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BEST COMONOTONE APPROXIMATION

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ABSTRACT. A general theory of best comonotone approximation in $C[a, b]$ by elements of an n-dimensional extended Chebyshev subspace is described. In particular, theorems on the existence, (in general) nonuniqueness, and characterization of best comonotone approximations are established.

1. INTRODUCTION

Let $C[a, b]$ denote the Banach space of all real-valued continuous functions on $[a, b]$ endowed with the uniform norm: $\|f\| = \sup\{|f(t)| | t \in [a, b]\}$. Let $C^r[a, b]$ be the subspace of $C[a, b]$ consisting of all r-times continuously differentiable functions on $[a, b]$. A function $f \in C^{r-1}[a, b]$ is said to have a *zero of multiplicity* r at a point $t \in [a, b]$, if $f^{(i)}(t) = 0$, $i = 0, \dots, r-1$. An n-dimensional subspace M of $C^{(n-1)}[a, b]$ is called an *extended Chebyshev subspace* if every nontrivial function in M has at most $n-1$ zeros (counting multiplicities). The prototype of such a subspace is Π_{n-1}, the subspace of polynomials of degree at most $n-1$. For another example, let $\alpha_1 < \cdots < \alpha_n$ be real numbers. Then $E_n = \{\sum_{i=1}^n c_i e^{\alpha_i t} |\ c_1, \dots, c_n$ are real numbers$\}$ is an n-dimensional extended Chebyshev subspace.

A function $f \in C[a, b]$ is called *piecewise monotone* if there is a partition of $[a, b]$ (into a finite number subintervals) such that f alternately increases and decreases on these subintervals. A function p is said to be *comonotone* with f if p increases (resp., decreases) on those subintervals where f *strictly* increases (resp., *strictly* decreases). Note that there is no restriction on p for those subintervals on which f is constant. Equivalently, p is comonotone with f if for each $t \in [a, b]$ which is the end point of a subinterval I_t on which f is strictly monotone, then p is monotone on I_t (in the same sense as f).

We should mention that the term "piecewise monotone" was defined by Newman, Passow, and Raymon [7] as follows: $f \in C[a, b]$ is "piecewise monotone" if f has only a finite number of local extrema. But this is equivalent to saying that $[a, b]$ can be partitioned into subintervals on which f is alternately *strictly* increasing and *strictly* decreasing. Thus, such a function is piecewise monotone in our sense as well. The term "comonotone" was also defined in [7] for functions which are "piecewise monotone" in the sense of [7]: a polynomial p is "comonotone" with f if it increases and decreases simultanously

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with f. Again such a p is comonotone in our sense as well. The main reason we wish to use our somewhat more general definitions of piecewise monotone and comonotone is that we can prove characterization theorems in this larger class. Note, in particular, that a constant function is piecewise monotone in our sense, but not in the sense of [7].

For a given positive integer n, let M be an n-dimensional extended Chebyshev subspace of $C[a, b]$ containing constant functions. Let f be piecewise monotone and

$$C_M(f) := \{p \in M \mid p \text{ is comonotone with } f\}.$$

Note that $C_M(f) \neq \varnothing$ since $C_M(f)$ contains all constant functions. An element $p \in C_M(f)$ is called a *best comonotone approximation* to f provided that

$$\|f - p\| = d(f, C_M(f)) := \inf\{\|f - q\| : q \in C_M(f)\}.$$

In the case that $M = \Pi_{n-1}$, various "Jackson-type" results (i.e., upper bounds on $d(f, C_M(f)))$ were investigated in [7], [8], [9], and [6].

In this paper we study the problems of the characterization and uniqueness of best comonotone approximations. In §2 we give two such characterizations (see Theorem 2.5 and Theorem 2.6). Such characterizations should be useful for the actual computation of best comonotone approximations. Less detailed characterizations may be obtained from the results of Deutsch and Maserick [2] and Dunham [3]. In §3 we show that these characterization theorems can be significantly improved in the special case when $M = \Pi_{n-1}$. In §4 we show that best comonotone approximations are *not unique* in general. This is in marked contrast to the case of best monotone approximations (see [4]).

2. Characterization of best comonotone approximations

Throughout this section, we assume that M is an n-dimensional extended Chebyshev subspace of $C[a, b]$ which contains constant functions.

We first note that the *existence* of best comonotone approximations to each piecewise monotone function $f \in C[a, b]$ is always guaranteed. This follows from the fact that $C_M(f)$ is a closed convex cone in the finite-dimensional subspace M.

If $C_M(f)$ consists of only constant functions, then the problem of best comotone approximation reduces to the problem of best (unconstrained) approximation from the subspace of constant functions. Since this case is well-known and easy, we shall assume henceforth that $C_M(f)$ contains nonconstant functions. Let $M_0(f) = \operatorname{span}\{C_M(f)\}$. Assume that $\Phi_1, \dots, \Phi_k$ are linearly independent elements in $C_M(f)$ so that $M_0(f) = \operatorname{span}\{\Phi_1, \dots, \Phi_k\}$. It is obvious that $\dim M_0(f) = k \geq 2$. Throughout this paper, we shall always assume that $\Phi = \Phi_1 + \cdots + \Phi_k$.

