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3-PRIMARY v_1-PERIODIC HOMOTOPY GROUPS OF F_4 AND E_6

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ABSTRACT. We compute the 3-primary v_1-periodic homotopy groups of the exceptional Lie groups F_4 and E_6. The unstable Novikov spectral sequence is used for the most delicate part of the analysis.

1. MAIN THEOREM

In this paper we compute the 3-primary v_1-periodic homotopy groups of the exceptional Lie groups F_4 and E_6.

The p-primary v_1-periodic homotopy groups of a space X, denoted $v_1^{-1}\pi_*(X;p)$, were defined in [15]. They are a localization of the actual homotopy groups, telling roughly the portion which are detected by K-theory and its operations. If X is a compact Lie group, each $v_1^{-1}\pi_i(X;p)$ is a direct summand of some actual homotopy group of X, and so summands of v_1-periodic homotopy groups of X give lower bounds for the p-exponent of X.

After the second author computed $v_1^{-1}\pi_*(SU(n);p)$ for odd p in 1989, Mimura proposed the goal of calculating $v_1^{-1}\pi_*(X;p)$ for all compact simple Lie groups X. This has now been achieved in the following cases (X, p):

- X a classical group and p odd ([14]).
- $X = SU(n)$ or $Sp(n)$ and $p = 2$ ([8], [9]).
- X an exceptional Lie group with $H_*(X;\mathbf{Z})$ p-torsion-free ([10]).
- $(G_2, 2)$ ([16]) (F_4 and E_6, 3) (the current paper).

The only cases remaining then are $(E_8, 5)$, (E_7 or E_8, 3), $(SO(n), 2)$, (F_4 or E_6 or E_7 or E_8, 2). Many of these appear tractable. The unstable Novikov spectral sequence (UNSS) has played an important role in every case except $(G_2, 2)$ and the cases where p is large enough that $X_{(p)}$ is a product of p-local spheres.

Since $v_1^{-1}\pi_*(X\times Y)\approx v_1^{-1}\pi_*(X)\oplus v_1^{-1}\pi_*(Y)$, the following known 3-local splitting theorem will be useful in both the statement and proof of our main result. From now on, all of our spaces are localized at the prime 3.

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Theorem 1.1. (1) ([17], [18, 4.4.1]) *There is a* 3-*equivalence*

$$F_4 \approx K \times B(11, 15),$$

where K *is the finite complex constructed by Harper in* [17] *satisfying*

$$H^*(K; \mathbf{Z}_3) = \Lambda(x_3, x_7) \otimes \mathbf{Z}_3[x_8]/(x_8^3),$$

with $x_7 = \mathscr{P}^1 x_3$ *and* $x_8 = \beta x_7$. *Also,* $B(11, 15)$ *is an* S^{11}-*bundle over* S^{15} *with* $\mathscr{P}^1 x_{11} = x_{15}$ *in* $H^*(B(11, 15); \mathbf{Z}_3)$.

(2) ([19]) *There is a* 3-*equivalence*

$$E_6 \approx F_4 \times (E_6/F_4).$$

Now we state our main theorem. We frequently abbreviate $v_1^{-1}\pi_*(X; 3)$ as $v_*(X)$, and denote by $\nu(n)$ the exponent of 3 in the integer n.

Theorem 1.2.

$$v_1^{-1}\pi_*(F_4; 3) \approx v_*(K) \oplus v_*(B(11, 15))$$

and

$$v_1^{-1}\pi_*(E_6; 3) \approx v_*(K) \oplus v_*(B(11, 15)) \oplus v_*(E_6/F_4),$$

where

$$(1.3) \qquad v_{2i}(K) \approx v_{2i-1}(K) \approx \begin{cases} \mathbf{Z}/3^{\min(12, \nu(i-11-2\cdot 3^{10})+1)} & \text{if } i \text{ is odd}, \\ 0 & \text{if } i \text{ is even}, \end{cases}$$

$$(1.4) \qquad v_{2i}(B(11, 15)) \approx \begin{cases} \mathbf{Z}/3^{\min(5, \nu(i-5)+2)} & \text{if } i \equiv 3 \text{ or } 5 \bmod 6, \\ \mathbf{Z}/3^{\min(7, \nu(i-7)+2)} & \text{if } i \equiv 1 \bmod 6, \\ 0 & \text{if } i \text{ even}, \end{cases}$$

and $|v_{2i-1}(B(11, 15))| = |v_{2i}(B(11, 15))|$;

$$(1.5) \qquad v_{2i}(E_6/F_4) \approx \begin{cases} \mathbf{Z}/3^{\min(8, \nu(i-8)+2)} & \text{if } i \equiv 0 \text{ or } 2 \bmod 6, \\ \mathbf{Z}/3^{\min(4, \nu(i-4)+2)} & \text{if } i \equiv 4 \bmod 6, \\ 0 & \text{if } i \text{ odd}, \end{cases}$$

and $|v_{2i-1}(E_6/F_4)| = |v_{2i}(E_6/F_4)|$.

As with much of our earlier work, the one blemish on these results is our inability to determine the group structure of $v_1^{-1}\pi_j(X)$ for many odd values of j; in these cases we only assert the order of the group.

An immediate corollary of this work is a lower bound for the 3-exponent of F_4 and E_6. Recall that the p-exponent of a space X, denoted $\exp_p(X)$, is the largest e such that $\pi_*(X)$ has an element of order p^e.

Corollary 1.6. *The* 3-*exponents of* F_4 *and* E_6 *satisfy*

$$\exp_3(F_4) = \exp_3(E_6) \geq 12.$$

Proof. If X is a compact Lie group, then $v_1^{-1}\pi_i(X; 3) = \varinjlim_k \pi_{i+4k\cdot 3^e}(X)$ for appropriate e. Hence an element of order 3^{12} in $v_1^{-1}\pi_*(F_4)$ corresponds to an element of order 3^{12} in some $\pi_i(F_4)$. The 3-exponents of F_4 and E_6 are equal since, by Theorem 1.1(2) and Proposition 3.1,

$$\exp_3(F_4) \le \exp_3(E_6) \le \max(\exp_3(F_4), \exp_3(S^9) + \exp_3(S^{17})).$$

The equality then follows since, by [12], $\exp_3(S^{2n+1}) = n$, and $\max(\exp_3(F_4), 12) = \exp_3(F_4)$. □

We conjecture that the lower bound given by Corollary 1.6 is sharp, but the best upper bound we can prove now is

$$\exp_3(F_4) \le \exp_3(S^3) + \exp_3(S^7) + \exp_3(S^7) + \exp_3(S^{23}) = 18.$$

This follows from Theorem 1.1(1), Proposition 2.9, (2.3), and [12].

The groups $v_1^{-1}\pi_*(B(11, 15); 3)$ were determined in [10, 1.3(2)], and so the second part of Theorem 1.2 is read off from that. We will prove the first part in Section 2, and the third part in Section 3. We thank John Harper for a last-minute correction to our lemmas.

2. v_1-PERIODIC HOMOTOPY GROUPS OF K

In this section, we show that $v_1^{-1}\pi_*(K; 3)$ is as claimed in the first of the three parts of Theorem 1.2.

