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THE BROWN-PETERSON HOMOLOGY OF MAHOWALD'S X_k SPECTRA

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ABSTRACT. We compute the Brown-Peterson homology of Mahowald's X_k spectrum which is the Thom spectrum induced from $\Omega J_{2^k-1}S^2 \to \Omega^2 S^3 - \mathrm{BO}$, and the edge homomorphism of the Adams-Novikov spectral sequence for $\pi_*(X_k)$. We then compute the nonnilpotent elements of $\pi_*(X_k)$.

The main theorem of the beautiful paper *Nilpotence and stable homotopy theory*. I [12], states that for an arbitrary ring spectrum X, the kernel of the Brown-Peterson-Hurewicz homomorphism consists of nilpotent elements. The more standard way to describe the Brown-Peterson-Hurewicz homomorphism is that, firstly, one needs to describe the Brown-Peterson homology of X, BP_*X, as well as the comodule structure of BP_*X over Quillen's algebra $\mathrm{BP}_*\mathrm{BP}$. Secondly one needs to compute the edge-homomorphism of the Adams-Novikov differentials. In the present paper, we will describe BP_*X_k and the BP-Hurewicz homomorphism explicitly for Mahowald's noncommutative ring spectra X_k.

Let $\eta\colon S^1 \to \mathrm{BO}$ represent the generator of $\pi_1(\mathrm{BO}) = \mathbb{Z}/2$. Since BO is a double loop space there is an induced map $r\colon \Omega^2 S^3 \to \mathrm{BO}$. Then one takes the composite map $\Omega J_{2^k-1}S^2 \to \Omega^2 S^3 \xrightarrow{r} \mathrm{BO}$, where J_i is the James construction. These maps result in Thom spectra which will be denoted by X_k due to Mahowald [8, 17, 18, 19]. The interest of this X_k spectrum is that this X_k spectrum realizes part of the dual Steenrod algebra, that is,

$$H_*(X_k, Z/2) \cong \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_k].$$

The basic references for Brown-Peterson theory are [1, 26, 31]. Let BP be the 2-primary Brown-Peterson spectrum. Then the coefficient ring is

$$\mathrm{BP}_* = \mathbb{Z}_{(2)}[v_1, v_2, v_3, \dots], \qquad |v_i| = 2(2^i - 1).$$

Quillen's algebra is

$$\mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*[t_1, t_2, t_3, \dots], \qquad |t_i| = 2(2^i - 1).$$

We now state the main results of this paper.

Theorem A. *Let* BP *be the* 2*-primary Brown-Peterson spectrum and the invariant ideal* $I_k = (2, v_1, v_2, v_3, \dots, v_{k-1})$. *Then*

$$\mathrm{BP}_*X_k \cong (\mathrm{BP}_*/I_k)[t_1, t_2, t_3, \dots, t_k],$$

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as (left) BP$_*$BP *comodule algebra, where the* v_i *are the Hazewinkel generators and* t_i *are the standard generators in* BP$_*$BP.

Theorem B. *In the Adams-Novikov spectral sequence,*

$$\mathrm{Ext}^{0,*}_{\mathrm{BP_*BP}}(\mathrm{BP}, \mathrm{BP}_*X_k) = \mathbb{Z}/2[u_k, u_{k+1}, u_{k+2}, \dots, u_{2k}],$$

$|u_i| = 2(2^i - 1)$. *Moreover,*

(a) *Each* u_i *is a permanent cycle, hence*

$$\pi_*(X_k)/\mathrm{nil} = \mathbb{Z}/2[u_k, u_{k+1}, u_{k+2}, \dots, u_{2k}], \qquad |u_i| = 2(2^i - 1),$$

where nil *is the ideal generated by the nilpotent elements.*

(b) *If* $y_{k+i} \in \pi_*(X_k)$ *which is detected by* u_{k+i} *then* $y_{i+i_1} \cdot y_{k+i_2} = y_{k+i_2} \cdot y_{k+i_1}$ *in* $\pi_*(X_k)$ *for* $k - 1 \geq i_1 \geq i_2 \geq 0$.

Since X_k is a noncommutative ring spectrum, the difficulty of computing BP$_*X_k$ is the ring structure of BP$_*X_k$. So there are two steps to compute BP$_*X_k$:

(1) The commutativity of BP$_*X_k$.

Recall that in [32], there is a ring map of ring spectra $f\colon X_k \to M(2^{k+1} - 1)$ such that X_k is a retract of $M(2^{k+1} - 1)$, where $M(2^{k+1} - 1)$ is the Thom spectrum induced from the usual inclusion $\Omega(\mathrm{SU}(2^{k+1}-1)/\mathrm{SO}(2^{k+1}-1)) \to \mathrm{BO}$. Furthermore, in §2 we construct a ring map of ring spectra

$$h\colon T(k) \to M(2^{k+1} - 1),$$

where $T(k)$ is the 2-localization of Ravenel's spectrum $X(2^k)$ which is the Thom spectrum induced from the usual inclusion $\Omega\mathrm{SU}(2^k) \to \Omega\mathrm{SU} = \mathrm{BU}$. Hence we have

$$\mathrm{BP} \wedge T(k) \xrightarrow{1\wedge h} \mathrm{BP} \wedge M(2^{k+1} - 1) \xleftarrow{1\wedge f} \mathrm{BP} \wedge X_k,$$

that is,

$$\mathrm{BP}_*T(k) \xrightarrow{h_*} \mathrm{BP}_*M(2^{k+1} - 1) \xleftarrow{f_*} \mathrm{BP}_*X_k.$$

Then we use the Adams spectral sequence to prove that $\mathrm{Im}\, f_* \subseteq \mathrm{Im}\, h_*$. So BP$_*X_k$ is a commutative ring since $T(k)$ is a commutative ring spectrum and f_* is injective.

(2) The comodule structure of BP$_*X_k$.

Since X_k is a retract of $M(2^{k+1} - 1)$ [32], there is a map $p\colon M(2^{k+1} - 1) \to X_k$. So we have the composite map $T(k) \xrightarrow{h} M(2^{k+1} - 1) \xrightarrow{p} X_k$. Then combining this map with (1) and Ravenel's description

$$\mathrm{BP}_*T(k) \cong \mathrm{BP}_*[t_1, t_2, t_3, \dots, t_k],$$

where t_i are the standard generators in BP$_*$BP we prove Theorem A.

To prove Theorem B, we rely on the generalization of the Morava-Landweber theorem [26], that is,

$$\mathrm{Ext}^{0,*}_{\mathrm{BP_*BP}}(\mathrm{BP}_*, (\mathrm{BP}_*/I_k)[t_1, t_2, t_3, \dots, t_k]) = \mathbb{Z}/2[u_k, u_{k+1}, u_{k+2}, \dots, u_{2k}],$$

$|u_i| = 2(2^i - 1)$, where I_k is the invariant ideal $I_k = (2, v_1, v_2, v_3, \dots, v_{k-1})$ and t_i are the standard generators in

$$\mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*[t_1, t_2, t_3, \dots], \qquad |t_i| = 2(2^i - 1),$$

and the 2-primary Adams spectral sequence.

The paper is organized as follows: In §1 we recall and define the various Thom spectra. In particular, we recall the Thom splitting in [32]. In §2 we construct a map from Ravenel's $T(k)$ spectrum to Mahowald's X_k spectrum. In §3 we prove Theorem A. In §4 we compute the nonnilpotent part of $\pi_*(X_k)$ and prove Theorem B. In §5 we prove the various noncommutative ring spectra with commutative homology. In §6 we prove two vanishing lemmas which we need to prove Theorem B.

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1. The various Thom spectra

Given a connected H-space L and an H-map $\psi: L \to \mathrm{BO}$, the resulting Thom spectrum $\mathrm{Th}(\psi)$ is a ring spectrum with a two-sided unit [8, 19]. If L and BO admit associating homotopies compatible under ψ, then $\mathrm{Th}(\psi)$ is an associative ring spectrum, and if L has higher multiplicative structure compatible with BO under ψ, then $\mathrm{Th}(\psi)$ has analogous structure in the multiplication of $\mathrm{Th}(\psi)$. $\mathrm{Th}(\psi)$ is independent of the choice of the filtration and depends only on the homotopy class of ψ.

$\mathrm{Th}(\psi)$ is (-1)-connected, and $\pi_0(\mathrm{Th}(\psi))$ is either $\mathbb{Z}$ or $\mathbb{Z}/2$. If ψ is nonorientable, that is, $\psi^*(W_1) \neq 0$, then $\pi_0(\mathrm{Th}(\psi)) = \mathbb{Z}/2$. Otherwise $\pi_0(\mathrm{Th}(\psi)) = \mathbb{Z}$.

For our purpose a ring spectrum is a spectrum with a multiplication which is associative and with a two-sided unit, but not necessarily commutative. Furthermore "a ring map" will mean "a map between two ring spectra which is multiplicative and carries the unit", otherwise "a map" even between two ring spectra is not necessarily multiplicative.

From now on all unstated coefficient groups are $\mathbb{Z}/2$. A_* is the mod 2 dual Steenrod algebra [21, 30], and let ξ_i be the Milnor generator, $\deg \xi_i = 2^i - 1$. Then as an algebra $A_* = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots]$, and the coproduct is determined by $\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$.

We now define the various Thom spectra and recall the corresponding Thom splittings which we need later.

Recall Ravenel's construction of $T(k)$ [14, 26]. Let $X(2^k)$ be the Thom spectra of the maps $\Omega\mathrm{SU}(2^k) \to \Omega\mathrm{SU} \to \mathrm{BU}$, where the first map is the usual inclusion, and the second map is the homotopy equivalence of Bott. Since $\Omega\mathrm{SU}(2^k)$ is a commutative H-space compatible with BU under the above map, $X(2^k)$ is a commutative ring spectrum. Moreover, since $X(2^k)$ spectra filter MU, we have the following algebraic splitting.

Proposition 1.1 [26]. *As a left A_*-comodule algebra,*

$$H_*(X(2^k)) \cong \mathbb{Z}/2[\xi_1^2, \xi_2^2, \xi_3^2, \dots, \xi_k^2] \otimes_{\mathbb{Z}/2} \overline{L},$$

where $\overline{L}$ is concentrated in even dimensions.

Then Ravenel and Hopkins [14] proved the following splitting by applying the analogue of Quillen's idempotent.

Theorem 1.2 [14, 26]. *There is a* 2*-local commutative ring spectrum* $T(k)$ *such that* $X(2^k)_{(2)}$ *is a wedge of suspensions of* $T(k)$, *and*

$$H_*(T(k)) \cong \mathbb{Z}/2[\xi_1^2, \xi_2^2, \xi_3^2, \dots, \xi_k^2],$$

where $X(2^k)_{(2)}$ *is the* 2*-localization of* $X(2^k)$.

There is a real analogue of Ravenel's splitting in the complex case.

Let $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \to \Omega(\mathrm{SU}/\mathrm{SO}) \to \mathrm{BO}$ be the usual inclusion map, where the second map is the Bott equivalence. These maps yield the Thom spectra $M(n)$ [32]. Furthermore, there is a well-known inclusion $RP^{n-1} \to \Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$ such that

$$H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))) \cong \mathbb{Z}/2[b_1, b_2, b_3, \dots, b_{n-1}], \qquad |b_i| = i,$$

b_i is induced from $H_i(RP^{n-1})$, $1 \le i \le n-1$, and $H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))) \to H_*(\mathrm{BO})$ is injective. Via the Thom isomorphism,

$$H_*(M(n)) \cong \mathbb{Z}/2[\overline{b}_1, \overline{b}_2, \overline{b}_3, \dots, \overline{b}_{n-1}], \qquad |\overline{b}_i| = i,$$

and $H_*(M(n)) \to H_*(\mathrm{MO})$ is injective and isomorphic for $* \le n-1$. Hence one has the real analogue of the algebraic splitting in the complex case.

Proposition 1.3 [32, Proposition 3].

$$H_*(M(n)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_q] \otimes_{\mathbb{Z}/2} [\alpha_i/i \neq 2^l - 1],$$

$2^q \le n$, $i \le n-1$, $|\alpha_i| = i$, *and* α_i *is primitive over* A_*.

Recall Mahowald's X_k spectrum again. Let $\eta\colon S^1 \to \mathrm{BO}$ represent the generator of $\pi_1(\mathrm{BO}) = \mathbb{Z}/2$. Since BO is a double loop space, there is an induced map $r\colon \Omega^2 S^3 \to \mathrm{BO}$. Then one takes the composite map $\Omega J_{2^k-1} S^2 \to \Omega^2 S^3 \xrightarrow{r} \mathrm{BO}$, where J_i is the ith stage of the James construction. These maps result in Thom spectra which will be denoted by X_k [8, 17, 18, 19]. Then

Theorem 1.4 [8, 19]. *As a subcomodule algebra of* A_*,

$$H_*(X_k) \cong \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_k].$$

Then the main results in [32] are

Theorem 1.5. $M(2^{k+1}-1)$ *is a wedge of suspensions of* X_k.

We will give a brief idea how to prove this theorem. For more details see [32].

Lemma 1.6. *There is a ring map of ring spectra* $f\colon X_k \to M(2^{k+1}-1)$ *such that* $H_*(X_k)$ *is mapped isomorphically onto* $\mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_k]$ *of*

$$H_*(M(2^{k+1} = 1)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_k] \otimes_{\mathbb{Z}/2} [\alpha_i|\ i \neq 2^l - 1]$$

under f_*, *that is,* f_* *projects the* mod 2 *homology of* X_k *onto a wedge summand of the* mod 2 *homology of* $M(2^{k+1}-1)$.

