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FINITELY GENERATED KLEINIAN GROUPS IN 3-SPACE AND 3-MANIFOLDS OF INFINITE HOMOTOPY TYPE

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ABSTRACT. We prove the existence of a finitely generated Kleinian group $N \subset SO_{+}(1, 4)$ acting freely on an invariant component $\Omega \subset S^3$ without parabolic elements such that the fundamental group $\pi_1(\Omega/N)$ is not finitely generated.

Moreover, N is a finite index subgroup of a Kleinian group N_0 which has infinitely many conjugacy classes of elliptic elements.

1. INTRODUCTION

1.1. **Formulation of results.** We will consider finitely generated Kleinian (discontinuous) subgroups of the conformal group $M(n) \cong SO_{+}(1, n+1)$ acting on the n-dimensional sphere $S^n = R^n \cup \{\infty\}$. As usual, let $\Omega(G) \subset S^n$ denote the domain of discontinuity and $\Lambda(G) = S^n \setminus \Omega(G)$ denote the limit set of the group $G \subset M(n)$.

A Kleinian group $G \subset M(n)$ is called a function group if there exists a connected component $\Omega_G \subset \Omega(G)$ invariant under G. See [M2] for standard material on Kleinian group theory. In the present paper we prove the following:

Theorem A. *There exists a finitely generated function group $N \subset M(3)$, without parabolics, acting freely on an invariant component $\Omega_N \subset S^3$ such that the group $\pi_1(\Omega_N/N)$ is not finitely generated.*

To be hyperbolic in the sense of M. Gromov [G] is an important property of finitely generated groups. In particular, it follows from [Sc1] and [T2] that a finitely generated Kleinian group acting on $\mathbf{C}$ and containing no parabolic elements is Gromov hyperbolic. Let us notice that if $G \subset M(n)$ contains a parabolic subgroup of the rank greater than 1 then G is not Gromov hyperbolic [G, p. 108].

The following result shows that the situation in higher dimensions is quite different.

Theorem B. *The group N is a finite index subgroup of a group N_0 which contains infinitely many conjugacy classes of elements of finite order.*

Due to well-known properties of Gromov hyperbolic groups [G, p. 102] we have

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Corollary 1. *The groups N_0 and N are not Gromov hyperbolic.*

The following corollary immediately follows from the proof of Theorem B (see §2).

Corollary 2. *There exist a Gromov hyperbolic group $G^* \subset M(3)$, without torsion, containing N as a normal subgroup.*

1.2. **Historical remarks.** In [A], L. Ahlfors proved his famous finiteness theorem: that is a finitely generated nonelementary Kleinian group $G \subset M(2)$ has a factor-space $\Omega(G)/G$ consisting of a finite number of Riemann surfaces $S_1, \dots, S_n$ each having a finite hyperbolic area.

There are now topological approaches to finiteness statements for Kleinian groups acting on hyperbolic 3-space and its boundary. Let us point out some of them; P. Scott [Sc1] showed that each 3-manifold with a finitely generated fundamental group π_1, contains an embedded compact 3-manifold ("core") such that the embedding induces an isomorphism on the fundamental groups (as a corollary π_1 is finitely presented). By the well-known Selberg lemma [Se], each finitely generated Kleinian group $G \subset M(2)$ contains a torsion-free subgroup of finite index and hence, by Scott's "core" theorem, G is also finitely presented.

By using the "core" theorem, R. Kulkarni and P. Shalen obtained a weak version of Ahlfors' finiteness theorem [Ku-S]. D. McCullough and M. Feighn improved the last result in [Mc-F] by using a relative version of the "core" theorem, due to D. McCullough [Mc]. M. Feighn and G. Mess [F-M] showed that the number of conjugacy classes of finite subgroups of a finitely generated discontinuous group contained in $\mathrm{PSL}_2\,\mathbf{C}$ is finite.

It was proven by F. Bonahon [B] that the interior of a hyperbolic 3-manifold with indecomposable finitely generated fundamental group is homeomorphic to the interior of Scott's core.

In higher dimensions the situation is changed dramatically. We showed in [K-P] that even the weakest version of Ahlfors' finiteness theorem does not hold for Kleinian groups acting on S^3. Moreover, there exists a finitely generated Kleinian group which admits no finite presentation and, so, there is a hyperbolic 4-manifold without compact "core" [P].

M. Kapovich constructed [K] a finitely generated Kleinian group $\Gamma \subset M(3)$ which contains infinitely many Γ-conjugacy classes of maximal parabolic subgroups and elliptic elements. This result shows that Sullivan's finiteness theorem for cusps [Su2] (see also [Mc-F]) fails in higher dimensions.

The common point of all of these counterexamples to a version of the finiteness theorem in higher dimensions was the availability of parabolic elements. Moreover these constructions used accidental parabolics which do not correspond to cusp ends of the quotient of an invariant component [K-P].

An important problem arising here is to describe the class of finitely generated Kleinian groups in higher dimensions having a factor space of finite homotopy type. It was conjectured that all counterexamples to a version of the finiteness theorem in higher dimensions are connected either with the existence of parabolic elements or with algebraic structure of a Kleinian group (see discussion in [K]).

The construction of the group N given in Theorem A will be obtained in §3 and it uses new ideas compared with [K-P]. We give a sketch of the proof of Theorems A and B in §2.

Theorems A and B have been announced in [P].

As I learned after I finished this work, B. Bowditch and G. Mess constructed by a different method an interesting example of a finitely generated but infinitely presented Kleinian group in $M(3)$ without parabolic elements. Our Theorem A and the main result in [Bow-M] intersect but do not overlap.

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2. Outline of the proofs of Theorems A and B

We shall consider the family of Euclidean spheres $\Sigma_1, \dots, \Sigma_s, \dots, \Sigma_{2s}$ covering the square knot, K, which is a connected sum of trefoils along a sphere σ (see Figure 1). Each Σ_i is a result of a reflection τ_{i-1} in a plane Π_{i-1} of the previous sphere Σ_{i-1} and $\Sigma_{2s+1-i} = \tau_\sigma(\Sigma_i)$ where τ_σ is the reflection in σ $(i = 1, \dots, s)$. We suppose that a group $\Gamma_1 \subset \mathrm{PSL}_2(\mathbf{C})$ acts in the exterior of the sphere Σ_1 so that $\Lambda(\Gamma_1) = \Sigma_1$. The group Γ_1 will be commensurable to the reflection group determined by the faces of a right angled dodecahedron. The choice of Γ_1 is described more concretely in the next paragraph and is to be conditioned by the following:

(i) There is a normal finitely generated subgroup $F_1 \lhd \Gamma_1$ of infinite index.

(ii) There exists a subgroup $\Gamma_i' \subset \Gamma_i$ isomorphic to the fundamental group of a compact hyperbolic 3-manifold with geodesic boundary such that the group $G = \langle \Gamma_1', \Gamma_2', \dots, \Gamma_{2s}' \rangle$ is discontinuous. Here $\Gamma_i = \tau_{i-1}\Gamma_{i-1}\tau_{i-1}$ $(i = 2, \dots, 2s)$.

(iii) There is a normal, finitely generated, nontrivial subgroup $N \lhd G^*$ of infinite index in a group $G^* \subset G$ such that the index $|G : G^*| < \infty$.

We will construct an incompressible torus $T \subset R(G) \subset \Omega_G$ where $R(G)$ is a fundamental domain of the group G in a component Ω_G $(\infty \in \Omega_G)$. Due to the equality $\Omega(N) = \Omega(G) = \Omega(G^*)$ we have a diagram of coverings

$$\Omega_G \xrightarrow{p_1} M_N \xrightarrow{p_2} M_{G^*} \tag{1}$$

We will prove that G is geometrically finite and contains no parabolics and, hence, that the manifold M_{G^*} is compact.

Both coverings p_i are infinite; by the construction the group N is normal in G^*, it now easily follows that the group $\pi_1(M_N)$ is a normal subgroup of infinite index in $\pi_1(M_G^*)$, due to the fact that $p_{2_*}(\pi_1 M_N)$ contains the kernel of the natural epimorphism $\pi_1(M_{G^*}) \to G^*$. By [He, Theorem 11.1], either the group $\pi_1(M_N)$ is infinitely generated or is isomorphic to a surface group. But the second is impossible, since $\pi_1(M_N)$ is not virtually $\mathbf{Z} \oplus \mathbf{Z}$.

Informal description of the construction. [1] We will construct a 1-parameter family of convex cocompact groups $G(t)$ for $t \in [-\varepsilon, 0]$. There is a limiting group $G(0)$ which is geometrically finite but has a parabolic element, and the family

[1]This is an informal description of our construction; the reader who prefers exact explanation may skip this paragraph.

can be extended to give groups $G(t)$ for $t > 0$; for $t = \pi/2n$ (n is sufficiently large) these groups are discrete. Although the actual construction produces just one of these groups, it will help in following the construction to think of this group as the end of a process.

We start with quasifuchsian group G_s having the limit set $\Lambda(G_s)$ which is a 2-dimensional quasisphere embedded in S^3. The group G_s contains a fuchsian subgroup $\mathscr{H}_s$ leaving invariant the plane Π_s and all spheres containing the circle $\Lambda(\mathscr{H}_s)$—its limit set. We first consider the action of G_s on its interior invariant component Ω'_s not containing ∞. There is a family of spheres $P(\theta)$ ($\theta_0 \leq \theta \leq \pi/2$) between $\Pi_s = P(\theta_0)$ and $P(\pi/2)$ such that $P(\theta)$ makes an angle θ with ∂B_s, and all $P(\theta)$ contains $\Lambda(\mathscr{H}(G))$.

We choose θ to be close to $\pi/2$ and set $\Omega_s^- = \Omega \backslash G \cdot (\mathrm{int}(P(\theta)))$, which is a manifold with boundary $\partial\Omega_s^-$, $\Omega_s^- \backslash \partial\Omega_s^-$ is an open ball. A subgroup $G'_s \subset G_s$ which keeps Ω_s^- invariant has a fundamental domain $R_1 \subset \Omega_s^-$. Denote by R_2 the double of R_1 across $P(\theta)$. It is the fundamental domain for a group $G(-\varepsilon)$, which is a double $G'_s *_{\mathscr{H}_s} G'_s$ of the group G'_s along the subgroup $\mathscr{H}_s$ (the other copy of G'_s is the conjugate of G'_s by reflection in the plane $P(\theta)$).

Roughly speaking, both R_2 and the closed quasiball $\mathrm{cl}\, G(-\varepsilon) \cdot R_2$ look like a horseshoe with a trefoil tied in each arm, and bent so the arms are quite close. The construction ensures that ∂R_2 will be near to the limit set $\Lambda(G(-\varepsilon))$, which is quasisphere $\partial[\mathrm{cl}\, G(-\varepsilon) \cdot R_2]$.

