possible atoms.

A partition π of a Lebesgue space X into measurable blocks is called a *measurable partition* if there exist countably many measurable sets M_i which are unions of blocks, such that every block of π is an intersection of some of these and their complements. Then the factor space X/π with the measure induced from X becomes again a Lebesgue space. A measurable set which is equal (mod 0) to a union of blocks of π we will call a *π-set*. The π-sets form a complete sub-Boolean algebra of B, which is naturally identified with the lattice of measurable sets (mod 0) of X/π. Conversely, every complete sub-Boolean algebra of B (called a *factor*) arises from a measurable partition.

By definition, measurable partitions π and π' are equal (mod 0) if the corresponding factors are the same, i.e., the π-sets are the same as the π'-sets. We will work only with measurable partitions (mod 0). Strictly speaking, therefore, we should phrase everything in terms of factors, but for the most part we will use the more intuitive language of partitions.

The measurable partitions are ordered by refinement: $\pi \leq \pi'$ if every π'-set is a π-set. Under this ordering they form a complete lattice, with $\bigvee \pi_\alpha$ defined by intersecting the corresponding factors F_α and $\bigwedge \pi_\alpha$ by taking the factor generated by the F_α.

Given a measurable partition π and $b \in B$, let b^π denote the smallest π-set containing b. Roughly, this is the image of b under $X \to X/\pi$. More precisely, since a set of measure 0 can have an image of positive measure, b^π is the smallest image of a representative of b (mod 0), which is well-defined (mod 0). A measurable $b \subseteq X$ is *one-sheeted* if it contains at most one element from each block of π. The one-sheeted sets (mod 0) are those $b \in B$ with the property $a^\pi < b^\pi$ for all $a < b$. A measurable $b \subseteq X$ is a *system of representatives* if it contains exactly one element from each block of π. These (mod 0) are just the maximal one-sheeted sets; in particular they always exist.

Finally, let us call a measurable partition π of a subset $b \in B$ a *subpartition* and b its *support*. If b is the disjoint union of π-sets $b_1, b_2, \ldots$ we shall say that $\pi = \pi|_{b_1} + \pi|_{b_2} + \cdots$ is the *sum* of the subpartitions $\pi|_{b_i}$, where $\pi|_{b_i}$ denotes the restriction of π to b_i.

Definition. A subpartition π is *of class* k if the projection mapping its support b onto the factor space b/π is isomorphic (mod 0) to the projection of a product space $K \times Y$ on the factor Y, where Y is a space of total measure $\mu(b)$ and K consists of k atoms of measure $1/k$ each.

Subpartitions of class k are easily constructed: let $b = a_1 \cup \cdots \cup a_k$ where the a_i are disjoint and of equal measure and choose arbitrary measure-preserving isomorphisms $\phi_i \colon a_1 \to a_i$ for $2 \leq i \leq k$. Then $\pi = \{\{x, \phi_2(x), \ldots, \phi_k(x)\} \mid x \in a\}$ is a measurable partition of b of class k. Indeed, every subpartition of class k has this form.

Note that a subpartition of class k is not merely a subpartition in which each block has k elements (mod 0). In the definition of class k we are requiring that measurable transformations permuting points within blocks are actually *measure-preserving*. In particular, it is clear that:

Lemma 2.1. *Every measurable system of representatives for a subpartition π of class k has measure $\mu(b)/k$.*

Definition. The subpartition π with support b whose only π-sets are 0 and b is *of class* ∞.

In the class ∞ situation, the partition π has just the one block b (mod 0) and any system of representatives has measure 0, so Lemma 2.1 in effect extends to this case as well.

Definition. A *continuous partition* is a measurable partition $\pi = \pi_\infty + \pi_1 + \pi_2 + \cdots$ where each π_k is of class k, or is zero. The set of continuous partitions of a Lebesgue space X, partially ordered by refinement, will be denoted $\Pi(X)$, or just Π when X is understood.

Lemma 2.2. *The decomposition* $\pi = \pi_\infty + \pi_1 + \pi_2 + \cdots$ *of a continuous partition is unique.*

Proof. Writing b_k for the support of π_k, we see that $\pi|_a$ is of class k for every π-set a contained in b_k. Since a nonzero subpartition of class k cannot also be of class $l \neq k$ (e.g., by Lemma 2.1), in any other decomposition $\pi = \pi'_\infty + \pi'_1 + \cdots$ we must have $b_k \cap b'_l = 0$ for $k \neq l$. This shows the b'_k form the same partition of $1 = X$ as the b_k, proving the lemma. □

Corollary 2.3. *Every measurable system of representatives* s *of a continuous partition* π *has the same measure* $\mu(s) = \sum \mu(b_k)/k$, *where* b_k *is the support of* π_k.

Definition. The *rank* of a continuous partition $\pi \in \Pi$ is $r(\pi) = 1 - \mu(s)$ where s is a measurable system of representatives. More generally, for $\pi \in \Pi(X)$ where X has total measure different from 1, we define $r(\pi) = \mu(X) - \mu(s)$, so that in every case $r(0) = 0$ and $r(1) = \mu(X)$.

This definition is analogous to the rank function in a finite partition lattice, given by (number of elements) − (number of blocks). Here the measure of a system of representatives plays the role of the number of blocks. The definition of continuous partition is essentially forced by this analogy, as we now pause to show.

Proposition 2.4. *Let* π *be a measurable partition such that* (i) *every system of representatives for* π *has the same measure, and* (ii) *there is at most one nonzero* π*-set having a system of representatives of measure* 0. *Then* π *is a continuous partition.*

Proof. There is clearly a largest π-set b having a system of representatives of measure 0, and condition (ii) makes it the support of a subpartition of class ∞. Thus we are concerned only with $\pi|_{b^c}$, in which no set of positive measure has a measure zero system of representatives.

By Rohlin's theorem on decomposition of measure [3], each block c of π can be made into a Lebesgue space with a probability measure μ_c, in such a way that for all measurable $a \subseteq X$, $a \cap c$ is measurable in c for almost all c, and

$$\mu(a) = \int_{X/\pi} \mu_c(a \cap c)\, d\mu_{X/\pi}.$$

In this context, the condition that $\pi|_{b^c}$ is a sum of finite class subpartitions is precisely that for almost all c, μ_c is the equal probability measure concentrated on a finite set (of k points for c a block of π_k). If the set C of all c for which μ_c is not purely atomic has $\mu(C) > 0$, then taking the union over all c of the support of the continuous part of μ_c gives a set of positive measure having a measure zero system of representatives, contradicting (ii). Thus almost all μ_c are atomic, and the projection $b^c \to b^c/\pi$ is isomorphic (mod 0) to a projection $\tau: Z \to Y$, where Y is a Lebesgue interval, Z is a subset of a countable union of copies of Y, and the measure on Z is defined by integrating (with respect to the measure lifted from Y) a nonzero measurable function M whose sum is 1 on each fiber $\tau^{-1}(y)$. From this model it is clear that unless M defines an equal probability measure on almost all fibers, there are systems of representatives with unequal measures. □

We now have the essential definitions. In the rest of this section we show that $(\Pi, \leq)$ is a complete lattice and that the rank function r is strictly increasing and semimodular, and we obtain partition-lattice-like structural properties of Π, including identification of its modular elements.