Sketch of the proof. By using Serre's spectral sequence, we prove that the usual inclusion $\mathrm{SU}(2^{k+1}-1)/\mathrm{SO}(2^{k+1}-1) \to \mathrm{SU}/\mathrm{SO}$ induces an isomorphism on π_* for $* \le 2^{k+1}-2$ and an epimorphism for $* = 2^{k+1}-1$ when $k \ge 1$. It is known that the CW-complex $J_{2^k-1}S^2$ has dimension $2^{k+1}-2$, hence we have the following lifting (up to homotopy):

$$\begin{array}{ccc} & & SU(2^{k+1}-1)/SO(2^{k+1}-1) \\ & \nearrow & \downarrow \\ J_{2^k-1}S^2 & \longrightarrow & SU/SO, \end{array}$$

where the bottom map is the delooping map of the composite

$$\Omega J_{2^k-1}S^2 \to \Omega^2 S^3 \xrightarrow{r} BO = \Omega(SU/SO).$$

Then loop the diagram and thomify the resulting diagram to get

$$\begin{array}{ccc} & & M(2^{k+1}-1) \\ & \overset{f}{\nearrow} & \downarrow \\ X_k & \longrightarrow & MO. \end{array}$$

This ring map f is the desired one. Since each map in the above diagram is a ring map and

$$H_*(X_k) \to H_*(MO), \qquad H_*(M(2^{k+1}-1)) \to H_*(MO)$$

are injective, it follows that f_* maps $H_*(X_k)$ onto $\mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_k]$. For more details see [32].

Sketch of the proof of Theorem 1.5. Since

$$\pi_0(M(2^{K+1}-1)) = \pi_0(MO) = \mathbb{Z}/2,$$

not only are $M(2^{k+1}-1)$ and MO 2-local but also $\pi_*(M(2^{k+1}-1))$, $\pi_*(MO)$ are of characteristic 2. Furthermore, since the natural ring map $M(2^{k+1}-1) \to MO$ induces an isomorphism on $H_*(\;)$ for $* \leq 2^{k+1}-2$,

$$\pi_*(M(2^{k+1}-1)) \to \pi_*(MO)$$

is an epimorphism for $* \leq 2^{k+1}-2$, so in the splitting

$$H_*(M(2^{k+1}-1)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\alpha_2, \alpha_4, \dots, \alpha_{2^{k+1}-2}],$$

each polynomial generator in $\mathbb{Z}/2[\alpha_2, \alpha_4, \dots, \alpha_{2^{k+1}-2}]$ is stably spherical. Finally, construct ring spectra $L_{2^{k+1}-1}$ out of suitable wedges of spheres, satisfying

$$H_*(L_{2^{k+1}-1}) \cong \mathbb{Z}/2[\alpha_2, \alpha_4, \dots, \alpha_{2^{k+1}-2}].$$

Then using Lemma 1.6, one has the map

$$X_k \wedge L_{2^{k+1}-1} \to M(2^{k+1}-1) \wedge M(2^{k+1}-1) \to M(2^{k+1}-1),$$

where the last map is the multiplication map. By the construction, it follows that the above map induces isomorphism on mod 2 homology, hence the map is an equivalence since $X_k \wedge L_{2^{k+1}-1}$, $M(2^{k+1}-1)$ are 2-local. This completes the proof.

We finish this section by pointing out a remark we need later.

Remark 1.7. Under the splitting of Theorem 1.5, that is, $M(2^{k+1}-1)$ is a wedge of suspensions of X_k, one has

$$X_k \xrightarrow{f} M(2^{k+1}-1) \xrightarrow{p} X_k,$$

where f is the ring map in Lemma 1.6 and $pf \simeq 1$. Then

(a) p carries the unit.

(b) The following diagram commutes

$$\begin{array}{ccccc} X_k & \xrightarrow{f} & M(2^{k+1}-1) & \xrightarrow{p} & X_k \\ \downarrow & & \downarrow & & \downarrow \\ H(\mathbb{Z}/2) & \longrightarrow & \mathrm{MO} & \longrightarrow & H(\mathbb{Z}/2), \end{array}$$

where the map $X_k \to H(\mathbb{Z}/2)$ is induced by $\Omega J_{2^k-1}S^2 \to \Omega^2 S^3$ since in [17, 25], we know the Thom spectrum induced from the map $\Omega^2 S^3 \to \mathrm{BO}$ is the Eilenberg-Mac Lane spectrum $H(\mathbb{Z}/2)$, and $\mathrm{MO} \to H(\mathbb{Z}/2)$ is the Thom class.

(c) $p_*\colon H_*(M(2^{k+1}-1)) \to H_*(X_k)$ is a ring homomorphism.

2. A map from $T(k)$ to X_k

In order to describe the comodule structure BP_*X_k over $\mathrm{BP}_*\mathrm{BP}$ we need the following map.

Theorem 2.1. *There is a map $g\colon T(k) \to X_k$ which carries the unit. Furthermore, g induces an algebra map on* mod 2 *homology which sends ξ_i^2 to ξ_i^2, $1 \le i \le k$.*

The proof uses the following lemmas.

Lemma 2.2. *If a map $g\colon T(k) \to X_k$ carries the unit, then the map g induces an algebra map on* mod 2 *homology which sends ξ_i^2 to ξ_i^2, $1 \le i \le k$.*

Proof. Consider the composite map $T(k) \xrightarrow{g} X_k \to H(\mathbb{Z}/2)$ which carries the unit, where the second map is induced from $\Omega J_{2^k-1}S^2 \to \Omega^2 S^3$. Since

$$H^0(T(k)) \cong [T(k), H(\mathbb{Z}/2)]_0 \cong \mathrm{Hom}^0_{\mathbb{Z}/2}(H_*(T(k)), \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

this composite map is the unique essential map from $T(k)$ to $H(\mathbb{Z}/2)$. Furthermore $H(\mathbb{Z}/2)$ has a complex orientation of degree 2^k, that is, $\Sigma^{-2}\mathbf{CP}^{2^k} \to H(\mathbb{Z}/2)$. Hence this orientation extends to a ring map $\Sigma^{-2}\mathbf{CP}^{2^k} \to X(2^k) \to H(\mathbb{Z}/2)$. So a ring map $T(k) \to H(\mathbb{Z}/2)$ which induces the natural embedding on mod 2 homology. This map is also the unique essential map from $T(k)$ to $H(\mathbb{Z}/2)$. So the above composite map $T(k) \to X_k \to H(\mathbb{Z}/2)$ induces the natural embedding on mod 2 homology. But the second map in the composite map also induces the natural embedding on mod 2 homology. This completes the proof.

Lemma 2.3. *The following diagram commutes up to homotopy*

$$\begin{array}{ccc} \mathrm{BU} & \longrightarrow & \mathrm{BO} \\ \downarrow & & \downarrow \\ \Omega\mathrm{SU} & \longrightarrow & \Omega(\mathrm{SU}/\mathrm{SO}), \end{array}$$

where the two vertical maps are the Bott homotopy equivalences, the top map is the usual forgetful map, and the bottom map is the map induced by looping the quotient map.

Proof. Let E_i for $i = 1, 2$ be complex vector spaces and $E = E_1 \oplus E_2$. Note that $E \otimes_R \mathbb{C} \cong E_1 \oplus \overline{E}_1 \oplus E_2 \oplus \overline{E}_2 \cong E \oplus \overline{E}$ as complex vector spaces. Then this lemma follows from the following homotopy commutative diagram:

$$\begin{array}{ccc} U(E)/U(E_1) \times U(E_2) & \longrightarrow & O(E)/O(E_1) \times O(E_2) \\ \Phi_0 \downarrow & & \downarrow \Phi_2 \\ \Omega \mathrm{SU}(E) & \xrightarrow{\oplus \mathrm{id}} \Omega \mathrm{SU}(E \oplus \overline{E}) \longrightarrow & \Omega(\mathrm{SU}(E \otimes_R \mathbb{C})/\mathrm{SO}(E)). \end{array}$$

Here the vertical maps are the Bott maps Φ_0 and Φ_2 in [7] defined by

$$\Phi_0([T_1]) = \alpha(-\theta) \cdot T_1 \cdot \alpha(\theta) \cdot T_1^{-1},$$

and

$$\Phi_2([T_2]) = \alpha(\theta/2) \cdot T_2 \cdot \alpha(-\theta/2) \cdot T_2^{-1},$$

where $[T_1]$ is a coset in $U(E)/U(E_1) \times U(E_2)$ represented by T_1 in $U(E)$, $[T_2]$ is a coset in $O(E)/O(E_1) \times O(E_2)$ represented by T_2 in $O(E)$, and $\alpha(\overline{\theta})$ is the linear transformation from $F_1 \oplus F_2$ into itself, which is defined by $\alpha(\overline{\theta})(X, X') = (Xe^{i\overline{\theta}}, X')$, where $0 \leq \overline{\theta} \leq 2\pi$, $X \in F_1$, $X' \in F_2$, and F_i is a complex vector space for $1 \leq i \leq 2$. Then it is easy to see the diagram commutes up to homotopy. This completes the proof.

Proposition 2.4. *There is a ring map of ring spectra* $\overline{h}\colon X(2^k) \to M(2^{k+1} - 1)$.

Proof. Since we have the commutative diagram

$$\begin{array}{ccc} \Omega \mathrm{SU}(2^k) & \longrightarrow & \Omega(\mathrm{SU}(2^{k+1} - 1)/\mathrm{SO}(2^{k+1} - 1)) \\ \downarrow & & \downarrow \\ \Omega \mathrm{SU} & \longrightarrow & \Omega(\mathrm{SU}/\mathrm{SO}), \end{array}$$

where the bottom map is the map induced by looping the quotient map $\mathrm{SU} \to \mathrm{SU}/\mathrm{SO}$, the left vertical map is the usual inclusion, the right vertical map is the map induced by looping the usual inclusion, and the top map is the map induced by looping the composite map

$$\mathrm{SU}(2^k) \to \mathrm{SU}(2^k)/\mathrm{SO}(2^k) \to \mathrm{SU}(2^{k+1} - 1)/\mathrm{SO}(2^{k+1} - 1),$$

the first map is the quotient map and the second map is the usual inclusion, and by Lemma 2.3 the bottom map is homotopic to the usual forgetful map, we have a ring map of ring spectra $\overline{h}\colon X(2^k) \to M(2^{k+1} - 1)$. This completes the proof.

Corollary 2.5. *There is a ring map of ring spectra* $h\colon T(k) \to M(2^{k+1} - 1)$.

Proof. By Proposition 2.4, we have a ring map of ring spectra $\overline{h}\colon X(2^k) \to M(2^{k+1} - 1)$. Then since $M(2^{k+1} - 1)$ is a 2-local spectrum, this induces a ring map of ring spectra $\overline{h}_{(2)}\colon X(2^k)_{(2)} \to M(2^{k+1} - 1)$. Via Ravenel's splitting, we have the ring map of ring spectra $h\colon T(k) \to X(2^k)(2) \to M(2^{k+1} - 1)$. This completes the proof.

Proof of Theorem 2.1. Take g to be the composite map $T(k) \xrightarrow{h} M(2^{k+1}-1) \xrightarrow{p} X_k$, where h is from Corollary 2.5 and P is coming from Remark 1.7. Since h and p both carry the units respectively, g also carries the unit. Then by Lemma 2.2, the map g induces the mod 2 homology as described. This completes the proof.

3. Computation of BP_*X_k

The difficulty to compute BP_*X_k is the ring structure and comodule structure. We will prove that BP_*X_k is a commutative ring in §5. Here we will compute the comodule structure over the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$. Actually one can easily compute BP_*X_k by using only the 2-primary Adams spectral sequence but it is hard to understand the comodule structure. So we rely on two inputs: Ravenel's description of $\mathrm{BP}_*T(k)$ and the map $g: T(k) \to X_k$ in Theorem 2.1. That is, the map $g: T(k) \to X_k$ induces a homomorphism from the Adams spectral sequence for $\mathrm{BP}_*T(k)$ to the Adams spectral sequence for BP_*X_k. Hence we can choose the generators of BP_*X_k corresponding to the generators of $\mathrm{BP}_*T(k)$. Then this will tell us how to describe the comodule structure of BP_*X_k. However, we have to be very careful in one point since g is not even a ring map. We do not know if $g_*: \mathrm{BP}_*T(k) \to \mathrm{BP}_*X_k$ is a ring homomorphism.

Recall the 2-primary Adams spectral sequence. Let $\mathbb{Z}_{(2)}$ be the integers localized at prime 2. The Adams spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{A_*}^{*,*}(\mathbb{Z}/2, H_*(X)) \Rightarrow \pi_*(X) \otimes Z_{(2)}$$

can be used to compute $\mathrm{BP}_*X = \pi_*(\mathrm{BP} \wedge X)$.

Let $E = E(\xi_1, \xi_2, \xi_3, \dots)$ be the exterior algebra on the Milnor generators. By a well-known change of rings isomorphism [16] we can replace

$$\mathrm{Ext}_{A_*}^{*,*}(\mathbb{Z}/2, H_*(\mathrm{BP} \wedge X)) \quad \text{with} \quad \mathrm{Ext}_E^{*,*}(\mathbb{Z}/2, H_*(X)).$$

So the Adams spectral sequence we use is $\mathrm{Ext}_E^{*,*}(\mathbb{Z}/2, H_*(X)) \Rightarrow \mathrm{BP}_*X$.

Recall that $|\xi_i| = 2^i - 1$ and that

$$\mathrm{Ext}_E^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[e_1, e_2, e_3, \dots],$$

where $\mathrm{bideg}(e_i) = (1, |\xi_i|)$. We will denote this ring by F.

We turn now to the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$, where BP is the 2-primary Brown-Peterson spectrum. Thus $H_*(\mathrm{BP}, \mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}[m_1, m_2, m_3, \dots]$ for canonical generators m_i of dimension $2(2^i - 1)$, under the Hurewicz map $\pi_*(\mathrm{BP})$ is embedded in $H_*(\mathrm{BP}, \mathbb{Z}_{(2)})$ as $\mathbb{Z}_{(2)}[v_1, v_2, \dots]$, where the v_i are Hazewinkel's generators [1, 26, 31], described inductively by

$$v_n = 2m_n - \sum_{i=1}^{n-1} v_{n-i}^{2^i} m_i,$$

and $\mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*[t_1, t_2, \dots] = \mathrm{BP}_*[m_1, m_2, \dots]$, $|t_i| = |m_i| = 2(2^i - 1)$. The structure maps are $\eta_R: \mathrm{BP}_* \to \mathrm{BP}_*\mathrm{BP}$ given by

$$\eta_R(m_n) = \sum_{i+j=n} m_i t_j^{2^i}, \qquad t_0 = m_0 = 1.$$

The coproduct $\Delta\colon \mathrm{BP}_*\mathrm{BP} \to \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}$ is

$$\sum_{i+j=n} m_i(\Delta t_j)^{2^i} = \sum_{i+j+l=n} m_i t_j^{2^i} \otimes t_l^{2^{i+j}},$$

and the conjugation $C\colon \mathrm{BP}_*\mathrm{BP} \to \mathrm{BP}_*\mathrm{BP}$ is

$$\sum_{i+j+l=n} m_i t_j^{2^i} (C t_l)^{2^{i+j}} = m_n.$$

We also recall Ravenel's work on $\mathrm{BP}_*T(k)$ [26].

