THE DYNAMICS OF CONTINUOUS MAPS OF FINITE GRAPHS THROUGH INVERSE LIMITS

MARCY BARGE AND BEVERLY DIAMOND

ABSTRACT. Suppose that $f: G \to G$ is a continuous piecewise monotone function on a finite graph G. Then the following are equivalent:
(i) f has positive topological entropy;
(ii) there are disjoint intervals I_1 and I_2 and a positive integer n with

$$I_1 \cup I_2 \subseteq f^n(I_1) \cap f^n(I_2);$$

(iii) the inverse limit space constructed by using f on G as a single bonding map contains an indecomposable subcontinuum.
This result generalizes known results for the interval and circle.

1. INTRODUCTION

Suppose that $f: X \to X$ is a continuous function on a space X. One expects that certain dynamical properties of f will be mirrored by topological properties of the inverse limit space (X, f) with bonding map f, and vice versa. In particular, if X is fairly simple and f is dynamically complicated, (X, f) should have a complicated topology.

A result along these lines is the following [BM₁]: if $f: I \to I$ is a piecewise monotone map of the interval I, then f has a periodic point whose period is not a power of 2 if and only if the inverse limit space (I, f) contains an indecomposable subcontinuum that is invariant under some power of the induced homeomorphism $\hat{f}: (X, f) \to (X, f)$. A similar theorem is proved for circle maps in [BR], and Roe proves in [Ro] that if G is a finite graph and $f: G \to G$ has a certain periodic structure (depending on G) then (G, f) contains an indecomposable subcontinuum.

The main result of this paper generalizes the above to maps on finite graphs. One reason for our interest in inverse limits of maps on finite graphs is that they occur quite naturally as attractors for diffeomorphisms. R. F. Williams proved in [Wi] that if a manifold diffeomorphism F has a one-dimensional hyperbolic attractor Λ (with associated stable manifold structure) then F restricted to Λ is topologically conjugate with the induced homeomorphism $\hat{f}$ on an inverse limit of a piecewise monotone map f of some finite graph. For certain f, Williams also proved a converse (in S^4) and it can be shown (along the lines of

[BM$_2$]) that for any map $f\colon G \to G$ of a finite graph there is a homeomorphism $F\colon R^3 \to R^3$ with an attractor on which F is conjugate with $\hat{f}$.

We prove

Main result. Suppose that $f\colon G \to G$ is a continuous piecewise monotone function on a finite graph G. The following are equivalent:

(i) the topological entropy of f is positive;

(ii) the inverse limit space (G, f) contains an indecomposable subcontinuum;

(iii) f has a horseshoe; and

(iv) there are positive integers r and M such that for $m \geq M$, f has a periodic point of prime period rm.

Some of the implications are known and some hold without the hypothesis of piecewise monotonicity. The equivalence of (i) and (iv) for continuous functions on finite graphs is 3.2 of [Bl]. That (i) implies (iii) appears in [Mi] for continuous functions on the interval or circle. In recent work, Misiurewicz and Llibre have extended this result to maps on finite graphs by techniques different from ours. It is easy to see that (iii) implies (i) for any continuous function f on any compact metric space X. That (ii) does not imply the other conditions in the absence of piecewise monotonicity, even for maps of the interval, is demonstrated by the pseudoarc example of Henderson [H]. In case G is an arc or a circle our theorem generalizes the previously mentioned results in [BM$_1$] and [BR] in that the indecomposable subcontinuum in the inverse limit is not required to be invariant under some power of the induced homeomorphism $\hat{f}$.

Note that if $f\colon G \to G$ is not onto, then $(G, f) = (G', f)$ where $G' = \bigcap_{n\geq 0} f^n[G]$. Since G' is a finite graph, it suffices to prove the main result for surjections.

Let $f\colon G \to G$ be a continuous surjection on the finite graph G. The proof of the main result is as follows. In §2 we prove that (a) if f has a horseshoe, then (G, f) contains an indecomposable subcontinuum (2.4), and (b) if f is piecewise monotone and (G, f) contains an indecomposable subcontinuum, then (G, f) contains an indecomposable subcontinuum that is invariant under $\hat{f}^n$ for some $n \geq 1$ (2.11). In §3 we show that if f is piecewise monotone and (G, f) is indecomposable, then f has a horseshoe and thus has positive entropy (3.4). Combining the main results of §§2 and 3, we conclude that, for piecewise monotone maps, if (G, f) contains an indecomposable subcontinuum, then f has a horseshoe (3.5). Finally, in §4 we show that if f has positive entropy, then (G, f) contains an indecomposable subcontinuum (4.3). In the remainder of this section we introduce some terminology and known results.

All functions are assumed continuous, and a map is a continuous surjection. A *continuum* X is a compact, connected metric space, and is *indecomposable* if X consists of more than one point and is not the union of two of its proper subcontinua. A continuum X is *irreducible* between x, $y \in X$ if no proper subcontinuum of X contains both x and y; if X is indecomposable, then there is an uncountable subset S of X such that X is irreducible between any two points of S (3-48 of [HY]). Given a continuum X with metric d and a continuous function $f\colon X \to X$, the associated inverse limit space (X, f) is defined by

$$(X, f) = \{\underline{x} = (x_0, x_1, \dots)|x_n \in X, f(x_{n+1}) = x_n \text{ for } n = 0, 1, 2, \dots\}$$

with metric $\underline{d}$ given by

$$\underline{d}(\underline{x}, \underline{y}) = \sum_{n=0}^{\infty} \frac{d(x_n, y_n)}{2^n}.$$

The space (X, f) is a continuum, and the map $\hat{f}: (X, f) \to (X, f)$ defined by $\hat{f}((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots)$ is called the *induced homeomorphism*. The *projection maps* $\Pi_n: (X, f) \to X$ given by $\Pi_n(\underline{x}) = x_n$ are continuous; if f is onto, the maps Π_n are onto. If H is a subcontinuum of (X, f), H_n will denote $\Pi_n[H]$. Note that $f \circ \Pi_{n+1} = \Pi_n$ and $\Pi_n = \Pi_{n+1} \circ \hat{f}$.

A *finite graph* is a connected union of finitely many arcs. A map f on a finite graph G is *piecewise monotone* if there is a set $A = \{a_1, a_2, \dots, a_n\}$ such that f is monotone on each component of $G \backslash A$. That is, $(f|_C)^{-1}(x)$ is connected for each $x \in G$ and each component C of $G \backslash A$. Suppose that f is piecewise monotone; for each $x \in G$, let C_x denote the component of $f^{-1}f(x)$ containing x. Then x is a *turning point* for f if f is not monotone on any neighbourhood of C_x (i.e., if the graph of f "folds" at x). Abusing standard terminology slightly, we say that f has *finitely many turning points* if there is a finite set $\{x_1, \dots, x_n\}$ such that no point of $G \backslash \bigcup_{i=1}^{n} C_{x_i}$ is a turning point for f. The function f is piecewise monotone if and only if f has finitely many turning points. A continuous function f on the finite graph G has a (one-dimensional) *horseshoe* if there are disjoint subcontinua I_0 and I_1 of G and $n_0 \in Z^+$ such that $f^{n_0}(I_0) \supseteq I_0 \cup I_1$ and $f^{n_0}(I_1) \supseteq I_0 \cup I_1$ (where Z^+ is the set of positive integers).

We make use of the following construction, which is a generalization of a much earlier construction for maps of the interval.

Theorem 1.1 [Ro]. *Suppose that* $f: G \to G$ *is a continuous surjection and that* (G, f) *contains no indecomposable subcontinuum with nonempty interior. Then there is an upper semicontinuous decomposition* $\mathscr{G}$ *of* (G, f) *into disjoint subcontinua such that*

(a) $\mathscr{G}$, *with the quotient topology, is a finite graph*;
if $g: (G, f) \to \mathscr{G}$ *is the quotient map, then*

(b) *the subcontinuum* $g(\underline{x})$ *of* (G, f) *has empty interior for each* $\underline{x} \in (G, f)$, *and*

(c) *the map* $f: \mathscr{G} \to \mathscr{G}$ *defined by* $F(g(\underline{x})) = g(\hat{f}(\underline{x}))$ *is well defined and a homeomorphism.*

As in (b) above, we shall think of $\xi \in \mathscr{G}$ both as an element of the space $\mathscr{G}$ and as a subcontinuum of (G, f). The decomposition elements are precisely the maximal nowhere dense subcontinua of (G, f). Equivalently, for $\underline{x} \in (G, f)$, the decomposition element containing $\underline{x}$ equals the intersection of all subcontinua of (G, f) which contain in their interior a subcontinuum containing $\underline{x}$ in its interior.

2. Horseshoe → indecomposable subcontinuum

Throughout this paper, G will denote a finite graph, and $f: G \to G$ will be a continuous surjection on G.

In the first part of this section we prove that if $f: G \to G$ has a horseshoe, then (G, f) contains an indecomposable subcontinuum (2.4). The map f need not be piecewise monotone.

Let n and l be positive integers. A continuum X is called a $\Theta_{n,l}$ continuum provided: (i) for each subcontinuum H of X, $X \backslash H$ has no more than n components and (ii) if $L^1, \ldots, L^{l+1}$ are subcontinua of X with $\bigcup_{i=1}^{l+1} L^i \neq \varnothing$, then, for some $i \in \{1, \ldots, l+1\}$, $L^i \subseteq \bigcup \{L^j: j \neq i, 1 \leq j \leq l+1\}$. It is evident that each finite graph is a $\Theta_{n,l}$ continuum for some $n, l \in N$. That this is also the case for an inverse limit space based on a finite graph is proved in [Ro]. We require some minor extensions of this fact.