We begin by computing $v_1^{-1}\pi_*(W; 3)$, where W denotes the Cayley plane. This is related to $v_1^{-1}\pi_*(K; 3)$ via Proposition 2.9, but may be of independent interest. The following description is certainly well-known.

Proposition 2.1. *After looping once, the Cayley plane is 3-equivalent to the space, often denoted $\widehat{S}^8$ in EHP-theory, which is defined to be the 16-skeleton of ΩS^9. That is, $\Omega W \simeq_{(3)} \Omega\widehat{S}^8$.*

Proof. Both W and $\widehat{S}^8$ are 2-cell complexes with cells of dimension 8 and 16. They are not 3-equivalent, since the suspension of the attaching map in W is essential when localized at 3 [13, 4.1], while the suspension of the attaching map $\widehat{S}^8$ is inessential.

We use the following 3-equivalence of [16, 3.1],

$$(*) \qquad \Sigma\Omega W \simeq S^8 \cup_u e^{23} \vee \bigvee_{i\ge1} (S^{22i+8} \vee S^{22i+23}).$$

We use this splitting to construct a map $\Sigma\Omega W \xrightarrow{g} \widehat{S}^8$ which has degree 1 on the bottom cell. Since $\widehat{S}^8$ first differs from ΩS^9 in dimension 24, and since $\Sigma u = 0$, where $u \in \pi_{22}(S^8)_{(3)}$, the identity map of S^8 extends to a map $S^8 \cup_u e^{23} \to \widehat{S}^8$. Sending the other spheres in the splitting of $(*)$ trivially yields the desired map g.

The adjoint of g is our desired 3-equivalence. To see that it is a 3-equivalence, we use the integral cohomology algebras. For both loop spaces, this is an exterior algebra on a class of degree 7 tensored with a divided polynomial algebra on a class of degree 22. The 7-class maps across by construction, and so does the 22-class since the EHP-sequence and [20, p. 168] imply that the 22-cell in both of $\Omega\widehat{S}^8$ and ΩW have the same nontrivial attaching map. The product structure then implies the desired isomorphism. □

Now we can compute $v_1^{-1}\pi_*(W;3)$ just proved.

Proposition 2.2. *Let*

$$M_i = \min(3, \nu(i-1)+1) \quad \text{and} \quad M_i' = \min(11, \nu(i-5)+1).$$

Then

$$v_1^{-1}\pi_{4i+\epsilon}(W;3) \approx \begin{cases} \mathbf{Z}/3^{M_i} & \text{if } \epsilon = 3, \\ \mathbf{Z}/3^{1+\max(M_i, M_i')} & \text{if } \epsilon = 2, \\ \mathbf{Z}/3^{M_i'} & \text{if } \epsilon = 1, \\ 0 & \text{if } \epsilon = 0. \end{cases}$$

Proof. We use the EHP fibrations

$$S^7 \to \Omega W \to \Omega S^{23} \tag{2.3}$$

and

$$\widehat{S}^8 \to \Omega S^9 \to \Omega S^{25} \tag{2.4}$$

established in [23]. We recall from [22] that

$$v_{2i}(S^{2n+1}) \approx v_{2i-1}(S^{2n+1}) \approx \begin{cases} \mathbf{Z}/3^{\min(n, \nu(i-n)+1)} & \text{if } i \equiv n \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

The exact sequence from (2.3) yields isomorphisms $v_{4i+3}(W) \approx v_{4i+2}(S^7)$, $v_{4i+1}(W) \approx v_{4i+1}(S^{23})$, $v_{4i}(W) = 0$, and an exact sequence

$$0 \to v_{4i+1}(S^7) \to v_{4i+2}(W) \to v_{4i+2}(S^{23}) \to 0. \tag{2.5}$$

Our task is to show that the extension in this exact sequence is cyclic.

Analysis of the exact sequence of (2.4) shows that if $v_{4i+3}(S^{25}) \approx \mathbf{Z}/3^{12}$, then $v_{4i+3}(S^9) \to v_{4i+3}(S^{25})$ must be injective and $v_{4i+3}(W) \to v_{4i+4}(S^9)$ is an isomorphism of $\mathbf{Z}/p$'s, and hence $v_{4i+4}(S^{25}) \to v_{4i+2}(W)$ is bijective, establishing the cyclicity in this case. A similar analysis shows that if $v_{4i+3}(S^9) \approx \mathbf{Z}/3^4$, then $v_{4i+2}(W)$ maps bijectively to it.

So now we restrict our attention to the remaining cases, $\nu(|v_{4i+3}(S^{25})|) < 12$ and $\nu(|v_{4i+3}(S^9)|) < 4$. Then (2.4) yields a short exact sequence

$$0 \to v_{4i+4}(S^{25}) \to v_{4i+2}(W) \to v_{4i+3}(S^9) \to 0. \tag{2.6}$$

Note that in both (2.5) and (2.6), at least one of the outer groups is $\mathbf{Z}/3$, and if $i \equiv 0 \bmod 3$, then both of the outer groups are $\mathbf{Z}/3$. Also note that, in the cases under consideration, $v_{4i+2}(S^{23})$ and $v_{4i+4}(S^{25})$ are abstractly isomorphic, as are $v_{4i+1}(S^7)$ and $v_{4i+3}(S^9)$. We will need the following lemma.

Lemma 2.7. *If* $\nu(|v_{4i+3}(S^9)|) < 4$, *then the composite*

$$v_{4i+1}(S^7) \to v_{4i+2}(W) \to v_{4i+3}(S^9)$$

is multiplication by 3. *If* $\nu(|v_{4i+3}(S^{25})|) < 12$, *then the composite*

$$v_{4i+4}(S^{25}) \to v_{4i+2}(W) \to v_{4i+2}(S^{23})$$

is multiplication by 3.

Proof. The first composite is just the double suspension homomorphism Σ^2. It follows from Thompson's proof [22] that if $j \equiv n \mod 2$, and $\nu(|v_{2j+1}(S^{2n+3})|) < n+1$, then $\Sigma^2 : v_{2j-\epsilon}(S^{2n+1}) \to v_{2j+2-\epsilon}(S^{2n+3})$ is multiplication by 3 if $\epsilon = 1$ and is bijective if $\epsilon = 0$. (Indeed, it is the homomorphism $J_{4i-1-\epsilon}(B^{4n}) \to J_{4i-1-\epsilon}(B^{4n+4})$ induced by the inclusion map for appropriate integer i. Here B^m is the m-skeleton of the 3-localization of $B\Sigma_\infty$. The homomorphism $\ell_{4i-1}(B^{4n}) \to \ell_{4i-1}(B^{4n+4})$ induced by the inclusion is $\mathbf{Z}/3^n \hookrightarrow \mathbf{Z}/3^{n+1}$. Then for $m = n$ or $n+1$

$$J_{4i-1}(B^{4m}) = \ker(3^a : \ell_{4i-1}(B^{4m}) \to \ell_{4i-1}(B^{4m}))$$

with $a \le n$, which implies the bijection when $\epsilon = 0$. Similarly,

$$J_{4i-2}(B^{4m}) = \operatorname{coker}(3^a : \ell_{4i-1}(B^{4m}) \to \ell_{4i-1}(B^{4m}))$$

with $a \le n$, which implies the $\cdot 3$ when $\epsilon = 1$.)

