# RIGIDITY OF ERGODIC VOLUME-PRESERVING ACTIONS
## OF SEMISIMPLE GROUPS OF HIGHER RANK
## ON COMPACT MANIFOLDS

### GUILLAUME SEYDOUX

ABSTRACT. Let $M$ be a compact manifold, $H$ a semisimple Lie group of higher rank (e.g., $H = SL(n, \mathbf{R})$ with $n \geq 3$) and $a \in \mathscr{A}(H, M)$ an ergodic $H$-action on $M$ which preserves a volume $v$. Such an $H$-action is conjectured to be "locally rigid": if $a'$ is a sufficiently $C^1$-small perturbation of $a$, then there should exist a diffeomorphism $\Phi$ of the manifold $M$ which conjugates $a'$ to $a$. This conjecture would imply that if $\omega$ is an $a$-invariant geometrical structure on $M$, then there should exist an $a'$-invariant geometrical structure $\omega'$ on $M$ of the same type. Using Kazhdan property, superrigidity for cocycles, and Sobolev spaces techniques we prove, under suitable conditions, two such results with $\omega = v$ and with $\omega$ a Riemannian metric along the leaves of a foliation of $M$.

## 1. INTRODUCTION

Let $M$ be a compact connected manifold and $H$ a Lie group acting on $M$. Unless otherwise specified we always assume that the manifold $M$, geometric structures, and $H$-actions on $M$ we consider are smooth. Let $\mathscr{A}(H, M)$ denote the set of $H$-actions on $M$, endowed for each integer $0 \leq k \leq \infty$ with the $C^k$-topology of uniform convergence of $k$-jets on compact subsets of $H$. In general, if $a \in \mathscr{A}(H, M)$, an arbitrary $C^\infty$-neighbourhood of $a$, however small, contains actions which are not conjugate to $a$: take $H = \mathbf{R}$, $M = T^2$, and let $a_\alpha \in \mathscr{A}(\mathbf{R}, T^2)$ denote the usual $\mathbf{R}$-action on $T^2$ by translations with slope $\alpha \in \mathbf{R}$. Then if $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of rationals such that $\alpha_n \xrightarrow[n \to \infty]{} \alpha$, $\alpha \notin \mathbf{Q}$, then although $a_{\alpha_n} \xrightarrow[n \to \infty]{C^\infty} a_\alpha$, the $a_{\alpha_n}$ are never conjugate to $a_\alpha$ (as, for instance, the $a_{\alpha_n}$ are not ergodic while $a_\alpha$ is). However we have the following conjecture:

**Main Conjecture 1.1** (Local rigidity of ergodic volume-preserving actions of semisimple Lie groups of higher rank and their lattice subgroups on compact manifolds). Let $H$ be a connected semisimple Lie group with finite center and such that the rank of each simple factor is at least two. Let $\Gamma \subset H$ be a lattice subgroup (i.e., a discrete subgroup such that the $H$-space $H/\Gamma$ admits a finite invariant measure). Assume that $a \in \mathscr{A}(H, M)$ (or $a \in \mathscr{A}(\Gamma, M)$)

---

is an ergodic action which preserves a volume. Then a sufficiently $C^1$-small perturbation $a'$ of $a$ is conjugate to $a$ in $\text{Diff}(M)$.

Local rigidity for the standard action of $\Gamma = SL(n, \mathbb{Z}) \subset SL(n, \mathbb{R})$ on $\mathbb{T}^n$ ($n \geq 4$) has been recently proved by A. Katok and J. Lewis [6, Theorem 1.3, p. 2]. The example of the $\mathbb{R}$-actions on the torus shows that local rigidity depends crucially on the group structure and not on the "richness" of the $H$-invariant geometric structure (since both the $a_{\alpha_n}$ and $a_\alpha$ leave the flat metric on $T^2$ invariant).

The actions dealt with in the Main Conjecture will be referred to as *actions of type a*. Canonical examples are obtained as follows: let $G = \Pi G_i$ be a finite product where each $G_i$ is a connected noncompact simple Lie group with finite center. Let $\Gamma \subset G$ be a cocompact lattice subgroup such that the transitive action of $G$ on $M = G/\Gamma$ is irreducible (we recall that if $M$ is a $G$-space with finite invariant volume, then the $G$-action is called irreducible if for every noncentral normal subgroup $N \subset G$, $N$ is ergodic on $M$). Let $H$ be a connected semisimple Lie group with finite center such that the rank of each simple factor is at least two, and let $h : H \to G$ be a homomorphism whose image $h(H) \subset G$ is closed and noncompact. By Moore's Ergodicity Theorem [1, Theorem 2.2.15, p. 21] the $H$-action on $M$ is ergodic, and thus of type $a$.

**Statement of the main results.** The Main Conjecture induces numerous "stability" properties for actions of type $a$. In particular if $a \in \mathscr{A}(H, M)$ is of type $a$, a $C^1$-small perturbation $a'$ of $a$, being conjugate to $a$, is also of type $a$, and if we moreover assume that $a$ preserves a $G$-structure $\omega$ then $a'$ must also preserve a $G$-structure of the same type. One might think of trying to prove such a stability result with $\omega$ a Riemannian metric on $M$, but we recall that the Lie algebra of a semisimple noncompact Lie group cannot be embedded into the Lie algebra of a compact Lie group, so that a connected semisimple noncompact Lie group $H$ cannot act faithfully on $M$ preserving a Riemannian metric. However, natural situations arise where $H$ preserves a foliation $\mathscr{F}$ on $M$ (i.e., $H$ permutes the leaves of the foliation) and a Riemannian metric on the leaves of this foliation: take for instance $M = (H \times G)/\Gamma$ where $G$ is a connected semisimple noncompact Lie group with finite center and $\Gamma \subset H \times G$ is a cocompact lattice (the leaf through $\overline{(h, g)}$ is then $\overline{(h, G)}$ on which we consider the $H$-invariant quotient of a right-invariant Riemannian metric on $G$). Such $H$-actions are moreover transverse to the foliation $\mathscr{F}$ (i.e., at each $m \in M$ we have $T_m M = T_m \mathscr{F} + T_m(Hm)$). The following theorem is thus clearly germane to the Main Conjecture, and one can actually show that it is implied by it.