There is also a fundamental domain R_3 for the action of $G(-\varepsilon)$ on the exterior of $\Lambda(G(-\varepsilon))$. The boundary of R_3 consists of isometric spheres, R_3 looks like a ball into which two wormholes have been dug. Each wormhole is knotted, but has a dead end. The tunnels fork slightly before their ends, so on ∂R_3 there are two pairs of faces: (F_{g_1}, F_{g_2}) and $(F_{g_1}^{-1}, F_{g_2}^{-1})$ where $F_{g_1} = I_{g_1(-\varepsilon)} \cap \partial R_3$ and $F_{g_2} = I_{g_2(-\varepsilon)} \cap \partial R_3$, and similarly for $F_{g_1}^{-1}$ and $F_{g_2}^{-1}$.

Given two spheres P, Q in S^3, we say that the conformal distance between P and Q is ε if ε is the distance between the totally geodesic hyperplanes which the spheres bound in $\mathscr{H}^4$ (if these are disjoint) or the negative of the angle between P and Q if the spheres meet.

Now let $R_2(t(\theta))$ be the double of R_1 across the sphere $P(t(\theta))$ for a function $t = t(\theta)$ to be defined. The domain $R_2(t(\theta))$ is fundamental inside a quasisphere $\Lambda(G(t))$, and there is a fundamental domain $R_3(t)$ for $G(t)$ in the exterior of $\Lambda(G(t))$, which has two faces $F_{g_1}(t)$, $F_{g_2}(t)$ and $F_{g_1^{-1}}(t)$, $F_{g_2^{-1}}(t)$ so that each pair has conformal distance $-t$. When t becomes zero the spheres $F_{g_1}(t)$ and $F_{g_2}(t)$ touch at a point p and the spheres through the faces $F_{g_1^{-1}}$, $F_{g_2^{-1}}$ touch at a point q. The points p and q belong to the limit set not to $R_3(0)$. When the faces of $R_3(0)$ are glued up, we obtain a noncompact 3-manifold. Note that $R_2(0)$ is a compact ball, and for sufficiently small t, $R_3(t)$ remains a compact ball. In $\mathrm{cl}(R_3(0))$, p has a neighbourhood $N(p)$ which is a solid of revolution: $N(p) \setminus \{p\} = S^1 \times V$ where $V \subset \mathbf{C}$ is the strip $y \geq 1$, $0 \leq x \leq c$ for some c. The domain $N(p) \setminus \{p\} \cup N(q) \setminus \{q\}$ projects to a cusp region $S^1 \times S^1 \times [0, \infty)$ conformally equivalent to a cusp end of a hyperbolic manifold. We can consider the manifolds $M(t)$ for $t < 0$ as the result of conformal Dehn surgery on the manifold $M(0)$; this conformal Dehn surgery generalizes Thurston's hyperbolic Dehn surgery. If we continue to deform the group, the pairs of faces meet at the angle t instead of having conformal distance $-t > 0$.

When $t = 2\pi/n$ and n is large enough, the group $G(t)$ is discrete. In the fundamental domain $R_3(t)$, the two knotted wormholes have met, so $R_3(t)$ is almost a "cube with a knotted hole"; in fact the boundary has genus 2, but if it is compressed we get a square knot exterior. When we have shown that the torus is incompressible, Theorem A will follow at the same time we will have constructed the example for Theorem B.

3. Preliminary results

Let us consider a piecewise linear curve K_1 representing the trefoil (Figure 1a). Let B_1 be a Euclidean ball centered on K_1 and planes Π_i are orthogonal to K_1. Denote $D_i = \Pi_i \cap B_1$ $(i = 1, 2)$.

We say that two disjoint Euclidean disks D_i are opposite if $D_i \subset B_1$ and the diameter d of B_1 which is the common perpendicular to D_i has $d \cap D_i$ as (Euclidean) center of D_i $(i = 1, 2)$. The following is purely geometrical.

Lemma 1. *Let us suppose that there exist three pairs (D_j, D'_j) of mutually disjoint opposite disks in B_1 such that the common perpendiculars d_j for each pair do not lie in a single 2-plane $(j = 1, 2, 3)$. Then there exists a covering*

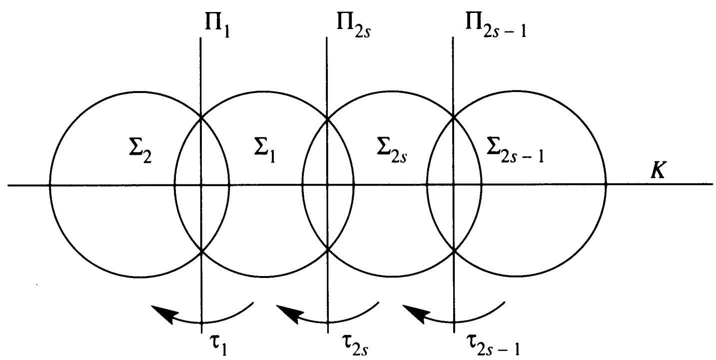

FIGURE 1a

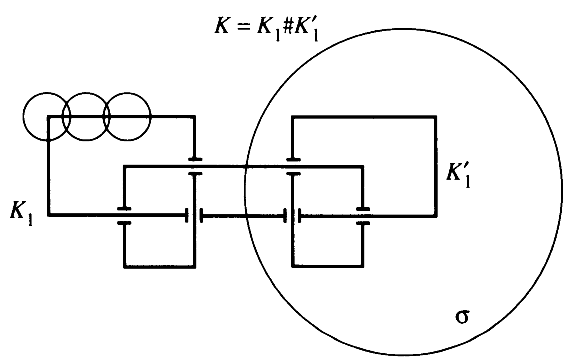

FIGURE 1b

$B = \{B_1, \dots, B_n\}$ of K_1 by balls, where each B_i is image under reflection τ_{i-1} in the plane Π_{i-1} of B_{i-1}. Here Π_1 and Π_0 are planes containing D_j or D'_j and $\Pi_i = \tau_{i-1}(\Pi_{i-2})$, $\Pi_n = \Pi_0$.

The proof is quite easy. Indeed the curve K_1 (Figure 1b) consists of four horizontal and six vertical intervals which lie on one plane containing vectors d_1 and d_2. There are also three fragments of K_1 having three linear sections. Each such a fragment belongs to the plane parallel either to the plane containing vectors (d_1, d_3) or to (d_2, d_3) depending on an over-crossing or under-crossing which we have on the diagram of K_1 (Figure 1b). We can use three directions, according to the initial diameters, to move along K_1 by using reflections in planes Π_i.

Choosing the length of each linear part of K_1 to be equal to a multiple of the distance between D_j and D'_j we get a system of equations in integers. One can check that there are many solutions and each of them gives parameters of possible coverings of K_1. The lemma is proved. Q.E.D.

The first example of a compact hyperbolic manifold fibered over S^1 is due to T. Jorgensen [J]; for our purposes we will use the following geometric construction which is due to W. Thurston.

We will consider a dodecahedron D all of whose dihedral angles are right angles. One can represent the dodecahedron D as a cube with six additional edges (Figure 2), and let $\Gamma \subset \mathrm{PSL}_2\,\mathbf{C}$ be the group generated by six isometries $\langle a_1, \dots, a_6\rangle$ identifying opposite faces of D. The quotient $\mathbf{H}^3/\Gamma$ is an orbifold whose underlying set is a 3-torus with three embedded circles (images of the six additional edges of D) as a branching locus of order two. This orbifold admits a fibration over a circle, so there is a subgroup $\Gamma_0 \in \Gamma$ of finite index such that the hyperbolic manifold $\mathbf{H}^3/\Gamma_0 = M_0$ is fiber bundle over the circle see e.g. [Sul, p. 196].

Let R be a reflection group determined by the faces of D. We have the following:

Lemma 2. *The groups Γ and R have a common finite-index subgroup R_0, $|\Gamma : R_0| < \infty$, $|R : R_0| < \infty$.*

Proof. Denote the generators of R by r_i, $i \in \{1, \dots, 12\}$. Now construct an epimorphism $\varphi_1 : R \to K = \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$ by mapping each of the four generators r_k of R corresponding to the top and bottom faces of D to the generator γ_1 of the first factor of K; map each of the four generators r'_j corresponding to

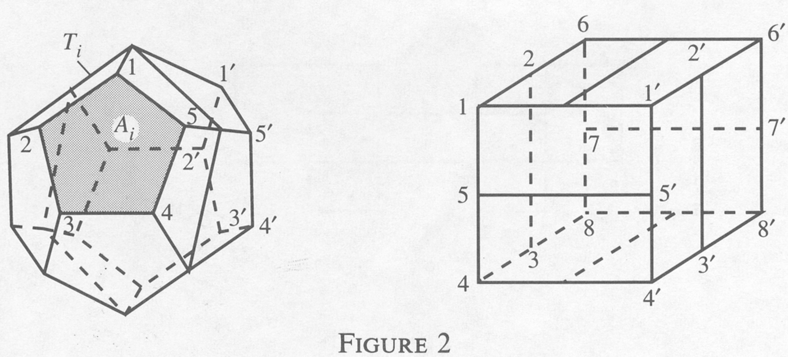

FIGURE 2

the front and back faces to the generator γ_2 of the second factor and map each of the four generators r_l'' corresponding to side faces to the generator γ_3 of the third factor. The group $R_0 = \operatorname{Ker}\varphi_1$ is generated by $r_i r_j$, $r_i' r_j'$ and $r_i'' r_j''$ $(i, j \in \{1, 2, 3, 4\})$. By analogy construct an epimorphism $\varphi_2 \colon \Gamma \to K$ so that the image of each pair of generators of Γ identifying opposite faces of the "cube" D is γ_i $(i = 1, 2, 3)$. By considering the identifications on D (see Figure 2) it is easy to see now that $\operatorname{Ker}\varphi_2 = \operatorname{Ker}\varphi_1 = R_0$, since all the generators of $\operatorname{Ker}\varphi_1$ belong to $\operatorname{Ker}\varphi_2$ and vice versa (e.g. $r_2'' r_1'' = a_6^{-1} a_5$, $r_3'' r_1'' = a_5^2$, $r_4'' r_2'' = a_6^2$, $r_4'' r_3'' = a_6 a_5^{-1}$ etc). The lemma is proved. Q.E.D.

Without loss of generality, we may assume that $\Gamma_0 = R_0$ and, moreover, that Γ_0 is normal in R; for otherwise we take the intersection over all conjugacy classes R/Γ_0.

Now we consider the action of R in both components B_1 and B_1^* of $S^3 \backslash \Sigma_1$, where $\Sigma_1 = \partial B_1$, $\infty \in B_1^*$.

Suppose that $D \subset B_1$ and the center of D coincides with the center of B_1.

Let us denote by W the collection of images of geodesic planes containing faces of D under elements of R.