Lemma 3.1 [26]. $\mathrm{BP}_*T(k) \cong \mathrm{BP}_*[t_1, t_2, t_3, \ldots, t_k] \subseteq \mathrm{BP}_*\mathrm{BP}$ *as a comodule algebra.*

Lemma 3.2 [26]. *In the Adams spectral sequence for* $\mathrm{BP}_*T(k)$,

$$E_2^{*,*} \cong \mathbb{Z}/2[e_1, e_2, e_3, \ldots] \otimes \mathbb{Z}/2[\beta_1^2, \beta_2^2, \beta_3^2, \ldots, \beta_k^2],$$

where β_i^2 *is represented by* $[\xi_i^2]$ *in the cobar complex and* $\mathrm{bideg}(\beta_i^2) = (0, |\xi_i^2|)$. *Moreover,* t_i $(1 \le i \le k)$ *is detected by* β_i^2 *module decomposables and* v_i *is detected by* e_{i+1}.

To prove Theorem A, recall that in [1], the comodule structure map

$$\mathrm{BP}_*X \to \mathrm{BP}_*(\mathrm{BP} \wedge X) \xleftarrow{\cong} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*X$$

for a ring spectrum X is defined by taking the BP_*-homology for the ring map $S^0 \wedge X \xrightarrow{\eta\wedge 1} \mathrm{BP} \wedge X$ in the first map, and taking the homotopy for the map

$$\mathrm{BP} \wedge \mathrm{BP} \wedge \mathrm{BP} \wedge X \xrightarrow{1\wedge u \wedge 1} \mathrm{BP} \wedge \mathrm{BP} \wedge X$$

in the second map. Then one can easily check the comodule structure map for a ring spectrum is a ring map and natural on X.

Proof of Theorem A. By Theorem 2.1, we have a map $g\colon T(k) \to X_k$ which carries the unit and induces the mod 2 homology as a ring map, that is, it is a natural ring embedding. So $1 \wedge g\colon \mathrm{BP} \wedge T(k) \to \mathrm{BP} \wedge X_k$, which is also denoted by g, also induces the mod 2 homology as a ring map. Both 2-primary Adams spectral sequences converge to $\mathrm{BP}_*T(k)$, and BP_*X_k respectively. From the usual change-of-rings, one has the following naturality

$$\begin{array}{ccc} E_2^{*,*} & \xrightarrow{\cong} & \mathrm{Ext}_E^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2[\xi_1^2, \xi_2^2, \ldots, \xi_k^2]) \\ \downarrow & & \downarrow \\ \overline{\overline{E}}_2^{*,*} & \xrightarrow{\cong} & \mathrm{Ext}_E^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2[\xi_1, \xi_2, \ldots, \xi_k]), \end{array}$$

so

$$\begin{array}{ccc} E_2^{*,*} & \xrightarrow{\cong} & \mathbb{Z}/2[e_1, e_2, e_3, \ldots] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\beta_1^2, \beta_2^2, \ldots, \beta_k^2] \\ \downarrow & & \downarrow \\ \overline{\overline{E}}_2^{*,*} & \xrightarrow{\cong} & \mathbb{Z}/2[e_{k+1}, e_{k+2}, \ldots] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\overline{\overline{\beta}}_1^2, \overline{\overline{\beta}}_2^2, \overline{\overline{\beta}}_3^2, \ldots, \overline{\overline{\beta}}_k^2], \end{array}$$

where $\overline{\overline{\beta}}_i^2$ $(1 \le i \le k)$ is represented by $[\xi_i^2]$ in the cobar complex and $\mathrm{bideg}(\overline{\overline{\beta}}_i^2) = (0, |\xi_i^2|)$.

From Lemmas 3.1 and 3.2, in the Adams spectral sequence for $\mathrm{BP}_*T(k)$, t_i $(1 \le i \le k)$ is detected by $\beta_i^2 +$ decomposables, and v_i is detected by e_{i+1} in the E_2-term. Moreover both Adams spectral sequences collapse at E_2-terms since $E_2^{s,t} = 0$ for $t-s =$ even, and there is no group extension problem in the Adams spectral sequence for BP_*X_k. Applying Theorem 5.4, under the map $g_*\colon \mathrm{BP}_*T(k) \to \mathrm{BP}_*X_k$, we make a choice of generators of BP_*X_k such that

$$\begin{array}{ccc} \mathrm{BP}_*T(k) & \xrightarrow{g_*} & \mathrm{BP}_*X_k \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{BP}_*[t_1, t_2, \dots, t_k] & \to & (\mathrm{BP}_*/I_k)[t_1, t_2, \dots, t_k], \end{array}$$

$g_*(t_i) = t_i$ for $1 \le i \le k$, and $g_*(v_i) = v_i$ for $i \ge k$.

Warning. We do not even know that $g_*(t_it_j) = t_it_j$, or $g_*(v_iv_j) = v_iv_j$, since g_* is not even a ring map. We only choose the generators t_i in BP_*X_k corresponding to the generators t_i in $\mathrm{BP}_*T(k)$, and v_i in BP_*X_k corresponding to v_i in $\mathrm{BP}_*T(k)$, this is via the E_∞-terms level.

However, since $g\colon T(k) \to X_k$ carries the unit, i.e.,

$$\begin{array}{ccc} \mathrm{BP} \wedge S^0 & \xrightarrow{1\wedge\eta} & \mathrm{BP} \wedge T(k) \\ \| & & 1\wedge g\downarrow \\ \mathrm{BP} \wedge S^0 & \xrightarrow{1\wedge\bar{\bar{\eta}}} & \mathrm{BP} \wedge X_k \end{array}$$

commutes, where $\eta\colon S^0 \to T(k)$, $\bar{\bar{\eta}}\colon S^0 \to X_k$ are units, so in $g_*\colon \mathrm{BP}_*T(k) \to \mathrm{BP}_*X_k$, $g_*(v_iv_j) = v_iv_j$ for $k \le i \le j$. Thus the BP_*-module structure is completely determined. To finish the proof, it remains to prove that the generators t_i in BP_*X_k have the correct coaction over $\mathrm{BP}_*\mathrm{BP}$ as the standard generators t_i in $\mathrm{BP}_*T(k)$.

Claim. In $g_*\colon \mathrm{BP}_*T(k) \to \mathrm{BP}_*X_k$, if $\deg(t_{i_1}t_{i_2}\cdots t_{i_m}) \le 2^{k+1} - 3$, then

$$g_*(t_{i_1}t_{i_2}\cdots t_{i_m}) = t_{i_1}t_{i_2}\cdots t_{i_m}.$$

Since g_* induces a ring map from the associated bigraded vector space of $\mathrm{BP}_*T(k)$ to the associated bigraded vector space of BP_*X_k, this is due to the fact that the homology is a ring map, so

$$g_*(t_{i_1}t_{i_2}\cdots t_{i_m}) = t_{i_1}t_{i_2}\cdots t_{i_m} + \text{ higher filtration.}$$

But in the Adams spectral sequence for BP_*X_k, $\bar{\bar{E}}_2^{s,*+s} = 0$ for $s \ge 1$ and $* \le 2^{k+1} - 3$, i.e., $F^1\mathrm{BP}_*X_k = F^2\mathrm{BP}_*X_k = \cdots = 0$ for $* \le 2^{k+1} - 3$, where $F^i\mathrm{BP}_*X_k$ $(i \ge 1)$ is the Adams ith filtration. The claim follows.

To finish the proof, we consider the naturality,

$$\begin{array}{ccc} \mathrm{BP}_*T(k) & \xrightarrow{g_*} & \mathrm{BP}_*X_k \\ \Delta\downarrow & & \downarrow \\ \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*T(k) & \xrightarrow{1\otimes g_*} & \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*X_k, \end{array}$$

the coproduct in $\mathrm{BP}_*T(k)$ is computed by,

$$\sum_{i+j=n} m_i(\Delta t_j)^{2^i} = \sum_{i+j+l=n} m_i t_j^{2^i} \otimes t_l^{2^{i+j}} \quad \text{for } n \le k,$$

hence

$$\Delta t_n = t_n \otimes 1 + 1 \otimes t_n + \sum \square \otimes (t_{i_1} t_{i_2} \cdots t_{i_m}),$$

$i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_m$, and $\deg(t_{i_1} t_{i_2} \cdots t_{i_m}) \leq 2(2^n - 1) - 2 = 2^{n+1} - 4 \leq 2^{k+1} - 3$ for $n \leq k$. So the coaction follows from the claim. This completes the proof of Theorem A.

4. The nonnilpotent part of $\pi_*(X_k)$

From the point of view of the Nilpotence Theorem [12], if one wants to compute all the nonnilpotent elements of $\pi_*(X_k)$, then one needs to compute the edge homomorphism of the Adams-Novikov spectral sequence for $\pi_*(X_k)$. So in this section we start to compute this edge-homomorphism. Then we use the 2-primary Adams spectral sequence for $\pi_*(X_k)$ to compute a certain range of the Adams E_2-term. Finally we rely on two vanishing lemmas to prove some elements are permanent cycles in the Adams first filtration, hence in the Novikov zero filtration.

Recall that one has the map of Hopf algebroid

$$\overline{f}\colon (\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}) \to (\mathrm{BP}_*, \Gamma(k+1)), \quad \Gamma(k+1) = \mathrm{BP}_*\mathrm{BP}/(t_1, t_2, \ldots, t_k),$$

and the only invariant prime ideals are $I_k = (2, v_1, v_2, \ldots, v_{k-1})$, that is, $I_k \cdot \mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*\mathrm{BP} \cdot I_k$.

Proposition 4.1.

$$\begin{aligned}\mathrm{Ext}^{0,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*X_k) &\cong \mathrm{Ext}^{0,*}_{\Gamma(k+1)}(\mathrm{BP}_*, \mathrm{BP}_*/I_k)\\ &\cong \mathbb{Z}/2[u_k, u_{k+1}, \ldots, u_{2k}],\end{aligned}$$

where $|u_i| = 2(2^i - 1)$.

If one knows the Comodule Algebra Structure Theorem [26, A1.1.17], then to compute the above Ext-group one needs only the change-of-rings isomorphism. For convenience to the reader, we compute it in the following:

Lemma 4.2. $\mathrm{BP}_*(T(k))/I_k \cong \mathrm{BP}_*\mathrm{BP}\square_{\Gamma(k+1)}(\mathrm{BP}_*/I_k)$ *as left* $\mathrm{BP}_*\mathrm{BP}$ *comodule algebras, where the right coaction of* $\mathrm{BP}_*\mathrm{BP}$ *over* $\Gamma(k+1)$ *is induced from the map* $f\colon \mathrm{BP}_*\mathrm{BP} \to \Gamma(k+1)$.

Proof. Since I_k is invariant, we have $\mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} (\mathrm{BP}_*/I_k) \cong \mathrm{BP}_*\mathrm{BP}/I_k$. Thus the isomorphism follows from the exact sequence and the coproduct formula of $\mathrm{BP}_*\mathrm{BP}$,

$$\begin{aligned}0 \to \mathrm{BP}_*\mathrm{BP}\square_{\Gamma(k+1)}(\mathrm{BP}_*/I_k) &\to \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} (\mathrm{BP}_*/I_k)\\ &\to \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \Gamma(k+1) \otimes_{\mathrm{BP}_*} \mathrm{BP}_*/I_k,\end{aligned}$$

where the second map is defined for the cotensor product. □

To use the change-of-rings isomorphism, one needs to use the Comodule Algebra Structure Theorem to prove the following lemma. But we minic the idea to prove it.

Lemma 4.3. *Give* $\mathrm{BP}_*\mathrm{BP}$ *the right* $\Gamma(k+1)$*-comodule structure induced by* $\overline{f}\colon \mathrm{BP}_*\mathrm{BP} \to \Gamma(k+1)$. *Then* $\mathrm{BP}_*\mathrm{BP}$ *is an extended* $\Gamma(k+1)$*-comodule.*

Proof. As a left BP_*-module, $\mathrm{BP}_*T(k)$ is a direct summand of $\mathrm{BP}_*\mathrm{BP}$. Let $\overline{\overline{f}}\colon \mathrm{BP}_*\mathrm{BP} \to \mathrm{BP}_*T(k)$ be a BP_*-linear splitting of the natural inclusion. Define g_1 by

$$g_1\colon \mathrm{BP}_*\mathrm{BP} \xrightarrow{\Delta} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \xrightarrow{\overline{\overline{f}}\otimes \overline{f}} \mathrm{BP}_*T(k) \otimes_{\mathrm{BP}_*} \Gamma(k+1),$$

where the first map is the coproduct map, and one regards $\mathrm{BP}_*T(k)$ as a right BP_*-module since BP_* is commutative. Hence g_1 is right BP_*-linear and g_1 is a right $\Gamma(k+1)$-comodule homomorphism due to the coassociative property of $\mathrm{BP}_*\mathrm{BP}$. It remains to be shown that g_1 is an isomorphism. Obviously g_1 is injective, and $t_i \in \mathrm{BP}_*\mathrm{BP}$, $1 \le i \le k$, has the trivial right coaction over $\Gamma(k+1)$, that is, $g_1(t_i) = t_i \otimes 1$ for $1 \le i \le k$. Now from the coproduct formula of $\mathrm{BP}_*\mathrm{BP}$, it is easy to see $g_1(t_{k+1}) = 1 \otimes t_{k+1}$. We then proceed by induction: Assume $\mathrm{BP}_*T(k) \otimes t_i^* \subseteq \operatorname{im} g_1$ for $i \le k+n$ (here t_i^* means $\sum t_{i_1}t_{i_2}\cdots t_{i_m}$, where the sum is taken over the sequence $i_1 \le i_2 \le i_3 \le \cdots \le i_m \le k+n$). To prove $\mathrm{BP}_*T(k) \otimes t_i^* \subseteq \operatorname{im} g_1$ for $i \le k+n+1$, it suffices to prove $1 \otimes t_{k+n+1} \in \operatorname{im} g_1$ since if $g_1(b) = 1 \otimes t_{k+n+1}$, then $g_1(ab^*) = a \otimes t_{k+n+1}^*$ for $a \in \mathrm{BP}_*T(k)$. But from the coproduct formula again, $g_1(t_{k+n+1}) = 1 \otimes t_{k+n+1} + \sum m' \otimes t_j^*$, where $k+n \ge j \ge k+1$, by the inductive assumption. It follows that $1 \otimes t_{k+n+1} \in \operatorname{im} g_1$. Thus g_1 is onto. This completes the proof.