This description of Σ^2 immediately implies the first half of the lemma, while the second half follows from this together with the fact that the composition

$$v_{4i+2}(S^{23}) \xrightarrow{\Sigma^2} v_{4i+4}(S^{25}) \to v_{4i+2}(W) \to v_{4i+2}(S^{23})$$

is multiplication by 3, since, by [23], it is induced by the degree-3 map. □

The following lemma is easily proved by diagram chasing.

Lemma 2.8. *If the vertical and horizontal sequences are exact in the diagram below with* $e > 1$, *and* $\beta \circ \alpha = \cdot 3$, *then* $G \approx \mathbf{Z}/3^{e+1}$.

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \mathbf{Z}/3^e & & & \\
 & & & \downarrow{\scriptstyle \alpha} & & & \\
0 \to \mathbf{Z}/3 & \xrightarrow{i} & & G & & \xrightarrow{\beta} & \mathbf{Z}/3^e \to 0 \\
 & & & \downarrow & & & \\
 & & & \mathbf{Z}/3 & & & \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}$$

Proof. Let g denote a generator of the top $\mathbf{Z}/3^e$. There exists $x \in G$ such that $\beta(\alpha(g) - 3x) = 0$. Thus $\alpha(g) - 3x \in \operatorname{im}(i)$. Hence $3^e x = 3^{e-1}\alpha(g) \neq 0$. □

The cyclicity of $v_{4i+2}(W)$ when $\nu(|v_{4i+3}(S^{25})|) < 12$ and $\nu(|v_{4i+3}(S^9)|) < 4$ and $i \not\equiv 0 \mod 3$ (so that $\nu(|v_{4i+3}(S^{25})|)$ and $\nu(|v_{4i+3}(S^9)|)$ are not both equal to 1) is now immediate from (2.5), (2.6), Lemma 2.7, and Lemma 2.8.

Finally we consider the case $i \equiv 0 \mod 3$. Let $x \in v_{4i+2}(W)$ project to a nonzero element of $v_{4i+2}(S^{23})$, and assume $3x = 0$. Then the Toda bracket $\langle x, 3, \alpha_1 \rangle$ can be formed. Here α_1 is an element in the 3-stem detected by $\mathscr{P}^1$. The Toda bracket $\langle -, 3, \alpha_1 \rangle$ is essentially multiplication by v_1, which acts nontrivially from $v_{4i+2}(S^{23})$ to $v_{4i+6}(S^{23})$. We deduced in the previous paragraph the nontrivial extension in $v_{4i+6}(W)$, and so we have $\langle x, 3, \alpha_1 \rangle 3 \neq 0$. By [24, 1.4], this equals $x\langle 3, \alpha_1, 3 \rangle$. But $\langle 3, \alpha_1, 3 \rangle$ is in $\pi_{k+4}(S^k) = 0$. This contradiction says that our assumption $3x = 0$ must have been false, establishing the nontrivial extension in $v_{4i+2}(W)$ when $i \equiv 0 \mod 3$, our final case. □

Now that we know $v_*(W)$, we will show how it is used in the determination of $v_*(K)$. The following result, although presumably well-known, was pointed out to us by Mimura.

Proposition 2.9. *There is a fibration* $B(3, 7) \to K \to W$.

We remind the reader that all spaces are localized at 3. The space $B(3, 7)$, similarly to $B(11, 15)$ considered earlier, is an S^3-bundle over S^7 with $\mathscr{P}^1(x_3) = x_7$. The spaces $B(3, 7)$ and $Sp(2)$ are 3-equivalent.

Proof. We form the following diagram of fibrations, in which the central horizontal fibration is due to Borel ([11]).

$$\begin{array}{ccccc}
X & \longrightarrow & K & \xrightarrow{g\circ i} & W \\
\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle ||} \\
Sp(4) & \xrightarrow{f} & F_4 & \xrightarrow{g} & W \\
\downarrow{\scriptstyle j\circ f} & & \downarrow{\scriptstyle j} & & \\
B(11, 15) & \xrightarrow{=} & B(11, 15) & &
\end{array}$$

Here X is the fibre of $j \circ f$, and, by the diagram, is also the fibre of $g \circ i$. Cohomological considerations and elementary obstruction theory imply that the inclusion $Sp(2) \to Sp(4)$ lifts to a 3-equivalence $Sp(2) \to X$. □

We will prove (1.3) by studying the exact sequence in $v_*(-)$ associated to the fibration of Proposition 2.9. We write $B = B(3, 7)$ for the remainder of this section. We recall from [10, 1.3] some information about $v_*(B)$. First, $v_{4i+4}(B) \approx v_{4i+3}(B) = 0$. Thus $v_{4i+4}(K) = 0$, since it sits between 0-groups in the exact sequence of 2.9. Second, $v_{4i+2}(B)$ and $v_{4i+1}(B)$ are isomorphic cyclic groups of order 3^e, with $e \le 4$. There is an injection from $\mathbf{Z}/3 = v_{4i+2}(S^3)$ to $v_{4i+2}(B)$, which is bijective if $i \not\equiv 1 \mod 3$. There is a surjection $v_{4i+1}(B) \to v_{4i+1}(S^7)$, which is bijective if $i \not\equiv 10 \mod 27$.

Our next step is the following result.

Proposition 2.10. (i) *The composite* $S^7 \to \Omega W \to B \to S^7$ *has degree* 3. *Here the first map is the inclusion of the bottom cell, which is part of* (2.3), *the second map is part of the fibre sequence of Proposition* 2.9, *and the third map is the projection used to define* B.

(ii) $v_{4i+3}(K) = 0$.

Proof. Part (i) follows from the fact that $x_8 = \beta(x_7)$ in $H^*(K; \mathbf{Z}_3)$, and the correspondence between the Bockstein β and the degree-3 map. Here x_8 corresponds to the bottom cell of W, and x_7 to the 7-cell of B.

To prove part (ii), it will suffice to show that $v_{4i+3}(W) \to v_{4i+2}(B)$ is injective. Since $v_{4i+2}(S^7) \to v_{4i+3}(W)$ is iso, it suffices to show that the composite, c, of the first two morphisms in part (i) induces an injection in $v_{4i+2}(-)$. If $\nu(|v_{4i+2}(S^7)|) > 1$, then this injectivity follows since, as noted above, $v_{4i+2}(B)$ is cyclic and maps with kernel $\mathbf{Z}/3$ to $v_{4i+2}(S^7)$. (An element in $\ker(\cdot 3)$ on $v_{4i+2}(S^7)$ will factor nontrivially through $v_{4i+2}(B)$.)