**Main Theorem** (Theorem 4.1.1). *Let $H$ be a connected semisimple Lie group with finite center which can be realized as the $\mathbb{R}$-points of an algebraic $\mathbb{R}$-group $\mathscr{H}$ and such that the rank of each simple factor is at least two. Assume that $a \in \mathscr{A}(H, M)$ is an irreducible $H$-action which preserves a smooth volume $v$, a foliation $\mathscr{F}$ and a smooth Riemannian metric $\omega$ on $T\mathscr{F}$. Assume moreover that $a$ is transverse to $\mathscr{F}$. Let $a'$ be a sufficiently $C^\infty$-small perturbation of $a$ which is irreducible, which preserves a smooth volume $v'$ and the foliation $\mathscr{F}$. Then for each integer $0 \leq r < \infty$ there exists an $a'$-invariant $C^r$-Riemannian metric on $T\mathscr{F}$.*

It would be nice to generalize this theorem to $H$-actions which are not necessarily transverse to $\mathscr{F}$. A partial result in this direction is the following:

**Theorem 3.1.1.** *Assume that $a \in \mathscr{A}(H, M)$ is an irreducible $H$-action ($H$ satisfies the hypotheses of the Main Theorem) which preserves a smooth volume $v$, a foliation $\mathscr{F}$, and which leaves a continuous Riemannian metric $\omega$ on $T\mathscr{F}$ invariant. Let $a'$ be a sufficiently $C^1$-small perturbation of $a$ which is irreducible and $\mathscr{F}$-preserving. Then $a'$ leaves a measurable Riemannian metric on $T\mathscr{F}$ a.e. invariant.*

Though the Main Conjecture implies that the irreducibility (or ergodicity) and the volume-preserving character of actions of type $a$ should be stable under small perturbations, as we have not succeeded proving such stability results, we assumed in the Main Theorem that the small perturbation $a'$ was both irreducible and volume-preserving. However, in §2, we prove the following theorem.

**Theorem 2.2.** *Assume that $H$ is a Kazhdan Lie group (e.g., $H$ satisfies the hypotheses of the Main Conjecture) and that $a \in \mathscr{A}(H, M)$ preserves some probability $\mu \in M_a(M)$ (where $M_a(M)$ denotes the set of measures on $M$ which are absolutely continuous with respect to one, and hence any, smooth volume). Fix a smooth volume $v_0$ on $M$, an $\epsilon > 0$, and let $a' \in \mathscr{A}(H, M)$ be a sufficiently $C^1$-small perturbation of $a$. Then:*

(1) *$a'$ preserves some probability $\mu' \in M_a(M)$ satisfying*

$$\left\| \frac{d\mu'}{dv_0} - \frac{d\mu}{dv_0} \right\|_{1, v_0} \leq 2\epsilon.$$

(2) *If $\mu$ is a smooth volume we can moreover insure that $\mu' \in M_s(M)$ (the smooth probability class on $M$).*

Our Main Theorem is an analog of Zimmer's Theorem [2, Theorem 10.1, p. 192] which deals with isometric actions of lattice subgroups $\Gamma \subset H$ on $M$. Most of the material in this paper is part of my Ph.D. dissertation written at the University of Chicago under the guidance of Professor Robert J. Zimmer. I am grateful to him for suggesting this line of research and guiding my first steps in the research process.

## 2. RIGIDITY OF THE VOLUME-PRESERVING CHARACTER OF ACTIONS OF TYPE $a$

Throughout this section $H$ denotes a Kazhdan Lie group. We recall the corresponding definitions.

**Definitions 2.1.**

(1) *Let $H$ be a Lie group and $(\Pi, \mathscr{H})$ be a unitary representation of $H$ on some Hilbert space $\mathscr{H}$. Let $\epsilon > 0$ and $K \subset H$ a compact subset. A unit vector $f \in \mathscr{H}$ is said to be $(\epsilon, K)$-invariant if $\|\Pi(h)f - f\| \leq \epsilon$ for each $h \in K$.*

(2) *We say that $\Pi$ almost has invariant vectors if for all $(\epsilon, K)$ there exists an $(\epsilon, K)$-invariant unit vector.*

(3) *The group $H$ is said to be Kazhdan if any unitary representation of $H$ which almost has invariant vectors actually has a nontrivial invariant vector.*

Let $M_a(M)$ denote the set of measures on $M$ which are absolutely continuous with respect to one (and hence any) smooth volume, and let $M_s(M)$ be the set of measures on $M$ which are in the same measure class as smooth volumes. As the Lie groups dealt with in the Main Conjecture 1.1 are Kazhdan [1, Theorem 7.1.4, p. 130], the following theorem, which is our first stability result, is clearly relevant to the Main Conjecture.

**Theorem 2.2.** *Assume that $H$ is a Kazhdan Lie group and that $a \in \mathscr{A}(H, M)$ preserves some probability $\mu \in M_a(M)$. Fix a smooth volume $v_0$ and $\epsilon > 0$. Let $a' \in \mathscr{A}(H, M)$ be a sufficiently $C^1$-small perturbation of $a$. Then:*

(1) *$a'$ preserves some probability $\mu' \in M_a(M)$ satisfying*

$$\left\| \frac{d\mu'}{dv_0} - \frac{d\mu}{dv_0} \right\|_{1, v_0} \leq 2\epsilon.$$

(2) *If we moreover assume that $\mu$ is a smooth probability volume we can moreover insure that $\mu' \in M_s(M)$.*