The reader is referred to the change-of-rings isomorphism in [12, Proposition 2.9] or [26, A1.3.12].

Proof of Proposition 4.1.

$$\begin{aligned}\mathrm{Ext}^{0,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*X_k) &\cong \mathrm{Ext}^{0,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*(T(k))/I_k)\\ &\cong \mathrm{Ext}^{0,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}\square_{\Gamma(k+1)}(\mathrm{BP}_*/I_k))\\ &\cong \mathrm{Ext}^{0,*}_{\Gamma(k+1)}(\mathrm{BP}_*, \mathrm{BP}_*/I_k);\end{aligned}$$

the first isomorphism is due to Theorem A, the second one is due to Lemma 4.2, and the last one is from the change-of-rings isomorphism. It remains to explain the generalized Morava-Landweber theorem. This follows directly from the cobar complex of the comodule BP_*/I_k over the coalgebra $\Gamma(k+1)$, since $v_i \in \mathrm{BP}_*/I_k$, $k \le i \le 2k$ are the primitive generators. □

Recall that $H_*(X_k) \cong \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_k]$ as the subcomodule algebra of the dual Steenrod A_*. Hence the Adams spectral sequence we use is

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{A_*}(\mathbb{Z}/2, \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots, \xi_k]) \Rightarrow \pi_*(X_k).$$

By a routine change-of-rings, we have

$$\mathrm{Ext}_{\mathbb{Z}/2[\xi_{k+1}, \xi_{k+2}, \xi_{k+3}, \dots]}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow \pi_*(X_k).$$

Proposition 4.4. *In the Adams spectral sequence for* $\pi_*(X_k)$,

(a) $E_2^{*,*} \cong \mathbb{Z}/2[h_{k+j,i}]$ *for* $t < 2^{2k+2}-1$. *Where* $1 \le j \le k+1$, $i \ge 0$, *and* $h_{k+j,i} \in E_2^{1,2^i(2^{k+j}-1)}$ *is represented by* $[\xi_{k+j}^{2^i}]$ *in the cobar complex.*

(b) $E_2^{0,t} = 0$ *for* $t > 0$.

(c) $E_2^{s,t} = 0$ *for* $s > 0$ *and* $s(2^{k+1}-1) > t$.

Note. Part (c) will give a great advantage in the computation of the Adams differentials.

Proof. To compute the Ext-group in the range $t < 2^{2k+2} - 1$ one actually concentrates on the computation of $\mathrm{Ext}^{*,*}_{\mathbb{Z}/2[\xi_{k+1}, \xi_{k+2}, \dots, \xi_{2k+1}]}(\mathbb{Z}/2, \mathbb{Z}/2)$. Then

$$\mathrm{Ext}^{*,*}_{\mathbb{Z}/2[\xi_{k+1}, \xi_{k+2}, \dots, \xi_{2k+1}]}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \bigotimes_{1\le j\le k+1} \mathrm{Ext}^{*,*}_{\mathbb{Z}/2[\xi_{k+j}]}(\mathbb{Z}/2, \mathbb{Z}/2)$$

$$\cong \bigotimes_{1\le j\le k+1} \mathbb{Z}/2[h_{k+j,i} | i \ge 0] \cong \mathbb{Z}/2[h_{k+j,i} | 1 \le j \le k+1,\ i \ge 0].$$

This proves part (a). Parts (b) and (c) follow form the cobar complex immediately.

Theorem 4.5. *In the Adams spectral sequence for* $\pi_*(X_k)$, *each* $h_{k+j,0}$ $(1 \le j \le k+1)$ *is a (nontrivial) permanent cycle.*

We reduce Theorem 4.5 to the following proposition which we will prove in §6.

Proposition 4.6. *In the Adams spectral sequence for* $\pi_*(X_k)$,

$$E_2^{r+1, (2^{k+j}-1)+(r-1)} = 0 \quad \text{for } 1 \le j \le k+1,\ 2 \le r \le 2^{j-1} - 1.$$

Proof of Theorem 4.5. In the Adams differentials,

$$E_r^{1, 2^{k+j}-1} \xrightarrow[d_r]{} E_r^{r+1, (2^{k+j}-1)+(r-1)}.$$

It follows from part (c) in Proposition 4.4, if $E_r^{r+1, (2^{k+j}-1)+(r-1)} \neq 0$, one has

$$(r+1)(2^{k+1} - 1) \le 2^{k+j} - 1) + (r-1).$$

So

$$r(2^{k+1} - 2) \le (2^{j-1} - 1)(2^{k+1} - 2) + 2^j - 3.$$

But $2^{k+1} - 2 > 2^j - 3$, hence $2 \le r \le 2^{j-1} - 1$. Thus from Proposition 4.6, $h_{k+j,0}$ $(1 \le j \le k+1)$ is a (nontrivial) permanent cycle. □

Note. In the above proof, we actually prove $h_{k+1,0}$ and $h_{k+2,0}$ are permanent cycles, hence we need only to discuss $2 \le r \le 2^{j-1}$, i.e., $j \ge 3$, $k \ge 2$.

In Theorem 4.5, each $h_{k+j,0}$ $(1 \le j \le k+1)$ is a (nontrivial) permanent cycle, represented by a homotopy class in $\pi_*(X_k)$ denoted by $\overline{y}_{k+j}$, $|\overline{y}_{k+j}| = 2^{k+j} - 2$. Furthermore from the algebraic structure of the Adams spectral sequence, one can easily see $h_{k+j_1,0} \cdot h_{k+j_2,0}$ $(1 \le j_1, j_2 \le k+1)$ is still a (nontrivial) permanent cycle, represented by $\overline{y}_{k+j_1} \cdot \overline{y}_{k+j_2}$ in $\pi_*(X_k)$. It follows from Proposition 5.2 that X_k is a noncommutative ring spectrum. However the mod 2 homology of X_k is a commutative ring, hence the associative bigraded ring from the Adams filtrations is a commutative ring. So if one wants to prove any two elements commute with each other in $\pi_*(X_k)$, all that is required is to check the Adams higher filtrations.

Theorem 4.7. *When* $k \ge 2$, $\overline{y}_{k+j_1} \cdot \overline{y}_{k+j_2} = \overline{y}_{k+j_2} \cdot \overline{y}_{k+j_1}$ *in* $\pi_*(X_k)$ *for* $k \ge j_1 > j_2 \ge 1$.

This theorem follows immediately from the following proposition since we need only check the Adams filtrations higher than 2.

Proposition 4.8. *In the Adams spectral sequence for* $\pi_*(X_k)$,

$$E^{r,2^{k+j_1}+2^{k+j_2}+(r-4)}=0$$

for $3\le r$ *and* $k\ge j_1>j_2\ge 1$.

We will prove this vanishing lemma in §6 and to prove Theorem B, we refer to the Nilpotence Theorem [12]:

Theorem 4.9. *Let* X *be an arbitrary ring spectrum. The kernel of the* BP *Hurewicz homomorphism* $\mathrm{BP}_*\colon \pi_*(X)\to \mathrm{BP}_*X$ *consists of nilpotent elements.*

Recall that the Thom map $\mathrm{BP}\to H(\mathbb{Z}/2)$ induces the homomorphism

$$\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*,\,\mathrm{BP}_*X)\to \mathrm{Ext}_{A_*}(\mathbb{Z}/2,\,H_*(X)),$$

for a 2-local (finite type) spectrum of X. Particularly, a homotopy class in $\pi_*(X)$ is detected by the Adams $E_2^{s,*}$ then detected by the Novikov $E_2^{s',*}$ for $s'\le s$.

Proof of Theorem B.

$$\mathrm{Ext}^{0,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*,\,\mathrm{BP}_*X_k)\cong \mathbb{Z}/2[u_k,\,u_{k+1},\,\dots,\,u_{2k}],$$

$|u_i|=2(2^{i-1})$, has been proved in Proposition 4.1. To prove each u_i $(k\le i\le 2k)$ is a permanent cycle, we use the Thom reduction. It follows from Theorem 4.5, $h_{k+j,0}$ $(1\le j\le k+1)$ is a permanent cycle in the Adams spectral sequence, represented by a homotopy class $\overline{y}_{k+j}$ $(1\le j\le k+1)$ in $\pi_*(X_k)$, $|\overline{y}_{k+j}|=2^{k+j}-2$. Furthermore each $\overline{y}_{k+j}$ is in the first Adams filtration, hence $\overline{y}_{k+j}$ is detected by the Novikov $E_2^{0,*}$ or $E_2^{1,*}$. But BP_*X_k is concentrated in even dimensions. Thus $\overline{y}_{k+j}$ must be detected by $E_2^{0,*}$. Obviously $\overline{y}_{k+1}$ is detected by $u_k\in E_2^{0,*}$. We will prove by induction $\overline{y}_{k+j}$ is detected by $u_{k+(j-1)}+$ decomposables in $E_2^{0,*}$ for $1\le j\le k+1$. Assume it is true for $1\le j\le m<k+1$. Then we need to prove $\overline{y}_{k+(m+1)}$ is detected by $u_{k+m}+$ decomposables in $E_2^{0,*}$. Suppose not, then $\overline{y}_{k+(m+1)}$ would be detected by decomposables in $E_2^{0,*}$. By the induction assumption, one would have $\overline{y}_{k+(m+1)}=$ decomposables $+$ Novikov higher filtration. However this contradicts the Adams spectral sequence. Hence each u_i $(k\le i\le 2k)$ is a permanent cycle in the Adams-Novikov spectral sequence, represented by a homotopy class $y_i\in\pi_*(X_k)$, i.e., $y_i=\overline{y}_{i+1}+$ decomposables. Thus by Theorem 4.9, each $y_i\in\pi_*(X_k)$ $(k\le i\le 2k)$ is nonnilpotent. It remains to prove $y_{i_1}y_{i_2}=y_{i_2}y_{i_1}$ in $\pi_*(X_k)$ for $k\le i_1<i_2\le 2k-1$. But this follows from Theorem 4.7. □

5. NONCOMMUTATIVITY

In this section we will make some remarks on the noncommutativity on the spectra X_k, $M(n)$, and $\mathrm{BP}\wedge X_k$ and on the commutativity on $H_*(X_k)$, $H_*(M(n))$, and BP_*X_k.

Recall the following theorem due to M. Hopkins.

Theorem 5.1 [14, 2.2.1]. *Let E be a commutative ring spectrum. If $\pi_*(E)$ contains an invertible element of order two then E is weakly equivalent to a wedge of suspensions of Eilenberg-Mac Lane spectra.*

Proposition 5.2. *X_k, $M(n)$, and $\mathrm{BP} \wedge X_k$ are noncommutative ring spectra for $K \geq 1$ and $n \geq 2$ respectively.*

Proof. We first prove that X_k $(k \geq 1)$ is a noncommutative ring spectrum. Since the pull-back of the first Stiefel-Whitney class is nonzero for $\Omega J_{2^k-1}S^2 \to \mathrm{BO}$ $(k \geq 1)$, then $\pi_0(X_k) = \mathbb{Z}/2$, that is, the unit is the invertible element of order two. If X_k were a commutative ring spectrum, then X_k would be a wedge of suspensions of Eilenberg-Mac Lane spectra by using Theorem 5.1. So the cohomology of X_k were a free module over the Steenrod algebra, and one has the ring map $X_k \to H(\mathbb{Z}/2)$, which would induce cohomology injective, a contradiction. An analogous argument holds for $M(n)$ $(n \geq 2)$. So it remains to prove that $\mathrm{BP} \wedge X_k$ is a noncommutative ring spectrum. For this, the analogous arguments as above can be applied to the map $\mathrm{BP} \wedge X_k \to H(\mathbb{Z}/2) \wedge H(\mathbb{Z}/2) \to H(\mathbb{Z}/2)$, where the maps $\mathrm{BP} \to H(\mathbb{Z}/2)$, $X_k \to H(\mathbb{Z}/2)$ are the usual maps respectively, and the second map is the usual multiplication map of the Eilenberg-Mac Lane spectrum $H(\mathbb{Z}/2)$. This completes the proof.

Proposition 5.3. *$H_*(X_k)$ and $H_*(M(n))$ are commutative rings for $k \geq 1$ and $n \geq 2$.*

Proof. Since $H_*(\Omega J_{2^k-1}S^2)$ and $H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ are commutative rings, via Thom isomorphism, the proof follows immediately.

Now it remains to prove that BP_*X_k is a commutative ring. As described in the introduction, the proof will rely on the following:

$$\mathrm{BP} \wedge T(k) \xrightarrow{1\wedge h} \mathrm{BP} \wedge M(2^{k+1}-1) \xrightarrow{1\wedge f} \mathrm{BP} \wedge X_k,$$

that is,

$$\mathrm{BP}_*T(k) \xrightarrow{h_*} \mathrm{BP}_*M(2^{k+1}-1) \xleftarrow{f_*} \mathrm{BP}_*X_k.$$

Then the commutativity of BP_*X_k will follow from $\mathrm{Im}\, f_* \subseteq \mathrm{Im}\, h_*$ since h_*, f_* are ring maps, $\mathrm{BP}_*T(k)$ is a commutative ring, and f_* is injective.