Lemma 2.5. *Let π be a subpartition of class $k < \infty$ with support b. Then any finer partition $\nu \leq \pi$ of b is a sum of subpartitions of class less than or equal to k.*

Proof. Identify $b \to b/\pi$ with $K \times Y \to Y$ in accordance with the definition of class k. Now ν is a partition of $K \times Y$ finer than the partition into fibers over Y. Given nonempty $J \subseteq K$, define v_J to be the union of all blocks of ν of the form $J \times y$. This v_J is measurable: for instance define $f \in L^2(K \times Y)$ to be α_j on $j \times Y$, where the α_j are real numbers linearly independent over $\mathbb{Q}$. Then the projection of f on $L^2((K \times Y)/\nu)$ takes as its value on each block c of ν the average of f over the points of c and this average is equal to $(1/|J|)\sum_{j \in J} \alpha_j$ exactly on v_J. Evidently, the v_J partition b and are ν-sets, so ν is the sum of the subpartitions $\nu|_{v_J}$. Such a subpartition is clearly of class $|J| \leq k$. □

We shall use later not only Lemma 2.5 itself but the subpartitions v_J constructed in the proof. In particular, we observe:

Corollary 2.6. *In the situation of Lemma 2.5, if ν is of class k then $\nu = \pi$.*

Proof. Only v_K is nonzero since all other v_J are of class less than k. This means $\nu = \pi$. □

Lemma 2.7. *If π is any measurable partition, then there is a unique least (= finest) continuous partition $\nu \geq \pi$.*

Proof. For each finite k, the sum of class k subpartitions is again class k, so there is a greatest π-set b_k for which $\pi_k = \pi|_{b_k}$ is of class k. Let ν be the sum of these π_k together with ν_∞ defined by $b_\infty = (\bigcup b_k)^c$. Clearly ν is a continuous partition and $\nu \geq \pi$. Suppose ν' is another, with subpartitions ν'_k supported on sets b'_k. By Lemma 2.5, $\pi|_{b'_k}$ is a sum of subpartitions of class k or less, so that $b'_k \subseteq b_1 \cup \cdots \cup b_k$ for finite k and, consequently, $b'_\infty \supseteq b_\infty$. Now $\nu = \pi|_{(b_\infty)^c} + \nu_\infty$ by definition, while $\nu' \geq \pi|_{(b'_\infty)^c} + \nu'_\infty$, showing that $\nu' \geq \nu$. □

Note that this proof not only ensures the existence of ν but provides an exact description of it.

Corollary 2.8. $(\Pi, \leq)$ *is a complete lattice in which* $\bigvee \pi_\alpha$ *is the least continuous partition greater than or equal to the join of the* π_α *in the lattice of measurable partitions.*

The meet as well as the join in Π has an explicit description.

Lemma 2.9. *For* $\pi_\alpha \in \Pi$, $\bigwedge \pi_\alpha$ *is the meet of the* π_α *as measurable partitions.*

Proof. The problem is to show that π, the meet of the π_α as measurable partitions, is a continuous partition. Let w_α be the support of $(\pi_\alpha)_\infty$. Then $\pi_\alpha|_{w_\alpha^c}$ is a sum of finite class subpartitions, and by Lemma 2.5 the same is true of $\pi|_{w_\alpha^c}$, and hence also $\pi|_{\bigcup w_\alpha^c}$. So we need to show that $\pi|_{(\bigcup w_\alpha^c)^c}$ is of class ∞ or is zero. But $(\bigcup w_\alpha^c)^c = \bigcap w_\alpha$. Since the restriction of π_α to w_α is the total partition, the same is true for each $\pi_\alpha|_{\bigcap w_\alpha}$, and also for $\pi|_{\bigcap w_\alpha}$, since a restriction of a meet of measurable partitions is the meet of their restrictions. Incidentally, the last assertion would be obvious for the set-theoretic meet of partitions, but is justified for measurable partitions only by appeal to the definition in terms of factors. □

Lemma 2.10. *The rank function* r *on* Π *is strictly increasing.*

Proof. For any continuous partition π define ρ_π to be the L^1 function taking the value $1 - 1/k$ on the support b_k of π_k. Then obviously $r(\pi) = \int \rho_\pi \, d\mu$. Now if $\nu \leq \pi$, then Lemma 2.5 says that $\rho_\nu \leq \rho_\pi$ and hence $r(\nu) \leq r(\pi)$. By Corollary 2.6, equality occurs only if $\pi_k' = \pi_k$ for all finite k, hence only if $\pi' = \pi$. □

Lemma 2.10 has a useful consequence we shall need in §4. In a complete lattice with a strictly increasing real rank function, the join or meet of any set of elements is the join or meet of countably many of them. This follows from the theorem that a subset of the reals well-ordered by $<$ is finite or countable.

In a finite partition lattice, an upper interval $[\pi, 1]$ is isomorphic to the lattice of partitions of the block set of π. The lower interval $[0, \pi]$ is the product of partition lattices on the individual blocks. With the next two lemmas, we establish analogous structural properties for the continuous partition lattice Π.

In order to describe the structure of a lower interval we will need a notion of "continuous product" of finite lattices, as $\pi \in \Pi$ may have continuously many finite blocks. Let (X, μ) be a Lebesgue space. Let L be a finite lattice. We wish to define the X-fold power L^X of L. The elements of L^X will be the measurable functions $f\colon X \to L$ (mod 0), where every subset of the finite set L is understood to be measurable. L^X is a lattice with operations defined pointwise; these are well-defined (mod 0). If L has a rank function r_L then we define a rank function r on L^X by $r(f) = \int r_L \circ f \, d\mu$. Evidently all linear inequalities valid for r_L hold for r too. In particular, r is modular or semimodular if r_L is. Note that if (X, μ) is a space consisting of n atoms of measure 1 each, then L^X is identical to L^n in the ordinary sense, and r is the correct rank function. The continuous analogue occurs when (X, μ) is an interval.

Definition. The *continuous power* L^t of a finite lattice L is the lattice L^X constructed as above when X is a nonatomic Lebesgue space of total measure t.

Lemma 2.11. *Let* $\pi \in \Pi$, $\pi = \pi_\infty + \pi_1 + \pi_2 + \cdots$. *The interval* $[0, \pi] \subseteq \Pi$ *is isomorphic as a lattice with rank function to the product* $\Pi(b_\infty) \times (\Pi_1)^{t_1} \times (\Pi_2)^{t_2} \cdots$, *where* b_k *is the support of* π_k, $\Pi(b_\infty)$ *is the lattice of continuous partitions of the subspace* b_∞, Π_k *is the lattice of partitions of a set of* k *elements with its usual rank function, and* $t_k = \mu(b_k)/k$.