Lemma 2.1. *Let f be an arbitrary piecewise monotone function and $S(f) = \{x \in [a, b] \mid f$ is strictly monotone in $(x - \delta, x]$ or $[x, x + \delta)$ for some $\delta > 0\}$. For any $q \in M_0(f)$,*

$$F(x) := \lim_{t \to x} \frac{q(t) - q(x)}{\Phi(t) - \Phi(x)}$$

exists for each $x \in S(f)$. Furthermore, F is continuous on $S(f)$.

Proof. Assume that $S(f) \neq \varnothing$. Then it is easy to see that Φ is not a constant function. Let $x \in S(f)$. Without loss of generality, we may assume that f is strictly monotone on $[x, x+\delta)$ for some $\delta > 0$. If $q \in M_0(f)$, we may write $q = \sum_{i=1}^{k} c_i\Phi_i$. Thus for all $t \in (x, x+\delta)$, we have

$$\begin{aligned}\left|\frac{q(t)-q(x)}{\Phi(t)-\Phi(x)}\right| &= \frac{|\sum_{i=1}^{k} c_i[\Phi_i(t)-\Phi_i(x)]|}{\sum_{i=1}^{k}|\Phi_i(t)-\Phi_i(x)|} \\ &\leq \sum_{i=1}^{k}|c_i|\frac{|\Phi_i(t)-\Phi_i(x)|}{\sum_{i=1}^{k}|\Phi_i(t)-\Phi_i(x)|} \leq \sum_{i=1}^{k}|c_i| < \infty.\end{aligned} \tag{1}$$

Let $g = \Phi - \Phi(x)$. Since M contains constant functions, $g \in M$. Now since M is an n-dimensional extended Chebyshev subspace and g is not a constant function, there is an integer $1 \leq w(x) \leq n-1$ so that $g^{(w(x))}(x) = \Phi^{(w(x))}(x) \neq 0$ and $g^{(i)}(x) = 0$ for $0 \leq i \leq w(x)-1$. It follows from (1) and the Mean Value Theorem that $q^{(i)}(x) = 0$ for $1 \leq i \leq w(x)-1$. Hence L'Hôpital's rule implies that

$$F(x) = \lim_{t\to x}\frac{q(t)-q(x)}{\Phi(t)-\Phi(x)} = \frac{q^{(w(x))}(x)}{\Phi^{(w(x))}(x)}.$$

Thus $F(x)$ exists for each $x \in S(f)$.

Let $x_j \in S(f)$ and $x_j \to x_0 \in S(f)$. Since $\Phi^{(w(x_0))}(x_0) \neq 0$, there is a neighborhood $U(x_0)$ of x_0 such that $\Phi^{(w(x_0))}(t) \neq 0$ for all $t \in U(x_0)$. Thus $w(x_j) \leq w(x_0)$ if j is large enough so that $x_j \in U(x_0)$. Since $1 \leq w(x_j) \leq n-1$, there is a subsequence of $\{x_j\}$, still denoted $\{x_j\}$, so that $w(x_j) = w$ for $j = 1, 2, \ldots$. It follows by L'Hôpital's rule that

$$\lim_{j\to\infty} F(x_j) = \lim_{j\to\infty}\frac{q^{(w(x_j))}(x_j)}{\Phi^{(w(x_j))}(x_j)} = \lim_{j\to\infty}\frac{q^{(w)}(x_j)}{\Phi^{(w)}(x_j)} = \frac{q^{(w(x_0))}(x_0)}{\Phi^{(w(x_0))}(x_0)} = F(x_0).$$

This means that for any sequence $\{x_j\}$ with $x_j \to x_0$, there is a subsequence $\{x_v\}$ of $\{x_j\}$ so that $\lim_{v\to\infty} F(x_v) = F(x_0)$. Thus F is continuous at x_0. This completes the proof. □

Now fix an arbitrary piecewise monotone function $f \in C[a, b]\backslash M$. For any $p \in C_M(f)$, we define certain subsets of $[a, b]$ as follows.

$$\begin{aligned} X_{+1}(p) &= \{x \in [a, b] \mid f(x) - p(x) = \|f-p\|\}, \\ X_{-1}(p) &= \{x \in [a, b] \mid f(x) - p(x) = -\|f-p\|\}, \\ X(p) &= \left\{x \in [a, b] \mid x \in S(f) \text{ and } \lim_{t\to x}\frac{p(t)-p(x)}{\Phi(t)-\Phi(x)} = 0\right\}. \end{aligned}$$

Next define a function $\sigma_1 = \sigma_{1,p}$ on $X_{+1}(p) \cup X_{-1}(p)$ as follows.

$$\sigma_1(t) = \sigma_{1,p}(t) = \begin{cases} 1 & \text{if } t \in X_{+1}(p), \\ -1 & \text{if } t \in X_{-1}(p). \end{cases}$$

Lemma 2.2. *The set*

$$C = \left\{\left(\lim_{t\to x}\frac{\Phi_1(t)-\Phi_1(x)}{\Phi(t)-\Phi(x)}, \cdots, \lim_{t\to x}\frac{\Phi_k(t)-\Phi_k(x)}{\Phi(t)-\Phi(x)}\right)\middle| x \in X(p)\right\}$$

is compact in $\mathbb{R}^k$.