Lemma 2.1. *If $H^1, H^2, \ldots, H^m$ are subcontinua of (G, f), then*
  (i) *$(G, f) \backslash \bigcup_{i=1}^m H^i$ has finitely many components, and*
  (ii) *$H^1 \cap H^2$ has finitely many components.*

Proof. In case $m = 1$, (i) is the result of 2.4 of [Ro]. Assume that (i) is true for $m \leq M$ and let $H^1, H^2, \ldots, H^{M+1}$ be subcontinua of (G, f). Without loss of generality, $H^i \cap H^j = \varnothing$ for $i \neq j$. Then H^{M+1} is contained in some component C of $(G, f) \backslash \bigcup_{i=1}^M H^i$, hence $H^{M+1} \subseteq \mathrm{cl}(C)$. To prove the theorem it is enough to prove that $\mathrm{cl}(C) \backslash H^{M+1}$ has finitely many components.

Let n and l be such that G is a $\Theta_{n,l}$ continuum and suppose that $V^1, \ldots, V^{l+1}$ are distinct components of $\mathrm{cl}(C) \backslash H^{M+1}$. Then $K^i = H^{M+1} \cup \mathrm{cl}(V^i)$, $i = 1, \ldots, l+1$, are subcontinua of (G, f) such that $\bigcup_{i=1}^{l+1} K^i = H^{n+1} \neq \varnothing$ and $K^i \not\subset \bigcup \{K^j: 1 \leq j \leq l+1, j \neq i\}$ for all i. It follows for sufficiently large k, the continua $\Pi_k[K^i]$, $i = 1, \ldots, l+1$, in G satisfy: $\bigcap_{i=1}^{l+1} \Pi_k[K^i] \neq \varnothing$ for each i, $\Pi_k[K^i] \not\subset \bigcup \{\Pi_k[K^j]: 1 \leq j \leq l+1, j \neq i\}$. But this contradicts the choice of l so that $\mathrm{cl}(C) \backslash H^{M+1}$ can have at most l components.

To prove (ii), note that there is $s \in Z^+$ such that the intersection of every pair of subcontinua of G has no more than s components. Suppose that $K^1, \ldots, K^{s+1}$ are distinct components of $H^1 \cap H^2$. Then for all sufficient large k, $\Pi_k[K^i] \cap \Pi_k[K^j] = \varnothing$ for $i \neq j$. Since $\bigcup_{i=1}^{s+1} \Pi_k[K^i] \subseteq \Pi_k[H^1 \cap H^2] \subseteq \Pi_k[H^1] \cap \Pi_k[H^2]$ and $\Pi_k[H^1] \cap \Pi_k[H^2]$ has at most s components, it must be the case that for some $i \neq j$, $\Pi_k[K^i]$ and $\Pi_k[K^j]$ are in the same component, say J_k, of $\Pi_k[H^1] \cap \Pi_k[H^2]$ for infinitely many k, say for $k_1 < k_2 < \cdots$. Then $f^{k_{i+1}-k_i}[J_{k_{i+1}}] \subseteq J_{k_i}$ for each i and $J = \{\underline{x}: x_{k_i} \in J_{k_i}, i \in Z^+\}$ is a subcontinuum of $H^1 \cap H^2$ that contains both K^i and K^j. Since this is not possible, $H^1 \cap H^2$ has at most s components. $\square$

The next result follows directly from 2.1.

Lemma 2.2. *Suppose that (G, f) contains no indecomposable subcontinuum with nonempty interior. Then for each $m \geq 1$ there are nonempty subcontinua $H^1, H^2, \ldots, H^m$ of (G, f) such that $(G, f) = \bigcup_{i=1}^m H^i$, and for each i, $\mathrm{int}(H^i) = (G, f) \backslash \bigcup_{j \neq i} H^j$.*

Lemma 2.3. *For any finite graph G, there is a positive integer m so that if $L_1, L_2, \ldots, L_m$ are m subcontinua of G with $\bigcap_{1 \leq i \leq m} L_i \neq \varnothing$, then for some i, $L_i \subseteq \bigcup_{i \neq j} L_j$.*

Theorem 2.4. *Suppose that there are disjoint subcontinua I_0 and I_1 of G and $n_0 \in Z^+$ such that $f^{n_0}(I_0) \supseteq I_0 \cup I_1$ and $f^{n_0}(I_1) \supseteq I_0 \cup I_1$. Then (G, f) contains an indecomposable subcontinuum.*

Proof. Since the spaces (G, f) and (G, f^{n_0}) are homeomorphic, we need only show that if there are disjoint subcontinua I_0 and I_1 of G such that $f(I_0) \supseteq I_0 \cup I_1$ and $f(I_1) \supseteq I_0 \cup I_1$, then (G, f) contains an indecomposable subcontinuum. Suppose that I_0 and I_1 are as above, and let $\Gamma = \mathrm{cl}(\bigcup_{n \geq 0} f^n(I_0 \cup I_1))$. Then Γ is a subcontinuum of G and $f(\Gamma) = \Gamma$. We show that $(\overline{\Gamma}, f)$ contains an indecomposable subcontinuum with nonempty interior.

Suppose not; let m be as in 2.3 for Γ and $l = \#[\mathrm{cl}(\Gamma \backslash I_0) \cap I_0]$. Choose $\{H^1, \ldots, H^{lm}\}$ as in 2.2. For each i there is a nonempty open set V^i of Γ and $N_i \in Z^+$ such that $\Pi_{N_i}^{-1}(V^i) \subseteq (\Gamma, f) \backslash (\bigcup_{j \neq i} H^j) = \mathrm{int}(H^i)$. Since $\Gamma = \mathrm{cl}(\bigcup_{n \geq 0} f^n(I_0 \cup I_1))$, for each i there is $p_i \in I_0 \cup I_1$ and $m_i \in Z^+$ such that $f^{m_i}(p_i) \in V^i$. Then $\Pi_{N_i + m_i}^{-1}(p_i) \subseteq \mathrm{int}(H^i)$. Since $I_0 \cup I_1 \subseteq f(I_0)$, there is $\underline{x}^i \in \mathrm{int}(H^i)$ such that $x_{N_i + m_i}^i = p_i$ and $x_{N_i + m_i + k}^i \in I_0$ for $k \geq 1$.

Since $\underline{x}^i \in \mathrm{int}(H^i)$, $\underline{x}^i \notin \bigcup_{j \neq i} H^j$. For each i, there is a positive integer k_i so that if $k \geq k_i$, $x_k^i \notin \bigcup_{j \neq i} \Pi_k(H^j)$. Let

$$N = \max(\{N_i + m_i + 1 : 1 \leq i \leq lm\} \cup \{k_i : 1 \leq i \leq lm\}).$$

Then for each i:
(i) $\Pi_N(H^i) \cap I_0 \neq \varnothing$, since $x_N^i \in I_0$;
(ii) $\Pi_N(H^i) \cap I_1 \neq \varnothing$, by reasoning identical to that allowing the construction of $\underline{x}^i$; and
(iii) $x_N^i \in \Pi_N(H^i) \backslash \bigcup_{j \neq i} \Pi_N(H^j)$, since $N \geq k_i$.

It follows from (i) and (ii) that for each i, the subcontinuum $\Pi_N(H^i)$ contains at least one of the l points of $[\mathrm{cl}(\Gamma \backslash I_0) \cap I_0]$. Since there are lm of these subcontinua, at least m of them have one point in common. But (iii) implies that no one of these subcontinua is contained in the union of others, contradicting the choice of m. □

The remainder of this section is devoted to showing that if $f: G \to G$ is piecewise monotone and (G, f) contains an indecomposable subcontinuum, then (G, f) contains an indecomposable subcontinuum invariant under $\hat{f}^n$ for some $n \neq 0$.

Lemma 2.5. *If H is an indecomposable subcontinuum of (G, f) with nonempty interior, then*
(i) $\mathrm{int}(H)$ *is connected, and*
(ii) $H = \mathrm{cl}(\mathrm{int}(H))$.

Proof. Since $\mathrm{int}(H) = (G, f) \backslash \mathrm{cl}[(G, f) \backslash H]$, it follows from 2.1 that $\mathrm{int}(H)$ has finitely many components. Let U be a component of $\mathrm{int}(H)$. Then U is open, hence $\mathrm{cl}(U)$ is a subcontinuum of H with nonempty interior. Since H is indecomposable, $\mathrm{cl}(U) = H$; (i) and (ii) follow. □

Lemma 8 of [BM] states that if f is a continuous surjection on the unit interval I, and there is a proper subinterval J of I such that for all $n \in Z^+$, $f^{-n}[J]$ is an interval, then (I, f) is decomposable. The proof of this result can easily be generalized to yield the following:

Lemma 2.6. *Suppose that f is a continuous surjection on the connected space X, and that H is a subcontinuum of (X, f). If there is a subcontinuum J of X such that for each $n \in Z^+$, $f^{-n}[J] \cap \Pi_n[H]$ is proper, connected, and has nonempty interior in $\Pi_n[H]$, then H is decomposable.*

Lemma 2.7. *Suppose that f is piecewise monotone, and that H is an indecomposable subcontinuum of (G, f) with nonempty interior. Then there is an $n \neq 0$ such that $\hat{f}^n[H] = H$.*

Proof. Since H is indecomposable, it follows from 2.6 that for any positive integer n and any proper subcontinuum J of $\Pi_n[H]$, there is a positive integer m such that $f^{-m}[J] \cap \Pi_{n+m}[H]$ is not connected. Then either $\Pi_j[H]$ contains a turning point of f for infinitely many j, or $\Pi_j[H]$ contains a simple closed curve for infinitely many j. Since there are only finitely many turning points for f and only finitely many simple closed curves, there is $p \in G$ and an infinite subset B of Z^+ such that $p \in \Pi_{n_i}[H]$ for $n_i \in B$.

Let m be as in 2.3 for G. Choose $\underline{x} \in \text{int}(H)$, $N_0 \in Z^+$ such that $\Pi_n^{-1}[x_n] \subseteq \text{int}(H)$ for $n \geq N_0$, and $n_i \geq N_0$. Since $\bigcup_{j=0}^{m-1} \Pi_{n_{i+j}}[H] \neq \varnothing$, there is $j \leq m-1$ such that $\Pi_{n_{i+j}}[H] \subseteq \bigcup \{\Pi_{n_{i+k}}[H] : 0 \leq k \leq m-1, k \neq j\}$. Now $\Pi_{n_{i+j}}^{-1}(x_{n_{i+j}}) \subseteq \text{int}(H)$, hence $x_{n_{i+j}} \in \Pi_{n_{i+s}}[H] = \Pi_{n_{i+j}}[\hat{f}^{n_{i+j}-n_{i+s}}[H]]$ for some $s \in \{0, 1, \ldots, m-1\} \setminus \{j\}$. It follows that $\hat{f}^w[H] \cap \text{int}(H) \neq \varnothing$, where $w = n_{i+j} - n_{i+s}$.