Proof. If $\nu \leq \pi$ then each b_k is a ν-set and ν is the sum of the subpartitions $\nu|_{b_k} \leq \pi_k$. Also, $\nu|_{b_\infty}$ is a continuous partition of b_∞.

Conversely, suppose given for each finite k a subpartition with support b_k finer than π_k, and a continuous partition of b_∞. By Lemma 2.5 the given subpartition for each finite k is a continuous partition of b_k containing only finite class summands, so the sum of all these subpartitions is a continuous partition $\nu \leq \pi$.

The preceding considerations give us a bijective correspondence between the interval $[0, \pi] \subseteq \Pi$ and the product $\Pi(b_\infty) \times P_1 \times P_2 \times \cdots$, where P_i is the set of continuous partitions of b_k finer than π_k, i.e., the interval $[0, \pi_k] \subseteq \Pi(b_k)$. The bijection is clearly order-preserving, and $r(\nu)$ is the sum of the ranks of the summands of ν in $\Pi(b_\infty)$ and the P_k. Hence it remains only to show that for finite k, P_k is isomorphic as a lattice with rank function to $(\Pi_k)^{\mu(b_k)/k}$, which is the special case of the lemma when π is a factor of class k.

Assume π is of class k, so that without loss of generality $X = K \times Y$ where K has k atoms of equal measure $1/k$ and π is the partition into fibers over Y. Recall from the proof of Lemma 2.5 the decomposition of $\nu \leq \pi$ as a sum of subpartitions with support v_J and of class $|J|$, where $J \subseteq K$ and v_J is the union of blocks of ν of the form $J \times y$ for some $y \in Y$. Let $\lambda = \{J_1, \ldots, J_n\}$ be a partition of K. Then $v_\lambda = \bigcap (v_{J_i})^\pi$ is the union of those fibers $K \times y$ which are partitioned by ν in accordance with the partition λ of K. The v_λ are measurable unions of fibers which partition X as λ ranges over $\Pi(K) = \Pi_k$. Associated with ν we thus have a measurable function $f\colon Y \to \Pi_k$ taking the value λ on fibers in v_λ. In other words, we have an element $f \in \Pi_k^Y$, and it is clear that the correspondence $P_k \to \Pi_k^Y$ given by $\nu \mapsto f$ is an isomorphism of partially ordered sets.

Now $(\Pi_k)^{1/k}$ is isomorphic to Π_k^Y with the rank function scaled by the factor $1/k$. Therefore it remains to prove that $r(\nu) = r(f)/k$. If in v_λ we choose a system of representatives S_λ for λ then the union s of the sets $v_\lambda \cap (S_\lambda \times Y)$ is clearly a system of representatives for ν and $r(\nu) = 1 - \mu(s)$. But $\mu(s) = \sum (|S_\lambda|/k)\mu(v_\lambda) = (1/k)\int (k - r_{\Pi_k} \circ f)\, d\mu = 1 - r(f)/k$, as required. □

Lemma 2.12. *Let* $\pi \in \Pi$. *The interval* $[\pi, 1]$ *is isomorphic to the lattice* $\Pi(Y)$ *of continuous partitions of a nonatomic Lebesgue space* Y *of total measure* $1 - r(\pi)$ *via an isomorphism* $\iota\colon [\pi, 1] \to \Pi(Y)$ *that is rank-preserving except for the additive constant* $r(\pi)$, *i.e.,* $r(\iota(\sigma)) = r(\sigma) - r(\pi)$.

Proof. In the first place, we can assume that $\pi_\infty = 0$. For in general, if b_∞ is the support of π_∞ then any $\sigma \geq \pi$ has σ_∞ supported on a set containing b_∞. Hence restriction to $(b_\infty)^c$ defines an order-preserving bijection carrying each continuous partition $\sigma \geq \pi$ of X to a continuous partition $\sigma' \geq \pi'$ of $(b_\infty)^c$. From the definition of rank in Π it is apparent that this bijection preserves rank up to an additive constant: $r(\sigma') = r(\sigma) - \mu(b_\infty)$. Hence the result for π follows from the result for π', whose class ∞ component is zero.

Now $\pi_\infty = 0$ means that X/π is a nonatomic Lebesgue space. We define a new measure μ^* on X/π by multiplying μ by $1/k$ on the support of π_k, so that $\mu^*(b) = \mu(b)/k$ for $b \subseteq b_k/\pi$.

Let Y stand for X/π with the measure μ^*. If $Y' \subseteq X$ is a measurable system of representatives for π, then, identifying π-sets b with subsets of X/π, we have $\mu^*(b) = \mu(b \cap Y')$. Thus associating each $y' \in Y'$ with its block in π gives an isomorphism $Y \cong Y'$. In particular, Y is a nonatomic Lebesgue space of total measure $1 - r(\pi)$. To construct the required isomorphism $\iota\colon [\pi, 1] \to \Pi(Y)$, we now show that a measurable partition $\sigma \geq \pi$ is a continuous partition of X iff it is a continuous partition of Y, when regarded as a measurable partition of $Y = (X/\pi, \mu^*)$.

Note that to make precise sense of this, we must observe that the Lebesgue lattice B_Y of Y is nothing but the factor corresponding to π, though with a different measure. Measurable partitions $\sigma \geq \pi$ of X correspond, by definition, to factors contained in $B(Y)$, as do measurable partitions of Y.

A decomposition of $\sigma \geq \pi$ as a sum of subpartitions in X is the same thing as a decomposition in Y, and a subpartition is of class ∞ in X iff it is of class ∞ in Y, since these concepts do not depend on the measure. Therefore we need only show that every subpartition of $\sigma \geq \pi$ which is of finite class in X is a sum of subpartitions of finite class in Y, and vice versa.

First let σ be a subpartition of class k in X, supported on the π-set b, and coarser than $\pi|_b$. Since $\pi|_b \leq \sigma$, as in the proof of Lemma 2.5 we can express b as $K \times Z$, with σ the partition into fibers over Z and $\pi|_b$ determined by the decomposition $b = \bigcup v_J$, $J \subseteq K$, where v_J is the union of all blocks of $\pi|_b$ of the form $J \times z$. As in the proof of Lemma 2.11, we then have another decomposition $b = \bigcup v_\lambda$, $\lambda \in \Pi(K)$, where now v_λ is a σ-set: the union of all fibers $K \times z$ partitioned by π in accordance with λ.