Theorem 5.4. *BP_*X_k is a commutative ring.*

The proof uses the following lemma.

Lemma 5.5. *The 2-primary Adams spectral sequence for each of*

$$\mathrm{BP}_*M(2^{k+1}-1),$$

and BP_*X_k collapses at the E_2-term. Moreover,

(a) *The E_2-term for $\mathrm{BP}_*M(2^{k+1}-1)$ is*

$$\overline{E}_2^{*,*} \cong \mathbb{Z}/2[e_{k+1}, e_{k+2}, \dots] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\overline{\beta}_1^2, \overline{\beta}_2^2, \overline{\beta}_3^2, \dots, \overline{\beta}_k^2] \otimes_{\mathbb{Z}/2} \overline{M},$$

where $\overline{\beta}_i^2$ $(1 \leq i \leq k)$ is represented by $[\xi_i^2]$ in the cobar complex, $\mathrm{bideg}(\overline{\beta}_i^2) = (0, |\xi_i^2|)$, $\overline{M} = \mathbb{Z}/2[\overline{\alpha}_i / i \neq 2^l - 1,\ \deg \overline{\alpha}_i = i$, *and* $i \leq 2^{k+1} - 2]$, *$\overline{\alpha}_i$ is represented by $[\alpha_i]$ in the cobar complex, and* $\mathrm{bideg}(\overline{\alpha}_i) = (0, i)$.

(b) *The* E_2*-term for* BP_*X_k *is*

$$\overline{\overline{E}}_2^{*,*} \cong \mathbb{Z}/2[e_{k+1}, e_{k+2}, \dots] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\overline{\overline{\beta}}_1^2, \overline{\overline{\beta}}_2^2, \overline{\overline{\beta}}_3^2, \dots, \overline{\overline{\beta}}_k^2],$$

where $\overline{\overline{\beta}}_i^2$ $(1 \le i \le k)$ *is represented by* $[\xi_i^2]$ *in the cobar complex, and* $\mathrm{bideg}(\overline{\overline{\beta}}_i^2) = (0, |\xi_i^2|)$.

Proof. Applying Proposition 1.3 and Theorem 1.4 to a direct computation of the desired Ext-groups for $\mathrm{BP}_*M(2^{k+1}-1)$ and BP_*X_k respectively, (a) and (b) follow immediately. Since $\overline{\overline{E}}_2^{s,t} = 0$ for $t-s=$ even, the Adams spectral sequence for BP_*X_k collapses. It remains to show that the Adams spectral sequence for $\mathrm{BP}_*M(2^{k+1}-1)$ collapses at the E_2-term. Since $\mathrm{BP} \wedge M(2^{k+1}-1)$ is a ring spectrum, the Adams spectral sequence for $\mathrm{BP}_*M(2^{k+1}-1)$ has a corresponding ring structure. So to finish the proof, it suffices to show each of $\overline{\beta}_i^2$ and $\overline{\alpha}_i$ is a permanent cycle. Nevertheless by Lemma 1.6, we have a ring map of ring spectra $f\colon X_k \to M(2^{k+1}-1)$, hence a homomorphism $f_*\colon \overline{\overline{E}}_2^{*,*} \to \overline{E}_2^{*,*}$. So $\overline{\beta}_i^2$ is a permanent cycle, which is corresponding to $\overline{\overline{\beta}}_i^2$. Finally that $\overline{\alpha}_i$ is a permanent cycle follows from the natural ring map $M(2^{k+1}-1) \to \mathrm{MO}$ induces an epimorphism for $* \le 2^{k+1}-2$ on

$$\pi_*(M(2^{k+1}-1)) \to \pi_*(\mathrm{MO}),$$

that is, each $\alpha_i \in H_*(M(2^{k+1}-1))$ is stably spherical, hence each $\overline{\alpha}_i$ is a permanent cycle. This completes the proof.

Proof of Theorem 5.4. Consider the following commutative diagram, where each map is a ring map:

$$\begin{array}{ccccc}
\mathrm{BP} \wedge S^0 & = & \mathrm{BP} \wedge S^0 & = & \mathrm{BP} \wedge S^0 \\
\Big\downarrow{\scriptstyle 1\wedge\eta} & & \Big\downarrow{\scriptstyle 1\wedge\overline{\eta}} & & \Big\downarrow{\scriptstyle 1\wedge\overline{\overline{\eta}}} \\
\mathrm{BP} \wedge T(k) & \xrightarrow{1\wedge h} & \mathrm{BP} \wedge M(2^{k+1}-1) & \xleftarrow{1\wedge f} & \mathrm{BP} \wedge X_k.
\end{array}$$

Here $\eta\colon S^0 \to T(k)$, $\overline{\eta}\colon S^0 \to M(2^{k+1}-1)$, and $\overline{\overline{\eta}}\colon S^0 \to X_k$ are units, and the ring maps h and f are from Corollary 2.5 and Lemma 1.6 respectively. Then by Lemma 5.5, it is not difficult to solve the group extension problem in the Adams spectral sequence for $\mathrm{BP}_*M(2^{k+1}-1)$ and we have

$$\begin{aligned}
\mathrm{BP}_*M(2^{k+1}-1) &\cong \mathbb{Z}/2[v_k, v_{k+1}, \dots] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\overline{t}_1, \overline{t}_2, \dots, \overline{t}_k] \otimes_{\mathbb{Z}/2} \overline{N} \\
&\cong (\mathrm{BP}_*/I_k) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\overline{t}_1, \overline{t}_2, \overline{t}_3, \dots, \overline{t}_k] \otimes_{\mathbb{Z}/2} \overline{N}
\end{aligned}$$

as $\mathbb{Z}/2$-vector spaces, where $\overline{t}_i$ $(1 \le i \le k)$ is detected by $\overline{\beta}_i^2$, $\overline{N} = \mathbb{Z}/2[\overline{\overline{\alpha}}_i / i \ne 2^l - 1,\ \deg\overline{\overline{\alpha}}_i = i, \text{ and } i \le 2^{k+1}-2]$, and $\overline{\overline{\alpha}}_i$ is detected by $\overline{\alpha}_i$.

Also by Theorem 2.1 and Lemma 5.5, as in the proof of Theorem A, we still make a choice of generators of BP_*X_k corresponding to the generators of $\mathrm{BP}_*T(k)$. So

$$\mathrm{BP}_*X_k \cong (\mathrm{BP}_*/I_k)[t_1, t_2, t_3, \dots, t_k]$$

as $\mathbb{Z}/2$-vector spaces, where t_i $(1 \le i \le k)$ is detected by $\overline{\overline{\beta}}_i^2$.

Warning. Both above isomorphisms are only as $\mathbb{Z}/2$-vector spaces. Hence by Theorem 2.1, Corollary 2.5, and Lemmas 1.6, 3.1, and 3.2, one has

$$\begin{array}{ccc} \mathrm{BP}_*T(k) & \xrightarrow{\cong} & \mathrm{BP}_*[t_1, t_2, t_3, \dots, t_k] \\ \big\downarrow h_* & & \big\downarrow h_* \\ \mathrm{MP}_*M(2^{k+1}-1) & \xrightarrow{\cong} & (\mathrm{BP}_*/I_k) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k] \otimes_{\mathbb{Z}/2} \overline{N} \\ \big\uparrow f_* & & \big\uparrow f_* \\ \mathrm{BP}_*X_k & \xrightarrow{\cong} & (\mathrm{BP}_*/I_k)[t_1, t_2, \dots, t_k], \end{array}$$

where $h_*(v_i) = v_i$ for $i \geq 0$, $f_*(v_i) = v_i$ for $i \geq k$, $h_*(t_i) = \bar{t}_i$ + higher filtration, and $f_*(t_i) = \bar{t}_i$ + higher filtration.

Now since $\mathrm{BP}_*T(k)$ is a commutative ring, h_* and f_* are ring homomorphisms, and f_* is injective (by Theorem 1.5), to finish the proof, it suffices to prove that $\mathrm{Im}\, f_* \subseteq \mathrm{Im}\, h_*$ and this follows from the following two steps.

Step 1. $(\mathrm{BP}_*/I_k) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k] \subseteq \mathrm{Im}\, h_*$. Since h_* is a ring map, it suffices to prove that $\bar{t}_i \in \mathrm{Im}\, h_*$ for $1 \leq i \leq k$. However in the Adams spectral sequence for $\mathrm{BP}_*M(2^{k+1}-1)$, $\overline{E}_2^{s, *+s} = 0$ for $s \geq 1$ and $* \leq 2^{k+1} - 3$ (by Lemma 5.5), that is,

$$\overline{F}^1\mathrm{BP}_*M(2^{k+1}-1) = \overline{F}^2\mathrm{BP}_*M(2^{k+1}-1) - \overline{F}^3\mathrm{BP}_*M(2^{k+1}-1) = \cdots = 0$$

for $* \leq 2^{k+1} - 3$, where $\overline{F}^i\mathrm{BP}_*M(2^{k+1}-1)$ is the Adams ith filtration of $\mathrm{BP}_*M(2^{k+1}-1)$. So combining this with $h_*(t_i) = \bar{t}_i$ + higher filtration, we have $h_*(t_i) = \bar{t}_i$ for $i \leq k-1$ since $2(2^{k-1}-1) \leq 2^{k+1} - 3$. Furthermore, for $\bar{t}_k$, we still have $h_*(t_k) = \bar{t}_k$ + higher filtration. Since $\overline{E}_2^{1, 2^{k+1}} = \mathbb{Z}/2$ and

$$\overline{E}_2^{2, 2^{k+1}} = \overline{E}_2^{3, 2^{k+1}+1} = \cdots = \overline{E}_2^{l+2, 2^{k+1}+l} = \cdots = 0,$$

that is, $\overline{F}^1\mathrm{BP}_{2^{k+1}-2}(M(2^{k+1}-1)) = \mathbb{Z}/2$, so $h_*(t_k) = \bar{t}_k$ or $h_*(t_k) = \bar{t}_k + v_k$. For the second case, we have $h_*(t_k + v_k) = \bar{t}_k$. This completes the proof of step 1.

Step 2. $f_*(\mathrm{BP}_*X_k) \subseteq (\mathrm{BP}_*/I_k) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k]$. By an argument analogous to that in step 1, one has $f_*(t_i) = \bar{t}_i$ for $i \leq k-1$, and $f_*(t_k) = \bar{t}_k$ or $f_*(t_k) = \bar{t}_k + v_k$. This completes the proof of Theorem 5.4.

6. The vanishing lemmas

In this final section we will prove the two vanishing lemmas, Propositions 4.6 and 4.8, by Proposition 4.4 and a very basic argument.

Lemma 6.1.

$$\sum_{p=1}^{r+1} 2^{i_p}(2^{k+j_p} - 1) \neq 2^{k+j} + (r-2)$$

for $1 \leq j \leq k+1$, $2 \leq r \leq 2^{j-1} - 1$, $i_p \geq 0$, $k+1 \geq j_p \geq 1$, *and* $2^{i_p}(2^{k+j_p} - 1) < 2^{2k+2} - 1$ *for each* p.

Proof of Proposition 4.6. Since $1 \leq j \leq k+1$, $2 \leq r \leq 2^{j-1} - 1$,

$$(2^{k+j} - 1) + (r-1) \leq (2^{2k+1} - 1) + (2^k - 1) < 2^{2k+2} - 1.$$

Hence $E_2^{r+1,(2^{k+j}-1)+(r-1)}$ is in our range for Proposition 4.4. So Proposition 4.6 follows from Lemma 6.1 and Proposition 4.4. □

The proof of Lemma 6.1 will be by contradiction.

Proof of Lemma 6.1. First we discuss $r \geq 3$. Suppose

$$(*) \qquad \sum_{p=1}^{r+1} 2^{i_p}(2^{k+j_p}-1) = 2^{k+j} + (r-2),$$

for $1 \leq j \leq k+1$, $2 \leq r \leq 2^{j-1}-1$, $i_p \geq 0$, $k+1 \geq j_p \geq 1$, and $2^{i_p}(2^{k+j_p}-1) < 2^{2k+2}-1$ for each p. So one has the following easy facts:

(a) $r+1 \leq 2^{j-1}$.

(b) $\max(i_1, i_2, i_3, \dots, i_{r+1}) \leq j-2$ since

$$2^{j-1}(2^{k+1}-1) + (2^{k+1}-1) + \cdots > 2^{k+j} + 2^k - 3 \geq 2^{k+j} + (r-2).$$

(c) If $(r-2) = a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0$, $a_i = 0$ or 1 for each i, then there exists a_i such that $a_i = 1$ $(r \geq 3)$ and $l \leq j-2$ (since $2^{j-1}-1 = 2^{j-2} + 2^{j-3} + \cdots + 2 + 1$).

Claim. Let $u = \min(i_1, i_2, i_3, \dots, i_{r+1})$ in the whole sum of $(*)$. Then $r+1 \leq 2^{j-u-1}-1$. Furthermore if $r+1 = 2^{j-u-1}-1$, then in the whole sum of $(*)$

(a) $\forall p$, $2^{i_p}(2^{k+j_p}-1) = 2^u(2^{k+j}-1)$ or

(b) the number of $2^u(2^{k+1}-1)$'s is $(2^{j-u-1}-2)$ and there is only one p such that $2^{i_p}(2^{k+j_p}-1) = 2^{u+1}(2^{k+1}-1)$ or $2^u(2^{k+2}-1)$. That is to say, no p such that $2^{i_p}(2^{k+j_p}-1) = 2^{u+2}(2^{k+1}-1)$, or $2^u(2^{k+3}-1)$, or $2^{u+3}(2^{k+1}-1)$, or $2^u(2^{k+4}-1)$, ..., etc.

Proof. We prove $r+1 \leq 2^{j-u-1}-1$.