We first construct a unitary representation $\Pi_{a'}^{v_0} : H \longrightarrow U(L_{v_0}^2(M))$: the $H$-action $a'$ on $M$ induces an $H$-action on $M_a(M)$ defined by

$$(a'(h, \mu))(A) = \mu(a'(h^{-1}, A)), \qquad A \subset M.$$

By the Radon-Nikodym Theorem the map $\Phi_{v_0} : M_a(M) \longrightarrow L_{v_0}^{2;+}(M) = \{f \in L_{v_0}^2(M) \mid f \geq 0\}$ defined by $\mu \mapsto (\frac{d\mu}{dv_0})^{1/2}$ is a bijection so that the $H$-action $a'$ on $M$ also yields through this bijection an $H$-action on $L_{v_0}^{2;+}(M)$. In order to find an explicit formula for this action we need to express $(\frac{d(a'(h,\mu))}{dv_0})^{1/2}$ as a function of $(\frac{d\mu}{dv_0})^{1/2}$. We have

$$\frac{d(a'(h, \mu))}{dv_0}(m) = \frac{d\mu}{dv_0}(a'(h^{-1}, m)) \frac{d(a'(h, v_0))}{dv_0}(m)$$

and thus introduce the smooth strictly positive function

$$J_{a'}^{v_0}(h, m) = \left( \frac{d(a'(h, v_0))}{dv_0}(m) \right)^{1/2}$$

and define the representation $\Pi_{a'}^{v_0} : H \longrightarrow U(L_{v_0}^2(M))$ of $H$ on $L_{v_0}^2(M)$ by setting

$$(\Pi_{a'}^{v_0}(h)f)(m) = f(a'(h^{-1}, m)) J_{a'}^{v_0}(h, m).$$

By construction the representation $\Pi_{a'}^{v_0}$ is unitary, $\mu \in M_a(M)$ is $a'$-invariant if and only if $(\frac{d\mu}{dv_0})^{1/2} \in L_{v_0}^2(M)$ is $\Pi_{a'}^{v_0}$-invariant, and conversely, as $J_{a'}^{v_0} > 0$, if $f \in L_{v_0}^2(M)$ is $\Pi_{a'}^{v_0}$-invariant then $|f|$ is also $\Pi_{a'}^{v_0}$-invariant and thus $\mu = f^2 v_0 / \|f\|_{2, v_0}^2$ is an $a'$-invariant probability on $M$.

*Proof of Theorem 2.2.* Let $f = (\frac{d\mu}{dv})^{1/2}$ be the $\Pi_a^{v_0}$-invariant function in $L_{v_0}^2(M)$ associated to the $a$-invariant probability $\mu$, let $(a_n)_{n \in \mathbb{N}}$ be a sequence of $H$-actions which converge to $a$ in the topology of $C^1$-uniform convergence on compact subsets of $H$, and let $(\Pi_{a_n}^{v_0})_{n \in \mathbb{N}}$ be the associated sequence of unitary representations on $L_{v_0}^2(M)$.

We first prove (1), namely that for $n$ large enough there exist $\Pi_{a_n}^{v_0}$-invariant unit functions $f_n$ in $L_{v_0}^2(M)$ which satisfy $\|f_n - f\|_{2,v_0} \leq \epsilon$ (as indeed if we then set $\mu_n = f_n^2 v_0$ we do have $\|\frac{d\mu_n}{dv_0} - \frac{d\mu}{dv_0}\|_{1,v_0} = \int_M |f_n^2 - f^2| \, dv_0 = \int_M |f_n + f| |f_n - f| \, dv_0 \leq \|f_n + f\|_{2,v_0} \|f_n - f\|_{2,v_0} \leq 2\epsilon$ ). It is well known that the definition of a Kazhdan group can be reformulated as follows.

**Lemma 2.3** [5, Proposition 16, p. 15]. *There exist a compact subset $K_0 \subset H$ (depending only on $H$) and a $\delta > 0$ (depending only on $\epsilon$ and $H$) such that for any unitary representation $(\Pi, \mathscr{H})$ of $H$ on a Hilbert space $\mathscr{H}$ and given any $(\delta, K_0)$-invariant unit vector $f \in \mathscr{H}$ there exists an $H$-invariant unit vector $g \in \mathscr{H}$ with $\|g - f\| < \epsilon$.*

Let $\delta > 0$ and $K_0 \subset H$ as in Lemma 2.3: it is enough, by this lemma, to show that $f$ is a $(\delta, K_0) - \Pi_{a_n}^{v_0}$-invariant vector for $n$ large enough. But if we set

$$A = \sup_{[m \in M, \, n \in \mathbf{N}, \, h \in K_0]} J_{a_n}^{v_0}(h, m)$$

we have:

$$\|\Pi_{a_n}^{v_0}(h) f - f\|_{2,v_0} = \|[\Pi_{a_n}^{v_0}(h) - \Pi_a^{v_0}(h)] f\|_{2,v_0}$$
$$\leq A \|f(a_n(h^{-1}, .)) - f(a(h^{-1}, .))\|_{2,v_0}$$

which converges uniformly on $K_0$ to 0. This proves (1).

In order to prove (2) we can assume without loss of generality that $\mu = v_0$ (so that the associated $\Pi_a^{v_0}$-invariant function in $L_{v_0}^2(M)$ is $f = 1$, and $J_a^{v_0} = 1$). There exists an integer $n_0$ such that for all $n \geq n_0$ and all $h \in K_0$, $\|\Pi_{a_n}^{v_0}(h)1 - 1\|_{2,v_0} = \|J_{a_n}^{v_0}(h, .)1 - 1\|_{2,v_0} \leq \|J_{a_n}^{v_0}(h, .) - 1\|_\infty < \delta$ (the compact $K_0 \subset H$ and $\delta > 0$ are again yielded by Lemma 2.3).

From now on we fix such an integer $n \geq n_0$ and show that there exists an $a_n$-invariant $\mu_n \in M_s(M)$.

Again let $g$ be the $\Pi_{a_n}^{v_0}$-invariant function in $L_{v_0}^2(M)$ satisfying $\|g - 1\|_{2,v_0} < \epsilon$ yielded by Lemma 2.3. By replacing $g$ by $|g|$ we can assume without loss of generality that $g \geq 0$. We would like to have the additional property $g > 0$. Set $Z_g = \{m \in M \mid g(m) = 0\}$. As $J_{a_n}^{v_0} > 0$, $Z_g$ is $a_n$-invariant. As $\|g - 1\|_{2,v_0} < \epsilon$, necessarily $v_0(Z_g) < \epsilon^2$. If $v_0(Z_g) = 0$ we are done. Assume thus that $Z_g$ is not a null set. We can then consider the "restricted representation" $\Pi_{a_n}^{v'} : H \times L_{v'}^2(Z_g) \longrightarrow L_{v'}^2(Z_g)$ which is unitary (we set $v' = v_0/v_0(Z_g)$).