If $\nu(|v_{4i+2}(S^7)|) = 1$, its generator is α_{i-1}. Since the 7-skeleton of B is $S^3 \cup_{\alpha_1} e^7$, the map g in the diagram represents an element of $v_{4i+2}(S^3)$ for which $i_*([g]) = c_*(\alpha_{i-1})$.

$$\begin{array}{ccccc} S^{4i+2} & \xrightarrow{\alpha_{i-1}} & S^7 & & \\ \downarrow g & & \downarrow c & \overset{3}{\searrow} & \\ S^3 & \xrightarrow{i} & S^3 \cup_{\alpha_1} e^7 & \longrightarrow & S^7 \end{array}$$

By [24, 1.8], $[g] \in \langle \alpha_1, 3, \alpha_{i-1} \rangle$. The indeterminacy here is 0, at least as far as v_1-periodic classes are concerned. Hence [24, 13.4] implies that the Toda bracket equals α_i, which is the nonzero element of $v_{4i+2}(S^3)$. Thus in this case, c_* is an isomorphism of $\mathbf{Z}/3$'s. □

We now continue our study of the exact sequence in $v_*(-)$ of the fibration of Proposition 2.9 to determine $v_{4i+2}(K)$ and $v_{4i+1}(K)$. We consider first the case $i \not\equiv 1 \mod 3$. In this case $v_{4i+\epsilon}(B) \approx v_{4i+\epsilon}(S^7) \approx \mathbf{Z}/3$ for $\epsilon = 1$ and 2. Since $v_{4i+3}(W) \to v_{4i+2}(B)$ is a bijection of groups of order 3, the exact sequence becomes

$$(2.11) \qquad 0 \to v_{4i+2}(K) \to \mathbf{Z}/3^{1+M_i'} \xrightarrow{\phi} \mathbf{Z}/3 \to v_{4i+1}(K) \to \mathbf{Z}/3^{M_i'} \to 0,$$

where $M_i' = \min(11, \nu(i-5)+1)$ as in Proposition 2.2, and the homomorphism ϕ is $v_{4i+2}(W) \to v_{4i+1}(B)$. The following key proposition will be proved later in this section, using the UNSS.

Proposition 2.12. *The homomorphism* ϕ *in* (2.11) *is nonzero unless* $i - 10 \equiv 3^5 \mod 3^{11}$, *in which case it is zero.*

The results for $v_{4i+2}(K)$ and $v_{4i+1}(K)$ are then immediate from (2.11), except for the extension in $v_{4i+1}(K)$ in the rare case that $\phi = 0$. Here we have

Proposition 2.13. *If* $i - 5 \equiv 3^{10} \mod 3^{11}$, *then* $v_{4i+1}(K) \approx \mathbf{Z}/3^{12}$.

Proof. The proof follows exactly the method of the proof of [10, 2.24]. We have

$$\circ\alpha_1 \neq 0 : v_{4i-2}(W) \to v_{4i+1}(W)$$

and

$$\partial \neq 0 : v_{4i-2}(W) \to v_{4i-3}(B).$$

Let G denote a generator of $v_{4i-2}(W)$, and let $Y \in v_{4i+1}(K)$ project to $G \circ \alpha_1$, which is the element of order 3 in $v_{4i+1}(W)$. Then

$$3Y = i_*(\langle \partial G, \alpha_1, 3 \rangle) = \partial(G) \cdot v_1 \neq 0,$$

where $i : B \to K$ is the map of 2.9. □

The next case is when $i \equiv 1 \mod 3$ and $i \not\equiv 10 \mod 27$, so that, by [10, 1.3(1)],

$$v_{4i+2}(S^7) \approx v_{4i+1}(S^7) \approx v_{4i+2}(B) \approx v_{4i+1}(B) \approx \mathbf{Z}/3^e$$

with $e = 2$ or 3. In this case, again $v_{4i+3}(W) \to v_{4i+2}(B)$ is bijective, and so the exact sequence becomes

$$(2.14) \qquad 0 \to v_{4i+2}(K) \to \mathbf{Z}/3^{e+1} \xrightarrow{\phi} \mathbf{Z}/3^e \to v_{4i+1}(K) \to \mathbf{Z}/3 \to 0.$$

Using part (i) of Proposition 2.10, the homomorphism ϕ fits into

$$\begin{array}{cccccc} v_{4i+1}(S^7) & \hookrightarrow v_{4i+2}(W) & \xrightarrow{\phi} & v_{4i+1}(B) & \xrightarrow{\approx} & v_{4i+1}(S^7) \\ \| & \| & & \| & & \| \\ \mathbf{Z}/3^e & \mathbf{Z}/3^{e+1} & & \mathbf{Z}/3^e & & \mathbf{Z}/3^e, \end{array}$$

with the composite being multiplication by 3. This implies that ϕ is surjective, and the claim of (1.3) that $v_{4i+2}(K) \approx v_{4i+1}(K) \approx \mathbf{Z}/3$ in this case is immediate from (2.14).

Finally, we consider the case $i \equiv 10 \mod 27$, so that $v_{4i+2}(B) \approx v_{4i+1}(B) \approx \mathbf{Z}/3^4$. In this case, the exact sequence of (2.9), starting with $v_{4i+3}(K) = 0$ and ending with $v_{4i}(B) = 0$, becomes

(2.15)

$$0 \to \mathbf{Z}/3^3 \hookrightarrow \mathbf{Z}/3^4 \to v_{4i+2}(K) \to \mathbf{Z}/3^4 \xrightarrow{\approx} \mathbf{Z}/3^4 \to v_{4i+1}(K) \to \mathbf{Z}/3 \to 0.$$

The $\xrightarrow{\approx}$ here has been deduced using part (i) of Proposition 2.10. Indeed, it is the only morphism ϕ that makes the following composite, which is $v_{4i+1}(-)$ of the composite of that proposition, equal to $\cdot 3$.

$$\mathbf{Z}/3^3 \hookrightarrow \mathbf{Z}/3^4 \xrightarrow{\phi} \mathbf{Z}/3^4 \xrightarrow{\text{epi}} \mathbf{Z}/3^3.$$

The exact sequence (2.15) immediately gives $v_{4i+2}(K) \approx v_{4i+1}(K) \approx \mathbf{Z}/3$ in this case, as claimed in (1.3).

It remains to prove Proposition 2.12. We will use the UNSS based on the Brown-Peterson spectrum BP. Most of our work is very similar to that of [10], although Lemma 2.16 is novel.