A pair $(\rho, \rho') \subset W \times W$ will also be called opposite if the diameter of B_1 which is the common perpendicular connects the centers of ρ and ρ'.

We are able to consider three pairs of nonintersecting opposite geodesic planes $(\rho_i, \rho_i') \in W \times W$ and associate with each pair (ρ_i, ρ_i') the pair (χ_i, χ_i') of Euclidean planes such that

$$\chi_i \cap B_1 = \rho_i \cap B_1, \quad \chi_i' \cap B_1 = \rho_i' \cap B_1.$$

By Lemma 1 we can construct a covering of K_1 by a family of balls $\mathscr{B} = \{B_1, \dots, B_n\}$, where each B_i is an image of B_{i-1} under reflection τ_{i-1} in the plane Π_{i-1} (see Figure 1a). Here (Π_1, Π_n) is a pair of planes (χ_i', χ_i) mentioned above $i \in \{1, 2, 3\}$.

In what follows, we will denote by $\operatorname{Stab}(A, G)$ the stabilizer of a set A, i.e., $\operatorname{Stab}(A, G) = \{g \in G \colon gA = A\}$.

The groups $L_i = \operatorname{Stab}(\chi_i, \Gamma_0)$ (respectively $L_i' = \operatorname{Stab}(\chi_i', \Gamma_0)$) are Fuchsian groups of the first kind which are subgroups in a reflection group determined by sides of a right pentagon which is a face of D.

In the proof of the following lemma, we will use

Definition (see [Se2]). The group H is a separable subgroup of Γ if for each $\gamma \in \Gamma \backslash H$ there exists a subgroup $\Gamma^* \subset \Gamma$ of finite index such that $H \subset \Gamma^*$ and $\gamma \notin \Gamma^*$.

Notice that P. Scott proved in [Se2] that any geometrically finite subgroup H of a discrete group $\Gamma \in \mathrm{PSL}_2 C$ is separable if Γ is commensurable with the group R generated by reflections in the faces of a compact polyhedron $P \in \mathbf{H}^3$ whose dihedral angles are right angles.

Also we denote by I_γ an isometric sphere of an element $\gamma \in R$ and by $\mathscr{P}(\Gamma)$ an isometric fundamental domain of Γ which is the intersection of exteriors of all isometric spheres of Γ. Let χ_i^- be the half-space of $\overline{R}^3 \setminus \chi_i$ not containing the center of B_1.

We will use the notation $[\Gamma_0, \Gamma_0]$ for the commutator subgroup Γ_0.

Main Lemma. *There exists a subgroup* $\Gamma_1 \subset \Gamma_0$ *of finite index such that the following conditions hold*:

(a) *The boundary of the isometric fundamental domain* $\mathscr{P}(\Gamma_1)$ *lies in some regular* ε*-neighbourhood of* Σ_1 $(\varepsilon > 0)$.

(b) *There exists three pairs of opposite planes* $(\rho_i, \rho_i') \in W \times W$ *such that* $\partial\mathscr{P}(H_i)$ *and* $\partial\mathscr{P}(H_i')$ *are all disjoint. Here* $H_i = L_i \cap \Gamma_1$, $H_i' = L_i' \cap \Gamma_1$ $(i = 1, 2, 3)$.

(c) *There exists an element* $g_1 \in [\Gamma_0, \Gamma_0] \cap H_1' \subset \Gamma_1$ *and a plane* π *orthogonal to* K_1 *such that* $I_{g_1} \cap I_{g_1^{-1}} = \varnothing$, $I_{g_1} \cap \pi = l_1$, $I_{g_1^{-1}} \cap \pi = l_2$, $l_2 = g_1(l_1)$ *and* $\pi \cap I_\gamma = \pi \cap \Sigma_1 = \varnothing$ *for each* $\gamma \in \Gamma_1 \setminus \{g_1, g_1^{-1}\}$ (*see Figure* 3).

(d) *The group* $H = \langle H_1, H_1', \dots, H_3' \rangle$ *is a separable subgroup of* Γ_1 *and there exists a fundamental domain* $R(\Gamma_1) \subset B_1^*$ *such that* $R(\Gamma_1) \cap (\chi_i \cup \chi_i') = \mathscr{P}(H) \cap (\chi_i \cup \chi_i')$, $(i \in \{1, 2, 3\})$.

(e) *There exists a normal finitely generated subgroup* $F_1 \triangleleft \Gamma_1$ *such that* $\Gamma_1 / F_1 \cong \mathbf{Z}$.

Proof. (a)–(b) We start by fixing some ε-neighbourhood N_1 of Σ_1. There exists at most a finite number of elements $\gamma_i \in \Gamma_0$ $(i \in I$, I is a finite set) such that $I_{\gamma_i} \cap \mathrm{cl}(B_1^* \setminus N_1) \neq \varnothing$, where $\mathrm{cl}(\ )$ is a closure of a set.

Let us now consider nonintersecting opposite geodesic planes $(\rho_i, \rho_i') \in W \times W$. Evidently such pairs do exist because, by reflection in opposite faces of D, we can get infinitely many opposite elements of W.

Let $N(\rho_i)$ (respectively $N(\rho_i')$) be a regular neighbourhood of ρ_i (resp. ρ_i') in $\mathrm{cl}\, B_1$ such that $\bigcap_{i=1}^3 (N(\rho_i) \cap N(\rho_i')) = \varnothing$, (where $N(\rho_i) \cong \{z \in \mathbf{C} : |z| \leq 1\} \times [0, 1]$). Again there is at most a finite number of elements $\gamma_j \in L_i$ (resp. $\gamma_j \in L_i'$) for which $I_{\gamma_j} \notin N(\rho_i)$ (resp. $I_{\gamma_j} \notin N(\rho_i')$), where $j \in J$ and J is a finite set, $i \in \{1, 2, 3\}$.

By residual finiteness [Ma], we can choose a subgroup $\Gamma_1' \subset \Gamma_0$ of finite index such that $\gamma_l \notin \Gamma_1'$ $(l \in I \cup J)$.

(c) Let us consider a plane π' tangent to the sphere Σ_1 at the point $x_1 = \Sigma_1 \cap K_1$ and orthogonal to K_1 (Figure 3). We begin with the following:

Claim. There exists an element $g_1 \in [L_1^*, L_1^*]$ such that

$$I_{g_1} \cap \pi' \neq \varnothing, \quad L_1^* = \mathrm{Stab}(\rho_1', \Gamma_1'). \tag{2}$$

Indeed, we can suppose that $K_1 \cap B_1$ coincides with a diameter B_1 orthogonal to ρ_1'. Take any primitive element $g_0 \in [L_1^*, L_1^*] \subset [\Gamma_0, \Gamma_0]$ which can be represented by a simple dividing loop on the surface ρ_1'/L_1^*. Now if (2) does not hold for g_0 we can consider a sequence $g_n = \xi^n g_0 \xi^{-n}$ $(n \in \mathbf{Z})$ where $\xi = \tau_{\rho_1'} \tau_{\rho_1} \in R$ (here τ_{ρ_1} is a reflection in ρ_1 and ρ_1, ρ_1' are opposite faces).

Evidently, $g_n \in [\xi^n L_1^* \xi^{-n}, \xi^n L_1^* \xi^{-n}]$.

Now we claim that $\xi^n(I_{g_0}) \cap \pi' \neq \varnothing$ for all $n \geq N > 0$. The simplest way to see this is to conjugate our action to an action in the upper-half space by an element $b \in \mathrm{PSL}_2\mathbf{C}$ such that $b(K_1 \cap \pi') = \infty$; and so that $\gamma = b\xi b^{-1} = \lambda z$ $(\lambda \in \mathbf{R}, z \in \mathbf{C})$. The last is possible since ξ is a loxodromic element having two fixed points in $K_1 \cap \Sigma_1$. It is not clear that $\gamma^n(b(I_{g_0})) \cap b(\pi') \neq \varnothing$ for $n \geq N$. Returning to B_1 we get the desired property.

Moreover, $\xi^n(I_{g_0}) = I_{\xi^n g_0 \xi^{-n}}$, since ξ preserves each plane containing $B_1 \cap K_1$.

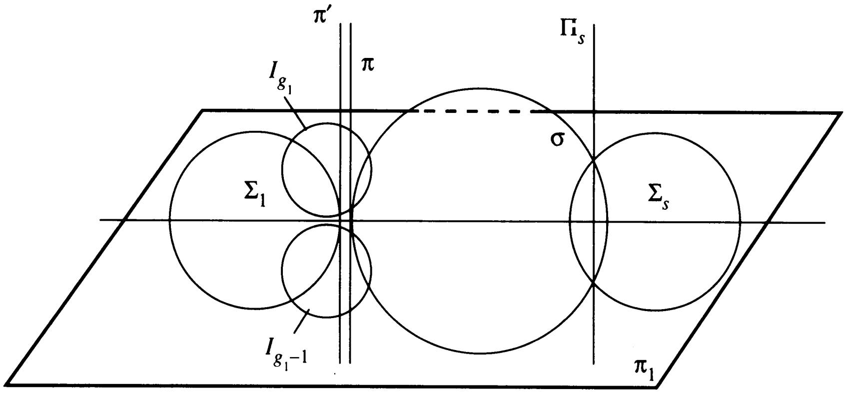

FIGURE 3

Simple assertion. If Γ_0 is normal in R then (obviously) $[\Gamma_0, \Gamma_0]$ is normal in R as well.

Clearly, we have that $g_0 \in [\Gamma_0, \Gamma_0]$ and so is $g_n \in [\Gamma_0, \Gamma_0]$. Planes $\xi^n(\rho_1')$, $\xi^{-n}(\rho_1)$ are opposite and assertions (a), (b) are valid for them. Therefore, keeping the notations L_1^*, ρ_1 and ρ_1' for $\xi^n L_1^* \xi^{-n}$, $\xi^{-n}(\rho_1)$ and $\xi^n(\rho_1')$ we can find a primitive element $g_1 \in [L_1^*, L_1^*]$ such that (2) is valid. The claim is proved.

Now let π be a plane parallel to π' such that $\pi \cap \Sigma_1 = \varnothing$, $\pi \cap I_{g_1} = l_1$ and $\pi \cap I_{(g_1)^n} = \varnothing$ $(n > 1)$ (Figure 3).

In what follows, we will denote by τ_S the reflection in a sphere S.

Now we claim that $\pi \cap I_{g_1^{-1}} = l_2$ and $l_2 = g_1(l_1)$. Indeed $g_1 = \tau_{\pi_1} \cdot \tau_{I_{g_1}}$ where π_1 is some plane containing $K_1 \cap B_1$ and orthogonal to π (Figure 3). We obtain $g_1(l_1) = g_1(I_{g_1} \cap \pi) = \tau_{\pi_1}(I_{g_1} \cap \pi) = I_{g_1^{-1}} \cap \pi = l_2$. Obviously $I_{g_1^{-m}} \cap \pi = \varnothing$, $m > 1$.