We claim that $\sigma|_{v_\lambda}$ is a subpartition of class $|\lambda|$ in Y, which is to say, in $(X/\pi, \mu^*)$. Indeed, let $z_\lambda = \{z | K \times z \subseteq v_\lambda\} \subseteq Z$, and let $y_\lambda \subseteq Y$ be the set of blocks of π contained in v_λ. Then in an obvious way $y_\lambda \cong \lambda \times z_\lambda$ as sets. Let $J \in \lambda$ be a block and let $u \subseteq z_\lambda$ be a measurable subset. Since $J \times u$ is contained in $v_J \subseteq b_{|J|}$, we have $\mu^*(\{J\} \times u) = \mu(J \times u)/|J| = \mu(u)/k$. Therefore $(y_\lambda, \mu^*) \cong \lambda \times z_\lambda$ as a measure space, where λ is given the uniform measure $1/|\lambda|$ on each block and z_λ is given the measure $(|\lambda|/k) \cdot \mu$. In other words, y_λ supports a subpartition of class $|\lambda|$ in Y.

Next let σ be a subpartition of of class k in Y. Again let b be its support (a subset of Y, now) and write $b \cong K \times Z$ with σ the partition into fibers. Here each $(i, z) \in K \times Z$ corresponds to an element of Y and as such to a block of π. Define $g(i, z)$ to be the class j of the subpartition π_j to which this block belongs. For each z, then, $g(-, z)$ is a function from K to $\mathbb{N}$ and as z varies these define a measurable function $g\colon Z \to \mathbb{N}^K$. Given $f\colon K \to \mathbb{N}$, let $z_f = \{z \in Z | g(-, z) = f\}$ and let v_f be the union of the blocks of π corresponding to (i, z) with $z \in z_f$.

We claim that v_f supports a subpartition of class $n = \sum_i f(i)$ in X. Indeed, let each subpartition π_j be given by the partition into fibers of $b_j \cong \langle j \rangle \times W_j$, where $\langle j \rangle$ is a space of j atoms carrying equal measure $1/j$. Then the elements of each block of π_j correspond to the elements of $\langle j \rangle$ for the relevant value of j, and so the elements of the blocks in the fiber over $z \in z_f$ correspond to pairs (i, j) where $i \in K$, $j \in \langle f(i) \rangle$. Let N be the set of such pairs; we have $v_f \cong N \times z_f$ as sets. Let $(i, j) \in N$ and let $u \subseteq z_f$ be measurable. We have $\mu(\{(i, j)\} \times u) = \mu_{X/\pi}(\{i\} \times u)/f(i) = \mu^*(\{i\} \times u) = \mu^*(K \times u)/k$. Therefore assigning measure $1/n = 1/|N|$ to each element of N and giving z_f the measure $(n/k) \cdot \mu^*$ yields $v_f \cong N \times z_f$ as measure spaces, so that v_f supports a subpartition of class n.

To complete the proof, we turn to the rank function. We are to show that as a continous partition of X, $\sigma \geq \pi$ has the same corank as it has as a continuous partition of Y. Consider again the isomorphism $Y \cong Y'$, where Y' is a system of representatives for π. Evidently, $s \subseteq Y$ is a system of representatives for σ as a partition of Y iff the corresponding $s' \subseteq Y'$ is a system of representatives for σ as a partition of X. But as we already noted, $\mu^*(s) = \mu(s')$, and by definition these are the coranks in question. □

By examining the preceding proof we can be a bit more precise about the isomorphism $[\pi, 1] \cong \Pi(Y)$ provided by Lemma 2.12.

Corollary 2.13. *Let Y be a measurable system of representatives for $\pi \in \Pi$. Then the isomorphism $[\pi, 1] \cong \Pi(Y)$ is realized by the map sending $\sigma \in [\pi, 1]$ to its restriction to the subspace $Y \subseteq X$.*

With the aid of the above lemmas on the structure of intervals, we proceed to establish our first major theorem about Π. First we need one additional technical lemma which can be deduced with ease from Corollary 2.13.

Lemma 2.14. *Let $a \in B$ and let $\pi, \sigma \in \Pi$. Assume that a^c is contained in the support of π_1. If $\pi \vee \sigma = 1$ in Π, then $\pi|_a \vee \sigma|_a = 1$ in $\Pi(a)$.*

Proof. We are to show that the join of $\pi|_a$ and $\sigma|_a$ as measurable partitions has no summand of class $k < \infty$. Suppose for contradiction that b is the nonzero support of such a summand ν'.

Since $\pi \vee \sigma = 1$, the class 1 component of π cannot contain (mod 0) any blocks of σ belonging to finite-class subpartitions. Indeed, if $\sigma|_g$ is of class $k < \infty$ for a nonzero σ-set g contained in the support of π_1, then every σ-subset of g is also a π-subset (since a class 1 subpartition is the partition into singletons), hence $\sigma|_g$ is a class k summand of the measurable partition join of π and σ, contradicting the description of $\pi \vee \sigma$ given in the proof of Lemma 2.7.

It follows immediately that σ has a system of representatives Y disjoint from the support of π_1, so that $Y \subseteq a$. This same Y is then also a system of representatives for $\sigma|_a$. Hence we obtain from Corollary 2.13 an isomorphism between the intervals $[\sigma, 1] \subseteq \Pi$ and $[\sigma|_a, 1] \subseteq \Pi(a)$, where continuous partitions correspond if they restrict to the same continuous partition of Y.

Consider the continuous partition $\nu = \nu' + \nu'' \in \Pi(a)$ where ν'' is the class ∞ subpartition with support $a \setminus b$. We have $\sigma|_a \leq \nu < 1$, so $\nu = \tau|_a$ for a unique continuous partition $\tau \in \Pi$ with $\sigma \leq \tau < 1$. Every subset of the support of π_1 is a π-set, so in particular a^c and a are π-sets. It follows that all $\pi|_a$-sets are π-sets, so if c is a τ-set, then $c \cap a$ is a ν-set, hence a $\pi|_a$-set, hence a π-set. Since every subset of a^c is also a π-set, c is a π-set, and we have shown $\pi \leq \tau$ and hence $\pi \vee \sigma \leq \tau < 1$, contrary to assumption. □

Theorem 1. *The rank function* r *on* Π *is semimodular:* $r(\pi) + r(\sigma) \geq r(\pi \wedge \sigma) + r(\pi \vee \sigma)$.

Proof. By Lemmas 2.11 and 2.12, the interval $[\pi \wedge \sigma, \pi \vee \sigma]$ is isomorphic as a lattice with rank function to a product of continuous powers of finite partition lattices and a continuous partition lattice. Since finite partition lattices are semimodular, we need only examine the continuous partition lattice factor, where we only need the special case of the theorem in which $\pi \wedge \sigma = 0$ and $\pi \vee \sigma = 1$. Then we are to show $r(\pi) + r(\sigma) \geq 1$.