Proof. Since $X(p)$ is a compact subset of $S(f)$ and for each $i = 1, \ldots, k$,

$$\lim_{t \to x} \frac{\Phi_i(t) - \Phi_i(x)}{\Phi(t) - \Phi(x)}$$

is continuous on $S(f)$ by the preceding lemma, C is compact. □

Lemma 2.3. *Let $f \in C[a, b]$ be piecewise monotone and $p \in C_M(f)$. If $q \in M_0(f)$ satisfies*

$$F(x) := \lim_{t \to x} \frac{q(t) - q(x)}{\Phi(t) - \Phi(x)} > 0 \quad \text{for all } x \in X(p),$$

then there exists $\lambda_0 > 0$ such that $p + \lambda q \in C_M(f)$ for all $0 < \lambda \leq \lambda_0$.

Proof. Let $q_\lambda = p + \lambda q$. Let $\{[a_i, b_i]\}_{i \in D}$ be the set of all the intervals on which f is constant, i.e., $f(t) = c_i$ for some constant c_i and for $t \in [a_i, b_i]$, but $f(t) \neq c_i$ if $t \in (a_i - \delta, a_i) \cup (b_i, b_i + \delta)$ for some small $\delta > 0$. Denote $T(f) = \{a_i, b_i\}_{i \in D}$. We consider the following five cases.

Case 1. If $x \in X(p) \backslash T(f)$, then f is strictly monotone on both a left neighborhood and a right neighborhood of x and $F(x) > 0$. Since F is continuous on $S(f)$, there exists a neighborhood $\Delta_x = (x - \delta, x + \delta)$ of x so that $F(y) > 0$ for all $y \in \Delta_x$ and f is strictly monotone on $(x - \delta, x)$ and $(x, x + \delta)$. From this and the fact that f is comonotone with Φ on Δ_x, we see that q is comonotone with f on Δ_x. This implies that q_λ is comonotone with f on Δ_x for all $\lambda > 0$.

Case 2. If $x \in X(p) \cap T(f)$, without loss of generality, we may assume that f is strictly monotone on a left neighborhood of x and is constant on a right neighborhood of x, and $F(x) > 0$. By continuity of F on $S(f)$, there exists a neighborhood $\Delta_x = (x - \delta, x + \delta)$ of x so that $F(y) > 0$ for all $y \in (x - \delta, x)$, f is strictly monotone on $(x - \delta, x)$, and f is constant on $(x, x + \delta)$. From this and the fact that f is comonotone with Φ on $(x - \delta, x)$, we get that q is comonotone with f on $(x - \delta, x)$. On the other hand, since f is constant on $(x, x + \delta)$, q is comonotone with f on $(x, x + \delta)$. This implies that q_λ is comonotone with f on Δ_x for all $\lambda > 0$.

Case 3. Denote $\Delta_i = (a_i, b_i)$ for $i \in D$. Since f is constant on $\bigcup_{i \in D} \Delta_i$, q_λ is comonotone with f on $\bigcup_{i \in D} \Delta_i$ for all $\lambda > 0$.

Case 4. If $x \in ([a, b] \backslash [\bigcup_{i \in D} \Delta_i \cup \{\Delta_x : x \in X(p)\}]) \backslash T(f)$, then f is strictly monotone on both a left neighborhood and a right neighborhood of x and

$$Q(x) := \lim_{t \to x} \frac{p(t) - p(x)}{\Phi(t) - \Phi(x)} > 0.$$

By continuity of F and Q on $S(f)$, there exist a neighborhood Δ_x of x and $\lambda_x > 0$ so that

$$Q(y) + \lambda F(y) > 0$$

for all $y \in \Delta_x$, $0 < \lambda \leq \lambda_x$, and f is strictly monotone on Δ_x. It follows that

$$\lim_{t \to y} \frac{q_\lambda(t) - q_\lambda(y)}{\Phi(t) - \Phi(y)} > 0$$

for all $y \in \Delta_x$ and $0 < \lambda \leq \lambda_x$. Thus q_λ is comonotone with f on Δ_x for all $0 < \lambda \leq \lambda_x$.

Case 5. If $x \in ([a, b] \backslash [\bigcup_{i \in D} \Delta_i \cup \{\Delta_x : x \in X(p)\}]) \cap T(f)$, without loss of generality, we may assume that f is strictly monotone on a left neighborhood of x and is constant on a right neighborhood of x, and $Q(x) > 0$. Since F and Q are continuous on $S(f)$, there exist a neighborhood $\Delta_x = (x-\delta, x+\delta)$ of x and $\lambda_x > 0$ so that $Q(y) + \lambda F(y) > 0$ for all $y \in (x-\delta, x)$, $0 < \lambda \le \lambda_x$, and f is strictly monotone on $(x-\delta, x)$ and is constant on $(x, x+\delta)$. It follows that

$$\lim_{t \to y} \frac{q_\lambda(t) - q_\lambda(y)}{\Phi(t) - \Phi(y)} > 0$$

for all $y \in (x-\delta, x)$ and $0 < \lambda \le \lambda_x$. Thus q_λ is comonotone with f on $(x-\delta, x)$ for all $0 < \lambda \le \lambda_x$. Since f is constant on $(x, x+\delta)$, q_λ is comonotone with f on $(x, x+\delta)$. This implies that q_λ is comonotone with f on Δ_x for all $0 < \lambda \le \lambda_x$.