The element g_1 which was chosen to be primitive in the group L_1^* is also primitive in Γ_0, so g_1 generates a maximal cyclic subgroup of Γ_1'. As before, there is a finite set $M \subset \mathbf{Z}$ for which $I_{g_m} \cap \pi \neq \varnothing$, $m \in M$, $g_m \in \Gamma_1'$. According to separability of maximal infinite cyclic subgroups we can choose a subgroup $\Gamma_1'' \subset \Gamma_1'$ of finite index such that $\langle g_1 \rangle \subset \Gamma_1''$ and $g_m \notin \Gamma_1''$ $(m \in M)$ [L].

Denote $H_i = \operatorname{Stab}(\rho_i, \Gamma_1'')$; $H_i' = \operatorname{Stab}(\rho_i', \Gamma_1'')$. By construction $g_1 \in H_1' \cap [\Gamma_0, \Gamma_0] \subset \Gamma_1''$.

(d) By part (b) $H = \langle H_1, \dots, H_3' \rangle$ is a result of the Klein combination of H_i, H_i' and so is geometrically finite. The domain $\mathscr{P}(H) = \bigcap_{i=1}^3 \mathscr{P}(H_i) \cap \bigcap_1^3 \mathscr{P}(H_i')$ is fundamental for $H = H_1 * \cdots * H_3'$. Let us consider a compact domain $Z = \mathscr{P}(H) \cap B_1^* \cap (\bigcup_{i=1}^3 \chi_i \cup \bigcup_{i=1}^3 \chi_i')$. Again there exists at most a finite number of elements $\gamma_l \in \Gamma_1''$ such that $\gamma_l Z \cap Z = \varnothing$, $l \in L$.

The group H is a geometrically finite subgroup of Γ_1'' which is commensurable with a reflection group in a right angled polyhedron, so by the theorem of P. Scott [Sc2] mentioned above we can state that H is a separable subgroup of Γ_1''. So, there is a subgroup $\Gamma_1 \subset \Gamma_1''$ of finite index for which $H \subset \Gamma_1$ and $\gamma_l \notin \Gamma_1$ for all $l \in L$. Evidently, $g_1 \in H \subset \Gamma_1$ and the assertions of all previous steps are valid for Γ_1 as well.

Set $A = \bigcup_{i=1}^{3}(\chi_i \cup \chi_i')$; if $\gamma A \cap A \neq \varnothing$ for $\gamma \in \Gamma_1 \setminus H$ then $\gamma_0 h_1 Z \cap h_2 Z \neq \varnothing$, $h_i \in H$, which is impossible.

Moreover, due to compactness of surfaces χ_i/H_i and χ_i'/H_i', we can choose regular neighbourhoods V_i and V_i' of $\chi_i \cap B_i^*$ and $\chi_i' \cap B_1^*$ such that $H_i V_i = V_i$, $H_i' V_i' = V_i'$ and $\gamma V \cap V = \varnothing$ for $V = \bigcup_{i=1}^{3}(V_i \cup V_i')$, $\gamma \in (\Gamma_1 \setminus H)$.

The domain $W = \mathscr{P}(\Gamma_1) \cap (B_1^* \setminus \Gamma_1 V)$ is fundamental for the action of Γ_1 on $B_1^* \setminus \Gamma_1 V$ and evidently we get the required domain $R(\Gamma_1) = W \cup (\mathscr{P}(H) \cap \mathrm{cl}\, V)$.

(e) The group Γ_1 is a finite-index subgroup of Γ_0 and therefore the group $F_1 = F_0 \cap \Gamma_1$ is a finitely generated normal subgroup in Γ_1. We observe that $g_1 \in [\Gamma_0, \Gamma_0] \cap H_1' \subset F_0 \cap H_1' \subset F_1 \cap H_1'$, since $[\Gamma_0, \Gamma_0] \subset F_0$. The lemma is proved. Q.E.D.

Let us introduce some terminology and notation which we will use later.

Suppose Γ_i $(i = 1, 2)$ are Kleinian groups in $M(3)$, $U = \Gamma_1 \cap \Gamma_2$ is a common subgroup and there exists a compact surface $D \subset S^3$ such that $S^3 \setminus D = B_1 \cup B_2$, $D \cap \Omega(U) \subset \Omega_{\Gamma_1} \cap \Omega_{\Gamma_2}$.

We will say that D is strongly invariant under U in Γ_i iff $g(B_j) \subset B_i$, $gD \cap D = \varnothing$, $hD = D$, $g \in \Gamma_i \setminus U$, $h \in U$, $i \neq j$, $i, j \in \{1, 2\}$.

It follows from Maskit's Combination Theorem that, if $(\mathrm{int}\, D)/U$ is a compact surface in $\Omega_{\Gamma_i}/\Gamma_i$ and D is strongly invariant under U in Γ_i, then the group $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is discontinuous and isomorphic to $\Gamma = \Gamma_1 *_U \Gamma_2$ [M1, M2].

4. Proof of Theorems A and B

Recall that we have a trefoil represented by piecewise linear curve $K_1 \subset S^3$ and the covering $\mathscr{B}_1 = \{B_1, \dots, B_n\}$ of K_1 by balls centered on K_1. Let us denote by $\Lambda_i = B_i \cap B_{i+1} \cap \Pi_i$. Now we change this construction in the following way. According to the Main Lemma, we can find a sphere $\sigma \subset S^3$ centered on K_1 such that $\sigma \subset \pi^-$, $\sigma \cap I_{g_1} = l_1$ and $\sigma \cap \Sigma_s = \Lambda_s$ for some $s < n$ (see Figure 3), where π^- is the right half space of $S^3 \setminus \pi$.

Moreover (without loss of generality, due to the slight moving of σ) we can suppose that the angle between σ and I_{g_1} is equal to $\pi/(2n)$, $n \in \mathbf{Z}$. Simple geometric considerations show that σ can be chosen in such a way that $\sigma \cap \Sigma_1 = \varnothing$; namely this is true for sufficiently large radius σ and lengths of linear sections K_1.

By assertion (c) of the Main Lemma, it follows that $\sigma \cap I_{g_1^{-1}} = l_2 = g_1(l_1)$ and $\sigma \cap I_\gamma = \varnothing$, $\gamma \in \Gamma_1 \setminus \{g_1, g_1^{-1}\}$.

Let $K = K_1 \# \tau_\sigma(K_1)$ be a double knot, i.e. a connected sum of two trefoils along σ (Figure 1b). Consider the the family of balls $\mathscr{B} = \mathscr{B}_1' \cup \mathscr{B}_2'$ where $\mathscr{B}_1' = (B_1, \dots, B_s)$, $\mathscr{B}_2' = \tau_\sigma(\mathscr{B}_1')$.

Let $\Gamma_{k+1} = \tau_k \Gamma_k \tau_k$, $B_{k+1}^* = \tau_k(B_k^*)$ $(k = 1, \dots, s-1)$ and let $\partial\mathscr{B}_1$ denote the boundary $\partial(\bigcap_{k=1}^{s-1} \tau_k(B_k^*))$.

Consider the plane $\Pi_1 = \chi_1$ and group $\Gamma_1' = \mathrm{Stab}((B_1^* \setminus \Gamma_1 \Pi_1^-), \Gamma_1)$ where Π_1^- is a component of $B_1^* \setminus \Pi_1$ left invariant under $\mathrm{Stab}(\Pi_r, \Gamma_r)$.

Lemma 3. *There exists a subgroup $\Gamma_i' \subset \Gamma_i$ isomorphic to the fundamental group of a compact hyperbolic 3-manifold with geodesic boundary such that the following hold.*

1. *Group* $G_s = \langle \Gamma'_1, \dots, \Gamma'_s \rangle$ *is a discontinuous group acting freely and cocompactly on a component* $\Omega_{G_s} \subset S^3$ $(\infty \in \Omega_{G_s})$.

2. *The group* G_s *is a result of the Maskit combination of groups* Γ'_s *and* G'_{s-1} *along the sphere* $\Pi_s \cup \{\infty\}$ $(G'_1 = \Gamma'_1$ *and* G'_{s-1} *is to be defined inductively).*

3. *There exists a fundamental domain* $R(G_s)$ *such that* $\partial R(G_s)$ *is isotopic to* $\partial \mathscr{B}_1$.

4. *There exists a nontrivial finitely generated normal subgroup* $F_s \triangleleft G_s^*$ *in a subgroup* G_s^* *of* G_s *and* $|G_s : G_s^*| < \infty$.

Proof. Let us prove this lemma by induction on k. For $k = 1$ the assertion is true, namely we set $G_1 = \Gamma_1$ and all conclusions now follow from the Main Lemma.

Now we suppose that it is true for $k < r$, $2 \le r \le s$, i.e. the group $G_{r-1} = \langle \Gamma'_1, \dots, \Gamma'_{r-1} \rangle$ is discontinuous and has the fundamental domain $R(G_{r-1})$ such that $R(G_{r-1}) \cap \Pi_{r-1}$ is the fundamental domain for the action of $\mathscr{H}_{r-1} = \mathrm{Stab}(\Pi_{r-1}, \Gamma'_{r-1})$ on Π_{r-1}.

By construction the plane Π_{r-1} is the image of one of the planes χ_i, χ'_i under composition $\tau_{r-2} \cdots \tau_1$ and the group $\mathscr{H}_{r-1} = \mathrm{Stab}(\Pi_{r-1}, G_{r-1})$ is conjugate to one of H_i or H'_i $(i = 1, 2, 3)$ (see Main Lemma). Now we have to define groups G'_{r-1} and Γ'_r which take part in the process of combination.

By the remark above, there is a component Π^-_{r-1} of $B^*_{r-1} \setminus \Pi_{r-1}$ which is strongly invariant under $\mathscr{H}_{r-1}$ in G_{r-1}. Denote $\Pi^+_{r-1} = \tau_{r-1}(\Pi^-_{r-1})$. Consider the domains $\Omega^-_{G_{r-1}} = \Omega_{G_{r-1}} \setminus G_{r-1}(\Pi^-_{r-1})$ and $B^-_r = B^*_r \setminus \Gamma_r(\Pi^+_{r-1})$ and the groups $G'_{r-1} = \mathrm{Stab}(\Omega^-_{G_{r-1}}, G_{r-1})$, $\Gamma'_r = \mathrm{Stab}(B^-_r, \Gamma_r)$. Evidently, $\mathscr{H}_{r-1} = G_{r-1} \cap \Gamma'_r$.

Claim 1. All conditions of Maskit's Combination Theorem are satisfied for $\langle G'_{r-1}, \Gamma'_r \rangle = G_r$.