Assume $\pi \vee \sigma = 1$. We define a decreasing sequence $a_0 \supseteq a_1 \supseteq \cdots$ in B as follows. Take $a_0 = 1 = X$. Given a_i, let $\pi^i, \sigma^i \in \Pi(a_i)$ be the restrictions $\pi|_{a_i}$ and $\sigma|_{a_i}$. Form a_{i+1} by removing from a_i the supports of the class 1 subpartitions of π^i and σ^i. Using Lemma 2.14 we see by induction that $\pi^i \vee \sigma^i = 1$ in $\Pi(a_i)$ for all i.

As in the proof of Lemma 2.14, it follows that σ^i possesses a system of representatives disjoint from the support of $(\pi^i)_1$; likewise π^i has a system of representatives disjoint from $(\sigma^i)_1$. On the other hand every system of representatives for π^i contains the support of $(\pi^i)_1$ and likewise for σ^i. Thus deleting these supports from a_i leaves $r(\pi^{i+1}) + r(\sigma^{i+1}) = r(\pi^i) + r(\sigma^i) - \mu(a_i \setminus a_{i+1})$. By induction it follows that $r(\pi^i) + r(\sigma^i) = r(\pi) + r(\sigma) - \mu(a_i^c)$ for all i.

From Corollary 2.3 it is clear that the rank of any continuous partition is at least $1/2$ the total measure of the supports of its subpartitions of class greater than 1. In particular, $r(\pi^i) + r(\sigma^i) \geq \mu(a_{i+1})$, whence $r(\pi) + r(\sigma) \geq 1 - \mu(a_i \setminus a_{i+1})$ for all i. Since the a_i descend, we have $r(\pi) + r(\sigma) \geq 1$ in the limit. □

We conclude this section with a discussion of the modular elements of Π. An element ε of a finite semimodular lattice is said to be *modular* [2] if it obeys the three equivalent conditions:

(M1) $r(\varepsilon) + r(\pi) = r(\varepsilon \wedge \pi) + r(\varepsilon \vee \pi)$ for all π.

(M2) $(\alpha \vee \varepsilon) \wedge \beta = \alpha \vee (\varepsilon \wedge \beta)$ for all $\alpha \leq \beta$.

(M3) $(\gamma \vee \alpha) \wedge \varepsilon = \gamma \vee (\alpha \wedge \varepsilon)$ for all $\gamma \leq \varepsilon$ and all α.

Now (M1) ⇒(M2) depends only on r being strictly increasing, and (M1) ⇒ (M3) follows from the semimodular inequality. The reverse implications require that r is the actual rank function of the finite lattice and have no continuous analogues in general. For Π, however, we shall be able to establish that certain elements are modular in sense (M1), and hence in all three senses, while no other elements are modular in any of the three senses. Thus (M1), (M2), and (M3) are in fact equivalent for elements of Π and we will characterize the elements for which they hold.

Definition. For each $a \in B$ let ε_a denote the continuous partition which is class ∞ on a and class 1 on a^c. The following are obvious: a^c is the unique system of representatives (mod 0) for ε_a, $r(\varepsilon_a) = \mu(a)$, the ε_a form a sublattice of Π, and $a \mapsto \varepsilon_a$ defines an isomorphism of ranked lattices from B onto this sublattice.

Lemma 2.15. *The elements* $\varepsilon_a \in \Pi$ *are modular in the three senses* (M1), (M2), *and* (M3).

Proof. As noted, (M1) suffices. Let $\pi \in \Pi$. We describe $\varepsilon_a \wedge \pi$ and $\varepsilon_a \vee \pi$.

First, $\varepsilon_a \wedge \pi$ is the sum of the class 1 partition (i.e., the 0, or trivial partition) on a^c and the restriction $\pi|_a$ on a. This is clear from Lemma 2.9. Let r be a system of representatives for $\pi|_a$. Then $r \cup a^c$ is a system of representatives for $\varepsilon_a \wedge \pi$.

Second, $\varepsilon_a \vee \pi$ is the sum of the class ∞ partition on $b = a^\pi \cup b_\infty$ and the restriction $\pi|_{(b^c)}$ on b^c, where b_∞ is the support of π_∞. This is not hard to see from the description of the join in the proof of Lemma 2.7. Let s be a system of representatives for $\pi|_{(b^c)}$. Then s is also a system of representatives for $\varepsilon_a \vee \pi$.

Since r is a system of representatives for $\pi|_a$, it is also a system of representatives for $\pi|_{(a^\pi)}$ and thus for $\pi|_b$. Since b is a π-set, $r \cup s$ is a system of representatives for π. The intersections $r \cap s$ and $r \cap a^c$ are empty, so we have $r(\varepsilon_a) + r(\pi) = \mu(a) + 1 - (\mu(r) + \mu(s)) = 1 - (\mu(a^c) + \mu(r)) + 1 - \mu(s)$ $= r(\varepsilon_a \wedge \pi) + r(\varepsilon_a \vee \pi)$. □

Restricting a partition π to a subspace a is in effect the same thing as computing the meet $\varepsilon_a \wedge \pi$. From this viewpoint, the technical Lemma 2.14 is not as mysterious as it might otherwise appear. Translated, it says if $\pi \leq \varepsilon_a$ and $\pi \vee \sigma = 1$ then $\pi \vee (\sigma \wedge \varepsilon_a) = \varepsilon_a$, which is a special case of modularity in sense (M3). Since (M1)$\Rightarrow$(M3) is a consequence of semimodularity, our technical device used in the proof of Theorem 1 is really a very special case of the theorem itself.

We defer the proof that the ε_a are all the modular elements of Π until later (Lemma 4.4) because it is most easily understood using the embeddings $\Pi_n \hookrightarrow \Pi$ of finite partition lattices which will be discussed in §4.

3. Properties of Π as a metric space

Following Björner, we now define a metric on the semimodular lattice Π.

Definition. For $\pi, \sigma \in \Pi$, $d(\pi, \sigma) = 2r(\pi \vee \sigma) - r(\pi) - r(\sigma)$.

Lemma 3.1 (Björner [1]). *The function d is a metric on Π and the join operation is uniformly continuous in this metric. These facts depend only upon the rank function r being strictly increasing and semimodular.*

Our goal in this section is to show that Π is a complete metric space. The idea is this: given a Cauchy sequence of continuous partitions, the sequence of their class k subpartitions, for k fixed and finite, will be Cauchy in a nice metric on subpartitions, leading to a class k subpartition for the limit. Anything not so accounted for we throw into the class ∞ subpartition to construct the limit of the original sequence. Let us begin by defining the metric we need on subpartitions.

Definition. Let α, β be subpartitions of X. A set $x \in B$ is a *set of agreement* between α and β if x is both an α-set and a β-set, and $\alpha|_x = \beta|_x$. Since the union of sets of agreement is again a set of agreement, there is a largest such set a, the *domain of agreement.*

Lemma 3.2. *Define a function $\delta(\alpha, \beta)$ for subpartitions α, β as follows. Let b_α, b_β be the supports of the respective subpartitions and $a_{\alpha\beta}$ their domain of agreement. Set $\delta(\alpha, \beta) = \mu((b_\alpha \cup b_\beta) \setminus a_{\alpha\beta})$. Then δ is a metric.*

Proof. δ is symmetric by definition, and $\delta(\alpha, \beta) = 0$ means both supports are equal to the domain of agreement, so $\alpha = \beta$.