Combining Cases 1, 2, 3, 4, and 5, we have that for any $x \in [a, b]$, there exist a neighborhood Δ_x of x and $\lambda_x > 0$ such that q_λ is comonotone with f on Δ_x for all $0 < \lambda \le \lambda_x$.

Now from the open cover $\{\Delta_x\}_{x \in [a, b]}$ of the closed interval $[a, b]$, we may select a finite subcover $\{\Delta_{x_1}, \dots, \Delta_{x_s}\}$.

Let $\lambda_0 = \min\{\lambda_{x_1}, \dots, \lambda_{x_s}\}$. Then it is easy to see that q_λ is comonotone with f on $[a, b]$ for all $0 < \lambda \le \lambda_0$.

Therefore $q_\lambda \in C_M(f)$ for all $0 < \lambda \le \lambda_0$. □

Lemma 2.4. *Let $f \in C[a, b]$ be piecewise monotone and $p \in C_M(f)$. Then p is a best comonotone approximation to f if and only if there exist s distinct points $x_1, \dots, x_s \in X_{-1}(p) \cup X_{+1}(p)$, where $1 \le s \le n+1$, and s positive constants $\lambda_1, \dots, \lambda_s$ such that*

$$\sum_{i=1}^{s} \lambda_i \sigma_1(x_i)[q(x_i) - p(x_i)] \le 0 \quad \textit{for all } q \in C_M(f).$$

Proof. The lemma follows immediately from Corollary 2.6 in [2] and the well-known fact that the extremal points of the unit ball in the dual space of $C[a, b]$ are (plus or minus) the point evaluations. □

Theorem 2.5. *Let $f \in C[a, b]$ be piecewise monotone and $p \in C_M(f)$. Then the following statements are equivalent*:

(1) *p is a best comonotone approximation to f.*

(2) *The origin of $\mathbb{R}^k$ belongs to the convex hull of the set*

$$B = \{\sigma_1(x)(\Phi_1(x), \dots, \Phi_k(x)) \mid x \in X_{+1}(p) \cup X_{-1}(p)\} \cup \left\{ \left(\lim_{t \to x} \frac{\Phi_1(t) - \Phi_1(x)}{\Phi(t) - \Phi(x)}, \dots, \lim_{t \to x} \frac{\Phi_k(t) - \Phi_k(x)}{\Phi(t) - \Phi(x)} \right) \middle| \, x \in X(p) \right\}.$$

(3) *There exist s constants $\lambda_i > 0$, $i = 1, \dots, s$, and s distinct elements $\{x_i\}_{i=1}^{s_0} \subset X_{+1}(p) \cup X_{-1}(p)$, $\{x_i\}_{i=s_0+1}^{s} \subset X(p)$, where $s_0 \ge 1$ and $s \le k+1$, such that*

$$\sum_{i=1}^{s_0} \lambda_i \sigma_1(x_i) q(x_i) + \sum_{i=s_0+1}^{s} \lambda_i \lim_{t \to x_i} \frac{q(t) - q(x_i)}{\Phi(t) - \Phi(x_i)} = 0$$

for all $q \in M_0(f)$.

Proof. (1) ⇒ (2): Assume that the origin of $\mathbb{R}^k$ does not belong to the convex hull of B. By Lemma 2.2, B is a compact set in $\mathbb{R}^k$. Therefore it follows from the Theorem on Linear Inequalities [1, p. 19] that there is a $q \in M_0(f)$ such that

$$\sigma_1(x)q(x) > 0 \quad \text{if } x \in X_{+1}(p) \cup X_{-1}(p),$$

and

$$\lim_{t \to x} \frac{q(t) - q(x)}{\Phi(t) - \Phi(x)} > 0 \quad \text{if } x \in X(p).$$

Let $q_\lambda(x) = p(x) + \lambda q(x)$. By Lemma 2.3, there exists $\lambda_0 > 0$ such that $q_\lambda \in C_M(f)$ for all $0 < \lambda \leq \lambda_0$. Now since $\sigma_1(x)[q_\lambda(x) - p(x)] > 0$ for all $x \in X_{-1}(p) \cup X_{+1}(p)$, it follows from Lemma 2.4 that p is not a best comonotone approximation to f.

(2) ⇒ (3): Let the origin of $\mathbb{R}^k$ belong to the convex hull of B. By the Theorem of Carathéodory [1, p. 17], there exist s constants $\lambda_i > 0$, $i = 1, \ldots, s$, and s distinct elements $\{x_i\}_{i=1}^{s_0} \subset X_{+1}(p) \cup X_{-1}(p)$ and $\{x_i\}_{i=s_0+1}^{s} \subset X(p)$, where $s \leq k+1$, such that

$$\sum_{i=1}^{s_0} \lambda_i \sigma_1(x_i) q(x_i) + \sum_{i=s_0+1}^{s} \lambda_i \lim_{t \to x_i} \frac{q(t) - q(x_i)}{\Phi(t) - \Phi(x_i)} = 0$$

for all $q \in M_0(f)$. Since $\Phi \in M_0(f)$ and

$$\lim_{t \to x_i} \frac{\Phi(t) - \Phi(x_i)}{\Phi(t) - \Phi(x_i)} = 1 > 0 \quad \text{for all } i = s_0 + 1, \ldots, s,$$

it follows from the preceding equation that $s_0 \geq 1$.