Proof of the claim. Consider $\delta_{r-1} = R(G_{r-1}) \cap \Pi_{r-1}$ which is the boundary of the fundamental domain $\mathscr{P}(\mathscr{H}_{r-1}) \cap \Pi_{r-1}$ for the action on Π_{r-1}. By construction $\mathrm{cl}(R(\Gamma_r))$ is homeomorphic to a ball, so δ_{r-1} bounds an embedded disk $V_{r-1} \subset \Pi^-_{r-1} \cap R(\Gamma_r)$. Let $U_{r-1} = \bigcup_{h \in \mathscr{H}_{r-1}} hV_{r-1}$, which is a surface strongly invariant under $\mathscr{H}_{r-1}$ in the groups G'_{r-1} and Γ'_r. Moreover $U_{r-1} \subset \Omega_{G'_{r-1}} \cap B^*_r$ and $U_{r-1}/\mathscr{H}_{r-1}$ is homeomorphic to a $\Pi_{r-1}/\mathscr{H}_{r-1}$.

Now we have to prove that $g(\mathrm{cl}(U_{r-1})) \cap \mathrm{cl}(U_{r-1}) = \varnothing$, $g \in (G'_{r-1} \cup \Gamma'_r) \setminus \mathscr{H}_{r-1}$. The last will follow from $g(\mathrm{cl}(\Pi^-_{r-1})) \cap \mathrm{cl}\, \Pi^-_{r-1} = \varnothing$. Suppose first that there exists $\gamma_0 \in G'_{r-1} \setminus \mathscr{H}_{r-1}$ such that $\gamma_0(\Pi^-_{r-1}) \cap \Pi^-_{r-1} \neq \varnothing$. Let us show that $\gamma_0 \in \Gamma'_{r-1}$, indeed there exist elements $h_i \in \Gamma'_{r-1}$ such that $h_1 \gamma_0 h_2(\mathrm{int}(R(G'_{r-1}))) \cap \mathrm{int}(R(G'_{r-1})) \neq \varnothing$. Hence, $\gamma_0 \in (\Gamma'_{r-1} \cap G'_{r-1}) = \mathscr{H}_{r-1}$ what is impossible. The same arguments work when $g \in \Gamma'_r \setminus \mathscr{H}_{r-1}$.

Suppose now that $\gamma_0(\mathrm{cl}(\Pi^-_{r-1})) \cap \mathrm{cl}(\Pi^-_{r-1}) = \{x_0\} \in \Lambda(\mathscr{H}_r)$ and consider the action of Γ'_{r-1} on a sphere Σ_{r-1}.[2]

Let w_1 be a component of $\Sigma_{r-1} \setminus \Lambda(\mathscr{H}_{r-1})$ such that $w_1 \subset \Pi^-_{r-1}$. We get that $w_1 \cap w_2 = \{x_0\} \in \Lambda(\mathscr{H}_{r-1})$ where $w_2 = \gamma_0(w_1)$.

Group $\mathscr{H}_{r-1}$ is Fuchsian, so each limit point of $\mathscr{H}_{r-1}$ is an approximation point [Be-M]. Now consider a model of the hyperbolic plane in $\Sigma_{r-1} \setminus \omega_1$. There exists a geodesic l with end points $\{x_0, y_0\} \in \Lambda(\mathscr{H}_{r-1})$ such that for

[2]These considerations concluding the proof of Claim 1 are suggested by M. Kapovich.

some sequence $\{h_n\} \in \mathscr{H}_{r-1}$ we have $h_n l \cap T \neq \varnothing$ for a compact subset $T \subset (\Sigma_{r-1} \setminus \omega_1)$. Evidently, $h_n(\omega_2) \cap h_{n_0}(\omega_2) \neq \varnothing$ $(n \geq n_0)$ because

$$\lim_{n\to\infty} h_n \omega_2 = \Sigma_1 \setminus w_1 .$$

We have deduced that $\gamma_0^{-1} h_{n_0}^{-1} h_n \gamma_0(\Pi_{r-1}^-) \cap \Pi_{r-1}^- \neq \varnothing$ which is impossible. The same arguments show that $\gamma(\mathrm{cl}(\Pi_{r-1}^+)) \cap \mathrm{cl}(\Pi_{r-1}^+) = \varnothing$, $\gamma \in \Gamma_r' \setminus \mathscr{H}_{r-1}$. Claim 1 is proved.

By Maskit's Combination Theorem we have

$$G_r = G_{r-1}' *_{\mathscr{H}_{r-1}} \Gamma_r' . \tag{3}$$

To prove assertion 3 of the lemma, we notice that the domain $R(\Gamma_r') = \tau_{r-1} \cdots \tau_1(R(\Gamma_1'))$ (here $R(\Gamma_1') = R(\Gamma_1) \cap B_1^-$) is fundamental for the action of Γ_r' on B_r^- which follows from part (d) of the Main Lemma. There exists a neighbourhood V_{r-1} of $\Pi_{r-1} \cap B_r^*$ such that $R(G_{r-1}) \cap V_{r-1} = R(\Gamma_r') \cap V_{r-1} = \mathscr{P}(\mathscr{H}_{r-1}) \cap V_{r-1}$ (see proof of part (d) of the Main Lemma). Evidently, domain $R(G_{r-1}') = R(G_{r-1}) \cap \Omega_{G_{r-1}}^-$ is fundamental for the action of G_{r-1}' on $\Omega_{G_{r-1}}^-$. Again using Maskit's Combination Theorem, we get that $R(G_r) = R(G_{r-1}') \cap R(\Gamma_r')$ is a fundamental domain for G_r. Obviously, $\delta_r = R(G_r) \cap \Pi_r$ bounds a fundamental domain for $\mathscr{H}_r$ on Π_r and $\partial R(G_r)$ is isotopic to $\partial(\bigcap_{k=1}^r \tau_k(B_k^*))$.

The assertion 4 of the present lemma will follow from the following.

Claim 2. The group G_r is isomorphic to a subgroup of $R \subset \mathrm{Isom}(\mathbf{H}^3)$ of finite index.

Proof. First of all we prove that there exists a monomorphism $i_r: G_r \to R$. Let us suppose that it is true for $k \leq (r-1)$. The group Γ_1 has a fundamental domain $R(\Gamma_1) \subset \mathbf{H}^3$ such that $R(\Gamma_1) \cap \Pi_1$ is a fundamental one for the action of $\mathscr{H}_1$ on Π_1^-. Remember that the first combination was along the plane Π_1 and that there exists a geodesic plane $\xi_1 = \rho_1' \in W$ for which $\Pi_1 \cap \Sigma_1 = \partial \xi_1$ where $H_1 = \mathrm{Stab}(\Pi_1, \Gamma_1') = \mathrm{Stab}(\xi_1, \Gamma_1')$ (see Main Lemma).

Consider group $\widetilde{G}_2 = \langle \Gamma_1', \widetilde{\Gamma}_2 \rangle$ where $\widetilde{\Gamma}_2 = \tau_{\xi_1} \Gamma_1' \tau_{\xi_1}$ and τ_{ξ_1} is a reflection in ξ_1. Obviously, there exists an isomorphism $i_2: G_2 \to \widetilde{G}_2$. In fact, G_2 is a result of a bending deformation of $\widetilde{G}_2$ along ξ_1 [T1].

Recall that τ_i means reflection in the plane Π_i. We will repeat our construction of G_3 by modelling it in $\mathbf{H}^3$. There is a subgroup $\widetilde{\mathscr{H}_2}$ isomorphic to $\mathscr{H}_2$ which stabilizes the plane $\xi_2 = \tau_{\xi_1}(\rho_1) \in W$. Let us consider groups $\widetilde{\Gamma}_3 = \tau_{\xi_2} i_2(\tau_2 \Gamma_3' \tau_2) \tau_{\xi_2}$ and $\widetilde{G}_3 = i_2(G_2') *_{\widetilde{\mathscr{H}_2}} \widetilde{\Gamma}_3$. Evidently we have an isomorphism $i_3: G_3 \to \widetilde{G}_3$.

Let us suppose that isomorphisms $i_{r-1}: G_{r-1} \to \widetilde{G}_{r-1}$ have been constructed $(r \leq s)$.

There exists a geodesic plane $\xi_{r-1} \subset W \subset B_1$ which is invariant under $\widetilde{\mathscr{H}_{r-1}} = \mathrm{Stab}(\xi_{r-1}, \widetilde{G}_{r-1})$. Of course, by induction $\xi_{r-1} = \tau_{\xi_{r-2}}(\xi_{r-3})$.

Let $\widetilde{\Gamma}_r = \tau_{\xi_{r-1}}(i_{r-1}(\tau_{r-1}\Gamma_r' \tau_{r-1}))\tau_{\xi_{r-1}}$ and $\widetilde{G}_r = \langle i_{r-1}(G_{r-1}'), \widetilde{\Gamma}_r \rangle$.

Take a fundamental domain $R(\widetilde{G}_{r-1})$ such that $R(\widetilde{G}_{r-1}) \cap \xi_{r-1}$ is a fundamental domain for $\widetilde{\mathscr{H}_{r-1}}$. Planes in W are all disjoint, so by using the Maskit Combination Theorem we get that $\widetilde{G}_r = i_{r-1}(G_{r-1}') *_{\widetilde{\mathscr{H}}_{r-1}} \widetilde{\Gamma}_r$. An isomorphism $i_r: G_r \to \widetilde{G}_r$ have been constructed.

Now it remains to prove that Γ_0 and $\widetilde{G}_r$ are commensurable. Let us notice that $\mathbf{H}^3/\widetilde{G}_r$ is compact since $R(\widetilde{G}_r)$ is a compact subset of $\mathbf{H}^3 = B_1$. By the construction index $|R : \Gamma_0| < \infty$ and we have that the index $|\widetilde{G}_r : (\Gamma_0 \cap \widetilde{G}_r)|$ is also finite, hence the manifold $\mathbf{H}^3/(\Gamma_0 \cap \widetilde{G}_r)$ is compact. Now it follows that the covering $\mathbf{H}^3/(\Gamma_0 \cap \widetilde{G}_r) \to \mathbf{H}^3/\Gamma_0$ is finite and the groups $\widetilde{G}_r$ and Γ_0 are commensurable.

There is a finitely generated normal subgroup $F_0 \triangleleft \Gamma_0$ and therefore $i_s^{-1}(F_0 \cap \widetilde{G}_s) = F_s$ is a normal finitely generated subgroup of $i_s^{-1}(\Gamma_0 \cap \widetilde{G}_s) = G_s^*$, which is a finite index subgroup of G_s. Lemma 3 is proved. Q.E.D.

We shall denote by $\operatorname{int}(\sigma)$ the bounded component $S^3 \setminus \sigma$. Recall that we have shown that group G_s is discontinuous and G_s is the result of Maskit's combination of groups acting in hyperbolic 3-space. In fact, one can equivalently think of G_s as the result of bending deformation in initial group $\Gamma_1 \subset \operatorname{Iso}(\mathbf{H}^3)$ along the family of embedded totally geodesic planes constructed in the Main Lemma [T1].