Recall that, localized at 3, $BP_* \approx \mathbf{Z}_{(3)}[v_1, v_2, \dots]$ with $|v_i| = 2(3^i - 1)$, and $BP_*BP \approx BP_*[h_1, h_2, \dots]$ with $|h_i| = 2(3^i - 1)$. Also $BP_*(S^n)$ is a free BP_*-module on a single generator ι_n. The UNSS converging to $\pi_*(S^{2n+1})_{(3)}$ has $E_2^{s,t}$ equal to the homology of the unstable cobar complex $C^*(S^{2n+1})$ in homological degree s and total degree t. This complex has $C^s(S^{2n+1}) \subset (BP_*BP)^{\otimes s} \otimes BP_*(S^{2n+1})$ spanned by monomials satisfying an unstable condition which we illustrate when $s = 2$. Then a monomial $v^{I_1}h^{J_1} \otimes v^{I_2}h^{J_2} \otimes v^{I_3}\iota_{2n+1}$ satisfies the unstable condition if $|J_1| \le \frac{1}{2}|v^{I_2}h^{J_2}v^{I_3}| + n$ and $|J_2| \le \frac{1}{2}|v^{I_3}| + n$. Here $h^J = h_1^{j_1}h_2^{j_2}\cdots$ satisfies $|J| = \sum j_i$. All tensor products are over BP_*, and

$$h^J \otimes v^I T = h^J v^I \otimes T = \eta_R(v^I)h^J \otimes T$$

with $\eta_R(v_1) = v_1 - 3h_1$. We also recall, from [2], that all v_1-periodic elements in $E_2^{s,t}(S^{2n+1})$ occur in $E_2^{1,4j+2n+1}(S^{2n+1})$ and $E_2^{2,4j+2n+1}(S^{2n+1})$, and from [7] that

$$E_2^{s,t}(\Omega S^{2n+1}) \approx E_2^{s,t+1}(S^{2n+1}).$$

We will be somewhat cavalier in using the symbol E_2 to refer either to the UNSS or the localized UNSS of [2] since a group in the latter SS can be thought of as a direct summand of a group of the former.

A major step in the proof of Proposition 2.12 is the following lemma, which we prove in Section 4.

Lemma 2.16. *Let* $\Omega W \xrightarrow{h} \Omega S^{23}$, $B = B(3, 7) \xrightarrow{p} S^7$, *and* $\phi : v_{4i+2}(W) \to v_{4i+1}(B)$ *be the maps and morphism considered earlier. Suppose* $y \in v_{4i+1}(\Omega W)$

has $h_*(y)$ represented by $A\otimes\iota_{23}\in E_2^{1,4i+3}(S^{23})$. Then $p_*(\phi(y))$ is represented by $A\otimes v_1h_1^3\otimes\iota_7\in E_2^{2,4i+3}(S^7)$ mod terms that desuspend to S^5.

Proof of Proposition 2.12. Let $m=i-5$. The generator of $v_{4i+2}(W)$ projects to the generator of $v_{4i+2}(S^{23})$, which is represented by $\alpha_{m/e}$ with $e=\min(11,\nu(m)+1)$. Here, as in [10], $\alpha_{m/e}$ denotes the generator defined by $d(v_1^m)/2^e$ with $d(v_1^m)=\eta_R(v_1)^m-v_1^m$. As noted in [10, p. 122], $\alpha_{m/e}\otimes h^Jv^I\iota_7$ is an unstable monomial if $e\le\frac{1}{2}|h^Jv^I|+3$. By Lemma 2.16, it suffices to prove

$$\text{If } e<11,\ \text{ then } \alpha_{m/e}\otimes v_1h_1^3\iota_7\neq 0\in E_2^2(S^7), \tag{2.17}$$

and

(2.18)

$$\text{If } \nu(m)\ge 10,\ \text{ then } \alpha_{m/11}\otimes v_1h_1^3\iota_7=0\in E_2^2(S^7) \text{ iff } m/3^{10}\equiv 1 \pmod 3).$$

The proof of (2.17) is just like the proof in the middle paragraph of [10, p. 125]. Let $x=\alpha_{m/e}\otimes v_1h_1^3\iota_7$. Since $e<\frac{1}{2}|v_1h_1^3|+3$, the image $H'(x)$ under the double suspension Hopf invariant H' can be evaluated by reading off the portion to the left of $h_1^3\iota_7$. This was proved in [1, 5.3] and restated in [10, 2.12(2)]. Thus

$$H'(x)=\alpha_{m/e}v_1=sv_1^mh_1\neq 0\in E_2^1(M),$$

where s is a unit in $\mathbf{Z}/3$, and M denotes the mod 3 Moore spectrum. The second "=" and the "$\neq$" are given in [10, 2.11]. Since $H'(x)\neq 0$, then $x\neq 0$.

The proof of (2.18) is similar to the proof which begins on the second half of [10, p. 125]. Let $s=m/3^{10}$. Working mod elements that desuspend to S^5, we have

$$\begin{aligned}-\alpha_{m/11}\otimes v_1h_1^3\iota_7&=\sum_{j=1}^{m}\binom{m}{j}3^{j-11}h_1^j\otimes v_1^{m-j+1}h_1^3\iota_7\\&\equiv sh_1\otimes v_1^mh_1^3\iota_7+3v_1^{m-12}h_1^{12}\otimes v_1h_1^3\iota_7.\end{aligned}\tag{2.19}$$

Here we have noted that all terms in the sum except the first and the last desuspend. Indeed, for $j>1$, $\binom{m}{j}3^{j-11}$ has a factor of 3, which is used to write $3h_1^3$ as $v_1h_1^2-h_1^2v_1$. For terms with large j, we write $3^{j-11}h_1^j$ as $3(v_1-\eta_R(v_1))^{j-12}h_1^{12}$, leaving the 3 in order to make $h_1^3\iota_7$ desuspend.

We evaluate H' of the first term of (2.19) by the method used earlier, utilizing [1, 5.3] and [10, 2.11], and obtain $sv_1^mh_1\in E_2^1(M)$. Since H' is injective when $E_2^2(S^{2n+1})=\mathbf{Z}/3$, and $v_1^mh_1\neq 0\in E_2^1(M)$, we will be done once we have shown that H' of the second term of (2.19) is $-v_1^mh_1$.

The second term of (2.19) is not in unstable form, but can be written as

$$v_1^{m-11}h_1^{11}\otimes v_1h_1^3\iota_7-v_1^{m-12}h_1^{11}\otimes v_1^2h_1^3\iota_7, \tag{2.20}$$

using $3h_1=v_1-\eta_R(v_1)$. The second term T of (2.20) satisfies $H'(T)=-v_1^mh_1$, using [10, 2.11(4)]. We proceed to show that the first term of (2.20) desuspends to S^5, which will complete the proof.

In [10, 2.6(iii)], the relation

$$v_2 = 3h_2 - 26h_1^3v_1 + \eta_R(v_2) - 4v_1^3h_1 + 9a_1v_1^2h_1^2 + 27a_2v_1h_1^3 \tag{2.21}$$

is derived, with $a_i \in \mathbf{Z}$. We multiply this on the right by $h_1^8 \otimes h_1^3\iota_7$, and analyze the terms obtained one by one.