(3) ⇒ (1): It is easy to see that if $q \in C_M(f)$, then

$$\lim_{t \to x} \frac{q(t) - q(x)}{\Phi(t) - \Phi(x)} \geq 0 \quad \text{for all } x \in X(p). \tag{2}$$

Assume that there exist s constants $\lambda_i > 0$, $i = 1, \ldots, s$, and s elements $\{x_i\}_{i=1}^{s_0} \subset X_{+1}(p) \cup X_{-1}(p)$, $\{x_i\}_{i=s_0+1}^{s} \subset X(p)$, where $s_0 \geq 1$ and $s \leq k+1$, such that

$$\sum_{i=1}^{s_0} \lambda_i \sigma_1(x_i) q(x_i) + \sum_{i=s_0+1}^{s} \lambda_i \lim_{t \to x_i} \frac{q(t) - q(x_i)}{\Phi(t) - \Phi(x_i)} = 0 \tag{3}$$

for all $q \in M_0(f)$.

Then it follows from (2) and (3) that for any $q \in C_M(f)$, we have

$$\begin{aligned} &\sum_{i=1}^{s_0} \lambda_i \sigma_1(x_i)[q(x_i) - p(x_i)] \\ &\quad = -\sum_{i=s_0+1}^{s} \lambda_i \lim_{t \to x_i} \frac{[q(t) - p(t)] - [q(x_i) - p(x_i)]}{\Phi(t) - \Phi(x_i)} \\ &\quad = -\sum_{i=s_0+1}^{s} \lambda_i \lim_{t \to x_i} \frac{q(t) - q(x_i)}{\Phi(t) - \Phi(x_i)} \leq 0. \end{aligned}$$

By Lemma 2.4, p is a best comonotone approximation to f. □

Theorem 2.6. *Let $f \in C[a, b]$ be piecewise monotone and $p \in C_M(f)$. Then p is a best comonotone approximation to f if and only if there exist s elements*

$\{x_i\}_{i=1}^{s_0} \subset X_{+1}(p) \cup X_{-1}(p)$ *and* $\{x_i\}_{i=s_0+1}^{s} \subset X(p)$, *where* $s_0 \geq 1$ *and* $s \leq k+1$, *such that no* $q \in M_0(f)$ *satisfies the following equations*

$$\sigma_1(x_i)q(x_i) > 0 \quad \text{if } i = 1, \ldots, s_0,$$

and

$$\lim_{t \to x_i} \frac{q(t) - q(x_i)}{\Phi(t) - \Phi(x_i)} > 0 \quad \text{if } i = s_0 + 1, \ldots, s.$$

Proof. ($\Rightarrow$): This follows immediately from (1) $\Rightarrow$ (3) of Theorem 2.5.

($\Leftarrow$): This follows from (2) $\Rightarrow$ (1) of Theorem 2.5 and the Theorem of Carathéodory. □

3. Application of characterizations to Π_{n-1}

In this section we shall show that the characterization theorems in §2 have more detailed descriptions if $M = \Pi_{n-1}$. For brevity, let $C_n(f) := C_{\Pi_{n-1}}(f)$.

A piecewise monotone function f is called *s-piecewise monotone* if s is the minimal number of subintervals of $[a, b]$ on which f alternately increases and decreases. While s is uniquely determined by f, the $s+1$ partition points are not. For example, consider any function $f \in C[0, 3]$ which is strictly increasing on $[0, 1]$, constant on $[1, 2]$, and strictly decreasing on $[2, 3]$. This function is 2-piecewise monotone, but if $0 = t_0 < t_1 < t_2 = 3$ for any $t_1 \in [1, 2]$, then f increases on $[t_0, t_1]$ and decreases on $[t_1, t_2]$. Clearly, if f is s-piecewise monotone and $q \in C_n(f)$, then there exist $s-1$ points t_i in (a, b) such that $q'(t_i) = 0$. Since $C_n(f)$ contains nonconstant functions, $s \leq n-1$. Therefore we shall only be interested in approximating functions from the set

$$C_n[a, b] := \{f \in C[a, b] \mid f \text{ is } s\text{-piecewise monotone and } s \leq n-1\}.$$

Let $f \in C_n[a, b]$. Recall that f has a *strict local extremum* at $t_0 \in (a, b)$ if there exists a neighborhood U of t_0 such that either $f(t) > f(t_0)$ for all $t \in U \backslash \{t_0\}$ or $f(t) < f(t_0)$ for all $t \in U \backslash \{t_0\}$. (Such points were called "peaks" in [7].) For any $f \in C_n[a, b]$, let $E(f)$ denote the set of all strict local extrema of f. Then the cardinality of $E(f)$, $m := \operatorname{card} E(f)$, is at most $n-2$.