Now we wish to emphasize that the domain of discontinuity $\Omega(G_s)$ consists of two invariant components Ω_{G_s} and Ω'_{G_s} $(\infty \in \Omega_{G_s})$. To see that, one can use Tukia's theorem [Tu] stating that the limit set $\Lambda(G_s)$ is homeomorphic to the limit set of the group $\widetilde{G}_s$ (which is a sphere S^2 embedded in S^3) isomorphic to G_s and which was constructed in the Claim 2 of Lemma 3. Thus, $\Lambda(G_s)$ divides S^3 into two invariant components Ω_{G_s} and Ω'_{G_s}.

Remark. Although we will not use this fact later one can also prove that each invariant component of G_s is simply connected. For example, let us show this for the component Ω_{G_s}. We see that the groups $\pi_1(M_{G_s} = \Omega_{G_s}/G_s)$ and G_s are isomorphic (one needs to express the first group as a free product with amalgamation by Van Kampen theorem and the analogous expression can be obtained for the second group by Maskit's combination theorem). Further, we have an epimorphism $\tau : \pi_1(M_{G_s}) \to G_s$ induced by the covering projection and since G_s is Hopfian (G_s is residually finite as a subgroup of $M(3)$), it follows that Ω_{G_s} is simply connected. The proof is analogous for the component Ω'_{G_s}.

Consider now a fundamental domain $R'(G_s)$ for the action of G_s on the component Ω'_{G_s}, constructed in exactly the same way as we did for $R(G_s)$.

By our construction (see the beginning of §4) we have that $\sigma \cap R(G_s) = (\mathscr{P}(H_s) \cap \sigma \cap R(G_s)) \setminus (I_{g_1} \cup I_{g_1^{-1}})$ and $\sigma \cap R'(G_s) = \mathscr{P}(H_s) \cap \sigma \cap R'(G_s)$.

We shall identify $\mathbf{H}^4$ with the 4-ball $\mathbf{B}^4$ endowed with the Poincaré metric and we identify the boundary $\partial \mathbf{H}^4$ with the 3-sphere S^3 where all our groups act.

Now construct a 4-dimensional fundamental polyhedron $\mathscr{R}(G_s) \subset \mathbf{H}^4 \cup \Omega(G_s)$ by extending each sphere whose part is a face of $R(G_s)$ to a geodesic hypersurface in $\mathbf{H}^4$ orthogonal to $\partial \mathbf{H}^4$. Our polyhedron $\mathscr{R}(G_s)$ consists of points lying in the closure of the exterior of every one of these geodesic hypersurfaces. we see that $\mathscr{R}(G_s) \cap \Omega(G_s)$ has two components $R(G_s)$ and $R'(G_s)$ which form the fundamental domain for the action of the group G_s in S^3 (this gives an independent way of getting $R'(G_s)$).

We still have $\mathscr{R}(G_s) \cap \tilde{\sigma} = (\widetilde{\mathscr{P}}(H_s) \cap \tilde{\sigma}) \setminus (\widetilde{I}_{g_1} \cap \widetilde{I}_{g_1^{-1}})$ (here we use the notation $\widetilde{A}$ to denote the geodesic extension of a set A to $\mathbf{H}^4$). Indeed, if it were not so, then there would exist a 3-face $\tilde{f} \in \partial\mathscr{R}(G_s)$ such that $\tilde{f} \cap \tilde{\sigma} \neq \varnothing$ and $\tilde{f}$ does not belong to $\widetilde{I}_{g_1} \cup \widetilde{I}_{g_1^{-1}} \cup \partial\widetilde{\mathscr{P}}(H_s)$. Then projecting on $\partial\mathbf{H}^4$ we obtain $f \cap \sigma \neq \varnothing$ and $f \notin (I_{g_1} \cup I_{g_1^{-1}} \cup \partial\mathscr{P}(H_s))$ which is impossible by Main Lemma.

Now we are going to construct a bigger group G which will be our final group (modulo a finite-index subgroup). Set $\mathscr{R}^-(G_s)$ to be the closure of the set $\mathscr{R}(G_s) \cap \operatorname{ext}(\tilde{\sigma})$ where $\operatorname{ext}(\tilde{\sigma})$ is a half-space $\mathbf{H}^4 \setminus \tilde{\sigma}$ containing the origin. Now consider the polyhedron

$$(*) \qquad \mathscr{R} = \mathscr{R}^-(G_s) \cap \tau_{\tilde{\sigma}}(\mathscr{R}^-(G_s)),$$

where $\tau_{\tilde{\sigma}}$ is the reflection in $\tilde{\sigma}$.

We claim that faces of $\mathscr{R}$ are paired by transformations of the group $\operatorname{Iso}(\mathbf{H}^4)$. Recall that $\tilde{\sigma} \cap \widetilde{\mathscr{P}}(H_s)$ is a fundamental domain for the action of H_s on $\tilde{\sigma}$. Since $\tilde{\sigma}$ contains the limit set of H_s all isometric spheres bounding $\widetilde{\mathscr{P}}(H_s)$ are orthogonal to $\tilde{\sigma}$. Hence, they are invariant under the reflection $\tau_{\tilde{\sigma}}$. From the formula $\partial\mathscr{R} \cap \tilde{\sigma} = \tilde{\sigma} \cap (\partial\widetilde{\mathscr{P}}(H_s) \cup \widetilde{I}_{g_1} \cup \widetilde{I}_{g_1^{-1}})$, it follows that the only edges of $\mathscr{R}$ lying on $\tilde{\sigma}$ are the edges $\tilde{l}_1 = \mathscr{R} \cap \widetilde{I}_{g_1}$, $\tilde{l}_2 = \mathscr{R} \cap \widetilde{I}_{g_1^{-1}}$. There is a pairing of faces of $\mathscr{R}^-(G_s)$ by a system $\langle g_1, \gamma_1, \dots, \gamma_l\rangle$ of elements generating G_s which comes from the pairing of faces of the fundamental polyhedron $\mathscr{R}(G_s)$ [M2, p. 69]. Let us now consider the set of elements $S = \langle g_1, \gamma_1, \dots, \gamma_l, g_2, \gamma'_1, \dots, \gamma'_l\rangle$, where $\gamma'_i = \gamma_i^{\tau_\sigma}$, $g_2 = g_1^{\tau_\sigma}$. The above means that S forms a complete collection of face-pairing transformations of the polyhedron $\mathscr{R}$. Indeed, all sides of $\partial\mathscr{R}$ are either those of $\partial\mathscr{R}^-(G_s)$ or belong to $\tau_{\tilde{\sigma}}(\partial\mathscr{R}^-(G_s))$ and, so are paired by elements of S.

Let G be a group generated by the system S. Our goal is to prove that $\mathscr{R}$ is a fundamental polyhedron for G and hence G is discontinuous. By Poincaré's Polyhedron Theorem we have only to check that the sum of angles of each cycle of edges is $2\pi/q$ where q is the order of the generator of the stabilizer of the edge [M2, p. 69].

An arbitrary cycle in $\partial\mathscr{R}$ either consists of edges in $\partial(\mathscr{R}^-(G_s))$ and $\partial(\tau_{\tilde{\sigma}}(\mathscr{R}^-(G_s)))$ or is a cycle consisting of faces $\tilde{\zeta}_i$, $\tilde{\zeta}'_i$ lying on isometry spheres $\widetilde{I}_{g_i}$, $\widetilde{I}_{g_i^{-1}}$ $(i = 1, 2)$. If now $C = (e_1, \dots, e_t)$ is a cycle of the first type then the sum of the angles of C satisfies the condition above, since C is also a cycle of edges of one of the fundamental polyhedrons $\mathscr{R}(G_s)$ or $\tau_{\tilde{\sigma}}(\mathscr{R}(G_s))$. It remains to verify this condition for the cycle of the second type.

Claim. All conditions of Poincaré's Polyhedron Theorem are valid at the edges $\tilde{l}_i = \tilde{\zeta}_i \cap \tilde{\zeta}'_i$ $(i = 1, 2)$.

Proof. We will check that $\tilde{l}_1$ (respectively $\tilde{l}_2$) is the fixed axis of elliptic element $g = g_1^{-1}g_2$ (resp. $g_2g_1^{-1}$) of order n. Obviously one can check this by considering the action of g in S^3.

Let us first prove that $g|_{l_1} = g_1^{-1}g_2|_{l_1} = \mathrm{id}$. Indeed we have that $g_1 = i_{\pi_1} \cdot i_{I_{g_1}}$, where i_S denotes reflection in a sphere S (see Figure 3).

Using the fact that σ meets π_1 at right angles, a calculation gives us $g = (i_{I_{g_1}} \cdot \tau_\sigma)^2$ and $g|_{l_1} = \mathrm{id}$ because both $\tau_\sigma|_\sigma = \mathrm{id}$ and $i_{I_{g_1}}|_{l_1} = \mathrm{id}$.

Also, we remember that the angle $\alpha(\zeta_1, \zeta_2) = \alpha(\tilde{\zeta}_1, \tilde{\zeta}_2) = \alpha(\zeta'_1, \zeta'_2) = \pi/n$ (see the beginning of §4), hence $(i_{I_{g_1}}\tau_\sigma)^{2n} = 1$, so $g^n = 1$. The claim is proved. Q.E.D.

We have now proved the following.

Lemma 4. *The polyhedron $\mathscr{R}$ is a fundamental polyhedron for the group G acting in $\mathbf{H}^4$.*

Corollary 4.1. *The group G acts discontinuously in S^3 and has a fundamental domain which consists of two connected components $R(G)$ and $R'(G)$. Generators of G identify faces of $R(G)$ with faces of $R(G)$ and faces of $R'(G)$ with faces of $R'(G)$.*

Proof. From the construction of $\mathscr{R}$ it is easy to see that $\mathscr{R} \cap \Omega(G)$ consists of two components $R(G) = R(G_s) \cap \tau_\sigma(R(G_s))$ and $R'(G) = R'(G_s) \cap \tau_\sigma(R'(G_s))$ which form complete fundamental domain for the action of G in $S^3 = \partial\mathbf{H}^4$. From these formulas and the fact that the family S constructed in Lemma 4 generates G it follows that faces of $R(G)$ are matched by generators of G and that the same is true for $R'(G)$. The corollary is proved. Q.E.D.

Let $\langle\langle g\rangle\rangle_G$ be the normal closure of the infinite cycle group $\langle g\rangle$ in G.