But 1 is again a $(\delta, K_0) - \Pi_{a_n}^{v'}$-invariant vector so that there exists a unit $g_1 \in L_{v'}^2(Z_g)$ which is $\Pi_{a_n}^{v'}$-invariant and such that $v'(Z_{g_1}) < \epsilon^2$, i.e., $v_0(Z_{g_1}) < \epsilon^4$ and $\|1 - g_1\|_{2,v'} < \epsilon$. Proceed then inductively: we obtain a decreasing sequence of measurable sets $Z_g \supset Z_{g_1} \supset Z_{g_2} \supset \ldots \supset Z_{g_k} \supset \ldots$ such that $v(Z_{g_k}) < \epsilon^{2k}$ together with a sequence of functions $(M, g), (Z_g, g_1), (Z_{g_1}, g_2), \ldots$ Extend $g_i$ on $M$ by 0 and set $\phi = g + \sum_1^\infty g_k$. By construction $\phi$ is $\Pi_{a_n}^{v_0}$-invariant, $\phi \in L_{v_0}^2(M)$, $\|\phi - 1\|_{2,v_0} < 2\epsilon$, and $\phi > 0$. Thus $\mu_n = \phi^2 v_0 \in M_s(M)$ is an $a_n$-invariant probability which is solution of our problem. $\square$

We end this section with an example of a non-Kazhdan Lie group for which the conclusion of Theorem 2.2 fails: let $H = (\mathbf{R}_+^*, \times)$, let $M = S^2 \subset \mathbf{R}^3$ be the 2-sphere and $a : H \times S^2 \to S^2$ be the natural $H$-action on $S^2$ induced by

the rotation of $\mathbb{R}^3$ around the vertical axis with angle $\ln h$, which preserves the canonical volume of $S^2$. For any $h > 0$ let $\Phi_h$ be the diffeomorphism of $S^2$ that fixes the poles and which, viewed in the chart of the stereographic projection through the North pole (resp. the South pole) is a homothety of ratio $h$ (resp. $1/h$) and set

$$a_k(h) = \Phi_{h^{1/k}} \circ R_{\ln h}.$$

Though the sequence of $H$-actions $a_k$ converges to $a$ in the topology of $C^\infty$-uniform convergence on compact subsets of $H$, it is easy to see that none of the $a_k$ leaves a measure in $M_a(M)$ invariant.

### 3. RIGIDITY OF THE MEASURABLE METRIC PRESERVING CHARACTER OF ACTIONS OF TYPE $a$

**3.1. Introduction.** In this section we prove a stability result by applying the Superrigidity Theorem for Cocycles (Theorem 3.3.3), for which we need to assume that $H = \mathscr{H}_{\mathbb{R}}^0$ where $\mathscr{H}$ is a connected semisimple algebraic $\mathbb{R}$-group such that each simple factor has $\mathbb{R}$-rank at least 2: *from now on we make this additional assumption on $H$*. We recall that if $H$ satisfies the hypotheses of the Main Conjecture and is moreover center-free, then this additional assertion is redundant. As we said in the introduction, natural situations arise where such groups $H$ act on a compact manifold $M$ preserving a foliation and a Riemannian metric on the leaves of this foliation:

**Theorem 3.1.1.** *Assume that $H = \mathscr{H}_{\mathbb{R}}^0$ where $\mathscr{H}$ is a connected semisimple algebraic $\mathbb{R}$-group such that each simple factor has $\mathbb{R}$-rank at least 2. Let $a \in \mathscr{A}(H, M)$ be an irreducible $H$-action which preserves a smooth volume $v$, a foliation $\mathscr{F}$, and which leaves a continuous Riemannian metric $\omega$ on $T\mathscr{F}$ invariant. Let $a'$ be a sufficiently $C^1$-small perturbation of $a$ which is irreducible and $\mathscr{F}$-preserving. Then $a'$ leaves a measurable Riemannian metric on $T\mathscr{F}$ a.e. invariant.*

As Theorem 2.2 asserts that if $a'$ is a sufficiently $C^1$-small perturbation of $a$, then there exists an $a'$-invariant $\mu' \in M_s(M)$, we can assume without loss of generality that the perturbed action $a'$ we consider leave some $\mu' \in M_s(M)$ (depending on $a'$) invariant. *From now on we furthermore tacitly assume that the $H$-actions considered are irreducible and $\mathscr{F}$-preserving.*

**3.2. The algebraic hull of the ergodic $H$-action $a'$.** We will assume that the reader is familiar with the machinery of cocycles applied to measurable reductions of principal bundles as exposed, for instance, in [3, §2, pp. 251–256]. Set $d = \dim(M)$ and $p = \dim(\mathscr{F})$. Let $P$ be the $GL(p)$-principal fiber bundle of the frames of $\mathscr{F}$. As the $H$-action $a'$ on $M$ is assumed to be $\mathscr{F}$-preserving, it lifts to an $H$-action on $P$ by principal fiber bundle automorphisms, and as the $H$-action $a'$ on $M$ is assumed to be ergodic, one can canonically attach to $a'$ an algebraic subgroup of $GL(p)$ called *the algebraic hull of the $H$-action $a'$*: let $s : M \to P$ be a measurable section of $P$ and let $\Phi : M \times GL(p) \to P$ denote the corresponding measurable trivialization (defined by $\Phi(m, g) = s(m)g$). As $H$ acts on $P$ by principal bundle automorphisms we can describe the $H$-action on $P$ in the trivialization $\Phi$ by the cocycle $\alpha : H \times M \to GL(p)$ (defined by $h.\Phi(m, g) = \Phi(hm, \alpha(h, m)g)$).