Since $K = \hat{f}^w[H]$ is indecomposable, $K = \text{cl}(\text{int}(K))$, and so $\text{int}(K) = \text{int}(H) \neq \varnothing$. It follows from 2.1 that $K \cap H$ has finitely many components, so at least one, say C, has nonempty interior. Since H and K are indecomposable, $C = H$ and $C = K$. That is, $\hat{f}^w[H] = H$. □

Recall that a point x is *recurrent* under f if there is a sequence $n_i \to \infty$ such that $f^{n_i}(x) \to x$.

Lemma 2.8. *If h is a homeomorphism of the finite graph G and h has a recurrent nonperiodic point, then G is homeomorphic to the circle and h has no periodic points.*

Proof. Suppose that the set B of branch points of G is nonempty. Then B is finite and invariant under the homeomorphism h, hence if $x \in G$ is recurrent and nonperiodic, $x \in A$ for some component A of $G \setminus B$. If n is the smallest positive integer such that $h^n[A] = A$, then x is recurrent under $h^n|_A$. But A is an arc, and all recurrent points of a homeomorphism of an arc are periodic. Thus $B = \varnothing$, and G is either an arc or a circle. According to the above, G is a circle. If h has any periodic point of period k, then for each $x \in G$, $h^{2nk}(x)$ converges to a periodic point. But then the only recurrent points are periodic, and the lemma is proved. □

Lemma 2.9. *Suppose that $A \subseteq G$ is an arc containing no branch points. If $A \subseteq f[A]$ but $A \not\subseteq \text{int}(f[A])$ then f has a fixed point in A.*

Proof. Let $A = [a, b]$. Without loss of generality, $a \notin \text{int}(f[A])$. Let $x_1 = \inf\{x \in A : f(x) = a\}$.

Suppose that $b \in f[a, x_1]$, and let $x_2 = \sup\{x \in [a, x_1] : f(x) = b\}$. If $f[x_2, x_1]$ contains a point of $G \setminus A$, then either $a \in \text{int}(f[A])$, or there is $c \in (x_2, x_1)$ such that $c = f(b)$, and so $f[x_2, x_1] = [a, b]$. That is, $[x_2, x_1] \subseteq f[x_2, x_1]$, and f has a fixed point.

Similarly, if $b \in f[x_1, b]$, and $x_2 = \inf\{x \in [x_1, b] : f(x) = b\}$, then $[x_1, x_2] \subseteq f[x_1, x_2]$, and f has a fixed point. □

Lemma 2.10. *Suppose that H is an indecomposable subcontinuum of (G, f), and that (G, f) contains no indecomposable subcontinuum with nonempty interior. Let $\mathscr{G}$ be the decomposition space of (G, f) with induced homeomorphism $F : \mathscr{G} \to \mathscr{G}$. If $\xi = g(H)$ is the decomposition element of $\mathscr{G}$ containing H, then ξ is periodic under F.*

Proof. We first prove that ξ is recurrent under F. Suppose not; there is then an open neighbourhood U of ξ such that $F^{-n}(\xi) \subseteq C = \mathscr{G}\backslash U$ for all $n \geq 1$. Since $H \subseteq g^{-1}(\xi) = \xi$, H and $g^{-1}[C]$ are disjoint compact subsets of (G, f) and $\hat{f}^{-n}[H] \subseteq g^{-1}[C]$ for all $n \geq 1$. There is a positive integer N_0 such that if $k \geq N_0$ and $n \geq 1$, $\Pi_k[H] \cap \Pi_{k+n}[H] = \Pi_k[H] \cap \Pi_k[\hat{f}^{-n}[H]] \subseteq \Pi_k[H] \cap \Pi_k[g^{-1}[C]] = \varnothing$. Then the projections $\Pi_k[H]$, for $k \geq N_0$, are pairwise disjoint. But this contradicts the fact that, since H is indecomposable, there is, as in the proof of 2.7, an infinite subset B of Z^+ such that $\bigcap_{n_i \in B} \Pi_{n_i}[H] \neq \varnothing$. Therefore ξ is recurrent.

If ξ is not periodic under F, then according to 2.8, $\mathscr{G}$ must be a circle and F has no periodic points. Suppose that for every $n \in Z^+$, $\Pi_n(\xi)$ contains either a loop or a branch point. If $k = (\# \text{ of branch points of } G) + (\# \text{ of loops in } G) + 1$, then for each $n \in Z^+$, there are $i(n)$, $j(n) \in \{1, \ldots, k\}$ with $i(n) \neq j(n)$ and $\Pi_{nk+i(n)}(\xi) \cap \Pi_{nk+j(n)}(\xi) \neq \varnothing$. Choose i, $j \in \{i, \ldots, k\}$ such that $i \neq j$ and $i = i(n)$ and $j = j(n)$ for infinitely many n. Let $s = j - i$; then $\Pi_{N+nk+i}(\xi) \cap \Pi_{N+nk+j}(\xi) = \Pi_{N+nk+j}[\hat{f}^s(\xi)] \cap \Pi_{N+nk+j}(\xi) \neq \varnothing$ for infinitely many n. It follows that $\xi \cap \hat{f}^s(\xi) \neq \varnothing$, hence $\xi = \hat{f}^s(\xi)$ and ξ is periodic. So assume that for some positive integer r, $\Pi_r(\xi)$ is an arc containing no branch points.

Suppose that the infinite set B has been chosen so that $\varnothing \neq \bigcap_{n_i \in B} \Pi_{n_i}[H] \subseteq \bigcap_{n_i \in B} \Pi_{n_i}(\xi)$. Let m be as in 2.3 for G, and choose $n_s \in B$ so that $n_s \geq r$. For some $i \in \{0, \ldots, m - 1\}$,

$$\Pi_{n_{s+i}}(\xi) \subseteq \bigcup \{\Pi_{n_{s+j}}(\xi) : 0 \leq j \leq m - 1, i \neq j\}.$$

Let $l = n_{s+i} - r$. Then

$$\Pi_r(\xi) = f^l[\Pi_{n_{s+i}}(\xi)] \subseteq \bigcup \{f^l[\Pi_{n_{s+j}}(\xi)] : 0 \leq j \leq m - 1, i \neq j\}$$

$$= \bigcup \{[\Pi_{n_{s+j}-l}(\xi)] : 0 \leq j \leq m - 1, i \neq j\}.$$

(If $k < 0$, let $\Pi_k(\xi) = f^{|k|}[\Pi_0(\xi)]$.)

We consider two cases.

Case 1. $\Pi_r(\xi) \not\subset \text{int}[\bigcup \{\Pi_{n_{s+j}-l}(\xi) : 0 \leq j \leq m - 1, i \neq j\}]$.

Since $\Pi_r(\xi)$ is an arc containing no branch points, there is an integer $t = n_{s+j} - l$ for some $j \in \{0, \ldots, k - 1\}$, $j \neq i$, such that either $\Pi_r(\xi) \subseteq \Pi_t(\xi)$ (but $\Pi_r(\xi) \not\subset \text{int}(\Pi_t(\xi))$) or $\Pi_t(\xi) \subseteq \Pi_r(\xi)$. Note that $t \neq r$.

(1a) If $\Pi_r(\xi) \subseteq \Pi_t(\xi)$ and $t > r$, then $f^{t-r}[\Pi_t(\xi)] = \Pi_r(\xi)$, so $f^{t-r}[\Pi_r(\xi)] \subseteq \Pi_r(\xi)$ and f has a periodic point in G. Then $\hat{f}$ has a periodic point in (G, f) and F has a periodic point in $\mathscr{G}$. But F has no periodic points.

(1b) If $\Pi_r(\xi) \subseteq \Pi_t(\xi)$ and $t < r$, then $f^{r-t}[\Pi_r(\xi)] = \Pi_t(\xi) \supseteq \Pi_r(\xi)$. Since $\Pi_r(\xi) \not\subset \text{int}(f^{r-t}[\Pi_r(\xi)])$, 2.9 implies that f^{r-t} has a fixed point, and f has a periodic point. As in (1a), we are done.

(1c) If $\Pi_t(\xi) \subseteq \Pi_r(\xi)$ and $r > t$, then $f^{r-t}[\Pi_r(\xi)] = \Pi_t(\xi) \subseteq \Pi_r(\xi)$, and f has a periodic point.

(1d) If $\Pi_t(\xi) \subseteq \Pi_r(\xi)$ and $r < t$, then $f^{t-r}[\Pi_t(\xi)] = \Pi_r(\xi)$ and $\Pi_t(\xi)$ is a subarc of $\Pi_r(\xi)$. Hence f has a periodic point.

Case 2. $\Pi_r(\xi) \subseteq \text{int}[\bigcup\{\Pi_{n_{s+j}-l}(\xi): 0 \le j \le m-1, i \ne j\}]$. Since $\Pi_r(\xi)$ is an arc containing no branch points, and since $\bigcap_{j=0}^{m-1} \Pi_{n_{s+j}-l}(\xi) \ne \varnothing$, there are t_1, $t_2 \in \{n_{s+j} - l: 0 \le j \le m-1, j \ne i\}$ with $t_1 \ne t_2$ and such that $\Pi_r(\xi) \subseteq \text{int}[\Pi_{t_1}(\xi) \cup \Pi_{t_2}(\xi)]$. Then $\xi \subseteq \Pi_r^{-1}(\text{int}[\Pi_{t_1}(\xi) \cup \Pi_{t_2}(\xi)])$. Since the decomposition $\mathscr{G}$ is upper semicontinuous, there is an arc $I \subseteq \mathscr{G}$ such that $\xi \in I$, and for $\eta \in \text{int}(I)$, $\Pi_r(\eta) \subseteq \text{int}[\Pi_{t_1}(\xi) \cup \Pi_{t_2}(\xi)]$.