Corollary 4.2. *The group G is isomorphic to the group $X/(\langle\langle g^n\rangle\rangle_X)$, where $X = G'_s *_{\mathscr{H}_s} (\tau_\sigma G'_s \tau_\sigma)$.*

Proof. Each relation in the group G now follows from an edge relation in the polyhedron $\mathscr{R} \subset \mathbf{H}^4$. Edges in $\partial\mathscr{R} \setminus \{\tilde{l}_1, \tilde{l}_2\}$ give us a system of generating relations of the group X. Additional relations come from edges $\tilde{l}_i$. These are just relations of the form $g^n = 1$ and its consequences. The corollary is proved. Q.E.D.

Below we will use notations from Claim 2 of Lemma 3.

Lemma 5. *The group G contains a subgroup G_0 of finite index such that there is a normal nonelementary finitely generated subgroup $N_0 \triangleleft G_0$ with infinite factor group G_0/N_0.*

Proof. As we have shown in Lemma 3, the group G_s is isomorphic to a subgroup $\widetilde{G}_s \subset R \subset \mathrm{Isom}(\mathbf{H}^3)$. By the same method we now prove that there is an isomorphism $i\colon X \to \widetilde{X} \subset R \subset \mathrm{Isom}(\mathbf{H}^3)$ (for the definition of X see Corollary 4.2 above). Indeed, there is a subgroup $\widetilde{\mathscr{H}_s} \subset R$ stabilizing a plane $\xi_s \subset B_1$ and hence $\widetilde{X} = i_s(\widetilde{G}'_s) *_{\widetilde{\mathscr{H}_s}} \tau_{\xi_s}(i_s(\widetilde{G}'_s))\tau_{\xi_s}$ is the required group.

We can easily obtain that Γ_0 and $\widetilde{X}$ are commensurable (see Lemma 3) and therefore there is a finitely generated subgroup $\widetilde{Y} = \widetilde{X} \cap F_0$ which is normal in $\widetilde{Q} = \widetilde{X} \cap \Gamma_0$ and $\widetilde{Q}$ is a subgroup of finite index in $\widetilde{X}$, $\widetilde{Q}/\widetilde{Y} \cong \mathbf{Z}$.

We recall that the isomorphism $i\colon X \to \widetilde{X}$ was constructed in such a way that $i|\Gamma'_1 = \mathrm{id}$. Thus, $i(g_1) = g_1 \subset [F_0, F_0] \cap H'_1$ (see Main Lemma). Hence, $g_1 \in \widetilde{X} \cap [F_0, F_0] \subset \widetilde{Y}$. Let $\tilde{g}_2 = \tau_{\xi_s} g_1 \tau_{\xi_s}$. By using the normality of Γ_0 in R one has $\tilde{g}_2 \in F_0$ by the evident embedding $\tau([F_0, F_0])\tau^{-1} \subset F_0$ for any $\tau \in R$.

We can now say that $\langle\langle g_1, \tilde{g}_2\rangle\rangle_{\widetilde{X}} \subset \widetilde{Y}$ because each element of the form $\tilde{x} g_1 \tilde{x}^{-1}$ or $\tilde{x}\tilde{g}_2\tilde{x}^{-1}$ ($\tilde{x} \in \widetilde{X}$) belongs to $\widetilde{X} \cap [F_0, F_0]$. Let $Y = i^{-1}(\widetilde{Y})$, $Q = i^{-1}(\widetilde{Q})$, $g_2 = i^{-1}(\tilde{g}_2)$, $g = g_1^{-1} g_2$.

By the considerations above $\langle\langle g^n\rangle\rangle_X \subset \langle\langle g_1, g_2\rangle\rangle_X \subset Y$. Let $\psi\colon X \to G$ be a natural projection and $\mathrm{Ker}\,\psi = \langle\langle g^n\rangle\rangle_X \subset Y$. The following diagram is then commutative

$$
(4) \qquad \begin{array}{ccccc}
Y & \xrightarrow{i_1} & Q & \xrightarrow{i_2} & X \\
\big\downarrow{\scriptstyle \psi_2} & & \big\downarrow{\scriptstyle \psi_1} & & \big\downarrow{\scriptstyle \psi} \\
N_0 & \xrightarrow{j_1} & G_0 & \xrightarrow{j_2} & G
\end{array}
$$

Here i_k, j_k are inclusion maps $(k = 1, 2)$ and $\psi_1 = \psi|_Q$, $\psi_2 = \psi|_Y$ are restrictions of ψ.

Both groups Q and Y are finitely generated, so N_0 and G_0 are also finitely generated. Thus we have that $\operatorname{Ker}\psi = \operatorname{Ker}\psi_2 \subset Y$ and obviously $Q/Y \cong G_0/N_0 \cong \mathbf{Z}$. Lemma 5 is proved. Q.E.D.

Lemma 6. *The group G contains no parabolics.*

Proof. Each edge of $R(G)$ has a neighbourhood filled by a finite number of images of $R(G)$ under G (see the proof of Lemma 4). Thus, the stabilizer of each point of $\partial R(G)$ is at most finite.

Recall that the circle $l_1 = I_{g_1} \cap I_{g_2}$ is pointwise fixed under the elliptic element $g = g_1^{-1} g_2$ of order n. There exists a small neighbourhood $V_1 = N(l_1)$ of l_1, invariant under g for which $V_1 \cap I_\gamma = \varnothing$, $\gamma \in (\Gamma_1' \cup \tau_\sigma \Gamma_1' \tau_\sigma) \setminus \{g_1, g_2\}$.

Therefore, by construction, $R(G) \cap V_1 = \mathscr{P}(\langle g \rangle) \cap V_1$ where $\mathscr{P}(\langle g \rangle)$ is an isometric fundamental domain for the cyclic group $\langle g \rangle$. It now follows that $\gamma V_1 \cap V_1 = \varnothing$, $\gamma \in G \setminus \langle g \rangle$. Let us consider the domain $\Omega_G^- = \Omega_G \setminus G V_1$ and surface $\sigma' = \operatorname{cl}(\sigma \cap \Omega_G^-)$. It is not hard to see that σ' is strongly invariant under $\mathscr{H}_s$ in G, $\mathscr{H}_s = \operatorname{Stab}(\Pi_s, G_s)$.

One can write each $f \in G \setminus \mathscr{H}_s$ in a normal form, $f = f_1 \cdot f_2 \cdot \dots \cdot f_m$, where f_i belongs either to $G_s' \setminus \mathscr{H}_s$ or to $(\tau_\sigma G_s' \tau_\sigma) \setminus \mathscr{H}_s$, $G = \langle G_s', \tau_\sigma G_s' \tau_\sigma \rangle$. Suppose that f is of infinite order; we get $f(\operatorname{int}(\sigma')) \subset \operatorname{int}\sigma'$ if m is even. Evidently σ' divides Ω_G^-; let $\operatorname{int}\sigma'$ (respectively $\operatorname{ext}\sigma'$) be the bounded (resp. unbounded) component of $\Omega_G^- \setminus \sigma'$. We have $f^{2n}(\operatorname{int}\sigma') \subset f^{2n-2}(\operatorname{int}\sigma')$ and $f^{-2n}(\operatorname{ext}\sigma') \subset f^{-2n+2}(\operatorname{ext}\sigma')$. So $x = \lim_{n\to\infty}(f^{2n}(\sigma'))$ and $y = \lim_{n\to\infty} f^{-2n}(\sigma')$ are respectively the attractive and the repulsive fixed points of f $(n \in \mathbf{N})$.

Let us prove that $x \neq y$. Indeed, as we have seen, $\operatorname{cl} V_1 \subset \Omega(G)$ so $\{x, y\} \cap G(V_1) = \varnothing$. We conclude that the points x, y are separated by σ and $\operatorname{dist}(x, y) > 0$. The lemma is proved. Q.E.D.

Lemma 7. *The domain of discontinuity $\Omega(G)$ of the group G consists of two invariant components Ω_G and Ω_G' $(\infty \in \Omega_G)$.*

Proof. We have shown in Corollary 1 to Lemma 4 that the fundamental domain of the group G consists of two connected components $R(G)$ and $R'(G)$. Furthermore, generators of G independently match faces of $R(G)$ with faces of $R(G)$ and faces of $R'(G)$ with faces of $R'(G)$.

Write $\Omega_G = G \cdot R(G)$, $\Omega_G' = G \cdot R'(G)$ for the unions of G-translates of $R(G)$ and $R'(G)$. We obtain $\Omega_G/G = R(G)/\sim\, = O$ and $\Omega_G'/G = R'(G)/\sim\, = O'$, where the second quotient in each expression means an orbifold obtaining by identifying paired faces of the fundamental polyhedron.

The group G is geometrically finite, since its 4-dimensional fundamental polyhedron $\mathscr{R} \subset \mathbf{H}^4 \cup \Omega(G)$ has finitely many faces by the formula $(*)$ (see the proof of Lemma 4). By Lemma 6 the group G contains no parabolics, so we can conclude that both O and O' are compact orbifolds.

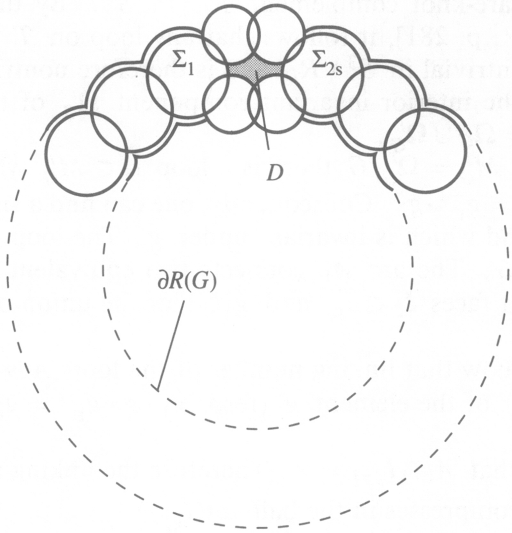

FIGURE 4

Since $R(G)$ and $R'(G)$ together form a fundamental domain for G in S^3, the domains Ω_G and Ω'_G are disjoint and $\Omega(G) = \Omega_G \cup \Omega'_G$. Notice that the domain Ω_G (respectively Ω'_G) is connected because faces of $R(G)$ (resp. $R(G)$) are equivalent under G to faces of $R(G)$ (resp. $R'(G)$).

Now standard arguments show that any boundary point of one of the components Ω_G and Ω'_G is a limit point. Indeed, let x be a boundary point of (say) Ω_G then there is a sequence of different points (x_n) in Ω_G tending to x. By compactness of O and the fact that $\Omega_G \to R(G)/\sim$ is a covering projection [M2, p. 68] we observe that all but finitely many x_n are equivalent under G. Consequently, $x \in \Lambda(G)$. The lemma is proved. Q.E.D.

Proof of Theorem B. In Lemma 5, we have obtained an element $g \in N_0$ of finite order (in fact $g \in [N_0, N_0]$).

