ON THE FORCE BETWEEN ROTATING COAXIAL BLACK HOLES

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ABSTRACT. We study the force between rotating coaxial black holes, as it was defined in [9 and 10]. We show that under a certain limit, the force is attractive, and in fact tends to infinity. This lends support to the conjecture that the force is always positive.

1. INTRODUCTION

The Einstein vacuum equations reduce in the stationary axially symmetric case to an axially symmetric harmonic map from $\mathbb{R}^3$ into the hyperbolic plane with prescribed blow-up on the axis and at infinity. This problem is used to model rotating coaxial black holes in equilibrium. The number of black holes is imposed by the choice of singular boundary conditions, see [9]. For one black hole, an explicit solution was discovered by Kerr [4], and proven to be unique by Robinson [8], based on work of Carter [2], see also [3].

The existence, regularity, and uniqueness of solutions of the reduced problem with two black holes has been established [9, 10]. However, when reconstructing the spacetime metric from a solution of the reduced equations, one additional regularity condition is to be checked: the absence of conical singularities on the axis of symmetry. This condition is easily verified on the two unbounded components of the axis. The value of the deficiency angle on the bounded component can be related to the gravitational force, see [1 and 9, §5]. Although it has been conjectured for some time that this force is always attractive [7], few results are known so far. These include the small angular momentum case, which is obtained by a continuity argument [5], and the case where the problem allows an involutive symmetry, see [6].

In this paper, we prove that the force is attractive, and in fact tends to infinity, in a limiting case, when one of the black holes becomes extreme, and the distance to one of the adjacent black holes then tends to zero. Phrased loosely, this says that rapidly rotating black holes always attract at close range. This result lends support to the conjecture that the force is always attractive, and implies a nonexistence result for the Einstein vacuum equations. The proof is an application of an a priori bound which was essentially obtained in the existence proof. We state all the results—the existence, uniqueness, and regularity, obtained in [10], and the new results—for any number N of black holes.

2. N-BLACK HOLE STATIONARY AXISYMMETRIC SPACETIMES

We summarize the notation and previous results. We present the N-black hole case. The proofs of the results in this section are the same as for the case $N = 2$ which was treated in [9, 10].

To begin, we pick $2N$ points $p_j \in \mathbb{A}$, the z-axis in $\mathbb{R}^3$, and consider the N intervals $I_j \subset \mathbb{A}$ from p_{2j-1} to p_{2j}. We define

$$\Gamma = \mathbb{A} \setminus \bigcup_{j=1}^{N} I_j = \bigcup_{j=1}^{N+1} \Gamma_j,$$

where Γ_j are the connected components of Γ. We let h_j be the potential of a uniform charge distribution on Γ_j, i.e.,

$$(2.1) \quad h_j = \begin{cases} \log(r_{2j-2} + z_{2j-2} - z) - \log(r_{2j-1} + z_{2j-1} - z) & \text{if } 2 \le j \le N, \\ \log(r_1 - z_1 + z) & \text{if } j = 1, \\ \log(r_{2N} + z_{2N} - z) & \text{if } j = N+1, \end{cases}$$

where r_j is the distance to p_j, and z_j the z-coordinate of p_j. Here and throughout, (ρ, z, ϕ) will denote cylindrical coordinates on $\mathbb{R}^3$. We let $h = \sum_{j=1}^{N+1} h_j$, and for future reference, we also write

$$h = 2 \log \rho + u = 2 \log \rho + \sum_{j=1}^{N} u_j,$$

where u_j is the potential of a uniform charge distribution on I_j. We define the norms:

$$\|x\|_1 = \left(\int_{\mathbb{R}^3} |\nabla x|^2 \right)^{1/2}, \qquad \|y\|_{1,h} = \left(\int_{\mathbb{R}^3} e^{-2h} |\nabla y|^2 \right)^{1/2},$$