Since F is a circle homeomorphism without periodic points, ξ is recurrent and $\xi \in \text{int}(I)$, there is a positive integer N such that $\bigcup_{i=0}^{N-1} F^i[I] = \mathscr{G}$. (To see that this is the case, let $n > 0$ be such that $F^n(\xi) \in I$ and let $G^\sim : \mathbf{R} \to \mathbf{R}$ be a lift of F^n such that $G^\sim(I^\sim) \cap I^\sim \ne \varnothing$ where $I^\sim$ is a component of $\Pi^{-1}(I)$. Let $\xi^\sim \in \Pi^{-1}(I)$. Since F has no periodic points, $\{F^{kn}(\xi^\sim) | k = 1, 2, \dots\}$ is unbounded, and hence for sufficiently large m, the arc $\bigcup_{k=0}^m F^{kn}(I^\sim)$ contains a fundamental domain. Let $N = nm + 1$.)

Since $\bigcup_{i=0}^{N-1} F^i[I] = \mathscr{G}$, $\Pi_r[\bigcup_{i=0}^{N-1} F^i[I]] = G$, and so $\Pi_r[\bigcup_{i=0}^{N-1} \hat{f}^i[I]] = G$. But then $\bigcup_{i=0}^{N-1} f^i[\Pi_r[I]] = G$. Since $\Pi_r[I] \subseteq \Pi_{t_1}(\xi) \cup \Pi_{t_2}(\xi)$,

$$\bigcup_{i=0}^{N-1} f^i[\Pi_{t_1}(\xi) \cup \Pi_{t_2}(\xi)] = G.$$

Without loss of generality $t_2 > t_1$, so that $f^{t_2-t_1}[\Pi_{t_2}(\xi)] = \Pi_{t_1}(\xi)$. Then $\bigcup_{i=0}^{N-1} f^i[f^{t_2-t_1}[\Pi_{t_2}(\xi)] \cup \Pi_{t_2}(\xi)] = G$, and so $\bigcup_{i=0}^{N-1} [\Pi_{t_2} \hat{f}^i[\hat{f}^{t_2-t_1}(\xi)] \cup \Pi_{t_2}(\hat{f}^i(\xi))] = G$. For $i = 0$ to $N-1$, let $\eta_i = \hat{f}^i(\xi)$ and $\eta_{N-1+i} = \hat{f}^i[\hat{f}^{t_2-t_2}(\xi)]$. Then the $2N$ elements $\eta_0, \eta_1, \dots, \eta_{2N-1}$ have the property that $\Pi_{t_2}[\bigcup_{i=0}^{2N-1} \eta_i] = G$. If $t_2 \ge 0$, then $\Pi_0[\bigcup_{i=0}^{2N-1} \eta_i] = G$. If $t_2 < 0$, then $\Pi_0[\bigcup_{i=0}^{2N-1} \hat{f}^{-t_2}(\eta_i)] = G$. In any case there are $2N$ elements of $\mathscr{G}$, say $\eta_0, \eta_1, \dots, \eta_{2N-1}$, whose union projects under Π_0 onto G. It follows that for any $n \ge 0$, $\Pi_k[\bigcup_{i=0}^{2N-1} \hat{f}^n(\eta_i)] = G$ for $0 \le k \le n$. That is, for $n \ge 0$, for any $x \in (G, f)$, there is $i \in \{0, 1, \dots, 2N-1\}$ such that $\underline{d}(x, \hat{f}^n(\eta_i)) < 1/2^n$.

Choose $\{U_i\}_{i=0}^{2N}$ to be a collection of disjoint open intervals in $\mathscr{G}$ with centers $\{a_i\}_{i=0}^{2N}$. Choose $\underline{x}^i \in g^{-1}(a_i)$, and let $\varepsilon = \min\{d(\underline{x}^i, (G, f) \backslash g^{-1}(U_i))\}_{i=0}^{2N}$. For any $n \ge 0$, for some $i \in \{0, \dots, 2N\}$,

$$U_i \cap \{F^n(\eta_0), F^n(\eta_1), \dots, F^n(\eta_{2N-1})\} = \varnothing$$

and thus $g^{-1}(U_i) \cap [\bigcup_{j=0}^{2N-1} \hat{f}^n(\eta_j)] = \varnothing$. But then for every $n \ge 0$, for some $i = \{0, \dots, 2N\}$, $\underline{d}(\underline{x}^i, \bigcup_{j=0}^{2N-1} \hat{f}^n(\eta_j)) \ge \varepsilon$, a contradiction to the last paragraph.

It follows that ξ must be periodic under F. $\square$

Note that the only property required of ξ in the proof of 2.10 is that there is an infinite set B such that $\bigcap_{n_i \in B} \Pi_{n_i}[\xi] \ne \varnothing$.

Theorem 2.11. *Suppose that* $f: G \to G$ *is piecewise monotone. If* (G, f) *contains an indecomposable subcontinuum then* (G, f) *contains an indecomposable subcontinuum that is invariant under* $\hat{f}^n$ *for some* $n \ne 0$.

Proof. Let H be an indecomposable subcontinuum of (G, f). As in the proof of 2.7, there is an infinite subset B of Z^+ such that $\bigcap_{n_i \in B} \Pi_{n_i}[H] \ne \varnothing$. Let

L_1 be a subcontinuum of (G, f) irreducible with respect to the properties: $\hat{f}[L_1] = L_1$ and $\hat{f}^n[H] \subseteq L_1$ for all $n \in Z$.

Let $f_1 = f|_{L_1}$ and $G_1 = \Pi_0[L_1]$. Suppose that $L_1 = (G_1, f_1)$ does not contain an indecomposable subcontinuum with nonempty interior; otherwise by 2.7 the theorem is proved. Let $\mathscr{G}_1 = \{g_1(\underline{x}) \colon x \in L_1\}$ be the decomposition space of (G_1, f_1) with quotient map g_1 and let $F_1 \colon \mathscr{G}_1 \to \mathscr{G}_1$ be the induced homeomorphism. Let $\xi_1 = g_1(\underline{x})$ for $\underline{x} \in H$.

According to 2.10, ξ_1 is periodic under F_1. If ξ_1 were fixed by F_1, then since $\mathscr{G}_1$ is nondegenerate, ξ_1 would be a proper subcontinuum of L_1 that is invariant under $\hat{f}$ and contains $\hat{f}^n[H]$ for all $n \in Z$, contradicting the definition of L_1. Then ξ_1 is of least period $k_1 > 1$ under F_1, and $\hat{f}_1^i(\xi_1) \cap \hat{f}_i^j(\xi_1) = \varnothing$ for $i, j \in \{0, \ldots, k_1 - 1\}$, $i \neq j$. It follows that the projections $\Pi_0(\xi_1), \ldots, \Pi_{k_1-1}(\xi_1)$ are pairwise disjoint. Hence $\Pi_0(H), \ldots, \Pi_{k_1-1}(H)$ are pairwise disjoint.

Let $L_1 \subseteq \xi_1$ be a subcontinuum irreducible with respect to the properties: $\hat{f}^{k_1}[L_2] = L_2$ and $\hat{f}^{nk_1}[H] \subseteq L_2$ for all $n \in Z$. Let $G_2 = \Pi_0[L_2]$, $f_2 = f_1^{k_1}|_{G_2} \colon G_2 \to G_2$, and $H^2 = \{\underline{x} \in (G_2, f_2) \colon x_n \in \Pi_{nk_1}[H]$ for $n \in Z^+\}$. Then H^2 is homeomorphic to H and hence is indecomposable. If $L_2 = (G_2, f_2)$ contains an indecomposable subcontinuum with nonempty interior, the theorem holds. Otherwise, let $\mathscr{G}_2$ be the decomposition space of (G_2, f_2), with induced homeomorphism $F_2 \colon \mathscr{G}_2 \to \mathscr{G}_2$, and let $\xi_2 = g_2(x)$ for $x \in H^2$. As before, ξ_2 must be periodic under F_2 of least period $k_2 > 1$. It follows that the projections $\Pi_0(\xi_2), \Pi_0[\hat{f}_2^{-1}(\xi_2)], \ldots, \Pi_0[\hat{f}_2^{1-k_2}(\xi_2)]$ are pairwise disjoint, and thus that $\Pi_0(\xi_2'), \Pi_0[\hat{f}_1^{-1}(\xi_2')], \ldots, \Pi_0[\hat{f}_1^{1-k_1k_2}(\xi_2')]$ are pairwise disjoint, where $\xi_2' = \{\underline{x} \in (G_1, f_1)|x_{nk_1} \in \Pi_n(\xi_2)$ for $n \geq 0\}$. Then $\Pi_0(H), \ldots, \Pi_{k_1k_2-1}(H)$ are pairwise disjoint.

Continuing, either we find an indecomposable continuum invariant under $\hat{f}^n$ for some $n \neq 0$, or we construct a sequence of integers $\{k_i\}_{i \in Z^+}$ with $k_i > 1$ for all i and such that $\Pi_0(H), \Pi_1(H), \ldots, \Pi_{k_1k_2\cdots k_s-1}(H)$ are pairwise disjoint for all s. Since this contradicts the fact that $\bigcap_{n_i \in B} \Pi_{n_i}[H] \neq \varnothing$, there is an indecomposable subcontinuum of (G, f) invariant under $\hat{f}^n$ for some $n \neq 0$. □

3. Indecomposable subcontinuum → Horseshore

In this section we prove that if f is piecewise monotone, and (G, f) contains an indecomposable subcontinuum, then f has a horseshoe (3.5). As indicated in the introduction, the result is not true without the hypothesis of piecewise monotonicity.

The following is proved in [Ku]; we include a proof for the sake of completeness.

A subset H of G is ε-dense in G if for each $x \in G$, $d(x, H) < \varepsilon$, where $d(x, H)$ is in terms of minimal arc length.