Let subpartitions α, β, γ be given and let $a_{\alpha\beta}$, $a_{\beta\gamma}$, $a_{\alpha\gamma}$ be the respective domains of agreement. We claim $a_{\alpha\gamma} \supseteq a_{\alpha\beta} \cap a_{\beta\gamma}$, from which the triangle inequality for δ follows easily.

For the claim, note that $a_{\alpha\beta}$ and $a_{\beta\gamma}$ are both β-sets, so $a = a_{\alpha\beta} \cap a_{\beta\gamma}$ is too. Since $a \subseteq a_{\alpha\beta}$, a is an α-set by definition of sets of agreement. Similarly, a is a γ-set and we have $\alpha|_a = \beta|_a = \gamma|_a$, so that a is a set of agreement between α and γ, hence $a \subseteq a_{\alpha\gamma}$ by definition. □

Lemma 3.3. *The limit of class k subpartitions in the metric δ is again of class k.*

Proof. Let π be such a limit, b_π its support. There are class k subpartitions arbitrarily close to π so their domains of agreement give class k summands of π with support arbitrarily close to b_π. But since a sum of class k subpartitions is of class k this implies π is of class k. □

Lemma 3.4. *The space of all subpartitions is complete in the metric δ.*

Proof. Let $\{\pi_i\}$ be a Cauchy sequence. Choose n_k so large that $\delta(\pi_i, \pi_j) < 2^{-k}$ for all $i, j \geq n_k$. Passing to the subsequence π_{n_k} we can assume $\delta(\pi_i, \pi_j) < 2^{-i}$ for all $i \leq j$.

Let b_i be the support of π_i; a_{ij} the domain of agreement between π_i and π_j. Let A_{ij} denote the common domain of agreement between all of $\pi_i, \pi_{i+1}, \ldots, \pi_j$, defined in the obvious way. It is easy to see that this is the same as the intersection of their pairwise domains of agreement and also that $A_{i,j+1} = A_{ij} \cap a_{j,j+1}$. We have $A_{ij} \subseteq b_j$ and $\mu(b_j \setminus a_{j,j+1}) \leq \delta(\pi_j, \pi_{j+1}) < 2^{-j}$ so that $\mu(A_{i,j+1}) > \mu(A_{i,j}) - 2^{-j}$. Starting with $A_{i,i} = b_i$ we find that $\mu(A_{ij}) > \mu(b_i) - 2^{1-i}$. With i fixed we have A_{ij} decreasing as j increases. Let $B_i = \bigcap_j A_{ij}$; we conclude that $\mu(b_i \setminus B_i) \leq 2^{1-i}$.

Now B_i is the common domain of agreement of all π_j for $j \geq i$. Hence $B_i \subseteq B_{i+1}$ and if we let σ_i be the restriction of π_i to B_i then σ_{i+1} extends σ_i for each i. That is, $\sigma_{i+1} = \sigma_i + \nu_i$ where ν_i is the restriction of σ_{i+1} to $B_{i+1} \setminus B_i$.

Define π to be the limit of the σ_i, which is to say, their union or equivalently the sum of the differences ν_i. Then the support of π is $\bigcup B_i$, and π agrees with π_i on B_i. Since $\mu(b_i \setminus B_i)$ and $\mu((\bigcup B_i) \setminus B_i)$ both tend to zero, π_i converges to π in the metric δ. □

Corollary 3.5. *The space of subpartitions of class k is complete in the metric δ.*

Lemma 3.6. *The map f_k sending $\pi \in \Pi$ to its class k component π_k is uniformly continuous.*

Proof. Given $\pi, \sigma \in \Pi$ let ν be the restriction of π_k and σ_k to their domain of agreement a. Let b denote the union of the supports of π_k and σ_k, so that $\delta(\pi_k, \sigma_k) = \mu(b \setminus a)$.

Next consider the continuous partition $\pi \vee \sigma$. The set b is contained in the support of the sum of the subpartitions of class $l \geq k$ in $\pi \vee \sigma$, with $l = k$ only on a. Define L^1 functions $\rho_{\pi\vee\sigma}$, ρ_π, ρ_σ to be $1 - 1/j$ on the support of the relevant class j components, as in the proof of Lemma 2.10. Then $2\rho_{\pi\vee\sigma} - \rho_\pi - \rho_\sigma \geq 1/k - 1/(k+1)$ on $b \setminus a$, giving

$$\begin{aligned} d(\pi, \sigma) &= 2r(\pi \vee \sigma) - r(\pi) - r(\sigma) \\ &= \int 2\rho_{\pi\vee\sigma} - \rho_\pi - \rho_\sigma \, d\mu \geq \left(\frac{1}{k} - \frac{1}{k+1}\right) \delta(\pi_k, \sigma_k). \quad \square \end{aligned}$$

Lemma 3.7. *Let $\{\pi^{(i)}\}$ be a sequence of continuous partitions and let $\pi \in \Pi$. Suppose for each finite k that the class k components $\pi_k^{(i)}$ converge to π_k in the metric δ. Then $\lim \pi^{(i)} = \pi$ in Π.*

Proof. Given K we can choose i so large as to bring the following three sets arbitrarily close to one another: (1) the support b of the components of class $k \leq K$ in π, (2) the support $b^{(i)}$ of the components of class $k \leq K$ in $\pi^{(i)}$, and (3) the domain of agreement a between the sum of these components in $\pi^{(i)}$ and their sum in π. The third set a will also be a set of agreement between π, $\pi^{(i)}$, and $\pi \vee \pi^{(i)}$, since it supports finite class subpartitions which are summands of both π and $\pi^{(i)}$.

Consider the functions ρ_π, $\rho_{\pi^{(i)}}$, and $\rho_{\pi\vee\pi^{(i)}}$. On the domain of agreement a, all three functions are equal. Off $b \cup b^{(i)}$ all three take values within $1/K$ of 1. The remaining subset $(b \cup b^{(i)}) \setminus a$ is small and the various functions $\rho_{(-)}$ are all bounded. Hence choosing first K and then i sufficiently large we make $d(\pi, \pi^{(i)}) = \int 2\rho_{\pi\vee\pi^{(i)}} - \rho_\pi - \rho_{\pi^{(i)}} \, d\mu$ arbitrarily small, as required. $\square$

Combining the preceding results we obtain:

Theorem 2. *The lattice Π is complete in the metric d.*

Proof. Let $\{\pi^{(i)}\}$ be a Cauchy sequence in Π. By Lemma 3.6, the class k components $\pi_k^{(i)}$ form a Cauchy sequence. By Corollary 3.5 they have a limit π_k which is again of class k. From the definition of δ it is easy to see that the map sending a subpartition σ to its support $b \in B$ is uniformly continuous (in the symmetric difference metric on B). Since for each i the supports of the $\pi_k^{(i)}$ are disjoint, it follows that the π_k have disjoint supports. Their sum, along with a class ∞ subpartition on the complement of their support, is then a continuous partition π and it is the limit of the π_i by Lemma 3.7. $\square$

We close this section with some remarks on maximal chains in Π.