and let $H_{1,0}(\mathbb{R}^3)$ and $H_{1,h,0}(\mathbb{R}^3)$ be the closure of $C_0^\infty(\mathbb{R}^3)$ under the norm $\| \cdot \|_1$, and $C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$ under the norm $\| \cdot \|_{1,h}$, respectively. We also use the norms $\| \cdot \|_\infty$, and $\|y\|_{\infty,h} = \|e^{-h} y\|_\infty$, and the corresponding spaces, $L^\infty(\mathbb{R}^3)$ and $L_h^\infty(\mathbb{R}^3)$. We pick a function $Y_0 \in C^\infty$ with $\|Y_0\|_{1,h} < \infty$, and $Y_0|_{\Gamma_j} = c_j$. Finally, we let $\mathbb{H}^2$ be the right half-plane model for the hyperbolic plane $\{(X, Y): X > 0\}$ with its metric $(dX^2 + dY^2)/X^2$.

The Einstein vacuum equations for a stationary, axially symmetric, asymptotically flat spacetime reduces to the problem of finding a harmonic map: $(X, Y): \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$, with the same asymptotic behavior as the map (e^h, Y_0), near Γ, and as $r \to \infty$. Conversely, suppose a solution of the following non-linear problem:

$$(2.2) \qquad \qquad \Delta \log X = -|\nabla Y|^2 / X^2,$$

$$(2.3) \qquad \qquad \text{div}(\nabla Y / X^2) = 0,$$

$$(2.4) \qquad x = \log X - h \in H_{1,0} \cap L^\infty, \qquad x \to 0 \quad \text{as } r \to \infty,$$

$$(2.5) \qquad y = Y - Y_0 \in H_{1,h,0} \cap L_h^\infty, \qquad e^{-h} y \to 0 \quad \text{as } r \to \infty,$$

is found. Then, a Ricci flat metric,

$$(2.6) \qquad ds^2 = -V\,dt^2 + 2W\,dt\,d\phi + X\,d\phi^2 + e^{2\lambda}(d\rho^2 + dz^2),$$

can be constructed on $\mathbb{R} \times (\mathbb{R}^3 \backslash \mathbb{A})$. To obtain the metric components, we integrate

$$(2.7) \qquad d\Omega = -\frac{\rho * dY}{X^2},$$

$$(2.8) \qquad d\beta = \frac{\rho}{2}\,\mathrm{Re}\left[\left((\partial u + \partial x)^2 + \frac{(\partial Y)^2}{X^2}\right)d\zeta\right],$$

with Ω, $\beta \to 0$, as $r \to \infty$. Here, we have used the Hodge dual operator given by $*d\rho = dz$, and $*dz = -d\rho$, the operator $\partial = \partial_\rho - i\partial_z$, and the 1-form $d\zeta = d\rho + i\,dz$. The 1-forms on the right-hand side in (2.7) and (2.8) are closed as a consequence of the harmonic map equations for (X, Y). We now take $2\lambda = \beta + u + x$, $W = X\Omega$, and $V = X^{-1}(\rho^2 - W^2)$. In view of the regularity result, see Theorem 2.1(b), it follows from (2.7) and (2.8) that Ω is constant on each I_j, and β is constant on each Γ_j. We define $\omega_j = -\Omega|_{I_j}$, and $b_j = \beta|_{\Gamma_j}$. We note that the problem is invariant under translation along the z-axis in $\mathbb{R}^3$, translation along the Y-axis in $\mathbb{H}^2$, and under the one-parameter scaling transformation given by

$$(2.9) \qquad \begin{aligned} X &\to a^2 X, & Y &\to a^2 Y, \\ \rho &\to a\rho, & z &\to az. \end{aligned}$$

In the case $N = 1$, the solution can be written explicitly. To write down the solution, we introduce the parameters $m = (z_2 - z_1)/2$, and $L = (c_2 - c_1)/8$, and change over to the parameters M and a defined by $m^2 + a^2 = M^2$, and $L = Ma$. Here M is the total mass, and L the total angular momentum. We define elliptical coordinates (s, θ), with $s > M + m$, by

$$\rho^2 = (s - M + m)(s - M - m)\sin^2\theta, \qquad z = (s - M)\cos\theta.$$

The Kerr solution is now given by

$$(2.10) \qquad X = \left(s^2 + a^2 + \frac{2Ma^2 s \sin^2\theta}{s^2 + a^2\cos^2\theta}\right)\sin^2\theta,$$

$$(2.11) \qquad Y = 2Ma\cos\theta\left(3 - \cos^2\theta + \frac{a^2\sin^4\theta}{s^2 + a^2\cos^2\theta}\right).$$