Lemma 3.1. *Suppose that* (G, f) *is irreducible between* $\underline{a}$ *and* $\underline{b}$ *and that* $\{J_n\}_{n \in Z^+}$ *is a sequence of subcontinua of* G *such that for all* n, $\{a_n, b_n\} \subseteq J_n$. *Then* $\lim_{n \to \infty} f^n(J_n) = G$ *(in the Hausdorff metric).*

That is, for $\varepsilon > 0$, *there is* $N_0 \in Z^+$ *such that for all* $n \geq N_0$, $f^n(J_n)$ *is* ε-*dense in* G.

Proof. Suppose that $\{J_n\}_{n\in Z^+}$ is such a sequence, and that $\lim_{n\to\infty} f^n(J_n) \neq G$. There is then an infinite subset B_0 of Z^+ and a proper subcontinuum J_0' of G such that $\{f^{n_i}(J_{n_i}): n_i \in B_0\}$ converges to J_0'. Certainly $\{a_0, b_0\} \subseteq J_0'$. There is a second infinite subset B_1 of B_0 and a subcontinuum J_1' such that $\{f^{n_i-1}(J_{n_i}): n_i \in B_1\}$ converges to J_1'. Then $f[J_1'] = J_0'$, and $\{a_1, b_1\} \subseteq J_1'$. Continuing in this way, we get a sequence of subcontinua $\{J_n': n \in Z^+\}$ with $f[J_{n+1}'] = J_n'$, and $\{a_n, b_n\} \subseteq J_n'$ for all $n \in Z^+$. Define $J = \{(z_0, z_1, \ldots)|z_n \in J_n', \ z_n = f(z_{n+1})\}$. Then J is a proper subcontinuum of (G, f) containing $\underline{a}$ and $\underline{b}$, contradicting the fact that (G, f) is irreducible between $\underline{a}$ and $\underline{b}$. $\square$

We need a slight generalization of the notion of horseshoe: for $n \geq 2$, the map f has an *n-horseshoe* if there are disjoint intervals $I_1, I_2, \ldots, I_n$ and a positive integer n_0 such that $f^{n_0}[I_j] \supseteq \bigcup_{i=1}^n I_i$ for $1 \leq j \leq n$.

Lemma 3.2. *Suppose that $I_1, I_2, \ldots, I_n$ are disjoint intervals in G.*

(a) *If $n \geq 3$ and there is $n_0 \in Z^+$ such that each of $I_1, I_2, \ldots, I_n$ maps under f^{n_0} over at least $n-1$ of $I_1, I_2, \ldots, I_n$, then f has an $(n-1)$-horseshoe.*

(b) *If $n \geq 2$ and there are positive integers $k_1, k_2, \ldots, k_n$ such that for each j, $f^{k_j}[I_j] \supseteq \bigcup_{i=1}^n I_i$, then f has an n-horseshoe.*

Proof. (a) Let A be the $n \times n$ matrix with ijth entry 1 if I_i maps over I_j under f^{n_0} and 0 otherwise. Then the ijth entry of A^2 is nonzero if and only if I_i maps over I_j under f^{2n_0}. Suppose that for some i and j, the ijth entry of A^2 is zero. Since each row of A contains at most one zero entry, the jth column of A contains at most one nonzero entry. Suppose that $a_{kj} = 1$ for some $k \neq i$. Then $a_{ij} = 0$, hence $a_{im} = 1$ for $m \neq j$. In particular, $a_{ik}a_{kj} = 1$ and the ijth entry of A^2 is nonzero. Then $a_{kj} = 0$ for $k \neq i$, hence $a_{km} = 1$ for $k \neq i$, $m \neq j$. Let B denote the $(n-1) \times (n-1)$ matrix obtained by deleting the jth row and jth column of B. Every entry of B is nonzero except for possibly one zero in the row corresponding to the original ith row of A. But then every entry of B^2 is nonzero. That is, for $k = 1$ to n, $k \neq j$, $f^{2n_0}[I_k] \supseteq \bigcup\{I_k: k = 1 \text{ to } n, \ k \neq j\}$.

(b) Let $w = \sum_{i=1}^n k_i$. For each i, $f^w[I_i] = f^{w-k_i}[f^{k_i}[I_i]] \supseteq f^{w-k_i}[\bigcup_{i=1}^n I_i] \supseteq f^{w-k_i}[I_j]$ (for some $j \neq i$) $\supseteq f^{w-k_i-k_j}[f^{k_j}[I_j]] \supseteq f^{w-k_i-k_j}[\bigcup_{i=1}^n I_i] \supseteq \cdots \supseteq \bigcup_{i=1}^n I_i$. $\square$

Lemma 3.3. *Suppose that (G, f) is indecomposable. Let $F = \{x \in G: |f^{-n}(x)| = 1$ for all $n \geq 0\}$. Then*

(i) *F is a finite set, so that if $a \in F$ then a is periodic.*

(ii) *Suppose that f is piecewise monotone and let n_0 be the period of $a \in F$. There is $m > 0$ and a base $\mathscr{V}$ of neighbourhoods of a such that for $V \in \mathscr{V}$, $f^{mn_0}[V] \subseteq V$.*

Proof. (i) Choose $\{\underline{x}^i\}_{i=1}^6 \subseteq (G, f)$ so that for a fixed edge E and an infinite subset B of positive integers, $\{x_{n_j}^i\}_{i=1}^6 \subseteq E$ for each $n_j \in B$, and such that (G, f) is irreducible between $\underline{x}^i$ and $\underline{x}^k$ if $i \neq k$. Since $\underline{x}^i \neq \underline{x}^k$ if $i \neq k$, we can assume that for $n_j \in B$, $x_{n_j}^i \neq x_{n_j}^k$. We can also assume that an orientation of E is fixed and that for $n_j \in B$, $\{x_{n_j}^i\}_{i=1}^6$ appears in the order $x_{n_j}^1 < x_{n_j}^2 < \cdots < x_{n_j}^6$, since otherwise some ordering appears infinitely often and we pass to a subsequence and relabelling.

Suppose that two distinct elements a_1 and a_2 of F lie in the same edge E' of G. Choose $\varepsilon > 0$ to be smaller than the minimum length of a minimal path from a_1 to a_2. According to 3.2, there is a positive integer N_0 so that for $n_j \in B$, $n_j \geq N_0$ and $i = 1$ to 5, $f^{n_j}[x_{n_j}^i, x_{n_j}^{i+1}]$ is ε-dense in G. Then one of a_1 and a_2 is in $f^{n_j}[x_{n_j}^i, x_{n_j}^{i+1}]$ for each i, and thus one of a_1 and a_2 is in $f^{n_j}[x_{n_j}^i, x_{n_j}^{i+1}]$ for at least three i-values. But at least two of the three corresponding intervals $[x_{n_j}^i, x_{n_j}^{i+1}]$ are disjoint, contradicting the fact that $|f^{-n_j}(a_1)| = 1 = |f^{-n_j}(a_2)|$. Then $|F| \leq$ # of edges of $G < \aleph_0$. It follows that if $a \in F$ then the backwards orbit of a is finite, hence a is periodic with period n_0 no larger than the number of edges of G.

(ii) Suppose that $a \in F$ has period n_0. Since $|f^{-n}(a)| = 1$ for $n > 0$ and f^{n_0} has finitely many turning points, a connected neighbourhood V of a can be chosen so that V contains no turning point of f^{n_0} and thus f^{n_0} is 1-1 on V. Then m can be chosen so that for each edge E extending from a, f^{mn_0} maps some initial segment of E into E.

Let $g = f^{nm_0}$, and fix an edge E extending from a. Since $|g^{-1}(a)| = 1$ and g has finitely many turning points, there is $c \in E$ such that $|g^{-1}(x)| = 1$ for all $x \in [a, c]$. Suppose that c can be chosen so that for all $x \in [a, c]$, $[a, x] \subseteq g[a, x]$. Then $g^{-1}[a, x] \subseteq [a, x]$, from which it follows that $g^{-n}[a, x]$ is an interval for all $n \in Z^+$. According to 2.6, this contradicts the indecomposability of (G, f). Then given c above, there is $x_1 \in [a, c]$ with $g[a, x_1] \subseteq [a, x_1]$. The neighbourhood of a is constructed in the obvious way.

Since the point x_1 used to construct the neighbourhood of a can be taken as close to a as desired, a has a base of such neighbourhoods. □

The following theorem can be strengthened to prove that f has an n-horseshoe for any n, a fact we need later. Since the proof of 3.4 is somewhat technical, we first prove the existence of a 2-horseshoe, and then indicate the details of the generalization (3.4b).

Theorem 3.4. *Suppose that f is piecewise monotone and that (G, f) is indecomposable. Then f has a horseshoe.*

Proof. Choose a set K consisting of 26 distinct elements of (G, f) with the property that there is a fixed edge E of G and an infinite subset A of Z^+ such that for $n_i \in A$, $\{x_{n_j}^i : 1 \leq i \leq 26\} \subseteq E$, and such that (G, f) is irreducible between any two elements of K.

Claim 1. Either (i) there is a subset $\{\underline{z}^i\}_{i=1}^6$ of K and an infinite subset B_1 of A such that for $i \in \{1, \ldots, 6\}$, $\{z_{n_j}^i : n_j \in B_1\}$ converges to a single point z_i, and $z_i \neq z_j$ if $i \neq j$, or

(ii) there is a subset $\{\underline{x}^i\}_{i=1}^6$ of K and an infinite subset B_2 of A such that for $i = \{1, \ldots, 6\}$, $\{x_{n_j}^i : n_j \in B_2\}$ converges to the same point x.

Proof of Claim 1. Choose $\underline{x}^1$ arbitrarily from K, and an infinite subset B_1^1 of A such that $\{x_{n_j}^1 : n_j \in B_1^1\}$ converges to a single point x_1. Either at least five elements of K have the property that $\{x_{n_j} : n_j \in B_1^1\}$ converges to x_1 also, or for at least 21 of the elements of K, $\{x_{n_j} : n_j \in B_1^1\}$ has a limit point different than x_1. In the latter case, choose $\underline{x}^2$ arbitrarily from this set of 21, and $B_1^2 \subseteq B_1^1$ to be infinite and such that $\{x_{n_j}^2 : n_j \in B_1^2\}$ converges to a single point $x_2 \neq x_1$. Continuing in this fashion, we get at least one of the two sets described above, and the claim is proved.