Lemma 3.1. *Let* $f \in C_n[a, b]$. *Then*

$$\Phi'(x) \neq 0 \quad \text{if } x \in S(f) \backslash E(f) \quad \text{and} \quad \Phi''(x) \neq 0 \quad \text{if } x \in E(f).$$

Proof. Let $[a, b]$ be partitioned into s subintervals $[z_0, z_1], \ldots, [z_{s-1}, z_s]$, where $a = z_0 < z_1 < \cdots < z_{s-1} < z_s = b$, so that $z_1, \ldots, z_{s-1}$ are not the end points of a subinterval on which f is constant and f alternately increases and decreases on these subintervals. Since $f \in C_n[a, b]$, $s \leq n-1$.

Let

$$\Psi(x) = \sigma \int_a^x \Pi_{i=1}^{s-1}(t - z_i)\, dt, \text{ where } \sigma = \pm 1.$$

Then $\Psi \in \Pi_{n-1}$. It is easy to see that if we choose $\sigma = +1$ or -1 appropriately, then Ψ is comonotone with f on $[a, b]$ and

$$\begin{aligned} &\Psi'(x) = \sigma \Pi_{i=1}^{s-1}(x - z_i) \neq 0 && \text{if } x \notin \{z_1, \ldots, z_{s-1}\}, \\ &\Psi''(x) \neq 0 && \text{if } x \in \{z_1, \ldots, z_{s-1}\}. \end{aligned}$$

It is clear that $E(f) \subset \{z_1, \ldots, z_s\}$. Hence

$$\Psi'(x) \neq 0 \quad \text{if } x \in S(f)\backslash E(f), \qquad \Psi''(x) \neq 0 \quad \text{if } x \in E(f).$$

It follows from Lemma 2.1 that

$$\Phi'(x) \neq 0 \quad \text{if } x \in S(f)\backslash E(f), \qquad \Phi''(x) \neq 0 \quad \text{if } x \in E(f). \quad \square$$

It follows immediately from Lemma 3.1 and the definition of $X(p)$ that for any $p \in C_n(f)$, if we define

$$\begin{aligned} X_{+2}(p) &= \{t \in [a, b]\backslash E(f) \mid p'(t) = 0 \text{ and } f \text{ is strictly increasing in} \\ &\qquad (t-\delta, t] \text{ or } [t, t+\delta) \text{ for some } \delta > 0\}, \\ X_{-2}(p) &= \{t \in [a, b]\backslash E(f) \mid p'(t) = 0 \text{ and } f \text{ is strictly decreasing in} \\ &\qquad (t-\delta, t] \text{ or } [t, t+\delta) \text{ for some } \delta > 0\}, \\ X_{+3}(p) &= \{t \in E(f) \mid p''(t) = 0 \text{ and } f \text{ has a local minimum at } t\}, \\ X_{-3}(p) &= \{t \in E(f) \mid p''(t) = 0 \text{ and } f \text{ has a local maximum at } t\}, \end{aligned}$$

then

$$X(p) = X_{+2}(p) \cup X_{-2}(p) \cup X_{+3}(p) \cup X_{-3}(p).$$

We would also like to define a function $\sigma_2 = \sigma_{2,p}$ on $X(p)$ as follows.

$$\sigma_2(t) = \sigma_{2,p}(t) = \begin{cases} 1 & \text{if } t \in X_{+2}(p) \cup X_{+3}(p), \\ -1 & \text{if } t \in X_{-2}(p) \cup X_{-3}(p). \end{cases}$$

It is easy to see that

$$\sigma_2(t) = \begin{cases} \operatorname{sgn} \Phi'(t) & \text{if } t \in X_{+2}(p) \cup X_{-2}(p), \\ \operatorname{sgn} \Phi''(t) & \text{if } t \in X_{+3}(p) \cup X_{-3}(p). \end{cases}$$

Lemma 3.2. *Let* $f \in C_n[a, b]$. *For any* $q \in \{q \in \Pi_{n-1} \mid q'(t) = 0$ *for all* $t \in E(f)\}$, *there exists* $\lambda_0 > 0$ *such that* $\Phi + \lambda_0 q \in C_n(f)$.

Proof. It is easy to see that $X(\Phi) = \varnothing$. A similar proof as in Lemma 2.3 gives the above result. $\square$

Lemma 3.3. *Let* $f \in C_n[a, b]$. *Then*

$$M_0(f) = \{q \in \Pi_{n-1} \mid q'(t) = 0 \textit{ for all } t \in E(f)\}.$$

As a consequence, $\dim M_0(f) = n - \operatorname{card} E(f)$.

Proof. Let

$$K = \{q \in \Pi_{n-1} \mid q'(t) = 0 \text{ for all } t \in E(f)\}.$$

It is obvious that $M_0(f) \subset K$. Conversely, for any $q \in K$, it follows from Lemma 3.2 that there exists $\lambda_0 > 0$ so that $\Phi + \lambda_0 q \in C_n(f) \subset M_0(f)$. Since $\Phi \in M_0(f)$ and $\lambda_0 \neq 0$, $q \in M_0(f)$. Thus $K \subset M_0(f)$. Therefore $M_0(f) = K$. $\square$

By Lemma 3.1, Lemma 3.3 and the fact that $X(p) = X_{+2}(p) \cup X_{-2}(p) \cup X_{+3}(p) \cup X_{-3}(p)$, Theorem 2.5 and Theorem 2.6 have the following improved versions.