We note that here $h = 2\log[(s - M + m)\sin\theta]$, and that $x \in L^\infty(\mathbb{R}^3)$. In fact,

$$1 \le e^x \le 1 + a^2/m^2.$$

Furthermore, we note that $a^2 < M^2$, i.e., $|L| < M^2$, and $M^2 \to |L|$ when $m \to 0$. In fact, a Kerr solution can also be written for $a^2 \ge M^2$, but for $a^2 > M^2$ it contains a naked singularity. When $a^2 = M^2$, the solution is called *extreme*. We will continue to call the limit $m_j \to 0$ the *extreme limit* when $N > 1$.

Given a choice of parameters for general N as above, we associate a *corresponding Kerr solution*: $(\Phi, \Psi)\colon \mathbb{R}^3 \backslash (\Gamma_1 \cup \Gamma_{N+1}) \to \mathbb{H}^2$, with $\Psi|_{\Gamma_j} = c_j$ for

$j = 1$ and $j = N + 1$. We write $\log \Phi = g + \varphi$, where $g = h_1 + h_{N+1} = h - f$, and $f = \sum_{j=2}^{N} h_j$.

The following theorem was essentially obtained in [10]. Although it was stated and proved only for the case $N = 2$, the same arguments work for any number of black holes.

Theorem 2.1. *For any choice of the parameters, we have*:

(a) *There exists a unique solution of* (2.2)–(2.5).

(b) *For this solution, we have*

$$(2.12) \qquad e^{\varphi - x} + e^{2f + x - \varphi} + e^{-2g - \varphi - x}(Y - \Psi)^2 \leq 2,$$

where $(e^{g + \varphi}, \Psi)$ *is the corresponding Kerr solution. Furthermore* $x, Y \in C^{\infty}(\mathbb{R}^3 \backslash \{p_1, \dots, p_{2N}\}) \cap C^1(\mathbb{R}^3)$, *and there is a constant* C *depending only on the parameters such that*

$$(2.13) \qquad 0 \leq x \leq C,$$

and

$$(2.14) \qquad |\nabla x|^2 + |\nabla Y|^2/X^2 \leq C.$$

(c) *The metric* (2.6) *is asymptotically flat.*

(d) *For* $j = 1, \dots, N + 1$, *we have*

$$(2.15) \quad b_{j+1} - b_j = -2 \left(\sum_{\substack{1 \leq k \leq N \\ k \neq j}} \{u_k(p_{2j}) - u_k(p_{2j-1})\} + x(p_{2j}) - x(p_{2j-1}) \right).$$

In the spacetime defined by (2.6), the intervals I_j are actually surfaces, and it is not difficult to check that, provided the spacetime is regular, these surfaces are event horizons, see [9]. There are several physically significant quantities arising as periods of closed 1-forms, and we shall discuss these now. If K is any Killing field on a vacuum spacetime, the quantity

$$(2.16) \qquad Q_K(S) = \int_S *dK$$

depends only on the homology class of S, as can easily be seen from Stokes' Theorem. For example if $K = \partial/\partial\phi$, then $L(S) = (1/16\pi)Q_K(S)$ is the angular momentum contained in S, and if $K = \partial/\partial t$, then $M(S) = (1/8\pi)Q_K(S)$ is the mass contained in S. Here, the second homology is generated by surfaces S_j, where S_j is any surface containing only the jth black hole, and we set $M_j = M(S_j)$, and $L_j = L(S_j)$, the mass and angular momentum of the jth black hole respectively. Strictly speaking, we have a vacuum spacetime only outside Γ, but it is not difficult to check that there is no contribution to (2.16) from integration along Γ. Possibly deforming S_j through a homotopy, we may take S_j axially symmetric, and we obtain