**Theorem 3.2.1** (algebraic hull of a cocycle) [3, Theorem 4.1, p. 260]. *Let* $\alpha$ : $H \times M \to GL(p)$ *be a cocycle. Then,*

(1) *There exists an algebraic $\mathbb{R}$-group $\mathscr{L} \subset GL(p, \mathbb{C})$ such that $\alpha$ is equivalent to a cocycle taking values in $L = \mathscr{L}_{\mathbb{R}}$ but $\alpha$ is not equivalent to a cocycle taking values in a proper subgroup of $L$ of the form $M = \mathscr{M}_{\mathbb{R}}$ where $\mathscr{M}$ is also an algebraic $\mathbb{R}$-group.*

(2) *$L$ is unique up to conjugacy. The conjugacy class of $L$ (or by abuse of language $L$ or $\mathscr{L}$) is called the algebraic hull of the cocycle $\alpha$.*

(3) *If $\alpha \sim \beta$ where $\beta(H \times M) \subset L'$ and $L' = \mathscr{L}'_{\mathbb{R}}$ where $\mathscr{L}'$ is an algebraic $\mathbb{R}$-group then $L'$ contains a conjugate of $L$.*

*Remark.* Since $a'$ canonically defines an equivalence class of cocycles, it does make sense to speak of *the algebraic hull of the $H$-action $a'$*. We will also use the following results:

**Theorem 3.2.2.**

(1) *[3, §2] There exists an a.e. $a'$-invariant measurable metric on $T\mathscr{F}$ if and only if $L$ is compact.*

(2) *[4, Theorem 1.1, p. 375] $L$ is a reductive group with compact center (this uses essentially the hypotheses we made on $H$).*

### 3.3. *H*-actions with noncompact algebraic hulls.

Fix a continuous Riemannian metric $\bar\omega$ on $M$ such that $\bar\omega|_{T\mathscr{F}} = \omega$ ($\omega$ is the continuous $a$-invariant Riemannian metric on $T\mathscr{F}$). By abuse of notation we will write $\bar\omega = \omega$ and we denote by $\| \ \|$ the associated norm on $TM$. In this section we fix an $H$-action $a'$ on $M$ with noncompact algebraic hull (which preserves some $\mu' \in M_s(M)$ with respect to which the action is irreducible) and prove

**Theorem 3.3.1.** *There exist an $h_0 \in H$ and a $\lambda > 1$, both independent of the fixed action considered (with noncompact algebraic hull), a $C > 0$ and a vector $v \in T\mathscr{F}$, such that for infinitely many integers $n$ we have $\|h_0^n v\| \geq C\lambda^n$.*

Let $L = \mathscr{L}_{\mathbb{R}}$ be the algebraic hull of the $H$-action on $P$. There exists a trivialization $\Phi_0 : M \times GL(p) \to P$ of $P$ (corresponding to the measurable section $s_0(m) = \Phi_0(m, e)$) in which the action is described by the cocycle $\alpha : H \times M \to L$. From now on we will work in this trivialization.

**Lemma 3.3.2.** *There exists a finite irreducible extension $p : \widetilde{M} \to M$ (i.e., all fibers are of a fixed finite cardinality), a nontrivial homomorphism $\tilde\rho : \widetilde{H} \to (\mathscr{L}^0)_{\mathbb{R}}$ (where $(\widetilde{H}, \pi)$ denotes the universal covering of $H$), a compact normal subgroup $K \lhd (\mathscr{L}^0)_{\mathbb{R}}$, a measurable map $\Phi : \widetilde{M} \to (\mathscr{L}^0)_{\mathbb{R}}$, and a map $\Psi : H \times \widetilde{M} \to K$, such that for all $h \in H$ and any $\tilde h \in \widetilde{H}$ satisfying $\pi(\tilde h) = h$, we have*

$$\alpha(h, m) = \Phi^{-1}(h\tilde m)\tilde\rho(\tilde h)\Phi(\tilde m)\Psi(h, \tilde m)$$

*for a.e. $m \in M$ and any $\tilde m \in \widetilde{M}$ such that $p(\tilde m) = m$.*

*Proof.* By [1, Proposition 9.2.6, p. 168] there exists a finite irreducible extension $p : \widetilde{M} \to M$ such that the cocycle defined by

$$\tilde\alpha(h, \tilde m) = \alpha(h, p(\tilde m))$$

has algebraic hull $(\mathscr{L}^0)_\mathbf{R}$. As $(\mathscr{L}^0)_\mathbf{R} \subset L$ has finite index and $L$ is assumed to be noncompact, $(\mathscr{L}^0)_\mathbf{R}$ is noncompact. As $Z(L) = (Z(\mathscr{L}))_\mathbf{R}$ is compact (by (2) of Theorem 3.2.2), $Z((\mathscr{L}^0)_\mathbf{R})$ is compact. As $\mathscr{L}$ is a reductive algebraic **R**-group (by (2) of Theorem 3.2.2), $\mathscr{L}^0$ is a connected reductive algebraic **R**-group so that we can write

$$\mathscr{L}^0/Z(\mathscr{L}^0) = \mathscr{L}_1 \times \mathscr{L}_2$$

where both $\mathscr{L}_1$ and $\mathscr{L}_2$ are connected center-free semisimple algebraic **R**-groups such that $L_1 = (\mathscr{L}_1)_\mathbf{R}$ is a compact semisimple Lie group, $\mathscr{L}_2 = \mathscr{S}_1 \times \cdots \times \mathscr{S}_n$ where each $\mathscr{S}_i$ is a connected simple center-free algebraic **R**-group such that each $S_i = (\mathscr{S}_i)_\mathbf{R}$ is a simple, center-free noncompact Lie group. Let $q$ and $q_i$ be the canonical homomorphisms of **R**-group defined over **R**,

$$q : \mathscr{L}^0 \to \mathscr{L}^0/Z(\mathscr{L}^0) \to \mathscr{L}_2, \qquad q_i : \mathscr{L}_2 \to \mathscr{S}_i,$$

and set

$$\widetilde{\alpha}_i = q_i \circ q \circ \widetilde{\alpha} : H \times \widetilde{M} \to S_i.$$