Case 1. If $r+1 > 2^{j-u-1}$, then

$$2^{j-u-1} \cdot 2^u(2^{k+1}-1) + (2^{k+1}-1) + \cdots > 2^{k+j} + 2^k - 3 \geq 2^{k+j} + (r-2).$$

Hence $r+1 \leq 2^{j-u-1}$.

Case 2. Suppose $r+1 = 2^{j-u-1} \geq 3$ $(r \geq 2)$. If there exists one $i_p \geq u+1$, then

$$\sum_{p=1}^{r+1} 2^{i_p}(2^{k+j_p}-1) \geq 2^{j-u-1} \cdot 2^u(2^{k+1}-1) + (2^{k+1}-1),$$

since $2^{u+1}(2^{k+1}-1) - 2^u(2^{k+1}-1) \geq 2^u(2^{k+1}-1)$. But

$$2^{j-u-1} \cdot 2^u(2^{k+1}-1) + (2^{k+1}-1) = 2^{k+j} - 2^{j-1} + (2^{k+1}-1) > 2^{k+j} + 2^k - 3,$$

and

$$2^{k+j} + 2^k - 3 \geq 2^{k+j} + (r-2) = 2^{k+j} + (2^{j-u-1}-3).$$

This would prove that $\sum_{p=1}^{r+1} 2^{i_p}(2^{k+j_p}-1) > 2^{k+j} + (r-2)$. Hence in case 2, $\forall p$ $i_p = u$. By an analogous argument one can also prove $\forall p$ $j_p = 1$. Therefore $2^{j-u-1} \cdot 2^u(2^{k+1}-1) = 2^{k+j} + (2^{j-u-1}-3)$. So

$$2^{k+j} - 2^{j-1} = 2^{k+j} + (2^{j-u-1}-3) \Rightarrow 3 = 2^{j-1} + 2^{j-u-1}$$

but $2^{j-u-1} \geq 3$, a contradiction.

Combining cases 1 and 2, we prove $r+1 \le 2^{j-u-1}-1$.

$$\begin{aligned}
&2^{k+1}(2^u-1) > 2^u-3 \quad \text{since } 0 \le u \le j-2 \le k-1\\
&\quad \Rightarrow 2^u \cdot 2^{k+1} > 2^{k+1} + 2^u - 3\\
&\quad \Rightarrow 2^{k+u+1} > 2^k + 2^{k-u} + 2^u - 3\\
&\quad \Rightarrow 2^{k+u+1} + 3\cdot 2^u > 2^{j-1} + 2^{j-u-1} + 4\cdot 2^u - 3 \quad \text{since } 1 \le j \le k+1\\
&\quad \Rightarrow 2^{k+j} - 2^{j-1} - 3\cdot 2^{u+k+1} + 3\cdot 2^u\\
&\qquad + 4\cdot 2^{k+u+1} - 4\cdot 2^u > 2^{k+j} + 2^{j-u-1} - 3\\
&\quad \Rightarrow (2^{j-u-1}-3)\cdot 2^u(2^{k+1}-1) + 2\cdot 2^{u+1}(2^{k+1}-1) > 2^{k+j} + (r-2).
\end{aligned}$$

This proves that in the whole sum of $(*)$, there is not more than one p such that

$$2^{i_p}(2^{k+j_p}-1) = 2^{u+1}(2^{k+1}-1).$$

Moreover, $2^{u+1}(2^{k+1}-1)$ is the smallest term in the whole sum of $(*)$ which is bigger than $2^u(2^{k+1}-1)$. This finishes the proof of the second statement. □

Consider a partition of the whole sum of $(*)$, i.e.,

$$(**)\qquad \begin{aligned}
\sum_{p=1}^{r+1} 2^{i_p}(2^{k+j_p}-1) &= \sum_{l_u} 2^u(2^{k+j_{u+1,*}}-1) + \sum_{l_{u+1}} 2^{u+\bar{i}_1}(2^{k+j_{u+2,*}}-1)\\
&\quad + \sum_{l_{u+2}} 2^{u+\bar{i}_2}(2^{k+j_{u+3,*}}-1) + \sum_{l_{u+3}} 2^{u+\bar{i}_3}(2^{k+j_{u+4,*}}-1)\\
&\quad + \cdots + \sum_{l_q} 2^{u+\bar{i}_{q-u}}(2^{k+j_{q+1,*}}-1)\\
&= 2^{k+j} + (r-2),
\end{aligned}$$

where $r+1 = l_u + l_{u+1} + l_{u+2} + \cdots + l_q$, $u + \bar{i}_{q-u} \le j-2$, and $1 \le \bar{i}_1 < \bar{i}_2 < \bar{i}_3 < \bar{i}_4 < \cdots < \bar{i}_{q-u} \le j-u-2$. Now from the claim and the partition, we come to our key observations:

(A) $\quad l_u \le [(2^{j-u-1}-1) - 2^{\bar{i}_1} l_{u+1} - 2^{\bar{i}_2} l_{u+2} - 2^{\bar{i}_3} l_{u+3} - \cdots - 2^{\bar{i}_{q-u}} l_{u+(q-u)}]$

or

(B) $\quad l_u \le [(2^{j-u-1}-1) - 1 - 2l'_{u+1} - 2^{\bar{i}_2} l_{u+2} - 2^{\bar{i}_3} l_{u+3} - \cdots - 2^{\bar{i}_{q-u}} l_{u+(q-u)}]$;
in this case $\bar{i}_1 = 1$, $l_{u+1} = l'_{u+1} + 1$, and $l'_{u+1} \ge 0$.

If the assumption were true, we would have a contradiction in the following cases: (i) case (B) $(\bar{i}_1 = 1)$, and

$$l_u = [(2^{j-u-1}-1) - 1 - 2l'_{u+1} - 2^{\bar{i}_2} l_{u+2} - 2^{\bar{i}_3} l_{u+3} - \cdots - 2^{\bar{i}_{q-u}} l_{u+(q-u)}],$$

$l_{u+1} = 1 + l'_{u+1}$. So

$$\begin{aligned}
-l_u &= 2 - 2^{j-u-1} + (2l'_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q)\\
&= (-2^{j-u-2} - 2^{j-u-3} - \cdots - 2^2 - 2)\\
&\quad + (2l'_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q).
\end{aligned}$$

The left side of $(**)$ is divisible by 2^u, so the right side is also divisible by 2^u. Since $0 \le u \le j-2$, $(r-2)$ is divisible by 2^u. Hence in $(**)$, after dividing by 2^u, one has

$$\begin{aligned}\left(\sum_{l_u} 2^{k+j_{u+1,*}}\right) &+ (-2^{j-u-2} - 2^{j-u-3} - \cdots - 2^2 - 2)\\ &+ (2l'_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q) + \sum_{l_{u+1}} 2^{\bar{i}_1}(2^{k+j_{u+2,*}} - 1)\\ &+ \sum_{l_{u+2}} 2^{\bar{i}_2}(2^{k+j_{u+3,*}} - 1) + \cdots + \sum_{l_q} 2^{\bar{i}_{q-u}}(2^{k+j_{q+1,*}} - 1)\\ &= 2^{k+j-u} + (a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0)/2^u,\end{aligned}$$

where $r-2 = a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 \cdot 2 + a_0 \ne 0$. Again the left side is divisible by 2 since $l_u > 0$, so $u < j-2$. Hence the right side is divisible by 2. In the above equality, after dividing by 2, and noting $l_{u+1} = l'_{u+1} + 1$, one has

$$\begin{aligned}\left(\sum_{l_u} 2^{k+j_{u+1,*}-1}\right) &+ (-2^{j-u-2}) + (2^{\bar{i}_2-1} l_{u+2} + 2^{\bar{i}_3-1} l_{u+3} + \cdots + 2^{\bar{i}_{q-u}-1} l_q)\\ &+ \left(\sum_{l_{u+1}} 2^{k+j_{u+2,*}}\right) + \left[\sum_{l_{u+2}} 2^{\bar{i}_2-1}(2^{k+j_{u+3,*}} - 1)\right]\\ &+ \left[\sum_{l_{u+3}} 2^{\bar{i}_3-1}(2^{k+j_{u+4,*}} - 1)\right] + \cdots + \left[\sum_{l_q} 2^{\bar{i}_{q-u-1}}(2^{k+j_{q+1,*}} - 1)\right]\\ &= 2^{k+j-u-1} + (a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0)/2^{u+1}.\end{aligned}$$

We are ready to get a contradiction from the above equality. If $j_* \ge 1$, then $k + j_* - 1 - (j-u-2) \ge u+1$, so the left side is divisible by 2^{j-u-2}. Hence the right side is also divisible by 2^{j-u-2}. However, in the right side $2^{k+j-u-1}$ is divisible by 2^{j-u-2} since $(k+j-u-1) - (j-u-2) \ge k+1$. It follows that $(a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0)/2^{u+1}$ would be divisible by 2^{j-u-2}. But $(a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0)/2^{u+1}$ is positive integer since $r \ge 3$, so in

$$r - 2 = a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_i 2^i + \cdots + a_1 2 + a_0,$$

$l \ge 1$, $a_i = 0$ for $i \le u$, and there exists $a_i = 1$ for some $i \ge u+1$. Moreover $r + 1 \le 2^{j-u-1} - 1$ from the claim, so $l \le j-u-2$. Thus in

$$\begin{aligned}&(a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0)/2^{u+1}\\ &\quad = a_l 2^{l-(u+1)} + a_{l-1}2^{l-1-(u+1)} + \cdots + a_{u+2}2 + a_{u+1},\end{aligned}$$

one has $l - (u+1) \le j - u - 2 - (u+1) \le j - 2u - 3$. This is a contradiction.

(ii) case (B) $(\bar{i}_1 = 1)$, and

$$0 < l_u < [(2^{j-u-1} - 1) - 1 - 2l'_{u+1} - 2^{\bar{i}_2} l_{u+2} - \cdots - 2^{\bar{i}_{q-u}} l_{u+(q-u)}],$$

i.e., there is a positive integer Q, $2^{j-u-1} - 2 > Q$, such that

$$0 < l_u = [(2^{j-u-1} - 1) - 1 - Q - 2l'_{u+1} - 2^{\bar{i}_2} l_{u+2} - \cdots - 2^{i_{q-u}} l_{u+(q-u)}].$$

Again the left side of $(**)$ is divisible by 2^u, so the right side is also divisible by 2^u. Hence in $(**)$, after dividing by 2^u, one has

$$\begin{aligned}&\left(\sum_{l_u} 2^{k+j_{u+1,*}}\right) + (-2^{j-u-2} - 2^{j-u-3} - \cdots - 2^2 - 2 + Q)\\ &\quad + (2l'_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_q} l_q) + \sum_{l_{u+1}} 2^{\bar{i}_1}(2^{k+j_{u+2,*}} - 1)\\ &\quad + \sum_{l_{u+2}} 2^{\bar{i}_2}\left(2^{k+j_{u+3,*}} - 1) + \cdots + \sum_{l_q} 2^{\bar{i}_{q-u}}(2^{k+j_{q+1,*}} - 1\right)\\ &= 2^{k+j-u} + (a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0)/2^u.\end{aligned}$$

Note. $-l_u < 0$, so $(-2^{j-u-2} - 2^{j-u-3} - \cdots - 2^2 - 2 + Q) < 0$. Let

$$(-2^{j-u-2} - 2^{j-u-3} - \cdots - 2^2 - 2 + Q) = -\bar{a}_s 2^s - \bar{a}_{s-1}2^{s-1} - \cdots - \bar{a}_1 2 - \bar{a}_0;$$

then $s \le j - u - 2$. And

$$(r-2) = a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0,$$

since $(r-2)/2^u$ is a positive integer and $l \le j - u - 2$, so in

$$(r-2)/2^u = a_l 2^{l-u} + a_{l-1}2^{l-1-u} + \cdots + a_{u+1}2 + a_u,$$

$l - u \le j - 2u - 2$, and there exists $a_i = 1$ for some $i \ge u$. Hence

$$\begin{aligned}&\left(\sum_{l_u} 2^{k+j_{u+1,*}}\right) + (-\bar{a}_s 2^s - \bar{a}_{s-1}2^{s-1} - \cdots - \bar{a}_1 2 - \bar{a}_0)\\ &\quad + (2l'_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q) + \sum_{l_{u+1}} 2^{\bar{i}_1}(2^{k+j_{u+2,*}} - 1)\\ &\quad + \sum_{l_{u+2}} 2^{\bar{i}_2}(2^{k+j_{u+3,*}} - 1) + \cdots + \sum_{l_q} 2^{\bar{i}_{q-u}}(2^{k+j_{q+1,*}} - 1)\\ &= 2^{k+j-u} + (a_l 2^{l-u} + a_{l-1}2^{l-1-u} + \cdots + a_{u+1}2 + a_u),\end{aligned}$$

where $s \le j - u - 2$ and $l - u \le j - 2u - 2$.

In the following, unless otherwise stated, we are using the following ambiguous notation: in

$$C_a 2^{a-b} + C_{a-1}2^{a-1-b} + C_{a-2}2^{a-2-b} + \cdots + C_{b+1}2 + C_b,$$

if $a \ge b$, then $C_i = 0$ or 1 for $b \le i \le a$, otherwise $a_i = 0$ for each i. For example in the above $(a_l 2^{l-u-1} + a_{l-1}2^{l-1-u-1} + \cdots + a_{u+1})$, if $l < u - 1$, then $a_i = 0$ for each i.

Now in the above equality, $\bar{a}_0$ depends on a_u, indeed

Case (a): If $a_u = 0$, then $\bar{a}_0 = 0$. Hence Q is an even number.

Case (b): If $a_u = 1$, then $\bar{a}_0 = 1$. Hence Q is an odd number. Then we move $\bar{a}_0$ into the right side of the above equality.