$$(2.17) \qquad M_j = \frac{1}{4} \int_{C_j} * \frac{X\, dV + W\, dW}{\rho},$$

$$(2.18) \qquad L_j = \frac{1}{8} \int_{C_j} * \frac{X\, dW - W\, dX}{\rho},$$

where C_j is any curve from Γ_j to Γ_{j+1}. It follows immediately from (2.7) that $L_j = (c_{j+1} - c_j)/8$. Furthermore, we have

$$(2.19) \qquad\qquad M_j = m_j + 2\omega_j L_j,$$

where $2m_j = z_{2j} - z_{2j-1}$. Indeed, a straightforward calculation yields

$$(X \, dV + W \, dW)/\rho = \Omega * dY - \rho(du + dx).$$

Equation (2.19) now follows, by letting C_j tend to I_j, and noting that $\rho u_\rho \to -2$ on I_j, see [10, §6]. The constant ω_j can be interpreted as the angular velocity of the jth black hole. Finally, we have the periods of $d\beta$:

$$b_{j+1} - b_j = \int_{C_j} d\beta.$$

Since the metric (2.6) is written in cylindrical coordinates, it extends across Γ only if there is no conical singularity on the axis. The deficiency angle on Γ_j is measured by b_j. It was shown in [9] that

$$(2.20) \qquad\qquad F_j = \tfrac{1}{4}(e^{-b_j/2} - 1)$$

could be interpreted as a gravitational force.

3. The force in the extreme limit

In this section, we prove that the force is attractive when one of the black holes is close to being extreme, and the distance to an adjacent black hole is small. We first consider the case where all the other black holes are on the same side of the one close to being extreme. This is always the case when $N = 2$.

Proposition 3.1. *For any solution of (2.2)–(2.5), we have*

$$(3.1) \qquad \begin{aligned} x(p_2) - x(p_1) \geq &-\log 4 + 2f(p_1) - 2g(p_2) - \varphi(p_1) - \varphi(p_2) \\ &+ 2\log|c_2 - \Psi(p_2)|. \end{aligned}$$

Proof. From part (b) of Theorem 2.1, we have

$$\begin{aligned} x(p_1) &\leq \log 2 - 2f(p_1) + \varphi(p_1), \\ x(p_2) &\geq -\log 2 - 2g(p_2) - \varphi(p_2) + 2\log|c_2 - \Psi(p_2)|. \end{aligned}$$

Summing up, we obtain (3.1). □

Note that on Γ_2 $-2\sum_{k=2}^{N}\{u_k(p_2) - u_k(p_2)\}$, the linear contribution to b_2 in (2.15), is always negative. We now show that the nonlinear contribution $-2(x(p_2) - x(p_1))$ is also negative in the extreme limit. Thus, we obtain that the force F_2, see (2.20), is positive in this case. Let $L = \sum_{j=1}^{N} L_j$ denote the total angular momentum.

Theorem 3.1. *Fix p_j, for $1 \leq j \leq 2N$. For any constant $C_1 > 0$, there is a constant C_2 such that if $|L| \leq C_1$, and $|L_1| > C_2$, then $x(p_2) - x(p_1) > 0$, for any solution of (2.2)–(2.5).*

Proof. Suppose $|L| \leq C_1$, then all the terms are bounded in the estimate (3.1) except for the term

$$2\log|c_2 - \Psi(p_2)|,$$

which tends to ∞ when $|c_1 - c_2| \to \infty$. The theorem follows. □

This theorem can be interpreted roughly as saying that the force between rapidly counterrotating black holes is always attractive. In order to remove the counterrotation assumption, we will have to assume that the black holes are not too far apart. However, note that if one wished to keep the angular momenta bounded in Theorem 3.1, as is the case in Theorem 3.2, one would have to rescale the solution using (2.9), which would again have the effect of bringing the black holes close together. Let $2d_j = z_{2j-1} - z_{2j-2}$ be the length of Γ_j, and $2m = z_{2N} - z_1$.