Theorem 3.4. *Let* $f \in C_n[a, b] \backslash \Pi_{n-1}$, $p \in C_n(f)$, *and* $m = \operatorname{card} E(f)$. *Then the following statements are equivalent:*

(1) p *is a best comonotone approximation to* f.

(2) *The origin of* $\mathbb{R}^{n-m}$ *belongs to the convex hull of the set*

$$\begin{aligned} B = &\{\sigma_1(x)(\Phi_1(x), \ldots, \Phi_{n-m}(x)) \mid x \in X_{+1}(p) \cup X_{-1}(p)\} \\ &\cup \left\{ \left(\frac{\Phi_1'(x)}{\Phi'(x)}, \ldots, \frac{\Phi_{n-m}'(x)}{\Phi'(x)} \right) \mid x \in X_{+2}(p) \cup X_{-2}(p) \right\} \\ &\cup \left\{ \left(\frac{\Phi_1''(x)}{\Phi''(x)}, \ldots, \frac{\Phi_{n-m}''(x)}{\Phi''(x)} \right) \mid x \in X_{+3}(p) \cup X_{-3}(p) \right\}, \end{aligned}$$

where $\{\Phi_1, \ldots, \Phi_{n-m}\}$ *is a basis for* $M_0(f)$.

(3) *The origin of* $\mathbb{R}^{n-m}$ *belongs to the convex hull of the set*

$$\begin{aligned} B_0 = &\{\sigma_1(x)(\Phi_1(x), \ldots, \Phi_{n-m}(x)) \mid x \in X_{+1}(p) \cup X_{-1}(p)\} \\ &\cup \{\sigma_2(x)(\Phi_1'(x), \ldots, \Phi_{n-m}'(x)) \mid x \in X_{+2}(p) \cup X_{-2}(p)\} \\ &\cup \{\sigma_2(x)(\Phi_1''(x), \ldots, \Phi_{n-m}''(x)) \mid x \in X_{+3}(p) \cup X_{-3}(p)\}. \end{aligned}$$

(4) *There exist* s *constants* $\lambda_i > 0$, $i = 1, \ldots, s$, *and* s *distinct elements* $\{x_i\}_{i=1}^{s_0} \subset X_{+1}(p) \cup X_{-1}(p)$, $\{x_i\}_{i=s_0+1}^{s_1} \subset X_{+2}(p) \cup X_{-2}(p)$, *and* $\{x_i\}_{i=s_1+1}^{s} \subset X_{+3}(p) \cup X_{-3}(p)$, *where* $s_0 \geq 1$ *and* $s \leq n + 1 - m$, *such that*

$$\sum_{i=1}^{s_0} \lambda_i \sigma_1(x_i) q(x_i) + \sum_{i=s_0+1}^{s_1} \lambda_i \sigma_2(x_i) q'(x_i) + \sum_{i=s_1+1}^{s} \lambda_i \sigma_2(x_i) q''(x_i) = 0$$

for all $q \in M_0(f)$.

Theorem 3.5. *Let* $f \in C_n[a, b] \backslash \Pi_{n-1}$, $p \in C_n(f)$, *and* $m = \operatorname{card} E(f)$. *Then* p *is a best comonotone approximation to* f *if and only if there exist* s *elements* $\{x_i\}_{i=1}^{s_0} \subset X_{+1}(p) \cup X_{-1}(p)$, $\{x_i\}_{i=s_0+1}^{s_1} \subset X_{+2}(p) \cup X_{-2}(p)$, *and* $\{x_i\}_{i=s_1+1}^{s} \subset X_{+3}(p) \cup X_{-3}(p)$, *where* $s_0 \geq 1$ *and* $s \leq n + 1 - m$, *such that no* $q \in M_0(f)$ *satisfies the following equations:*

$$\begin{aligned} \sigma_1(x_i) q(x_i) > 0 \quad & \textit{if } i = 1, \ldots, s_0, \\ \sigma_2(x_i) q'(x_i) > 0 \quad & \textit{if } i = s_0 + 1, \ldots, s_1, \textit{ and} \\ \sigma_2(x_i) q''(x_i) > 0 \quad & \textit{if } i = s_1 + 1, \ldots, s. \end{aligned}$$

Example 3.6. Take $C[a, b] = C[-1, 1]$ and $n = 2$. Let $f(x) = x^4$. Then $E(f) = \{0\}$, $C_n(f) = \{a + bx^2 \mid a, b \in \mathbb{R}, b \geq 0\}$, $M_0(f) = \{a + bx^2 \mid a, b \in \mathbb{R}\}$, and $\Phi_1(x) = 1$, $\Phi_2(x) = x^2$. For $p(x) = -1/8 + x^2 \in C_n(f)$, it is easy to check that $X_{+1}(p) = \{-1, 0, 1\}$, $X_{-1}(p) = \{1/\sqrt{2}, -1/\sqrt{2}\}$, and $X_{+2}(p) = X_{-2}(p) = X_{+3}(p) = X_{-3}(p) = \varnothing$. Therefore $B = \{(1, 0), (1, 1), (-1, -1/2)\}$. It is obvious that $(0, 0) \in \operatorname{co} B$. It follows from (1) $\Leftrightarrow$ (2) of Theorem 3.4 that p is a best comonotone approximation to f.