- $v_2h_1^8 \otimes h_1^3\iota_7$: $H'(-) = v_2h_1^8 = 0$, by [10, 2.11(3)].
- $3h_1^8h_2 \otimes h_1^3\iota_7$: $H'(-) = 3h_1^8h_2 = 0$, since $3E_2^1(M) = 0$.
- $26h_1^{11}v_1 \otimes h_1^3\iota_7$: This is a unit times our desired term. After we have shown that all other terms desuspend, it will imply this one does, too.
- $h_1^8v_2 \otimes h_1^3\iota_7$: $H'(-) = h_1^8v_2 = 0$, by [10, 2.11(3)].
- $4v_1^3h_1^9 \otimes h_1^3\iota_7$: This equals $d(4v_1^3h_2^3\iota_7)$ in the unstable cobar complex, mod terms that desuspend. To see this, we note that this boundary is

$$4(\eta_R(v_1^3) - v_1^3) \otimes h_2^3\iota_7 + 4v_1^3\overline{\psi}(h_2^3)\iota_7, \tag{2.22}$$

where $\overline{\psi}(x) = \psi(x) - x \otimes 1 - 1 \otimes x$. By [10, 2.6(i)],

$$\psi(h_2) = h_2 \otimes 1 + 1 \otimes h_2 + h_1 \otimes h_1^2v_1 + h_1^2 \otimes h_1v_1 + h_1^3 \otimes h_1.$$

When (2.22) is expanded, 33 terms are obtained, but all of them except $4v_1^3h_1^9 \otimes h_1^3\iota_7$ desuspend to S^5, using $H'(-)$ to see this in many of the cases.
- $9a_1v_1^2h_1^{10} \otimes h_1^3\iota_7 + 27a_2v_1h_1^{11} \otimes h_1^3\iota_7$: Using $3h_1 = v_1 - \eta_R(v_1)$, both terms satisfy the condition of [10, 2.12(2)] so that H' of each equals the part to the left of the $\otimes$ and hence is 0 mod 3. □

3. v_1-PERIODIC HOMOTOPY GROUPS OF E_6/F_4

In this section we compute $v_1^{-1}\pi_*(E_6/F_4; 3)$, thus proving the third part of Theorem 1.2. We use the following proposition, which is presumably well-known, but apparently not well-documented.

Proposition 3.1. *There is a fibration* (*localized at* 3)

$$S^9 \to E_6/F_4 \to S^{17}$$

with attaching map α_2.

Proof. It is proved in [13] that there is a 3-equivalence

$$\Sigma W \cup e^{26} \to E_6/F_4.$$

The attaching map in the 3-local Cayley plane W is α_2, and hence the same is true in E_6/F_4. The fibration can then be established by the argument used for F_4/G_2 at the prime 2 in [16, 1.1]. Here it depends on the fact that the S-dual of the attaching map of the top cell to the middle cell of the manifold E_6/F_4 is in the image of J in the 8-stem, and this is 0 when localized at 3. Thus when the bottom cell of E_6/F_4 is collapsed, the top one can be collapsed as well, yielding the desired map into S^{17}. □

To prove (1.5), we will again use the UNSS and will follow closely the method of [10]. We let $Q = E_6/F_4$ and study the following exact sequence of E_2-terms of the v_1-periodic UNSS of [2].

$$\begin{aligned}(3.2)\qquad &0 \to E_2^{1,4j+1}(S^9) \to E_2^{1,4j+1}(Q) \to E_2^{1,4j+1}(S^{17})\\ &\xrightarrow{\phi} E_2^{2,4j+1}(S^9) \to E_2^{2,4j+1}(Q) \to E_2^{2,4j+1}(S^{17}) \to 0.\end{aligned}$$

From [22] and [2], we have for $s = 1$ or 2

$$v_{4j+1-s}(S^{8m+1}) \approx E_2^{s,4j+1}(S^{8m+1}) \approx \mathbf{Z}/3^{\min(4m,\nu(j-2m)+1)}.$$

Note that $v_{4j+1-s}(S^9) \approx \mathbf{Z}/3$ unless $j \equiv 2 \mod 3$, and $v_{4j+1-s}(S^{17}) \approx \mathbf{Z}/3$ unless $j \equiv 1 \mod 3$, and so ϕ always involves at least one $\mathbf{Z}/3$. Our desired result, (1.5), will follow from the following two results, together with the result of [2] that $v_{4j+1-s}(Q) \approx E_2^{s,4j+1}(Q)$ for $s = 1$ or 2.

Proposition 3.3. *The homomorphism* ϕ *in* (3.2) *is* 0 *unless* $E_2^{1,4j+1}(S^9) \approx \mathbf{Z}/3^4$ *or* $E_2^{1,4j+1}(S^{17}) \approx \mathbf{Z}/3^8$. *In each of these cases,* $\phi \neq 0$.

Proposition 3.4. *The group* $E_2^{1,4j+1}(Q)$ *is cyclic.*

Note that (1.5) makes no claim about the structure of the group $v_{4j-1}(Q)$, only its order.

Proof of Proposition 3.3. Let $m = j - 4$.

Case 1: $j \not\equiv 1 \mod 3$, so that $m \not\equiv 0 \mod 3$. The generator of $E_2^{1,4j+1}(S^{17})$ is $\alpha_m = \alpha_{m/1}$, and by Proposition 3.1 $\phi(\alpha_m) = \alpha_m \otimes \alpha_2\iota_9$. Here we have also used that the E_2-class α_2 detects the homotopy class α_2, and the prescription for boundary homomorphisms given in [4, 4.9]. By [6, p. 246], α_2 is represented by v_1h_1, and α_m is represented by $v_1^{m-1}h_1$ mod terms that desuspend to S^1. Thus $\alpha_m \otimes \alpha_2\iota_9$ desuspends to $y = v_1^{m-1}h_1 \otimes v_1h_1\iota_3$, and $H'(y) = v_1^{m-1}h_1v_1 \neq 0$, by [10, 2.11(1)]. Thus $\phi(\alpha_m)$ is the image of a generator under the iterated suspension

$$E_2^{2,4j+1}(S^3) \to E_2^{2,4j+1}(S^9).$$

By the main result of [3], restated as [10, 2.12a], this morphism is nonzero if and only if $E_2^{2,4j+1}(S^9) \approx \mathbf{Z}/3^4$.

Case 2: $j \equiv 1 \mod 3$. Suppose $E_2^{1,4j+1}(S^{17}) \approx \mathbf{Z}/3^e$. If $e < 8$, then $\phi(\alpha_{m/e}) = \alpha_{m/e} \otimes \alpha_2\iota_9$ desuspends, and so is 0, since, by [10, 2.12a], $E_2^{2,4j-1}(S^7) \to E_2^{2,4j+1}(S^9)$ is 0 in these cases where the groups have order 3.

Now suppose that $e = 8$. Let $s = m/3^7$. We argue as in the previous section, or [10, p. 125].

$$\partial(\text{gen}) = -\alpha_{m/8} \otimes \alpha_2\iota_9 = \sum_{k=1}^{m} \binom{m}{k} 3^{k-8} h_1^k v_1^{m-k} \otimes \alpha_2\iota_9.$$

All terms in the sum desuspend except the term with $k = m$. Note that the analysis here is somewhat easier than that of the previous section or [10, p. 125] because $\alpha_2\iota_9$ desuspends, and so we do not have to worry about keeping a factor of 3 to make it desuspend. To see that terms in the sum with large k desuspend, write $3^{k-8}h_1^k$ as $(v_1 - \eta_R(v_1))^{k-8}h_1^8$. This substitution is also used for the $k = m$ term, for which we also note that all but the first term in this expansion desuspend.