Corollary 3.8. *Let $C \subseteq \Pi$ be a maximal chain. Then the rank function $r: C \to [0, 1]$ is an isometric bijection.*

Proof. It is clear that r is an isometric injection. Any increasing sequence in Π is Cauchy. Its limit is then its join, and thus belongs to C if the sequence does. Dually for a decreasing sequence. This shows $r(C) \subseteq [0, 1]$ is closed. Lemmas 2.11 and 2.12 imply that if $\pi < \sigma \in \Pi$ then Π contains elements strictly between π and σ. In turn, the same must be true of the maximal chain C and also of $r(C)$. But the only closed subset of $[0, 1]$ with this property and containing 0 and 1 is $[0, 1]$ itself. $\square$

This property of maximal chains was proved by von Neumann for continuous geometries as a justification of the postulate that they were genuinely "continuous dimensional". It is also noted by Björner for his continuous partition lattice. Actually, it is much more intimately related to Theorem 2 than it might appear, for one can prove that if L is any complete lattice with a strictly increasing semimodular rank function r, then L is complete in the associated metric d if and only if $r(C) \subseteq [0, 1]$ is closed for every maximal chain C.

4. Björner's Π_∞ as a sublattice of Π

Definition. Let $\Pi_{k+1} = \Pi(\{0, 1, \dots, k\})$ be the finite partition lattice of rank k. Normalize its rank function by the factor $1/k$ to lie in the range $\{0, 1/k, \dots, 1\}$. Let Π be the lattice of continuous partitions of the unit interval $(0, 1]$. We have a lattice embedding $\phi_k: \Pi_{k+1} \hookrightarrow \Pi$ which preserves the normalized rank function, defined as follows. For $1 \leq j \leq k$ let $I_j = (\frac{j-1}{k}, \frac{j}{k}]$. Given any subset $P = \{p_1, \dots, p_j\} \subseteq \{1, \dots, k\}$ we have a subpartition π_P of class j with support $b_P = I_{p_1} \cup \cdots \cup I_{p_j}$ defined by the translation isomorphisms between the intervals I_{p_i}. Equivalently, b is a π_P-set iff the intersections $b \cap I_{p_i}$ are translates of one another. Given $\pi \in \Pi_{k+1}$ we then take $\phi_k(\pi)$ to be the sum of the subpartitions π_P corresponding to blocks $P \in \pi$ for which $0 \notin P$, together with a class ∞ subpartition on the union $\bigcup_{j \in P_0 \setminus \{0\}} I_j$, where $0 \in P_0 \in \pi$.

One easily sees that when $m|n$, we have $\phi_m(\Pi_{m+1}) \subseteq \phi_n(\Pi_{n+1})$, yielding a normalized rank preserving lattice embedding $\phi_{mn} = \phi_n^{-1} \circ \phi_m: \Pi_{m+1} \to \Pi_{n+1}$. With these maps, the finite partition lattices form a direct system indexed by the positive integers, ordered by divisibility. Björner defines the maps ϕ_{mn} directly and thus constructs abstractly the following lattices:

Definition. The limit of the direct system $\phi_{mn}: \Pi_{m+1} \hookrightarrow \Pi_{n+1}$ is denoted $\Pi_{(\infty)}$. The completion of $\Pi_{(\infty)}$ in the metric of Lemma 3.1 is denoted Π_∞.

In our context, it is clear that:

Lemma 4.1. *The lattice $\Pi_{(\infty)}$ is the union of the sublattices $\phi_m(\Pi_{m+1}) \subset \Pi$. The lattice Π_∞ is the closure of $\Pi_{(\infty)}$ in Π. Since the join operation in Π is continuous, Π_∞ is a join-sublattice of Π.*

We even know that Π_∞ is a complete join-sublattice of Π. For by our remark following Lemma 2.10, we need only consider countable complete joins, and such a join is the limit of the sequence of finite partial joins, by Corollary 3.8.

What is not clear as yet is that Π_∞ is actually a (complete) sublattice of Π. To prove this is our principal purpose in this section. We do it by characterizing the elements of Π_∞ explicitly. In this way we are also able to study the automorphism group of Π_∞ and thus disprove a conjecture of Björner.

Throughout the sequel, the Lebesgue space X underlying Π will be the real unit interval. Given $b \in B$, we write $b + x$ for the translate of b by a real number x.

Definition. A subpartition of class $k < \infty$ with support b is a *fundamental rational* subpartition if its defining isomorphism $b \cong K \times Y$ can be chosen so that the isomorphisms $i \times Y \cong j \times Y$ between sections are translates by rational numbers. A sum of fundamental rational subpartitions is *rational*. A continous partition is *rational* if its subpartitions of finite class are.

Theorem 3. *The set of rational continuous partitions is exactly* Π_∞.

Proof. By construction, $\Pi_{(\infty)}$ contains only rational continuous partitions. To establish the same for Π_∞ we show that a limit of rational continuous partitions is rational.

Indeed, if $\pi = \lim \pi^{(i)}$, then for each k the domain of agreement between π_k and $\pi_k^{(i)}$ approaches the support b_k of π_k, by Lemma 3.6. If the $\pi^{(i)}$ are rational then π_k thus has rational summands whose supports converge to b_k, so π_k is rational.

We must also show that every rational continuous partition belongs to Π_∞. Since Π_∞ is a complete join sublattice, it suffices to do this for the partitions ε_a and for continuous partitions which are the sum of of a class 1 subpartition and a class $k < \infty$ fundamental rational subpartition.

If a is a finite union of intervals with rational endpoints, then ε_a belongs to Π_{n+1}, where n is a common denominator for the endpoints. Since any measurable a is a limit of such a's, $\varepsilon_a \in \Pi_\infty$ for all a.