Let us suppose that $z \in G_0 \setminus N_0$ projects to a generator of the group G_0/N_0. In N_0 there are infinitely many nonconjugate elements of the form $\gamma_m = z^m g z^{-m}$ $(m \geq 0)$. Indeed, each γ_m fixes pointwise a single circle $l_m \subset \Omega_{G_0} \subset S^3$. If γ_m was conjugate by $a \in N_0$ to γ_n, then the element a would move l_m to l_n and the same would be true for the element z^{-m+n}. This would mean that $(z^{-m+n})a^{-1}$ has finite order and so $z^p \in N_0$ for some $p \geq 0$ (since N_0 is normal in G_0), which is impossible. Theorem B is proved.

The last step in the proof of Theorem A is the following.

Lemma 8. *The component Ω_G of the domain of discontinuity $\Omega(G)$ $(\infty \in \Omega_G)$ contains an incompressible torus $T \subset \operatorname{int} R(G) \subset \Omega_G$.*

Proof. The boundary $\partial R(G)$ is a connected sum of two tori and so is a surface of genus 2 (see Figure 4). There is a compression disk $D \subset R(G)$ shown on the Figure 4. Now we denote by $V(D)$ the regular neighbourhood of D and let $R_D = \operatorname{cl}(R(G) \setminus V(D))$.

The proof of the lemma consists in showing that the boundary of the torus $T = \partial R_D$ is incompressible in Ω_G.

We have a square-knot complement R_D in S^3. By the property P for nonprime knots [R, p. 281], it follows that any loop on T different from the meridian m is nontrivial in $S^3 \setminus R_D$ and is therefore nontrivial in $\Omega_G \setminus R_D$.

Now consider the interior invariant component Ω'_G of the domain of discontinuity $\Omega(G) = \Omega_G \cup \Omega'_G$.

In the quotient $M'_G = \Omega'_G/G$ there is a loop $\alpha \subset M'_G$ which represents the elliptic element $g = g_1^{-1} \cdot g_2$. Consequently, one can find a simple loop $A \subset \Omega'_G$ which covers α and which is invariant under g. The loop A contains an arc A_1 projecting to α. The arc A_1 connects two equivalent points y_1 and y_2 lying on equivalent faces $\xi_2 \subset I_{g_2}$ and $g(\xi_2)$ of the union of two polyhedrons $R'(G) \cup g(R'(G))$.

Below we will show that linking number of the loop A with the fixed circle l_1 (respectively l_2) of the element g (resp. $g_1 \cdot g \cdot g_1^{-1} = g_2 \cdot g_1^{-1}$) is equal to 1 (resp. 0).

We can ensure that $A_1 \cap I_{g_1^{-1}} = \varnothing$. Therefore the linking number of A with l_2 is 0, since l_2 compresses in the ball $\operatorname{int}(I_{g_1^{-1}})$.

Let us now show that the linking number of A with l_1 is equal to 1. Consider the covering projection $S^3 \setminus l_1 \xrightarrow{\pi} (S^3 \setminus l_1)/\langle g \rangle = M_g$. The group $\pi_1(M_g)$ is an extension of $\mathbf{Z}_n$ by $\mathbf{Z}$ and is torsion-free. As the arc A_1 projects to nontrivial loop in M_g, the loop $\pi(A)$ is also nontrivial; otherwise, by connectedness of A, we would have an element of finite order in $\pi_1(M_g)$. It now follows that A is nontrivial in $S^3 \setminus l_1$ and thus that the linking number A with l_1 is equal to 1 (A is simple loop).

Therefore, the meridian m (which is freely homotopic to $l_1 \cdot l_2$) has linking number 1 with A. Observe that the meridian $m \subset \Omega_G$ and the loop $A \subset \Omega'_G$ are separated by the limit set $\Lambda(G)$ so they cannot be trivial in both of these components. We have shown that m is nontrivial in $\Omega_G \setminus R_D$; the lemma is proved. Q.E.D.

Proof of Theorem A. By the Selberg Lemma [Se], there exists a torsion-free subgroup $G^* \subset G$ of finite index; let $N = G^* \cap N_0$. We have an exact sequence of finitely generated groups

$$1 \to N \to G^* \to \mathbf{Z} \to 1.$$

and a diagram of infinite regular coverings

$$\Omega_G \to \Omega_G/N \to \Omega_G/G^*,$$

since $\Omega(G) = \Omega(G^*) = \Omega(N)$. The manifold $M_{G^*} = \Omega_G/G^*$ is compact so, if the group $\pi_1(M_N = \Omega_G/N)$ is finitely generated, then (by [H, Theorem 11.1]) $\pi_1 M_N$ is isomorphic to the fundamental group of a surface.

By Lemma 8, $\pi_1 M_N$ contains a subgroup isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$, so it should be a finite extension of the latter. This is impossible, since N is not an elementary group. Theorem A is proved. Q.E.D.

Appendix

Below we will present some complementary results. Recall that the group N_0 constructed in Theorem B contains infinitely many nonconjugate elliptic elements γ_m, all having the same order n. By our construction (see the beginning of §4) we can assume that n is prime.

Assertion 1. [3] *The group N is not finitely presented.*

Proof. Suppose to the contrary that the torsion-free subgroup $N \subset N_0$ is finitely presented. Then N_0 would also be finitely presented; here $N_0/N \cong K$ is a finite group. The group K acts on the manifold $M(N) = \mathbf{H}^4/N$, so that $M(N_0) = M(N)/K$. Consider the set L of fixed circles $\tilde{l}_n \subset \mathbf{H}^4$ of the elements γ_m. No elements of $\{\gamma_m\}$ are conjugate in N_0, so L projects to a set $L' \subset M(N)$ containing infinitely many different components, each fixed by an element of K whose order is n. Since K is finite, there is $k \in K$ of prime order which fixes infinitely many different circles in L'.

We now claim that the group $H_3(N, \mathbf{Z})$ is infinitely generated. Otherwise, all groups $H_i(M(N), \mathbf{Z})$ $(i = 1, \dots, 4)$ would be finitely generated, since the manifold $M(N)$ is not compact and N is finitely presented. By Smith theory, it follows that the cohomology groups of the fixed-point set of the element k of prime order n acting on $M(N)$ would all be finitely generated (see e.g. [Br, Theorem VII.10.5, p. 181–182]), so k could not fix infinitely many circles in L'. Therefore, $H_3(N, \mathbf{Z})$ is infinitely generated. Notice that the group $H_3(M(N), \mathbf{Z})$ cannot contain infinitely many linearly independent elements represented by nonseparated submanifolds of $M(N)$, otherwise, by duality, the group $H_1(M(N), \mathbf{Z})$ would also be infinitely generated. We would then conclude that the manifold $M(N)$ has infinitely many ends; let us show that this cannot happen.

The group N is a normal subgroup in the geometrically finite subgroup G having two invariant components of its action on S^3 (Lemma 7). There are two corresponding components (say L_1 and L_2) of the boundary of the convex hull $H_G \subset \mathbf{H}^4$ of the limit set $\Lambda(G)$. Between L_1 and Ω_G is an open region E_1; similarly we obtain E_2. The fundamental group of either component of the boundary of the convex hull maps onto G. It follows that there is a constant $\mathscr{C}$ such that, in the universal cover, every point of the convex hull is within distance $\mathscr{C}$ of each of these components of the boundary of the convex hull. Let us remove a compact set K from $M(N)$, then the unbounded components of $E_i/N \setminus K$ $(i = 1, 2)$ lie in the same component of the complement of K, because they can be joined by an arc of length $\mathscr{C}$ taken arbitrarily far from K. If there was another unbounded component of the complement then it would be separated from the component $M(N) \setminus K$ containing $E_i/N \setminus K$ by a compact surface S. We get a contradiction: points arbitrarily far away from S and, thus from the boundary of the convex hull lie in neither the set E_1/N nor E_2/N. The assertion is proved. Q.E.D.

In the following assertion we assume that $M = \mathbf{H}^3/\Gamma$ is a closed hyperbolic 3-manifold which fibers over the circle with a fiber – closed surface S_g $(g > 1)$.

Assertion 2. *Suppose that the group Γ above is commensurable with a reflection group R in faces of a polyhedron $D \subset \mathbf{H}^3$ all of whose dihedral angles are right angles. Then there is a a finite-index subgroup $\Gamma_0 \subset \Gamma$ and a 1-parameter family of discrete faithful representations $\rho(t)\colon \Gamma \to M(3) \cong SO_+(1, 4)$ $(t \in [-\varepsilon, 0])$ converging to a representation ρ_0 so that*

1. *All groups $G(t) = \rho_t(\Gamma_0)$ are convex cocompact.*
2. *The group $G(0) = \rho_0(\Gamma_0)$ is Kleinian with domain of discontinuity con-*

[3]The author wishes to thank the referee for pointing out to him the proof of Assertion 1.

sisting of two invariant components (noncontractible) $\Omega_{G(0)}$ *and* $\Omega^*_{G(0)}$ $(\infty \in \Omega_{G(0)})$.

3. *The normal subgroup* $F(0) = \rho(\Gamma_0 \cap \pi_1 S_g)$ *of* $G(0)$ *acts freely on* $\Omega_{G(0)}$ *so that the fundamental group* $\pi_1(\Omega_{G(0)})/F(0)$ *is infinitely generated. Moreover,* $F(0)$ *is isomorphic to the fundamental group of a closed surface but contains infinitely many* $F(0)$*-conjugacy classes of maximal parabolic subgroups.*

Proof. We start with the group G_s and its fundamental domain $R(G_s)$ from Lemma 3. Now take a family of spheres σ_t $(t \in [-\varepsilon, 0])$ such that $\sigma_t \cap \Sigma_s = \Lambda_s$, $\sigma_t \subset \pi^-$ for $t \leq 0$; and $\sigma_t \cap I_{g_i} = \varnothing$ $(t < 0)$, with σ_0 touching I_{g_i} in a point p_i $(i = 1, 2)$.

If now $t < 0$, we put $\Omega_s^- = \Omega_{G_s} \setminus G \cdot \mathrm{int}(\sigma_t)$ and $G_s' = \mathrm{Stab}(\Omega_s^-, G_s)$. The required group $G(t) = \langle G_s', \tau_{\sigma_t} G_s' \tau_{\sigma_t} \rangle$ is discontinuous and convex cocompact (by the proof of Lemma 3).

If $t = 0$, the construction of $G(0)$ is analogous to that of G from Lemma 4 (here we use Poincaré's Polyhedron Theorem to obtain $G(0)$). But the group $G(0)$ contains a parabolic element $g(0) = g^{-1} \cdot g_2(0)$, where $g_2(0) = \tau_{\sigma_0} \cdot g_1 \cdot \tau_{\sigma_0}$ fixes the point p. The assertion is proved. Q.E.D.

We notice that in [P1] we constructed limits of 2-quasifuchsian groups with properties 1–2 from assertion 2 with Γ satisfying somewhat weaker conditions.

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