Theorem 3.2. *Fix all the points p_j except p_2, and c_j for $1 \leq j \leq N + 1$, such that*

$$(3.2) \qquad\qquad |L_1| > 4D_2(m^2 + |L|)/(D_2 + m_2),$$

where $D_2 = m_1 + d_2$. Then, there is a constant $\varepsilon > 0$ such that if $m_1 < \varepsilon$, we have $x(p_2) - x(p_1) > 0$, for any solution of (2.2)–(2.5).

Proof. Note first that in the statement of the theorem D_2 is fixed although p_2 is not. Now, it follows from (2.10) that $e^\varphi \leq (1 + |L|/m^2)$. In view of (2.1), we have

$$f(p_1) = h_2(p_1) + \sum_{j=3}^{N} h_j(p_1) \geq \log\left(\frac{m_1}{d_2 + m_1}\right) + \log\left(\frac{m_1 + d_2 + m_2}{m}\right).$$

The last inequality follows from

$$\sum_{j=3}^{N} h_j(p_1) = \sum_{j=3}^{N} \log\left(\frac{r_{2j-2}(p_1)}{r_{2j-1}(p_1)}\right) \geq \log\left(\frac{r_4(p_1)}{r_{2N}(p_1)}\right) = \log\left(\frac{m_1 + d_2 + m_2}{m}\right).$$

Furthermore, again using (2.1), we find

$$g(p_2) = h_1(p_2) + h_{N+1}(p_2) = \log[16m_1(m - m_1)].$$

Substituting in (3.1), we obtain

$$x(p_2) - x(p_1) \geq 2\log\left[\frac{(D_2 + m_2)|c_2 - \Psi(p_2)|}{32mD_2(m - m_1)(1 + |L|/m^2)}\right].$$

When $m_1 \to 0$, we have $p_2 \to p_1$, hence the right-hand side tends to

$$2\log\left[\frac{(D_2 + m_2)|L_1|}{4m^2D_2(1 + |L|/m^2)}\right],$$

which is positive by (3.2). The theorem follows. $\square$

Note that (3.2) can be achieved by taking D_2 small enough. Thus, this theorem can be interpreted roughly as saying that extreme black holes always attract at close range. Furthermore, one sees from the proof that in the limit $D_2 \to 0$ the force tends to ∞.

We now treat the case where the extreme black hole is not the first or the last one along $\mathbb{A}$. Although the main idea is the same, we need a new estimate similar to (2.12). Let (X, Y) be an N-black hole solution of (2.2)–(2.5), and fix $2 \leq k \leq N - 1$. We will associate with this solution an $(N - 1)$-black hole solution (X', Y') of (2.2)–(2.5). We take

$$\Gamma' = \Gamma \backslash \Gamma_{k+1} = \bigcup_{\substack{1 \leq j \leq N+1 \\ j \neq k+1}} \Gamma_j, \quad h' = h - h_{k+1} = \sum_{\substack{1 \leq j \leq N+1 \\ j \neq k+1}} h_j,$$

and we use the existence theorem with $Y'|_{\Gamma_j} = c_j$ for $1 \leq j \leq N+1$, $j \neq k+1$. We set $\log X' = h' + x'$, and obtain from Theorem 2.1 part (b):

$$(3.3) \qquad\qquad 0 \leq x' \leq C',$$

where C' is independent of p_{2k}, p_{2k+1}, and c_{k+1}, see (2.13).

Proposition 3.2. *For the solutions* (X, Y) *and* (X', Y') *above, we have*

$$(3.4) \qquad e^{x'-x} + e^{2h_{k+1}+x-x'} + e^{-2h'-x'-x}(Y - Y')^2 \leq 2.$$

Proof. The proof is similar to the one used in [10] to obtain (2.12). Let $d = \text{dist}[(X', Y'), (X, Y)]$ be the distance in $\mathbb{H}^2$ between the two solutions. Since $\mathbb{H}^2$ is negatively curved, and (X', Y') and (X, Y) are both harmonic maps, it follows that d^2 is subharmonic in $\mathbb{R}^3 \backslash \Gamma$. Let $\delta = \log \cosh d + h_{k+1}$, then since $\log \cosh d$ is a convex function of d^2, $\Delta\delta \geq 0$ in $\mathbb{R}^3 \backslash \Gamma$. A straightforward calculation yields

$$\delta = \log \tfrac{1}{2}[e^{x'-x} + e^{2h_{k+1}+x-x'} + e^{-2h'-x'-x}(Y-Y')^2].$$