Suppose $\pi \in \Pi$ is the sum of a class 1 subpartition and a fundamental rational subpartition of class $k < \infty$. The class k part π_k of π is supported on a union of disjoint rational translates $a + t_1, \ldots, a + t_k$ of a measurable set a. Given $\epsilon > 0$, there is a finite union x of intervals with rational endpoints such that $\mu(a \Delta x) < \epsilon$. It may happen that the translates $x + t_j$ are not disjoint. If so, however, consider the set y of "bad" points $r \in x$ such that $r + t_i = s + t_j$ for some i, j and some $s \in x$. This y is a finite union of intervals with rational endpoints and $\mu(y)$ clearly approaches zero uniformly in ϵ. Replacing x with $x \setminus y$, we see that a can be approximated arbitrarily well by sets x which are finite unions of rational intervals and whose translates $x + t_j$ are disjoint. For such an x, $\bigcup_j x + t_j$ is the support of a fundamental rational subpartition π_ϵ with $\delta(\pi_\epsilon, \pi_k) < \epsilon$. Summing π_ϵ with the class 1 subfactor of complementary support, we obtain a continuous partition in $\Pi_{(\infty)}$ which approaches π as $\epsilon \to 0$. □

Lemma 4.2. *Let* π *be a rational subpartition of class* $k < \infty$ *with support* b. *Then any finer partition of* b *is also rational.*

Proof. We can assume π is fundamental rational. For the rest we refer the reader to the proof of the analogous Lemma 2.5. We only remark that the subpartitions v_J constructed there will obviously be fundamental rational when π is. □

Corollary 4.3. *If* $\pi, \sigma \in \Pi$ *are rational, so is* $\pi \wedge \sigma$.

Proof. Let b and c be the supports of the finite class components of π and σ, respectively. The support of the finite class component of $\pi \wedge \sigma$ is then $b \cup c$, and Lemma 4.2 shows that that $\pi \wedge \sigma$ has rational summands supported on b and c, hence also on $b \cup c$. □

By now we have proved most of:

Theorem 4. Π_∞ *is a complete sublattice of* Π.

Proof. The only aspect not yet covered is closure of Π_∞ under infinite meets. As remarked after Lemma 2.10, we need only consider countable meets, and passing to finite partial meets we need only consider the meet of a descending sequence. But such a sequence is automatically Cauchy, and its limit is its meet. The result follows since Π_∞ is closed. □

In order to discuss automorphisms of Π we turn our attention again to the modular elements ε_a of Lemma 2.15. Let us now prove they are the only modular elements.

Lemma 4.4. *Only the elements* $\varepsilon_a \in \Pi$ *are modular in any of the senses* (M1), (M2), *and* (M3).

Proof. If $\pi \notin \{\varepsilon_a\}$ is modular in any sense, then it is modular in the same sense as an element of any interval containing it. Now π must have a nonzero summand of class $1 < k < \infty$. Let b be its support. The continuous partitions $\varepsilon_{b^c} \wedge \pi$ and $\varepsilon_b \vee \pi$ agree with π on b^c and with 0 and 1 respectively on b. From Lemmas 2.11 and 2.12 and Corollary 2.13 we get an isomorphism $[\varepsilon_{b^c} \wedge \pi, \varepsilon_b \vee \pi] \cong \Pi(b)$ under which π corresponds to a pure class k partition. Thus we reduce to the case that π is of class $1 < k < \infty$.

Up to an automorphism of the space X, hence of Π, we can assume π is the image of the partition $\{\{0\}, \{1, 2, \dots, k\}\}$ under ϕ_k. Under $\phi_{k,2k}$ this corresponds to the partition $\{\{0\}, \{1, 3, \dots, 2k-1\}, \{2, 4, \dots, 2k\}\}$, which is not even modular as an element of Π_{2k+1} in any of the three senses. □

Since Π is constructed "naturally" from the Boolean algebra B of measurable sets (mod 0) is is clear that every automorphism of B, that is, every measure-preserving automorphism of X (mod 0), induces naturally an automorphism of Π. In fact this determines the automorphism group of Π.

Theorem 5. *The natural map* $\mathrm{aut}(B) \to \mathrm{aut}(\Pi)$ *is an isomorphism.*

Proof. By Lemma 4.4 each automorphism σ of Π restricts to an automorphism of the ranked sublattice $\{\varepsilon_a\} \cong B$. Moreover it is clear that the restriction map $\rho\colon \mathrm{aut}(\Pi) \to \mathrm{aut}(B)$ is left inverse to the natural map $\mathrm{aut}(B) \to \mathrm{aut}(\Pi)$. Thus both maps will be isomorphisms if ρ is injective, i.e., if each automorphism σ of Π is determined by its action on $\{\varepsilon_a\}$.

To see this, note that $\varepsilon_a \vee \pi \geq \varepsilon_b$ iff $a^\pi \cup b_\infty \supseteq b$, where b_∞ is the support of π_∞. Each $\pi \in \Pi$ is therefore determined by purely lattice-theoretic relationships with the $\{\varepsilon_a\}$. Specifically, b_∞ is the largest b such that $\pi \geq \varepsilon_b$, and a is a π-set if and only if $(\varepsilon_a \vee \pi)_\infty$ has support equal to $a \cup b_\infty$. This implies the result. □

Since $\{\varepsilon_a\} \subseteq \Pi_\infty$, each automorphism of Π_∞ restricts to an automorphism of B. Björner conjectured, based on the purely abstract construction of Π_∞, that $\rho\colon \mathrm{aut}(\Pi_\infty) \hookrightarrow \mathrm{aut}(B)$ would be an isomorphism. This is not so, as we now show.

Lemma 4.5. *The automorphism group* $\mathrm{aut}(\Pi_\infty)$ *is just the subgroup* $G \subseteq \mathrm{aut}(\Pi)$ *consisting of elements that map* Π_∞ *into itself.*

Proof. That $\mathrm{aut}(\Pi_\infty)$ "is just" G means the map restricting each $\sigma \in G$ to an automorphism of Π_∞ is an isomorphism. It is surjective because each automorphism of Π_∞ induces an automorphism of B which extends uniquely to Π_∞ and then to all of Π. It is injective because $\sigma \in G$ is determined by its action on $B \subseteq \Pi_\infty$. □

Lemma 4.6. *Every element of* Π *is conjugate via some automorphism of* Π *to an element of* Π_∞.

Proof. From the structure of a continuous partition it is clear that π and σ are conjugate just in case for each k their respective class k components have supports of the same measure. Thus it is only necessary to construct a rational continuous partition π with π_k supported on a set of prescribed measure m_k for each k. But it is trivial that there exists a rational class k subpartition supported on any interval of rational length and thus we can easily construct π by expressing each m_k as a countable sum of rational numbers. □

Corollary 4.7. *The natural embedding* $\mathrm{aut}(\Pi_\infty) \hookrightarrow \mathrm{aut}(B)$ *realizes* $\mathrm{aut}(\Pi_\infty)$ *as a proper subgroup of* $\mathrm{aut}(B)$.

Proof. Otherwise Lemmas 4.5 and 4.6 would imply $\Pi_\infty = \Pi$. But nonrational continuous partitions clearly exist: choose $0 \neq a \in B$ with an irrational translate $a + t$ disjoint from a, construct a class 2 subpartition ν supported on $a \cup (a + t)$ from the translation isomorphism between a and $a + t$, and let π be any continuous partition containing ν as a summand. The domain of agreement between ν and any rational subfactor is 0, so π cannot be rational. □

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