4. Nonuniqueness of best approximations

In this section we give an example showing that the uniqueness of best comonotone approximations in general does not hold.

Consider the spaces $C[-1, 2]$ and Π_3. Let $p_1(x) = 4x^3 - 6x^2 + 1$, $p_2(x) = 2p_1(x) \in \Pi_3$. It is easy to see that p_1 has three zeros at $\frac{1-\sqrt{3}}{2}$, $\frac{1}{2}$, and $\frac{1+\sqrt{3}}{2}$

in $[-1, 2]$ and $\|p_1\| = 9$. Let $f \in C[-1, 2]$ be defined by

$$f(x) = \begin{cases} p_1(x) - 9 & \text{if } -1 \le x \le \frac{1-\sqrt{3}}{2}, \\ \frac{38}{\sqrt{3}-1}x + 10 & \text{if } \frac{1-\sqrt{3}}{2} < x \le 0, \\ p_1(x) + 9 & \text{if } 0 < x \le \frac{1}{2}, \\ 2p_1(x) + 9 & \text{if } \frac{1}{2} < x \le \frac{1+\sqrt{3}}{2}, \\ p_1(x) + 9 & \text{if } \frac{1+\sqrt{3}}{2} < x \le 2. \end{cases}$$

It is easy to verify that p_1 and p_2 are comonotone with f, i.e., $p_1, p_2 \in C_n(f)$, and f has two strict local extrema 0 and 1 in $(-1, 2)$, i.e., $E(f) = \{0, 1\}$. Therefore

$$\begin{aligned} M_0(f) &= \{q \in \Pi_3 \mid q'(0) = q'(1) = 0\} \\ &= \{a + b(2x^3 - 3x^2) \mid a, b \in \mathbb{R}\}. \end{aligned}$$

Now we also have the following facts:

1. $\|f - p_1\| = 9$ and $\|f - p_2\| = 9$.
2. $X_{+2}(p_1) = X_{-2}(p_1) = X_{+3}(p_1) = X_{-3}(p_1) = \varnothing$, and $X_{+2}(p_2) = X_{-2}(p_2) = X_{+3}(p_2) = X_{-3}(p_2) = \varnothing$.

Since

$$f\left(\frac{1-\sqrt{3}}{2}\right) - p_1\left(\frac{1-\sqrt{3}}{2}\right) = -9, \quad f\left(\frac{1}{2}\right) - p_1\left(\frac{1}{2}\right) = 9,$$

$$f\left(\frac{1-\sqrt{3}}{2}\right) - p_2\left(\frac{1-\sqrt{3}}{2}\right) = -9, \quad f\left(\frac{1}{2}\right) - p_2\left(\frac{1}{2}\right) = 9,$$

we have $\frac{1-\sqrt{3}}{2} \in X_{-1}(p_1) \cap X_{-1}(p_2)$ and $\frac{1}{2} \in X_{+1}(p_1) \cap X_{+1}(p_2)$.

Next we shall show that no $q \in M_0(f)$ satisfies the following equations

$$q\left(\frac{1-\sqrt{3}}{2}\right) < 0, \tag{4}$$

$$q\left(\frac{1}{2}\right) > 0. \tag{5}$$

In fact, if $q \in M_0(f)$, then we may write $q(x) = a + b(2x^3 - 3x^2)$. Thus

$$q\left(\frac{1-\sqrt{3}}{2}\right) = q\left(\frac{1}{2}\right) = a - \frac{1}{2}b.$$

This implies that q does not satisfy equations (4) and (5). It follows from Theorem 2.6 that p_1 and p_2 are two best comonotone approximations to f. We should observe that this function f is also piecewise monotone in the sense of [7].

Remark. It is interesting to point out that *if $f \in C_n[a, b]$ does not have any strict local extremum, then f has a unique best comonotone approximation in Π_{n-1}.*

In fact, in this case, $E(f) = \varnothing$. Therefore, $M_0(f) = \Pi_{n-1}$ and $X_{+3}(p) \cup X_{-3}(p) = \varnothing$. Accordingly, from Theorem 3.5 we have the following special form of the characterization theorem.

Theorem 4.1. *Let* $f \in C_n[a, b]\backslash\Pi_{n-1}$, $E(f) = \varnothing$, *and* $p \in C_n(f)$. *Then* p *is a best comonotone approximation to* f *if and only if there exist* s *elements* $\{x_i\}_{i=1}^{s_0} \subset X_{+1}(p) \cup X_{-1}(p)$ *and* $\{x_i\}_{i=s_0+1}^{s} \subset X_{+2}(p) \cup X_{-2}(p)$, *where* $s_0 \geq 1$ *and* $s \leq n+1$, *such that no* $q \in \Pi_{n-1}$ *satisfies the following equations*

$$\begin{aligned} \sigma_1(x_i)q(x_i) &> 0 \quad \text{if } i = 1, \ldots, s_0, \text{ and} \\ \sigma_2(x_i)q'(x_i) &> 0 \quad \text{if } i = s_0+1, \ldots, s. \end{aligned}$$

Using this characterization theorem, the notion of "free" or "poised" matrices, and the corresponding Birkhoff interpolation problem in much the same way as in [4] or in [5], one can show that the best comonotone approximation to f is unique.

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