In both cases, we divide the resulting equality by 2. So

$$\left(\sum_{l_u} 2^{k+j_{u+1,*}-1}\right) + (-\bar{a}_s 2^{s-1} - \bar{a}_{s-1}2^{s-2} - \cdots - \bar{a}_2 2 - \bar{a}_1) + (-1)$$
$$+ (2^{\bar{i}_2-1} l_{u+2} + 2^{\bar{i}_3-1} 1_{u+3} + \cdots + 2^{\bar{i}_{q-u}-1} l_q) + \left(\sum_{l_{u+1}} 2^{k+j_{u+2,*}}\right)$$
$$+ \left[\sum_{l_{u+2}} 2^{\bar{i}_2-1}(2^{k+j_{u+3,*}} - 1)\right] + \cdots + \left[\sum_{l_q} 2^{\bar{i}_{q-u}-1}(2^{k+j_{q+1,*}} - 1)\right]$$
$$= 2^{k+j-u-1} + (a_l 2^{l-u-1} + a_{l-1}2^{l-1-u-1} + \cdots + a_{u+2}2 + a_{u+1} + \varepsilon),$$

where $\varepsilon = 0$ for case (a) and $\varepsilon = 1$ for case (b), and (-1) appears from $l_{u+1} = 1 + l'_{u+1}$.

Express

$$(-\bar{a}_s 2^{s-1} - \bar{a}_{s-1}2^{s-2} - \cdots - \bar{a}_2 2 - \bar{a}_1) + (-1) = -b_t 2^t - b_{t-1}2^{t-1} - \cdots - b_1 2 - b_0,$$

$b_i = 0$ or 1 for each i. If $\bar{a}_0 = 0$, i.e., Q is an even number, then

$$-\bar{a}_s 2^{s-1} - \bar{a}_{s-1}2^{s-2} - \cdots - \bar{a}_1 = (-2^{j-u-3} - 2^{j-u-4} - \cdots - 2 - 1 + Q/2).$$

So

$$\begin{aligned} &-\bar{a}_s 2^{s-1} - \bar{a}_{s-1}2^{s-2} - \cdots - \bar{a}_1 - 1 \\ &\quad = (-2^{j-u-3} - 2^{j-u-4} - \cdots - 2 - 1) + (Q/2 - 1) \\ &\quad = (-2^{j-u-2} + 1) + (Q/2 - 1). \end{aligned}$$

But $(Q/2 - 1) \geq 0$, and it follows that $t \leq j - u - 3$.

If $\bar{a}_0 = 1$, i.e., Q is an odd number, then let $Q = 2n + 1$, $n \geq 0$; then

$$-\bar{a}_s 2^s - \bar{a}_{s-1}2^{s-1} - \cdots - \bar{a}_1 2 = (-2^{j-u-2} - 2^{j-u-3} - \cdots - 2^2 - 2) + (2n + 1) + 1.$$

So

$$(-\bar{a}_s 2^{s-1} - \bar{a}_{s-1}2^{s-2} - \cdots - \bar{a}_2 2 - \bar{a}_1) + (-1) = -2^{j-u-2} + 1 + n,$$

so $t \leq j - u - 3$. Thus

(* * *)

$$\left(\sum_{l_u} 2^{k+j_{u+1,*}-1}\right) + (-b_t 2^t - b_{t-1}2^{t-1} - \cdots - b_1 2 - b_0)$$
$$+ (2^{\bar{i}_2-1} l_{u+2} + 2^{\bar{i}_3-1} l_{u+3} + \cdots + 2^{\bar{i}_{q-u}-1} l_q) + \left(\sum_{l_{u+1}} 2^{k+j_{u+2,*}}\right)$$
$$+ \left[\sum_{l_{u+2}} 2^{\bar{i}_2-1}(2^{k+j_{u+3,*}} - 1)\right] + \cdots + \left[\sum_{l_q} 2^{\bar{i}_{q-u}-1}(2^{k+j_{q+1,*}} - 1)\right]$$
$$= 2^{k+j-u-1} + (a_l 2^{l-u-1} + a_{l-1}2^{l-1-u-1} + \cdots + a_{u+2}2 + a_{u+1} + \varepsilon),$$

$t \leq j - u - 3$, $l - u - 1 \leq j - 2u - 3$, and $\varepsilon = 0$ or 1. Furthermore,

$$(-b_t 2^t - b_{t-1}2^{t-1} - \cdots - b_1 2 - b_0) < 0,$$
$$(a_l 2^{l-u-1} + a_{l-1}2^{l-1-u-1} + \cdots + a_{u+2}2 + a_{u+1} + \varepsilon) > 0.$$

We can get a contradiction in the following cases:

(1) If $j-2u-3 \le j-u-3 = 0$, then $b_i = 0$ for $t \ge i \ge 1$ and $a_i = 0$ for $l \ge i \ge u+2$. Hence $a_{u+1}+\varepsilon = 1$ and $b_0 = 1$. Then we move $b_0 = 1$ into the right side and divide the resulting equality by 2. A contradiction.

(2) If $j-u-3 \ge 1$ and $a_{u+1}+\varepsilon = 2$ or 1, then $b_0 = 0$ or 1 respectively. If $b_0 = 1$, one moves $b_0 = 1$ into the right side of $(***)$. On both cases we divide the resulting equalities by 2. Then one has

$$\left(\sum_{l_u} 2^{k+j_{u+1,*}-2}\right) + (-b_t 2^{t-1} - b_{t-1}2^{t-2} - \cdots - b_2 2 - b_1)$$
$$+ (2^{\bar{i}_2-2}l_{u+2} + 2^{\bar{i}_3-2}l_{u+3} + \cdots + 2^{\bar{i}_{q-u}-2}l_q) + \left(\sum_{l_{u+1}} 2^{k+j_{u+2,*}-1}\right)$$
$$+ \left[\sum_{l_{u+2}} 2^{\bar{i}_2-2}(2^{k+j_{u+3,*}}-1)\right] + \cdots + \left[\sum_{l_q} 2^{\bar{i}_{q-u}-2}(2^{k+j_{q+1,*}}-1)\right]$$
$$= 2^{k+j-u-2} + (a_l 2^{l-u-2} + a_{l-1}2^{l-u-3} + \cdots + a_{u+3}2 + a_{u+2} + 1).$$

Note. In the above equality, if $\bar{i}_2 - 2 = 0$, then l_{u+2} can be canceled by $[\sum_{l_{u+2}} 2^{\bar{i}_2-2}(2^{k+j_{u+3,*}}-1)]$. So one has $a_{u+2}+1 = 2$ or 1 depending on a_{u+2}, hence one can determine b_1. Then repeat the same argument as above, i.e., moving b_1 into the right side (if necessary), then dividing by 2 (if necessary). Hence we have a similar form on the right side of the resulting equality, i.e., $(a_l 2^{l-u-3}+a_{l-1}2^{l-u-4}+\cdots+a_{u+4}2+a_{u+3}+1)$. If we continue the above process, we would have

$$\left(\sum_{l_u} 2^{k+j_{u+1,*}-2-v}\right) + \text{even numbers} = 2^{k+j-u-2-v} + 2,$$

where $0 \le v = \max(t-1,\ 1-u-2) \le j-u-4$ (if $t<1$, $l-u<2$, then $v=0$). But $k+j-u-2-v \ge k+2 \ge 3$, $k+j_{u+1,*}-2-v \ge 2+u \ge 2$, and the even numbers in the left side is divisible by 2^2. Therefore, one divides the above equality by 2 again, contradiction.

(3) If $a_{u+1}+\varepsilon = 0$, then $j-u-3 \ge 1$. Let $w = \min\{i | u+2 \le i \le l,\ a_i = 1\}$. Then $a_w = 1$ since $(a_l 2^{l-u-1} + a_{l-1}2^{l-1-u-1} + \cdots + a_{u+2}2 + a_{u+1}+\varepsilon) > 0$. So in $(***)$, the right side is divisible by $2^{w-(u+1)}$ since $j-2u-3 < k+j-u-1$ and $w-(u+1) \le l-u-1 \le j-2u-3$, hence the left side is also divisible by $2^{w-(u+1)}$. Since $k+j_{u+1,*}-1-j+2u+3 \ge 2$, $(\sum_{l_w} 2^{k+j_{u+1,*}-1})$ is divisible by $2^{w-(u+1)}$. Hence in $(-b_t 2^t - b_{t-1}2^{t-1} - \cdots - b_1 2 - b_0)$, $b_i = 0$ for $i \le w-u-2$. In $(***)$, after dividing by $2^{w-(u+1)}$, one has

$$\left(\sum_{l_u} 2^{k+j_{u+1,*}-1-w+(u+1)}\right)$$
$$+ (-b_t 2^{t-w+(u+1)} - b_{t-1}2^{t-1-w+(u+1)} - \cdots - b_{w-u-1}) + \cdots$$
$$= 2^{k+j-u-1-w+(u+1)} + (a_l 2^{l-2u-1-w+(u+1)} + \cdots + a_{w+1}2 + 1).$$

Then by an analogous argument to that in case (1), one finds a contradiction.

(iii) case (A), i.e.,

$$l_u \le [(2^{j-u-1} - 1) - 2^{\bar{i}_1} l_{u+1} - 2^{\bar{i}_2} l_{u+2} - \cdots - 2^{\bar{i}_{q-u}} l_q],$$

so there is a nonnegative integer Q such that

$$0 < l_u = [(2^{j-u-1} - 1) - Q - 2^{\bar{i}_1} l_{u+1} - 2^{\bar{i}_2} l_{u+2} - \cdots - 2^{\bar{i}_{q-u}} l_q].$$

Hence

$$\begin{aligned} -l_u = & (-2^{j-u-2} - 2^{j-u-3} - \cdots - 2 - 1 + Q) \\ & + (2^{\bar{i}_1} l_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q). \end{aligned}$$

Again in $(**)$, the left side is divisible by 2^u, hence the right side is too. After dividing by 2^u, $(**)$ becomes

$$\begin{aligned} & \left(\sum_{l_u} 2^{k+j_{u+1,*}}\right) + (-2^{j-u-2} - 2^{j-u-3} - \cdots - 2 - 1 + Q) \\ & \quad + (2^{\bar{i}_1} l_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q) + \left[\sum_{l_{u+1}} 2^{\bar{i}_1}(2^{k+j_{u+2,*}} - 1)\right] \\ & \quad + \left[\sum_{l_{u+2}} 2^{\bar{i}_2}(2^{k+j_{u+3,*}} - 1)\right] + \left[\sum_{l_{u+3}} 2^{\bar{i}_3}(2^{k+j_{u+4,*}} - 1)\right] \\ & \quad + \cdots + \left[\sum_{l_q} 2^{\bar{i}_{q-u}}(2^{k+j_{q+1,*}} - 1)\right] \\ & = 2^{k+j-u} + (a_l 2^{l-u} + a_{l-1} 2^{l-1-u} + \cdots + a_{u+1} 2 + a_u). \end{aligned}$$

Express

$$-(2^{j-u-2} + 2^{j-u-3} + \cdots + 2 + 1) + Q = -\bar{a}_s 2^s - \bar{a}_{s-1} 2^{s-1} - \cdots - a_1 2 - a_0,$$

for each $a_i = 0$ or 1. It is easy to see $s \le j - u - 2$. By the same argument as before, in

$$(r-2)/2^u = a_l 2^{l-u} + a_{l-1} 2^{l-1-u} + \cdots + a_{u+1} 2 + a_u,$$

$l - u \le j - 2u - 2$ and there is an $a_i = 1$ for $i \ge u$. Then by an argument analogous to that in (3) of case (ii), we will have a contradiction. This finishes the proof of Lemma 6.1 when $r \ge 3$. The case $r = 2$ remains. From the note following the proof of Theorem 4.5, we need only discuss the case for $j \ge 3$, hence $k \ge 2$. When $r = 2$, $(*)$ becomes

$$2^{\bar{i}_1}(2^{k+j_1} - 1) + 2^{\bar{i}_2}(2^{k+j_2} - 1) + 2^{\bar{i}_3}(2^{k+j_3} - 1) = 2^{k+j},$$

$0 \le \bar{i}_1 \le \bar{i}_2 \le \bar{i}_3$, $3 \le j \le k + 1$. If $i_1 = k - 1$, then

$$2^{k-1}(2^{k+1} - 1) + 2^{k-1}(2^{k+1} - 1) + 2^{k-1}(2^{k+1} - 1) > 2^{k+j}$$

for $k \ge 1$, which is a contradiction. So $i_1 \le k - 2$, hence $i_1 = i_2$, and

$$\begin{aligned} & 2^{k+j_1} - 1 + 2^{k+j_2} - 1 + 2^{i_3 - i_1}(2^{k+j_3} - 1) = 2^{k+j-i_1} \\ & \quad \Rightarrow 2^{k+j_1-1} + 2^{k+j_2-1} - 1 + 2^{i_3-i_1-1}(2^{k+j_3} - 1) = 2^{k+j-i_1-1} \\ & \quad \Rightarrow 2^{k+j_1-1} + 2^{k+j_2-1} - 1 + (2^{k+j_3} - 1) = 2^{k+j-i_1-1} \quad (\text{i.e., } i_3 - i_1 - 1 = 0) \\ & \quad \Rightarrow 2^{k+j_1-2} + 2^{k+j_2-2} - 1 + 2^{k+j_3-1} = 2^{k+j-i_1-2}. \end{aligned}$$

But $k \geq 2$, $i_1 \leq k-2$, which is a contradiction. This completes the proof of Lemma 6.1. □

Lemma 6.2. *In the Adams spectral sequence for* $\pi_*(X_k)$, $E_2^{r, 2^{k+j_1}} + 2^{k+j_2} + (r-4) = 0$ *for* $3 \leq r < 2^k$ *and* $k \geq j_1 > j_2 \geq 1$.