The cocycle $\widetilde{\alpha}_i$ is not equivalent to a cocycle taking values in a proper subgroup of the form $G_i = (\mathscr{G}_i)_\mathbf{R}$ where $\mathscr{G}_i \subset \mathscr{S}_i$ is an algebraic **R**-subgroup (otherwise $\widetilde{\alpha}$ would be equivalent to a cocycle taking values in $[(q_i \circ q)^{-1}(\mathscr{G}_i)]_\mathbf{R}$ where $(q_i \circ q)^{-1}(\mathscr{G}_i) \subset \mathscr{L}^0$ is a proper algebraic **R**-subgroup, which is impossible as $(\mathscr{L}^0)_\mathbf{R}$ is the algebraic hull of the cocycle $\widetilde{\alpha}$), so that we can apply the following theorem with $\mathscr{S} = \mathscr{S}_i$ and $\beta = \widetilde{\alpha}_i$:

### 3.3.3. Superrigidity Theorem for Cocycles [1, Theorem 5.2.5, p. 98].
*Suppose that $\mathscr{H}$ is a connected semisimple algebraic **R**-group such that **R**-rank$(H) \geq 2$ and $H = \mathscr{H}_\mathbf{R}^0$ has no compact factors. Suppose that $\widetilde{M}$ is an irreducible H-space with a finite invariant measure $\widetilde{\mu}$. Let $\mathscr{S}$ be a connected simple algebraic **R**-group, set $S = \mathscr{S}_\mathbf{R}$ and suppose that $\beta : H \times \widetilde{M} \to S$ is a cocycle such that $\beta$ is not equivalent to a cocycle taking values in a subgroup $G = (\mathscr{G})_\mathbf{R}$ where $\mathscr{G} \subset \mathscr{S}$ is a proper algebraic **R**-subgroup. Then if $S$ is not compact, there exists a rational homomorphism of **R**-groups $\rho : \mathscr{H} \to \mathscr{S}$ defined over **R** and such that $\beta \sim \alpha_{\rho|_H}$ where $\alpha_{\rho|_H}(h, \widetilde{m}) = \rho|_H(h)$.*

Thus for each $i = 1, \ldots, n$ there exists a rational homomorphism of **R**-groups $\rho_i : \mathscr{H} \to \mathscr{S}_i$ defined over **R** and a measurable $\widetilde{\phi}_i : \widetilde{M} \to S_i$ such that for all $h \in H$ we have $\widetilde{\alpha}_i(h, \widetilde{m}) = \phi_i^{-1}(hm)\rho_i|_H(h)\phi_i(m)$ for a.e. $\widetilde{m} \in \widetilde{M}$. Equivalently there exists a measurable $\phi : \widetilde{M} \to L_2$ and a rational homomorphism of **R**-groups $\rho : \mathscr{H} \to \mathscr{L}_2$ defined over **R** such that for all $h \in H$ we have

$$q \circ \widetilde{\alpha}(h, \widetilde{m}) = \phi^{-1}(h\widetilde{m})\rho|_H(h)\phi(\widetilde{m}) \quad \text{for a.e. } \widetilde{m} \in \widetilde{M}$$

where $\rho|_H : H = \mathscr{H}_\mathbf{R}^0 \to L_2^0$ (as we assume $H$ is connected). Let $d\rho : \mathfrak{h} \to \mathfrak{l}_2$ denote the Lie algebra homomorphism corresponding to $\rho|_H$ and let $\mathfrak{l}$ be the Lie algebra of $(\mathscr{L}^0)_\mathbf{R}$. As $\mathfrak{h}$ is semisimple we can lift $d\rho$ to $d\widetilde{\rho} : \mathfrak{h} \to \mathfrak{l}$ and thus there exists a homomorphism of Lie groups $\widetilde{\rho} : \widetilde{H} \to (\mathscr{L}^0)_\mathbf{R}^0$ corresponding to $d\widetilde{\rho} : \mathfrak{h} \to \mathfrak{l}$ and covering $\rho|_H : H \to L_2^0$. Fix $r : L_2^0 \to (\mathscr{L}^0)_\mathbf{R}^0$ a measurable

section of $q$ and set $\Phi = r \circ \phi$. The following diagram commutes:

$$\begin{array}{ccc} \widetilde{H} & \xrightarrow{\tilde{p}} & (\mathscr{L}^0)^0_{\mathbf{R}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle q} \\ H & \xrightarrow{p|_H} & L^0_2 \xleftarrow{\phi} \widetilde{M} \end{array}$$

Let $K \lhd (\mathscr{L}^0)_{\mathbf{R}}$ be the normal Lie subgroup $K = q^{-1}(e)$. The compactness of $L_1$ and $(Z(\mathscr{L}^0))_{\mathbf{R}}$ implies that $K$ is compact. Let $\tilde{h} \in \widetilde{H}$ such that $\pi(\tilde{h}) = h$. As $q(a) = q(b) \Leftrightarrow a = bk$ for some $k \in K$ and $q(\Phi^{-1}(h\tilde{m})\tilde{p}(\tilde{h})\Phi(\tilde{m})) = \phi^{-1}(h\tilde{m})p|_H(h)\phi(\tilde{m}) = q \circ \tilde{\alpha}(h, \tilde{m})$ we have, for all $\tilde{h} \in \widetilde{H}$,

$$\tilde{\alpha}(h, \tilde{m}) = \Phi^{-1}(h\tilde{m})\tilde{p}(\tilde{h})\Phi(\tilde{m})\Psi(h, \tilde{m})$$

for a.e. $\tilde{m} \in \widetilde{M}$ and for some $\Psi : H \times \widetilde{M} \to K$. Clearly $\tilde{p}$ cannot be trivial (otherwise $\tilde{\alpha}$ and thus $\alpha$ would be equivalent to the cocycle $\Psi(h, \tilde{m})$ which takes its values in $K$ which is compact so that the algebraic hull of the $H$-action $L$ would be compact). As $\alpha(h, m) = \tilde{\alpha}(h, \tilde{m})$ we are done.  $\square$

At each $m \in M$ the measurable section $s_0 : M \to P$ defines a canonical linear isomorphism $s_0(m) : \mathbf{R}^p \to T_m\mathscr{F}$. Let $\|\ \|_{\mathbf{R}^p}$ denote the usual norm on $\mathbf{R}^p$ and $\pi : T\mathscr{F} \to M$ be the canonical projection. The following lemma is an easy exercise.