Suppose that (i) occurs. As in the proof of 3.3(i), we can assume that for $n_j \in B_1$, $z_{n_j}^i \neq z_{n_j}^k$. We can also assume that an orientation of E is fixed and that for $n_j \in B_1$, $\{z_{n_j}^i\}_{i=1}^6$ appears in the order $z_{n_j}^1 < z_{n_j}^2 < \cdots < z_{n_j}^6$.

Define $\varepsilon = \min\{d(z_i, z_k): i \neq k; \ i, k \in \{1, \ldots, 6\}\}$. According to 3.1, there is a positive integer N_0 such that if $n_j \geq N_0$ and $i \in \{1, \ldots, 5\}$, then $f^{n_j}[z_{n_j}^i, z_{n_j}^{i+1}]$ is $\varepsilon/2$-dense in G. In addition, N_0 can be chosen large enough so that if $n_j \geq N_0$ and $i \in \{1, \ldots, 6\}$, $d(z_i, z_{n_j}^i) < \varepsilon/4$. Then for $n_j \geq N_0$, the sets $\{[z_{n_j}^i, z_{n_j}^{i+1}]\}_{i=1,3,5}$ are pairwise disjoint intervals, and each maps under f^{n_j} over at least two of the three. It follows from 3.2 that f has a horseshoe.

Suppose then that (ii) occurs. Again we assume that for $n_j \in B_2$, $\{x_{n_j}^i\}_{i=1}^6$ appears in the order $x_{n_j}^1 < x_{n_j}^2 < \cdots < x_{n_j}^6$.

Claim 2. There is an infinite set $B_3 \subseteq B_2$ so that for at least one i-value i_0, the collection $\{[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]\}_{m_j \in B_3}$ is an infinite collection of pairwise disjoint intervals.

Proof of Claim 2. For each $n_j \in B_2$ and $i = 1$ to 5, the limit point x can be in at most two of the intervals $[x_{n_j}^i, x_{n_j}^{i+1}]$. It follows that for at least three i-values, $x \notin [x_{n_j}^i, x_{n_j}^{i+1}]$ for infinitely many n_j's in B_2. Let i_0 be an i-value for which this is the case and choose m_1 to be the smallest element of B_2 with $x \notin [x_{m_1}^{i_0}, x_{m_1}^{i_0+1}]$. Suppose that $m_1 < m_2 < \cdots < m_k$ have been inductively chosen so that $\{[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]\}_{j=1}^k$ is a pairwise disjoint collection of intervals and $x \notin \bigcup_{j=1}^k \{[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]\}$. Let $\delta_k = \min\{d(x, x_{m_k}^{i_0}), d(x, x_{m_k}^{i_0+1})\}$. Choose N_k so that if $n_j \geq N_k$ and $n_j \in B_2$, then $d(x, x_{n_j}^i) < \delta_k$ for $i = 1$ to 6. There is a least element m_{k+1} of B_2 satisfying $m_{k+1} \geq N_k$ and $x \notin [x_{m_{k+1}}^{i_0}, x_{m_{k+1}}^{i_01}]$. Then $[x_{m_{k+1}}^{i_0}, x_{m_{k+1}}^{i_0+1}] \cap [\bigcup_{j=1}^k [x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]] = \varnothing$, and $x \notin \bigcup_{j=1}^{k+1} [x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]$.

Let $B_3 = \{m_k\}_{k \in Z^+}$ defined above; Claim 2 is proved.

Claim 3. Suppose there are m_1, m_2, n_1, $n_2 \in B_3$ such that $n_1 \neq n_2$, $m_i \neq n_j$ for $i, j \in \{1, 2\}$ and

$$[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}] \not\subset f^{n_j - m_j}[x_{n_j}^{i_0}, x_{n_j}^{i_0+1}] \quad \text{for } j = 1, 2$$

(where i_0 is as in Claim 2). Then f has a horseshoe.

Proof of Claim 3. Let $I_{m_k} = [x_{m_k}^{i_0}, x_{m_k}^{i_0+1}]$ for all $m_k \in B_3$. Since $f^{n_j - m_j}(x_{n_j}^{i_0}) = x_{m_j}^{i_0}$ for $j = 1, 2$, $f^{n_j - m_j}[I_{n_j}] \supseteq E \backslash I_{m_j}$. In particular, for $j = 1, 2$, $I_{n_1} \cup I_{n_2} \subseteq f^{n_j - m_j}[I_{n_j}]$. It follows from 3.2(b) that f has a horseshoe and Claim 3 holds.

Recall that for $m_j \in B_3$, $\{x_{m_j}^i\}_{i=1}^6$ is contained in the fixed edge E. According to Claim 3, if f does not have a horseshoe, there is $m_j \in B_3$ such that for $k \geq i \geq j$, $I_{m_i} \subseteq f^{m_k - m_i}[I_{m_k}]$. Without loss of generality this holds for all m_i, $m_k \in B_3$.

Claim 4. Suppose that there is $n \in Z^+$ and $m_j \in B_3$ such that $E \subseteq f^n[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]$. Then f has a horseshoe.

Proof of Claim 4. Again let $I_{m_k} = [x_{m_k}^{i_0}, x_{m_k}^{i_0+1}]$. According to Claim 3, we can assume that $f^{m_{j+1} - m_j}[I_{m_{j+1}}] \supseteq I_{m_j}$ and $f^{m_{j+2} - m_j}[I_{m_{j+2}}] \supseteq I_{m_j}$. But then $E \subseteq f^{m_{j+1} - m_j + n}[I_{m_{j+1}}]$, and $E \subseteq f^{m_{j+2} - m_j + n}[I_{m_{j+2}}]$. It follows from 3.2(b) that f has a horseshoe.

Claim 5. There is $a \in E$ with $|f^{-n}(a)| = 1$ for all $n \in Z^+$.

Proof of Claim 5. For each $m_k \in B_3$ define

$$J_{m_k} = \bigcup_{m_j \geq m_k, \, m_j \in B_3} f^{m_j - m_k}[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}].$$

Then J_{m_k} is connected and so $\text{cl}(J_{m_k})$ is a subcontinuum of G.

For $n \in Z^+$, let m_k be the least element of B_3 such that $n \leq m_k$ and let $J_n = f^{m_k - n}[J_{m_k}]$. Then $\text{cl}(J_n)$ is a subcontinuum of G, and $f[J_{n+1}] = J_n$. Define $J = \{(z_0, z_1, \ldots) | z_n \in \text{cl}(J_n), \; z_n = f(z_{n+1})\}$. Then J is a subcontinuum of (G, f) containing $\{\underline{x}^i\}_{i=1}^6$, thus $J = (G, f)$. Since f is onto, $\Pi_n[J] = \text{cl}(J_n) = G$.

Suppose that for some n, $J_n \supseteq E$. Let m_k be the smallest element of B_3 with $m_k \geq n$. Then $f^{m_k - n}[J_{m_k}] = J_n \supseteq E$. But J_{m_k} is the union of nested continua so there is $m_j \in B_3$ with $m_j \geq m_k$ and $f^{m_k - n}[f^{m_j - m_k}[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]] \supseteq E$. But then $f^{m_j - n}[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}] \supset E$, contradicting the choice of i_0. So $E \not\subseteq J_n$ for any n. But J_n is connected and $\text{cl}(J_n) = G$, so for each n there is a set $F_n = G \backslash J_n$ with $|F_n| \leq k$, where k is the cardinality of the largest discrete set that can be removed from G without disconnecting the remaining subgraph. Since $f[J_{n+1}] = J_n$, $f^{-1}[F_n] \subseteq F_{n+1}$, and so $|F_n| \leq |F_{n+1}| \leq k$. Then there is a positive integer N_0 so that if $n \geq N_0$, $|F_n| = |F_{n+1}|$. That is, for $n \geq N_0$ and $a \in F_n$, $|f^{-1}(a)| = 1$, thus $|f^{-k}(a)| = 1$ for $k \geq 0$. Since $F_n \cap E \neq \varnothing$, Claim 5 holds.

According to the proof of 3.3(i), a is periodic and $f^{-n}(a) \cap E \subseteq \{a\}$ for all $n \geq 0$.

Claim 6. For $1 \leq i \leq 6$ and $n \in Z^+$, $x_{m_j}^i \neq a$.

Proof of Claim 6. Suppose that there are $j \leq 6$ and $n \in Z^+$ such that $x_n^j = a$. Since $f^{-n}(a) \cap E \subseteq \{a\}$ for all $n \in Z^+$, $x_{m_k}^i = a$ for any $m_k \in B_3$. It follows that if $i \neq j$, $x_n^i \neq a$ for any $n \in Z^+$, for otherwise $\underline{x}^j = \underline{x}^i$. Let n_0 be the period of a, and choose $m \in Z^+$ and V an open neighbourhood of a such that $f^{mn_0}[V] \subseteq V$. According to 3.3, the set V can be chosen small enough so that $V \cap \{x_k^i : i = 1 \text{ to } 6, \, i \neq j, \, 1 \leq k \leq mn_0\} = \varnothing$. Then for $i \neq j$, $x_{m_k}^i \notin V$ for $m_k \in B_3$. Recall that for each i, $\{x_{m_k}^i : m_k \in B_3\}$ converges to the same limit point x. Apparently $x = a$, and thus there is a positive integer N_0 so that $\{x_{m_k}^i : m_k \in B_3, \, m_j \geq N_0, \, i \leq 6\} \subseteq V$. This is a contradiction to the construction of V, and Claim 6 holds.

Then the open neighbourhood V of a can be chosen so that $f^{mn_0}[V] \subseteq V$ and $V \cap \{x_k^i : i = 1 \text{ to } 6, \, 1 \leq k \leq mn_0\} = \varnothing$. As above, $x_{m_j}^i \notin V$ for $m_j \in B_3$ and $i \leq 6$. That is, for each $m_j \in B_3$, $[x_{m_j}^i, x_{m_j}^{i+1}] \subseteq E \backslash V$ except for possibly one i-value, and for that i-value $V \subseteq [x_{m_j}^i, x_{m_j}^{i+1}]$.