Proof of Proposition 4.8. By Proposition 4.4, if $E_2^{r, 2^{k+j_1}+2^{k+j_2}+(r-4)} \neq 0$, then

$$\begin{aligned} r(2^{k+1}-1) &\leq 2^{k+j_1} + 2^{k+j_2} + (r-4) \\ &\Rightarrow r(2^{k+1}-2) \leq 2^{k+j_1} + 2^{k+j_2} - 4 \leq 2^{2k+1} - 4 \\ &\Rightarrow r(2^{k+1}-2) \leq 2^k(2^{k+1}-2) + 2^{k+1} - 4 \\ &\Rightarrow r \leq 2^k \quad \text{since } 2^{k+1} - 2 > 2^{k+1} - 4; \end{aligned}$$

if $r = 2^k$, then

$$\begin{aligned} 2^{k+1} &> 2^{k+1} - 4 \quad \text{for } k \geq 1 \\ &\Rightarrow 2^{2k+1} - 2^k > 2^{2k} + 2^{2k-1} + (2^k - 4) \\ &\Rightarrow 2^k(2^{k+1}-1) > 2^{2k} + 2^{2k-1} + (2^k - 4) \\ &\Rightarrow 2^k(2^{k+1}-1) > 2^{k+j_1} + 2^{k+j_2} + (2^k - 4), \end{aligned}$$

which is a contradiction. So $r < 2^k$. □

Remark. Of course, one wants to ask whether it is possible to change to the range in Theorem 4.7 to $k+1 \geq j_1 > j_2 \geq 1$ or not. This will require changing the condition in Lemma 6.2 to $3 \leq r < 2^{k+1}$. However our method of proving the following lemma will not work for this range.

Lemma 6.3.

$$\sum_{p=1}^{r} 2^{i_p}(2^{k+j_p} - 1) \neq 2^{k+j_1} + 2^{k+j_2} + (r-4)$$

for $k \geq j_1 > j_2 \geq 1$, $3 \leq r < 2^k$, $j_p \geq 0$, $k+1 \geq j_p \geq 1$, *and* $2^{i_p}(2^{k+j_p}-1) < 2^{2k+2} - 1$ *for each* p.

Proof of Lemma 6.2. Since both $2^{k+j_1} + 2^{k+j_2} + (r-4) \leq 2^{2k} + 2^{2k-1} + 2^k - 4$ and $2^{2k} + 2^{2k-1} + 2^k - 4 < 2^{2k+2} - 1$ for $k \geq 1$, Lemma 6.2 follows from Proposition 4.4 and Lemma 6.3. □

To prove Lemma 6.3, basically we will follow the proof of Lemma 6.1.

Proof of Lemma 6.3. First we discuss $(r-4) \geq 1$. Suppose

$$(*) \qquad \sum_{p=1}^{r} 2^{i_p}(2^{k+j_p} - 1) = 2^{k+j_1} + 2^{k+j_2} + (r-4),$$

for $k \geq j_1 > j_2 \geq 1$, $3 \leq r < 2^k$, $i_p \geq 0$, $k+1 \geq j_p \geq 1$, and $2^{i_p}(2^{k+j_p}-1) < 2^{2k+2} - 1$ for each p. So one has the following basic facts:

(a) $3 \leq r \leq 2^k - 1$,

(b) $\max(i_1, i_2, i_3, \ldots, i_r) \leq k-1$ since

$$\begin{aligned} 2^{2k-1} &> 2^{k+1} - 4 \quad \text{for } k \geq 1 \\ &\Rightarrow 2^{2k+1} - 2^k > 2^{2k} + 2^{2k-1} + (2^k - 4) \\ &\Rightarrow 2^k(2^{k+1}-1) > 2^{2k} + 2^{2k-1} + (2^k - 4) \geq 2^{k+j_1} + 2^{k+j_2} + (r-4). \end{aligned}$$

(c) If $(r-4) = a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_1 2 + a_0$, with $a_i = 0$ or 1 for each i, then there exists $a_i = 1$ $(r \geq 5)$ and $l \leq k-1$ (since $2^k - 1 = 2^{k-1} + 2^{k-2} + \cdots + 2 + 1$).

Claim. Let $u = \min(i_1, i_2, i_3, \ldots, i_r)$ in the whole sum of $(*)$. Then $r \leq 2^{k-u} - 1$. Furthermore if $r = 2^{k-u} - 1$, then in the whole sum of $(*)$, $\forall\, p\, 2^{i_p}(2^{k+j_p} - 1) = 2^u(2^{k+1} - 1)$.

Proof. Since

$$2^{k-u} \cdot 2^u(2^{k+1} - 1) = 2^{2k+1} - 2^k > 2^{2k} + 2^{2k-1} + 2^k - 4 \geq 2^{k+j_1} + 2^{k+j_2} + (r-4)$$

for $k \geq 1$, we have $r \leq 2^{k-u} - 1$. By the same argument as the claim in the proof of Lemma 6.1, it suffices to prove

$$(2^{k-u} - 1)2^u(2^{k+1} - 1) + 2^{u+1}(2^{k+1} - 1) > 2^{2k} + 2^{2k-1} + (2^{k-u} - 5).$$

But this follows from

$$\begin{aligned} 2^{2k-1} &> 2^{k+1} - 5 \geq 2^k + 2^{k-u} - 5 \quad \text{for } k \geq 1 \\ &\Rightarrow 2^{2k+1} - 2^k > 2^{2k} + 2^{2k-1} + (2^{k-u} - 5) \\ &\Rightarrow 2^{k+1} - 2^k - 2 \cdot 2^u 2^{k+1} \\ &\quad + 2 \cdot 2^u + 2^{u+1}2^{k+1} - 2^{u+1} > 2^{2k} + 2^{2k-1} + (2^{k-u} - 5). \end{aligned}$$

Consider a partition of the whole sum of $(*)$, i.e.,

(**)

$$\begin{aligned} \sum_{p=1}^{r} 2^{i_p}(2^{k+j_p} - 1) &= \sum_{l_u} 2^u(2^{k+j_{u+1,*}} - 1) + \sum_{l_{u+1}} 2^{u+\bar{i}_1}(2^{k+j_{u+2,*}} - 1) \\ &\quad + \sum_{l_{u+2}} 2^{u+\bar{i}_2}(2^{k+j_{u+3,*}} - 1) + \sum_{l_{u+3}} 2^{u+\bar{i}_3}(2^{k+j_{u+4,*}} - 1) + \cdots \\ &\quad + \sum_{l_q} 2^{u+\bar{i}_{q-u}}(2^{k+j_{q+1,*}} - 1) \\ &= 2^{k+j_1} + 2^{k+j_2} + (r-4), \end{aligned}$$

where $r = l_u + l_{u+1} + l_{u+2} + \cdots + l_q$, $1 \leq \bar{i}_1 < \bar{i}_2 < \bar{i}_3 < \bar{i}_4 < \cdots < \bar{i}_{q-1} \leq k-u-1$, and $u + \bar{i}_{q-1} \leq k-1$. From the claim and the partition, this time we have only one inequality:

$$l_u \leq [(2^{k-u} - 1) - 2^{\bar{i}_1} l_{u+1} - 2^{\bar{i}_2} l_{u+2} - \cdots - 2^{\bar{i}_{q-u}} l_q].$$

Hence there is a nonnegative integer Q such that

$$0 < l_u \leq [(2^{k-u} - 1) - Q - 2^{\bar{i}_1} l_{u+1} - 2^{\bar{i}_2} l_{u+2} - \cdots - 2^{\bar{i}_{q-u}} l_q].$$

So

$$-l_u = (-2^{k-u-1} - 2^{k-u-2} - \cdots - 2 - 1 + Q) + (2^{\bar{i}_1} l_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q).$$

Express $-(2^{k-u} - 1) + Q$ in the form $-b_s 2^s - b_{s-1}2^{s-1} - \cdots - b_1 2 - b_0$, where $b_i = 0$ or 1 for each i; then $s \leq k-u-1$. Moreover $r - 4 \leq 2^{k-u} - 5$, so if $(r-4) = a_l 2^l + a_{l-1}2^{l-1} + \cdots + a_l 2 + a_0$, where $a_i = 0$ or 1 for each i, then $l \leq k-u-1$, and some $a_i = 1$ $(r \geq 5)$.

The left side of $(**)$ is divisible by 2^u, hence the right side is also divisible by 2^u. Furthermore since $0 \leq u \leq k-1$, 2^{k+j_1}, 2^{k+j_2} are divisible by 2^u, hence $(r-4)$ is also. After dividing by 2^u. One has

$$\left(\sum_{l_u} 2^{k+j_{u+1,*}}\right) + (-b_s 2^s - b_{s-1}2^{s-1} - \cdots - b_1 2 - b_0)$$
$$+ (2^{\bar{i}_1} l_{u+1} + 2^{\bar{i}_2} l_{u+2} + \cdots + 2^{\bar{i}_{q-u}} l_q) + \left[\sum_{l_{u+1}} 2^{\bar{i}_1}(2^{k+j_{u+2,*}} - 1)\right]$$
$$+ \left[\sum_{l_{u+2}} 2^{\bar{i}_2}(2^{k+j_{u+3,*}} - 1)\right] + \left[\sum_{l_{u+3}} 2^{\bar{i}_3}(2^{k+j_{u+4,*}} - 1)\right]$$
$$+ \cdots + \left[\sum_{l_q} 2^{\bar{i}_{q-u}}(2^{k+j_{q+1,*}} - 1)\right]$$
$$= 2^{k+j_1-u} + 2^{k+j_2-u} + (a_l 2^{l-u} + a_{l-1}2^{l-1-u} + \cdots + a_{u+1}2 + a_u),$$

where $s \leq k-u-1$, $l-u \leq k-2u-1$, and $a_i = 1$ for some $l \geq i \geq u$. Then apply the argument analogous to (3) of case (ii) in the proof of Lemma 6.1. To finish the proof of Lemma 6.3, we need to prove the cases $r=4$, $r=3$.

When $r=4$, it suffices to prove

$$2^{i_1}(2^{k+\bar{j}_1} - 1) + 2^{i_2}(2^{k+\bar{j}_2} - 1) + 2^{i_3}(2^{k+\bar{j}_3} - 1) + 2^{i_4}(2^{k+\bar{j}_4} - 1) \neq 2^{k+j_1} + 2^{k+j_2}$$

for $0 \leq i_1 \leq i_2 \leq i_3 \leq i_4$, $\bar{j}_1, \bar{j}_2, \bar{j}_3, \bar{j}_4 \geq 1$, and $k \geq j_1 > j_2 \geq 1$. Since $r < 2^k$, $k \geq 3$. If $i_1 = k-1 \leq i_2 \leq i_3 \leq i_4$, then

$$2^{k-1}(2^{k+1}-1) + 2^{k-1}(2^{k+1}-1) + 2^{k-1}(2^{k+1}-1) + 2^{i_4}(2^{k+1}-1) > 2^{k+j_1} + 2^{k+j_2}.$$

If $i_1 = i_2 = i_3 = i_4 = k-2$, then

$$2^{k-2}(2^{k+1}-1) + 2^{k-2}(2^{k+1}-1) + 2^{k-2}(2^{k+1}-1) + 2^{k-2}(2^{k+1}-1)$$
$$= 3 \cdot 2^{2k-1} + 2^{2k-1} - 2^k > 2^{2k} + 2^{2k-1}$$

for $k \geq 2$. Hence $i_1 \leq k-3$.

Suppose

$$2^{i_1}(2^{k+\bar{j}_1} - 1) + 2^{i_2}(2^{k+\bar{j}_2} - 1) + 2^{i_3}(2^{k+\bar{j}_3} - 1) + 2^{i_4}(2^{k+\bar{j}_4} - 1) = 2^{k+j_1} + 2^{k+j_2}.$$

Divide by 2^{i_1}; then $i_1 = i_2$.

$$(2^{k+\bar{j}_1} - 1) + (2^{k+\bar{j}_2} - 1) + 2^{i_3-i_1}(2^{k+\bar{j}_3} - 1) + 2^{i_4-i_1}(2^{k+\bar{j}_4} - 1)$$
$$= 2^{k+j_1-i_1} + 2^{k+j_2-i_1}$$
$$\Rightarrow 2^{k+\bar{j}_1-1} + (2^{k+\bar{j}_2-1} - 1) + 2^{i_3-i_1-1}(2^{k+\bar{j}_3} - 1) + 2^{i_4-i_1-1}(2^{k+\bar{j}_4} - 1)$$
$$= 2^{k+j_1-i_1-1} + 2^{k+j_2-i_1-1}.$$

So $i_3 - i_1 - 1 = 0$.

$$2^{k+\bar{j}_1-2} + 2^{k+\bar{j}_2-2} + 2^{k+\bar{j}_3-1} - 1 + 2^{i_4-i_1-2}(2^{k+\bar{j}_4} - 1) = 2^{k+j_1-i_1-2} + 2^{k+j_2-i_1-2}.$$

So $i_4 - i_1 - 2 = 0$.

$$2^{k+\bar{j}_1-3} + 2^{k+\bar{j}_2-3} + 2^{k+\bar{j}_3-2} - 1 + 2^{k+\bar{j}_4-1} - 1 = 2^{k+j_1-i_1-3} + 2^{k+j_2-i_1-3},$$

which is a contradiction.

When $r = 3$, $3 < 2^k$, so $k \geq 2$.

Suppose

$$2^{i_1}(2^{k+\bar{j}_1} - 1) + 2^{i_2}(2^{k+\bar{j}_2} - 1) + 2^{i_3}(2^{k+\bar{j}_3} - 1) = 2^{k+j_1} + 2^{k+j_2} - 1,$$

for $0 \leq i_1 \leq i_2 \leq i_3$, $\bar{j}_1, \bar{j}_2, \bar{j}_3 \geq 1$, and $k \geq j_1 > j_2 \geq 1$. Hence $i_1 = 0$, so

$$2^{k+\bar{j}_1} + 2^{i_2}(2^{k+\bar{j}_2} - 1) + 2^{i_3}(2^{k+\bar{j}_3} - 1) = 2^{k+j_1} + 2^{k+j_2}.$$

If $k - 1 = i_2 = i_3$, then

$$2^{k+\bar{j}_1} + 2^{2k} - 2^{k-1} + 2^{2k} - 2^{k-1} > 2^{2k} + 2^{2k-1}.$$

Hence $i_2 \leq k - 2$, and $i_2 = i_3$. Divide the above equality by 2^{i_2} to get

$$2^{k+\bar{j}_1-i_2} + 2^{k+\bar{j}_2} - 1 + 2^{k+\bar{j}_3} - 1 = 2^{k+j_1-i_2} + 2^{k+j_2-i_2}.$$

Then divide by 2 to get a contradiction. This completes the proof of Lemma 6.3. □

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