**Lemma 3.3.4.** *There exists a measurable $\widetilde{A} \subset \widetilde{M}$ with $\tilde{\mu}(\widetilde{A}) > 0$ and a $C' > 0$ satisfying:*

    (1) $\|v\| \geq C' \|s_0(\pi(v))^* v\|_{\mathbf{R}^p}$ *for all $v \in \pi^{-1}(p(\widetilde{A}))$ (where $p : \widetilde{M} \to M$ is the finite irreducible extension introduced in Lemma 3.3.2),*

    (2) $\Phi(\widetilde{A}) \subset K'$ *for some compact $K' \subset L$ (where $\Phi$ is the map introduced in Lemma 3.3.2).*

**Lemma 3.3.5** (Poincaré recurrence) [1, Lemma 9.1.5, p. 165]. *Let $(X, \mu)$ be an $H$-space and assume that $\mu$ is $H$-invariant. Fix $h \in H$ and $A \subset X$ such that $\mu(A) \neq 0$. Then for almost all $x \in A$, $h^n x \in A$ for infinitely many positive $n$.*

*Proof of Theorem 3.3.1.* Let $\Pi = \tilde{p} : \widetilde{H} \to (\mathscr{L}^0)^0_{\mathbf{R}} \subset L \subset GL(p)$ be the canonical representation of $\widetilde{H}$ on $\mathbf{R}^p$ associated to $\tilde{p}$. As $\widetilde{H}$ is a simply connected semisimple Lie group such that the rank of each simple factor is greater than 1 (positive would actually be enough for what follows), there exists an $\tilde{h}_0 \in \widetilde{H}$ and a $\lambda > 1$, both depending only on $\widetilde{H}$, a $\zeta \in \mathbf{R}^p$ and a $\beta \in \mathbf{R}$ with $|\beta| > \lambda$ such that $\tilde{p}(\tilde{h}_0)\zeta = \beta\zeta$. Let $h_0 = \pi(\tilde{h}_0)$ and let $\widetilde{A} \subset \widetilde{M}$ as in Lemma 3.3.4. By Lemma 3.3.5 we can pick an $\tilde{m} \in \widetilde{M}$ such that $h_0^n \tilde{m} \in \widetilde{A}$ for infinitely many integers $n$ and the following diagram commutes (again set $m = p(\tilde{m})$):

$$\begin{array}{ccccccc} T_{s_0(m)}\mathscr{F} & \xrightarrow{h_0} & T_{s_0(h_0 m)}\mathscr{F} & \xrightarrow{h_0} & T_{s_0(h_0^2 m)}\mathscr{F} & \xrightarrow{h_0} & T_{s_0(h_0^3 m)}\mathscr{F} \\ {\scriptstyle s_0(m)}\uparrow & & {\scriptstyle s_0(h_0 m)}\uparrow & & {\scriptstyle s_0(h_0^2 m)}\uparrow & & {\scriptstyle s_0(h_0^3 m)}\uparrow \\ \mathbf{R}^p & \xrightarrow{\alpha(h_0, m)} & \mathbf{R}^p & \xrightarrow{\alpha(h_0, h_0 m)} & \mathbf{R}^p & \xrightarrow{\alpha(h_0, h_0^2 m)} & \mathbf{R}^p \end{array}$$

Set $v = s_0(m)_*((\Phi(\tilde{m}))^{-1}\xi) \in T_m\mathscr{F}$ where $\Phi : \widetilde{M} \to L \subset GL(p)$ is the map introduced in Lemma 3.3.2. By Lemma 3.3.2 we have

$$s_0(h_0^n m)^*(h_0^n v) = \alpha(h_0^n, m)((\Phi(\tilde{m}))^{-1}\xi)$$
$$= k_n(\Phi(h_0^n \tilde{m}))^{-1}\tilde{\rho}(\tilde{h}_0^n)\xi = k_n(\Phi(h_0^n \tilde{m}))^{-1}\beta^n \xi$$

for some sequence $(k_n)_{n \in \mathbb{N}}$ of elements of $K$ (we recall that $K \lhd (\mathscr{L}^0)_\mathbb{R}$). By Lemma 3.3.4 this implies that for infinitely many integers $n$ we have

$$\|h_0^n v\| \geq C'\|k_n(\Phi(h_0^n \tilde{m}))^{-1}\beta^n \xi\|_{\mathbb{R}^p} \geq C|\beta|^n$$

for some constant $C > 0$ and thus that for those integers we have $\|h_0^n v\| \geq C\lambda^n$. $\square$

### 3.4. Proof of Theorem 3.1.

The proof goes by contradiction: assume that there exists a sequence $(a_k)_{k \in \mathbb{N}}$ of $H$-actions on $M$ with noncompact algebraic hulls which converges to $a$ in the $C^1$-topology of uniform convergence on compact subsets of $H$. Let $h_0 \in H$ and $\lambda > 1$ be yielded by Theorem 3.3.1 and fix $\gamma \in \mathbb{R}$ such that $1 < \gamma < \lambda$.

As the sequence $(a_k)_{k \in \mathbb{N}}$ converges to $a$ and as $a$ preserves the continuous norm $\| \quad \|$, there exists an integer $k_0 \in \mathbb{N}$ such that

$$(*) \qquad k \geq k_0 \Rightarrow \text{for all } v \in T\mathscr{F}, \quad \|a_k(h_0, v)\| \leq \gamma\|v\|.$$

We now fix an integer $k \geq k_0$: by Theorem 3.3.1 there exist a $C > 0$ and a $v \in T\mathscr{F}$ (which we can assume to be unitary) such that for infinitely many integers $n$ we have

$$(**) \qquad \|a_k(h_0^n, v)\| \geq C\lambda^n.$$

But by $(*)$ we have $\|a_k(h_0^n, v)\| \leq \gamma^n$, so that by combining $(*)$ and $(**)$ we get $0 < C \leq (\frac{\gamma}{\lambda})^n$ for infinitely many integers $n$, which is impossible. $\square$

## 4. The main theorem

### 4.1. Introduction.

We now assume that the original $H$-action $a$ on $M$ leaves a *smooth* Riemannian metric $\omega$ on $T\mathscr{F}$ invariant. Let $a'$ be a sufficiently $C^\infty$-small perturbation of $a$. In this section we prove the existence, for each integer $r$, of an $a'$-invariant $C^r$-Riemannian metric on $T\mathscr{F}$. We will need the additional assumption that the original $H$-action $a$ is *transverse* to the foliation $\mathscr{F}$, namely that at each $m \in M$ we have $T_m\mathscr{F} + T_m(Hm) = T_mM$.