Thus $\partial(\text{gen})$ has reduced to $v_1^{m-8}h_1^8 \otimes \alpha_2\iota_9$. We omit writing v_1^{m-8} on the left, and use

$$\alpha_2 = \tfrac{1}{3}(\eta_R(v_1^2) - (3h_1 + \eta_R(v_1))^2)$$

to obtain the following form for the term being analyzed:

$$3h_1^8 \otimes h_1^2\iota_9 + 2h_1^8 \otimes h_1v_1\iota_9. \tag{3.5}$$

The first term of (3.5) desuspends, while, by [10, 2.13], the second term can be rewritten as $h_1^3 \otimes h_1^6v_1\iota_9$ mod terms that desuspend. (We are ignoring units in $\mathbf{Z}/3$ now.) If (2.21) is multiplied on the left by $h_1^3\otimes$ and on the right by $h_1^3\iota_9$, all terms desuspend except $h_1^3 \otimes h_1^6v_1\iota_9 + h_1^3 \otimes v_1^3h_1^4\iota_9$. Thus $\partial(\text{gen})$ has reduced to $z = h_1^3 \otimes v_1^3h_1^4\iota_9$. Here we have $H'(z) = h_1^3v_1^3 \neq 0$ by [10, 2.11]. □

Proof of Proposition 3.4. This proof is totally analogous to that of [10, 2.23]. Suppose $d(w\iota_9) = \alpha_m \otimes \alpha_2\iota_9$, and let $z = \alpha_m\iota_{17} - w\iota_9$. Then

$$3z = d(v_1^m\iota_{17}) + i_*((v_1^m\alpha_2 - 3w)\iota_9). \quad □$$

4. Proof of Lemma 2.16

In this section, we prove Lemma 2.16. The situation seems somewhat easier when $\nu(i-5) < 10$, and we begin by sketching a proof in that case, and then give a different, detailed proof that works for all values of i.

For either proof, it is useful to suspend everything once. The advantage of suspending is that, by [16, 3.1], $\Sigma\Omega W$ splits as $X \vee J$, where $X = S^8 \cup_u e^{23}$, and J is a wedge of spheres S^n with $n \geq 30$. Working with this 2-cell complex turns our question into one about an ordinary Toda bracket, provided $\nu(i-5) < 10$. The fact that we have suspended once is inconsequential since $\pi_*(S^7)\xrightarrow{E}\pi_{*+1}(S^8)$ is injective when localized at an odd prime. The inclusion map does not induce a surjection

$$v_{4i+1}(S^7 \cup_u e^{22}) \to v_{4i+1}(\Omega W)$$

if $\nu(i-5) \geq 10$, but it does after suspending once.

All we need to know about the attaching map u is that it is the suspension of a nontrivial element of $\pi_{21}(S^7) \approx \mathbf{Z}/3$, and this was established in [20, 7.4]. Let $s : \Sigma\Omega W \to X$ be a splitting map, and let $c : X \to S^{23}$ denote the collapse map. Let $h : \Omega W \to S^7$ denote the map of Propositon 2.10, and let $g : X \to S^8$ denote the restriction of Σh. Note that h induces the morphism $p_*\phi$ of Lemma 2.16, and that g is an extension of the degree-3 map of S^8. If $\nu(i-5) < 10$, then there is an element $y' \in v_{4i+1}(S^{22})$ such that $\Sigma y' = c_*(s_*(\Sigma y))$. Thus the element $g_*(s_*(\Sigma y)) \in v_{4i+2}(S^8)$ lies in the Toda bracket

$$\langle 3,\ u,\ y'\rangle \in v_{4i+2}(S^8). \tag{4.1}$$

We may write this Toda bracket as an element rather than a subset because of the following result. Toda brackets in v_1-periodic homotopy theory may be considered to be ordinary Toda brackets in the mapping telescope described in [15], whose ordinary homotopy groups equal the v_1-periodic homotopy groups of the space in question.

Proposition 4.2. *The indeterminacy of the Toda bracket* $\langle 3, u, y'\rangle \subset v_{4i+2}(S^8)$ *is* 0, *provided* $0 < \nu(i-5) < 10$.

Proof. The indeterminacy is $3v_{4i+2}(S^8) + v_{23}(S^8)\cdot y$. The first part is 0 because

$$v_{4i+2}(S^8) \approx v_{4i+1}(S^7) \oplus v_{4i+2}(S^{15}),$$

and both of these groups are $\mathbf{Z}/3$ since $i \equiv 2 \mod 3$. A generator of the second part of the indeterminacy is represented in the UNSS by the suspension of $\alpha_{m/e} \otimes v_1^3 h_1 \iota_7$ with $e < 11$. Such an element desuspends to S^5, and hence is 0 since we are in a situation ($\mathbf{Z}/3$'s on the 2-line) where the double suspension morphism is 0. □

The element u in the Toda bracket (4.1) is the suspension of a generator of $\pi_{21}(S^7)$. As explained in [1, p. 610(v)], the table in [1, p. 613] says that $\pi_{2i+15}(S^{2i+1})_{(3)}$ is $\mathbf{Z}/3$ for $1 \le i \le 3$ and 0 for $i > 3$, with

$$\Sigma^2 = 0 : \pi_{2i+13}(S^{2i-1}) \to \pi_{2i+15}(S^{2i+1}).$$

More explicitly, [3, 3.7] says that a generator of $\pi_{21}(S^7)$ is represented by $d(h_1^4)\iota_7$. Thus if $y' \in \pi_{4i+1}(S^{22})$ is represented by $A \otimes \iota_{22} \in E_2^{1,4i+2}(S^{22})$, then, by (4.1) and the UNSS analogue of [21, 1.2], our desired class is represented in the UNSS by the Massey product $\langle 3, d(h_1^4), A\rangle$. One can show, following Moss [21, 1.2], that the Massey product in the UNSS contains an element of the appropriate Toda bracket. Since the details are straightforward but tedious, and we have an alternate approach to the proof of Lemma 2.16, we prefer not to further complicate this paper with the details.

A cycle representative for this Massey product is given by $C = A \otimes 3h_1^4 - 3x$, where x is defined on S^7 and $d(x) = A \otimes d(h_1^4)$. Note that $x = A \otimes h_1^4$ does not work since it is not defined on S^7. There must be such an x since $A \otimes d(h_1^4)$ is defined on S^7, and all v_1-periodic cycles on the 3-line are boundaries. Note also that we write the cycle representation for the Massey product in order opposite to that of the Massey product. This first proof of Lemma 2.16 when $\nu(i-5) < 10$ is completed by noting that all terms in the cycle C desuspend to S^5 except $A \otimes v_1 h_1^3 \iota_7$. Here we have again used $3h_1^4 = v_1 h_1^3 - h_1^3 v_1$.

We remark again that the main problem with this argument when $\nu(i-5) \ge 10$ is that the element $y \in v_{4i+1}(\Omega W)$ does not factor through $v_{4i+1}(S^7 \cup_u e^{22})$ in this case, and so the bracket $\langle 3, u, y'\rangle$ cannot be formed. We now present the proof of Lemma 2.16 which works for any value of i. We begin with a preliminary result which should be useful in other applications of the UNSS.