Choose $\varepsilon > 0$ so that the ε-ball centered at a is contained in V, and choose N_1 large enough so that for $i = 1$ to 5, $m_j \in B_3$ and $m_j \geq N_1$, $f^{m_j}[x_{m_j}^i, x_{m_j}^{i+1}]$ is ε-dense in G. For a fixed m_j, the interiors of the intervals $[x_{m_j}^i, x_{m_j}^{i+1}]_{i+1}^5$ are disjoint, hence $a \in f^{m_j}[x_{m_j}^i, x_{m_j}^{i+1}]$ for at most one i-value. For the remaining four i-values, $E \backslash V \subseteq f^{m_j}[x_{m_j}^i, x_{m_j}^{i+1}]$, and according to the previous paragraph, for at least three of these, $[x_{m_j}^i, x_{m_j}^{i+1}] \subseteq E \backslash V$. Since at least two of the three intervals $[x_{m_j}^i, x_{m_j}^{i+1}]$ are disjoint, f has a horseshoe. $\square$

For maps of the interval, the existence of an n-horseshoe follows easily from the existence of a 2-horseshoe—given an interval I mapping over intervals J and K, there are disjoint subintervals I_1 and I_2 of I mapping over J and K respectively, etc. As is clear from the above proof, the latter statement need not hold for maps on circles or finite graphs, thus the existence of n-horseshoes and the "monotone" horseshoes of 3.5 requires some details.

Theorem 3.4b. *Suppose that f is piecewise monotone and that (G, f) is indecomposable. Then f has an n-horseshoe for any positive integer n.*

Proof. We trace the proof of 3.4 to ensure that at each point that a horseshoe is produced, an n-horseshoe can be produced. First, it is clear that the size of the original irreducible set can be as large as desired, and the cases (i) and (ii) in the first paragraph can be witnessed by sets of any finite size. Case (i) follows immediately from 3.2(a).

The appropriate generalization of Claim 3 is the following:

Claim 3b. Suppose that for the positive integer n, there are disjoint subsets $\{m_i : 1 \le i \le n\}$ and $\{n_i : 1 \le i \le n, \ n_j \ne n_i$ for $i \ne j\}$ of B_3 such that for each i,

$$[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}] \not\subseteq f^{n_i - m_i}[x_{n_i}^{i_0}, x_{n_i}^{i_0+1}].$$

Then f has an n-horseshoe.

As with Claim 3, this follows from 3.2(b).

Then if f does not have an n-horseshoe, there is $m_j \in B_3$ such that for $k \ge i \ge j$, $E \backslash I_{m_i} \subseteq f^{m_k - m_i}[I_{m_k}]$. Without loss of generality this holds for all m_i, $m_k \in B_3$.

Claim 4b. Suppose that there is $n_0 \in Z^+$ and $m_j \in B_4$ such that $E \subseteq f^{n_0}[x_{m_j}^{i_0}, x_{m_j}^{i_0+1}]$. Then f has an n-horseshoe.

Proof of Claim 4b. Again let $I_{m_k} = [x_{m_k}^{i_0}, x_{m_k}^{i_0+1}]$. As in the proof of 3.4, $f^{m_{j+i} - m_j + n_0}[I_{m_{j+i}}] \supseteq E$ for $1 \le i \le n$, and Claim 4b follows from 3.2(b).

The proofs of Claims 5 and 6 need no changes, and the neighbourhood V of the fixed point a can be constructed so that for some fixed m_j and as large a set of i-values as desired, $[x_{m_j}^i, x_{m_j}^{i+1}] \subseteq E \backslash V \subseteq f^{m_j}[x_{m_j}^i, x_{m_j}^{i+1}]$. Then f has an n-horseshoe. $\square$

Note that the horseshoe constructed in 3.4 or 3.4b has all intervals contained in a single edge without branch points.

An interval I *maps monotonely* over the interval J under f if there is a subinterval K of I such that $f[K] = J$. A horseshoe $\{I_1, I_2\}$ for f is *monotone* with respect to f^n if for $i, j = 1, 2$, there are subintervals I_{ij} of I_i such that $f^n[I_{ij}] = I_j$.

Corollary 3.5. *Suppose that f is piecewise monotone and (G, f) contains an indecomposable subcontinuum. Then f has a monotone horseshoe.*

Proof. Since (G, f) contains an indecomposable subcontinuum, (G, f) contains an indecomposable subcontinuum H invariant under $\hat{f}^n$ for some $n \in Z^+$. The inverse limit spaces (G, f) and (G, f^n) are homeomorphic; let H_1 be the image of H under the natural homeomorphism, and let $J = \Pi_0[H_1]$. Then J is invariant under f^n, and $g = f^n|_J$ is piecewise monotone, so according to 3.4b, g has a 3-horseshoe $\{I_1, I_2, I_3\}$ with $I_1 \cup I_2 \cup I_3 \subseteq E$ for some edge E of J having no branch points. Then each of I_i maps monotonely under g^m

over at least two of I_1, I_2 and I_3, where m is the positive integer witnessing the horseshoe. Since the property of mapping monotonely (under some iterate) is transitive, arguments identical to those in the proof of 3.2 imply that at least two of I_1, I_2, I_3 form a monotone horseshoe with respect to g^{2m}. But this monotone horseshoe will also be a monotone horseshoe for f on G. □

We can now deduce a "stability" result for piecewise monotone functions on finite graphs.

Corollary 3.6. *Suppose that $f: G \to G$ is piecewise monotone and that (G, f) contains an indecomposable subcontinuum. If g is any continuous map on G sufficiently close to f, (G, g) contains an indecomposable subcontinuum.*

Proof. According to 3.5, f has a horseshoe $\{I_1, I_2\}$ monotone with respect to f^n for some positive integer n. By using the monotonicity of the horseshoe repeatedly, one can find four subintervals J_1, J_2, J_3, J_4 of I_1 such that $f^{3n}[J_i] = I_1$ for $i = 1$ to 4. For some $\varepsilon > 0$ and i-values i_1, i_2, the distance from J_{i_j} to the endpoints of I_1 is larger than ε for $j = 1, 2$. Without loss of generality, if an edge E' of G is not contained in I_1, there is $x \in E'$ such that $d(x, I_1) \geq \varepsilon$, where $d(x, I_1)$ refers to minimal arc length. Suppose that $d(f^{3n}, g^{3n}) < \varepsilon$. Since $f^{3n}[J_{i_j}] = I_1$, $g^{3n}[J_{i_j}] \supseteq J_{i_1} \cup J_{i_2}$ for $j = 1, 2$. That is, g has a horseshoe. According to 2.5, (G, g) contains an indecomposable subcontinuum. □

4. POSITIVE ENTROPY → INDECOMPOSABLE SUBCONTINUUM

Suppose that $f: X \to X$ is a map of the compact metric space X with metric d. For $n \in N$ and $\varepsilon > 0$, a set $E \subseteq X$ is (n, ε)-separated under f provided that for distinct x, $y \in E$, there is a k such that $0 \leq k < n$ and $d(f^k(x), f^k(y)) > \varepsilon$. Given $A \subseteq X$, let $E(n, \varepsilon, A)$ be a subset of A of maximal cardinality that is (n, ε)-separated under f. Let $h(f, A, \varepsilon) = \limsup_{n \to \infty} (\log \#E(n, \varepsilon, A))/n$, and $h(f, A) = \lim_{\varepsilon \to 0} h(f, A, \varepsilon)$. The *topological entropy* of f is then $h(f) = h(f, X)$. (There are several equivalent definitions of entropy; see, for example, [AKM] or [Bo].) We review here a few facts about entropy:

(i) $h(f)$ is independent of (equivalent) metric;

(ii) for $n \geq 1$, $h(f^n) = nh(f)$;

(iii) if f is a homeomorphism, $h(f^{-1}) = h(f)$;

(iv) if $\hat{f}: (X, f) \to (X, f)$ is the induced homeomorphism then $h(\hat{f}) = h(f)$;

(v) if $f: G \to G$ is a homeomorphism of a finite graph G then $h(f) = 0$;

(vi) if $f: X \to X$ and $g: Y \to Y$ are maps of the compact spaces X and Y and $p: Y \to X$ is a continuous surjection such that $f \circ p = p \circ g$, then $h(g) \leq h(f) + \sup_{x \in X} h(g|_{p^{-1}(x)})$ and

(vii) $h(f) = h(f, \Omega(f))$, where $\Omega(f) = \{x \in X| \text{ for each neighbourhood } U \text{ of } x \text{ there is } n \geq 1 \text{ such that } f^n(U) \cap U \neq \varnothing\}$ is the *nonwandering set* of f.

Of the above, (i), (ii), (iii), (vi) and (vii) can be found in [Bo]; (iv) is in [G] and [Ba]; (v) follows from (ii) and the like results in case G is an arc or circle [AKM].

In the following, $\mathscr{G}$ will denote the decomposition space of Theorem 1.1 associated with (X, f) with induced homeomorphism $F: \mathscr{G} \to \mathscr{G}$.

Lemma 4.1. *If $h(f) > 0$ and (G, f) contains no indecomposable subcontinuum with nonempty interior, then there is $\xi \in \mathscr{G}$ that is periodic under F and for which $h(\hat{f}, g^{-1}(\xi)) > 0$.*

Proof. We have $0 < h(f) = h(\hat{f}) = h(\hat{f}^{-1}) \leq h(F^{-1}) + \sup_{\xi \in \mathscr{G}} h(\hat{f}^{-1}, g^{-1}(\xi))$ by (iii), (iv) and (vi) above. Since $h(F^{-1}) = 0$ ((v) above), there is $\xi \in \mathscr{G}$ such that $h(\hat{f}^{-1}, g^{-1}(\xi)) > 0$.