Using Lemma 1 in [10], we find that $\delta \in H_{1,\text{loc}}(\mathbb{R}^3)$, and therefore $\Delta\delta \geq 0$ weakly in all of $\mathbb{R}^3$. Since $\delta \to 0$ as $r \to \infty$, the maximum principle gives $\delta \leq 0$, and the proposition follows. □

We now prove

Theorem 3.3. *Fix* p_j *for* $1 \leq j \leq 2N$, $j \neq 2k$, *and* c_j *for* $1 \leq j \leq N+1$, *such that*

$$(3.4) \qquad\qquad d_k|L_k| > e^{C'} m^2 D_{k+1},$$

where $D_k = m_k + d_{k+1}$, *and with* C' *as in* (3.3). *Then, there is a constant* $\varepsilon > 0$ *such that if* $m_k < \varepsilon$, *we have* $x(p_{2k}) - x(p_{2k-1}) > 0$, *for any solution of* (2.2)–(2.5).

Proof. As before D_k is fixed although p_{2k} is not. From (3.4), we have

$$x(p_{2k-1}) \leq \log 2 - 2h_{k+1}(p_{2k-1}) + x'(p_{2k-1}),$$
$$x(p_{2k}) \geq -\log 2 - 2h'(p_{2k}) - x'(p_{2k}) + 2\log|c_{k+1} - Y'(p_{2k+1})|.$$

Summing up, and using (3.3), we obtain

$$(3.5) \qquad \begin{aligned} x(p_{2k}) - x(p_{2k-1}) \geq\ & -\log 4 + 2h_{k+1}(p_{2k-1}) - 2h'(p_{2k}) \\ & -2C' + 2\log|c_{k+1} - Y'(p_{2k})|. \end{aligned}$$

Now, using (2.1), we have

$$(3.6) \qquad\qquad h_{k+1}(p_{2k-1}) = \log(m_k/D_{k+1}),$$

and, since $h_j \leq 0$ for $2 \leq j \leq N$, we find

$$(3.7) \qquad h'(p_{2k}) \leq g(p_{2k}) + h_k(p_{2k}) \leq \log(4m^2 m_k/(m_k + d_k)).$$

The last inequality follows since g has its maximum $\log 4m^2$ on $\mathbb{A}$ at the midpoint between p_1 and p_{2N}. Substituting (3.6) and (3.7) into (3.5), we obtain

$$x(p_{2k}) - x(p_{2k-1}) \geq 2\log\left[\frac{(m_k + d_k)|c_{k+1} - Y'(p_{2k})|}{8e^{C'} m^2 D_{k+1}}\right].$$

When $m_k \to 0$, $p_{2k} \to p_{2k-1}$, and the right-hand side above tends to

$$2\log[|L_k|d_k/e^{C'}m^2 D_{k+1}],$$

which is positive by (3.4). The theorem follows. $\square$

Note that again (3.4) can be achieved by taking D_{k+1} small enough. Also note that when D_{k+1} is small, the linear contribution to b_{k+1} is negative too, so that the force F_{k+1} is positive. This implies a nonexistence result for the Einstein vacuum equations, see [9].

Corollary 3.1. *Fix m_j and d_{j+1} for $1 \leq j \leq N$, $j \neq k$, and L_j for $1 \leq j \leq N$, with $L_k \neq 0$. Then there are constants $\varepsilon_1 > \varepsilon_2 > 0$ such that if $m_k + d_{k+1} < \varepsilon_1$, and $m_k < \varepsilon_2$, we have for any solution of (2.2)–(2.5) $F_{k+1} > 0$. Consequently, there is no asymptotically flat, stationary, axially symmetric solution of the Einstein vacuum equations corresponding to these parameter values.*

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