Proposition 4.3. *There is a split short exact sequence of* E_2*-terms of the odd-primary UNSS*

$$0 \to E_2^{s,t-1}(S^{2n-1}) \to E_2^{s,t}(S^{2n}) \to E_2^{s-1,t-1}(S^{4n-1}) \to 0.$$

Proof. We argue as in the proof of [7, 7.1ii]. The reader should note that pages 389 and 390 of [7] are reversed, so that this proof occupies the bottom of page 388 and the top of page 390. The key observation is [7, 3.3], that the right derived functors of the primitives of $BP_*(S^{2n})$ have the property that $R^1P(BP_*(S^{2n}))$ is a free BP_*-module on a generator of degree $4n$, while

$R^sP(BP_*(S^{2n}))=0$ if $s>1$. The long exact sequence (with $M=BP_*(S^{2n})$)

$$\to \operatorname{Ext}_{\mathscr{U}}^{s,t}(PM)\to \operatorname{Ext}_G^{s,t}(M)\to \operatorname{Ext}_{\mathscr{U}}^{s-1,t}(R^1PM)\to$$

of [7, 5.2] reduces to the desired long exact sequence. It remains to construct a splitting morphism

$$E_2^{s-1,t-1}(S^{4n-1})\to E_2^{s,t}(S^{2n}).$$

The splitting morphism $s: E_2^{0,4n-1}(S^{4n-1})\to E_2^{1,4n}(S^{2n})$ sends ι_{4n-1} to $P(\iota_{4n+1})$, where P fits into the EHP sequence

$$E_2^{s,t}(S^{2n})\xrightarrow{E}E_2^{s,t+1}(S^{2n+1})\xrightarrow{H}E_2^{s,t+1}(S^{4n+1})\xrightarrow{P}E_2^{s+1,t}(S^{2n}),$$

which is derived similarly to [7, 7.1(i)]. This is a legitimate splitting morphism since one can show that the composite

$$E_2^{0,4n+1}(S^{4n+1})\xrightarrow{P}E_2^{1,4n}(S^{2n})\xrightarrow{H}E_2^{0,4n-1}(S^{4n-1})$$

is an isomorphism by chasing the diagram which involves the two relevant EHP sequences and the double suspension sequence of [7, 8.1].

As in [7, p. 385], $E_2(S^{2n})$ is the homology of the total complex $T^*(S^{2n})$ of the double complex $D^{*,*}(S^{2n})$, defined by

$$D^{p,q}(S^{2n})=U^qPG^{p+1}(BP_*(S^{2n})).$$

With s the section defined above, $s(\iota_{4n-1})$ is represented by a cycle w in $T^1(S^{2n})$. If $A\otimes\iota_{4n-1}$ is any cycle in a group $U^{p-1}(S^{4n-1})$ of the unstable cobar complex of S^{4n-1}, then $A\otimes w$ is a cycle in $T^p(S^{2n})$, and $H(\{A\otimes w\})=\{A\otimes\iota_{4n-1}\}$. This gives the desired splitting. □

By Proposition 4.3, we can view $E_2(S^8)$ as the homology of the complex $C^*(S^8)$, defined by

(4.4) $$C^p(S^8)_t=U^p(S^7)_{t-1}\oplus U^{p-1}(S^{15})_t,$$

where $U(-)$ is the unstable cobar complex, and there is no mixing of boundary morphisms. This is compatible with the odd-primary splitting $\Omega S^8\simeq S^7\times\Omega S^{15}$, but we could not use this splitting since we must deal with S^8 and not ΩS^8.

Now we let $X=S^8\cup_u S^{23}$ as above. Since the attaching map u has BP-filtration 2, a chain complex $C^*(X,S^8)$, whose homology is an E_2-term for the UNSS converging to $[X,S^8]_*$, can be defined by

(4.5) $$C^s(X,S^8)_t=C^s(S^8)_{t+23}\oplus C^{s-1}(S^8)_{t+7}.$$

This complex is similar to those considered in [5, §5]. Here $C^*(S^8)$ is the complex defined in (4.4). We will write an element of the right-hand side of (4.5) as

$$I_{-23}\otimes A\iota_8+J_{-7}\otimes B\iota_8,$$

where $|A|=t+15$ and $|B|=t-1$. The boundary ∂ in $C^*(X,S^8)$ satisfies

$$\partial(I_{-23}\otimes A\iota_8)=I_{-23}\otimes d(A)\iota_8,$$

$$\partial(J_{-7}\otimes B\iota_8)=I_{-23}\otimes B\otimes d(h_1^4)\iota_8+J_{-7}\otimes d(B)\iota_8,$$

where d is the boundary in the cobar complex. Note that J_{-7} has filtration 1, and contributes to homotopy degree $t-s=-8$. The $d(h_1^4)$ comes into play as the attaching map in X, as observed earlier this section.

We are going to use this complex to study the composite

$$S^{4i+2} \xrightarrow{y''} X \xrightarrow{g} S^8, \tag{4.6}$$

where y'' is the composite

$$S^{4i+2} \xrightarrow{\Sigma y} \Sigma\Omega W \xrightarrow{s} X.$$

We need to know that the map g is well-defined, which we show in the following lemma.

Lemma 4.7. *There is a unique map $g : X \to S^8$ whose restriction to S^8 has degree 3, and which is the suspension of a composite*

$$S^7 \cup_u e^{22} \to B(3, 7) \xrightarrow{p} S^7.$$

Proof. Since $B(3, 7)$ is the fiber of $S^7 \xrightarrow{\alpha_1} BS^3$, it suffices to show that $\pi_{22}(S^7) \xrightarrow{\alpha_{1*}} \pi_{21}(S^3)$ is injective. From the tables of [1], $\pi_{22}(S^7) \approx \mathbf{Z}/3$ with generator α_4. By [3], $\alpha_1\alpha_4$ is nonzero on S^3. □

The map $g : X \to S^8$ is represented in $C^*(X, S^8)$ by a cycle of the form $J_{-7} \otimes 3\iota_8 + I_{-23} \otimes B\iota_8$, for some B. Since $\partial(J_{-7} \otimes 3\iota_8) = I_{-23} \otimes 3d(h_1^4)\iota_8$, this B must equal $3h_1^4$ in order to create a cycle. We use the relation

$$3h_1^4 = v_1 h_1^3 - h_1^3 v_1$$

to write this cycle representing g as

$$J_{-7} \otimes 3\iota_8 + I_{-23} \otimes (v_1 h_1^3 - h_1^3 v_1)\iota_8. \tag{4.8}$$

The filtration-1 map $y'' : S^{4i+2} \to X$ induces a chain map

$$y''^* : C^*(X, S^8) \to C^{*+1}(S^{4i+2}, S^8).$$

Here we have written $C^s(S^{4i+2}, S^8)$ for $C^s(S^8)_{s+4i+2}$ to emphasize the naturality. Recall from the statement of Lemma 2.16 that the composite of y'' followed by the collapse map $X \to S^{23}$ is represented by $A\iota_{23}$. Thus y''^* sends the class (4.8) to $A \otimes (v_1 h_1^3 - h_1^3 v_1)\iota_8 + 3x\iota_8$, for some x such that $x\iota_8$ is defined. The terms $h_1^3 v_1 \iota_8$ and $3x\iota_8$ desuspend to S^5, establishing Lemma 2.16.

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