If it were that $\lim_{k \to \infty} \operatorname{diam}(\Pi_k[g^{-1}(\xi)]) = 0$, then

$$\lim_{n \to \infty} \operatorname{diam}(\hat{f}^{-n}[g^{-1}(\xi)]) = 0$$

and it would follow that $h(\hat{f}^{-1}, g^{-1}(\xi)) = 0$. Thus $\lim_{k \to \infty} \operatorname{diam}(\Pi_k[g^{-1}(\xi)]) \neq 0$, and there is a $p \in G$ such that $p \in \Pi_n[g^{-1}(\xi)]$ for infinitely many n. According to the remark following 2.10, ξ is periodic under F, say of period n. Then $0 < h(\hat{f}^{-1}, g^{-1}(\xi)) = (1/n)h(\hat{f}^{-n}|_{g^{-1}(\xi)}) = (1/n)h(\hat{f}^n|_{g^{-1}(\xi)}) = h(\hat{f}, g^{-1}(\xi))$. $\square$

By an *n-od* we will mean a space which is the union of n arcs $A_1, A_2, \ldots, A_n$ and which contains a point p such that $A_i \cap A_j = \{p\}$ for $i \neq j$. It is clear that if $f: X \to X$ is a homeomorphism of the n-od X and x is a periodic point of f then the period of x divides $n!$.

Lemma 4.2. *Suppose that $f: X \to X$ is a map of the n-od X and that $h(f) > 0$. Then (X, f) contains an indecomposable subcontinuum.*

Proof. According to 3.2 of [Bl], if $h(f) > 0$ there are positive integers r and M such that $\operatorname{Per}(f) \equiv \{m | f \text{ has a periodic point of period } m\} \supseteq \{mr | m \geq M\}$. In particular, f must have a periodic point, say x, of period pr where p is a prime larger than $n!$. Let $\underline{x}$ be a corresponding point in (X, f) of period pr under $\hat{f}$.

Suppose that (X, f) is hereditary decomposable. It is a consequence of Theorem A of [Ro] that $\mathscr{G}$ is a tree with no more than n endpoints, hence the period of every periodic point of F divides $n!$. In particular, if $\xi \in \mathscr{G}$ contains $\underline{x}$ then the period n_1 of ξ divides both $n!$ and pr. Since $p > n!$ and p is prime, n_1 must divide r.

Let $X_1 = \Pi_0[g^{-1}(\xi)]$, where $g: (X, f) \to \mathscr{G}$ is the quotient map, and let $f_1 = f^{n_1}|_{X_1}: X_1 \to X_1$. Then (X_1, f) contains a periodic point of period $p(r/n_1)$ under f_1 and X_1 is an l_1-od for some $l_1 \leq n$. Let $\mathscr{G}_2$ be the decomposition space of (X_1, f_1) and let $F_2: \mathscr{G}_2 \to \mathscr{G}_2$ be the homeomorphism induced by $\hat{f}_1$ (note that (X_1, f_1) is homeomorphic with the subcontinuum $g^{-1}(\xi)$ of (X, f) and thus is hereditarily decomposable). Let $\xi_2 \in \mathscr{G}_2$ contain a point of period $p(r/n_1)$ under $\hat{f}_1$. Then, as above, the period n_2 of ξ_2 under F_2 divides r/n_1.

Proceeding in this manner, we define a strictly decreasing sequence X_k of l_k-ods, $l_k \leq n$, and positive integers n_k such that $f_k \equiv f^{n_1 n_2 \cdots n_k}|_{X_k}$ maps X_k onto X_k and $n_1 \cdots n_k$ divides r. It must be the case that for some K, $n_k = 1$ for all $k \geq K$. Thus if $X_\infty = \bigcap_{k \geq 1} X_k$ then X_∞ is an l_∞-od for some positive integer $l_\infty \leq n$, $f_\infty \equiv f^{n_1 n_2 \cdots n_K}|_{X_\infty}$ maps X_∞ onto X_∞, and in X_∞ there is a periodic point of period $p(r/n_1 \cdots n_K)$ under f_∞.

This process can be continued with transfinite induction as long as the (X_α, f_α) are hereditarily decomposable. Since (X_α, f_α) is homeomorphic to a subcontinuum of (X, f) and thus is hereditarily decomposable, the process continues indefinitely. Now for $\alpha < \beta$, X_β is proper in X_α unless X_α is a single point. Thus there must be a (countable) ordinal α such that X_α is a single point. But X_α contains a periodic orbit of period some multiple of $p > 1$, and this is not possible. Thus (X, f) must contain an indecomposable subcontinuum. □

Theorem 4.3. *Suppose that $f \colon G \to G$ is a map of the finite graph G and that $h(f) > 0$. Then (G, f) contains an indecomposable subcontinuum.*

Proof. As in the proof of 4.2, suppose that $(G, f) \equiv (G_0, f_0)$ is hereditarily decomposable. Let $\mathscr{G}_0$ be the decomposition of (G_0, f_0) with quotient map $g \colon (G_0, f_0) \to \mathscr{G}_0$ and $F_0 \colon \mathscr{G}_0 \to \mathscr{G}_0$ the homeomorphism induced by $\hat{f}_0$. According to 4.1, there is a periodic (under F_0) point $\xi \in \mathscr{G}_0$ such that $h(\hat{f}_0, g^{-1}(\xi)) > 0$. Let n_1 be the period of ξ, let $G_1 = \Pi_0[g^{-1}(\xi)]$ and let $f_1 = f_0^{n_1}|_{G_1}$. Then G_1 is a proper subcontinuum of G_0, f_1 maps G_1 onto G_1, (G_1, f_1) is homeomorphic with $g^{-1}(\xi)$ and hence is hereditarily decomposable, and $h(f_1) = h(\hat{f}_1) = n_1 h(\hat{f}, g^{-1}(\xi)) > 0$. Continuing in this way we define a strictly decreasing sequence of graphs $G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$, positive integers $n_0 = 1$, $n_1, n_2, \ldots$, and maps $f_k = f^{n_1 n_2 \cdots n_k}|_{G_k} \colon G_k \to G_k$ such that $h(f_k) > 0$ for all k. For each $k \geq 1$, $G_k = \Pi_0[g^{-1}(\xi)]$ where ξ is an element of the decomposition space $\mathscr{G}_{k-1}$ of (G_{k-1}, f_{k-1}) periodic of period n_k under the induced homeomorphism $F_{k-1} \colon \mathscr{G}_{k-1} \to \mathscr{G}_{k-1}$. Thus the subcontinua $f_{k-1}^r(G_k)$ of G, $r = 0, \ldots, n_k - 1$, are pairwise disjoint and $f^{n_1 n_2 \cdots n_k}(G_k) = f_k(G_k) = G_k$.

Now if the sequence $\{n_1 n_2 \cdots n_k\}_{k=1}^\infty$ is unbounded there must be a k and an r, $0 \leq r < n_1 n_2 \cdots n_k$, such that $f_k^r(G_k) = I$ is an arc (G is a finite graph), and $f^{n_1 n_2 \cdots n_k}|_I \colon I \to I$ has positive entropy. But then $f^{n_1 n_2 \cdots n_k}$ has a horseshoe in I [Mi] and thus by 2.4 (G, f) has an indecomposable subcontinuum, contrary to our assumption.

Thus the sequence $\{n_1 n_2 \cdots n_k\}_{k=1}^\infty$ must be bounded and there is a K such that $n_k = 1$ for all $k \geq K$. Now $G_K \supseteq G_{K+1} \supseteq G_{K+2} \cdots$ is a strictly decreasing sequence of finite graphs, each invariant under f_K. Since G has only finitely many branch points, there are l and n such that for each $s \geq 1$, $G_{K+l} \backslash G_{K+l+s}$ is a finite disjoint union of half-open arcs A_i^s, $i = 1, \ldots, n$, with $A_i^s \subseteq A_i^r$ for $s \leq r$. Let $G_\infty = \bigcap_{k=1}^\infty G_k$. Then G_∞ is a subcontinuum of G invariant under f_K and $G_{K+l} \backslash G_\infty$ is the disjoint union of half-open arcs A_i, $i = 1, \ldots, n$, where $A_i = \bigcup_{s \geq 1} A_i^s$.

Let X be the quotient space G_{K+l}/G_∞ and let $t \colon X \to X$ be the map induced by $f_{K+l} \colon G_{K+l} \to G_{K+l}$. Note that X is an n-od. We consider two cases.

Case 1. $h(t) > 0$: Then (X, t) contains an indecomposable subcontinuum by 4.2. But $\hat{p} \colon (G_{K+l}, f_{K+l}) \to (X, t)$ defined by $\hat{p}(x_0, x_1, \ldots) = (p(x_0), p(x_1), \ldots)$, where $p \colon G_{K+l} \to X$ is the quotient map, is a monotone surjection. It follows that (G_{K+l}, f_{K+l}) must also contain an indecomposable subcontinuum, contrary to the assumption that (G, f) is hereditarily decomposable. (If $H \subseteq (X, t)$ is indecomposable, then $\hat{p}^{-1}(H)$ is a subcontinuum

of (G_{K+l}, f_{K+l}) and if C is a continuum in $\hat{p}^{-1}(H)$ irreducible with respect to the property that $\hat{p}(C) = H$, then C is indecomposable.)

Case 2. $h(t) = 0$: In this case, since $p: G_{K+l} \to X$ semiconjugates f_{K+l} with t and p is one-to-one off G_∞, it follows from property (vi) of entropy listed at the beginning of this section that $h(f_{K+l}|_{G_\infty}) > 0$. Let $f_\infty = f_{K+l}|_{G_\infty}: G_\infty \to G_\infty$. Then $h(f_\infty) > 0$ and we may inductively define subcontinua G_α and positive entropy maps $f_\alpha: G_\alpha \to G_\alpha$ for all ordinals α, as above. But for $\alpha < \beta$, G_β is proper in G_α unless G_α is a single point. Thus for some (countable) ordinal α, G_α is a single point. But then $h(f_\alpha) = 0$, a contradiction. Thus (G, f) must contain an indecomposable subcontinuum. $\square$

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DEPARTMENT OF MATHEMATICS, MONTANA STATE UNIVERSITY, BOZEMAN, MONTANA 59717

DEPARTMENT OF MATHEMATICS, COLLEGE OF CHARLESTON, CHARLESTON, SOUTH CAROLINA 29424