#### 4.1.1. Main Theorem.

*Let $H$ be a connected semisimple Lie group with finite center which can be realized as the $\mathbb{R}$-points of an algebraic $\mathbb{R}$-group $\mathscr{H}$ and such that the rank of each simple factor is at least two. Assume that $a \in \mathscr{A}(H, M)$ is an irreducible $H$-action which preserves a smooth volume $v$, a foliation $\mathscr{F}$, and a smooth Riemannian metric $\omega$ on $T\mathscr{F}$. Assume moreover that $a$ is transverse to $\mathscr{F}$. Let $a'$ be a sufficiently $C^\infty$-small perturbation of $a$ which is irreducible, which preserves a smooth volume $v'$ and the foliation $\mathscr{F}$. Then for each integer $0 \leq r < \infty$ there exists an $a'$-invariant $C^r$-Riemannian metric on $T\mathscr{F}$.*

By Theorem 3.1.1, if $a'$ is a sufficiently $C^1$-small perturbation of $a$, then $a'$ leaves a measurable Riemannian metric on $T\mathscr{F}$ a.e. invariant; it is moreover

easy to see that the transversality of $a$ implies the transversality of $C^1$-nearby actions: *from now on $a'$ will denote an irreducible $H$-action on $M$ which preserves the foliation $\mathscr{F}$, a smooth volume $v'$ (depending on $a'$), and which is contained in a sufficiently small $C^1$-neighbourhood of $a$ so that it is transverse to $\mathscr{F}$ and leaves a measurable Riemannian metric on $T\mathscr{F}$ a.e. invariant.*

*Sketch of the proof of the Main Theorem.* In §§4.2, 4.3, and 4.4 we build the machinery needed to show that if the perturbation $a'$ is sufficiently $C^\infty$-close to $a$, then for each integer $r$ there exists a nontrivial $\widetilde{a}'$-invariant function in $L^2(P) \cap C^r(P)$ (where again $P$ denotes the $GL(p)$-bundle of the frames of the foliation $\mathscr{F}$ and $\widetilde{a}'$ denotes the $H$-action on $P$ by principle bundle automorphisms induced by the $H$-action $a'$ on $M$). In §4.5 we show how the existence of such a function on $P$ implies the existence of an $a'$-invariant $C^r$-Riemannian metric on $T\mathscr{F}$.

*Remark.* As §§4.2, 4.3, and 4.4 mimic §§2, 4, and 8 of [2] we adopt a terse style and refer the reader to Professor Zimmer's original paper for details or proofs. However, the techniques involved in §4.5 differ significantly from the ones in [2] (which are inapplicable here as we do not have compactness of the isometry group $\mathrm{Is}(M, \omega)$): we develop them in full details.

### 4.2. The metrics $\widetilde{\omega}_k$ and $\widetilde{\eta'_k}$ on the vector bundles $J^k_{\mathscr{F}}(P, \mathbf{R})$.

Let $P = F(\mathscr{F})$ denote the $GL(p)$-bundle of frames of the foliation $\mathscr{F}$, $\pi$ the projection $P \to M$, and let $\mathscr{D}$ denote the $p$-dimensional integrable distribution on $M$ which yields the foliation $\mathscr{F}$. If $u \in P$ and $m = \pi(u)$, define $\widetilde{\mathscr{D}}_u = \pi^{-1}(\mathscr{D}_m) \subset T_u(P)$. This yields a $(p + p^2)$-dimensional distribution $\widetilde{\mathscr{D}}$ on $P$ which is $GL(p)$-invariant and integrable: let $\widetilde{\mathscr{F}}$ denote the corresponding $GL(p)$-invariant foliation. It is easy to see that an $H$-action on $M$ which is transverse to $\mathscr{F}$ (and $\mathscr{F}$-preserving) lifts to an $H$-action on $P$ transverse to $\widetilde{\mathscr{F}}$ (and $\widetilde{\mathscr{F}}$-preserving). Let $\mathrm{Met}(T\mathscr{F}) \to M$ and $\mathrm{Met}(T\widetilde{\mathscr{F}}) \to P$ denote the bundles whose sections are smooth Riemannian metrics on $T\mathscr{F}$ and $T\widetilde{\mathscr{F}}$ respectively. The following proposition enables us to lift metrics on $T\mathscr{F}$ to metrics on $T\widetilde{\mathscr{F}}$:

**Proposition 4.2.1.** *There exists a natural bundle map from $\mathrm{Met}(T\mathscr{F})$ into $\mathrm{Met}(T\widetilde{\mathscr{F}})$ sending $\xi$ to $\widetilde{\xi}$ such that:*

(1) *the Riemannian metric $\widetilde{\xi}$ is $GL(p)$-invariant,*

(2) *the map $\xi \mapsto \widetilde{\xi}$ is continuous, where $\mathrm{Met}(T\mathscr{F})$ has the topology of $C^l$-uniform convergence on compact sets and $\mathrm{Met}(T\widetilde{\mathscr{F}})$ has the topology of $C^{l-1}$-uniform convergence on compact sets $(l \geq 1)$,*

(3) *if $\Phi : M \to M$ is an $\mathscr{F}$-preserving diffeomorphism and $\widetilde{\Phi}$ denotes the corresponding automorphism of $P$ then $\widetilde{\Phi^*(\xi)} = (\widetilde{\Phi})^*(\widetilde{\xi})$.*

*Proof.* Take inspiration from [2, bottom of page 160 and top of page 161].

For each integer $k$, let $J^k_{\mathscr{F}}(P, \mathbf{R}) \to P$ denote the vector bundle of $k$-jets of smooth real valued functions on $P$ in the direction of $\widetilde{\mathscr{F}}$, i.e., $J^k_{\mathscr{F}}(P, \mathbf{R}) = C^\infty(P, \mathbf{R})/\sim$ where $C^\infty(P, \mathbf{R})$ denotes the space of smooth real-valued func-

tions on $P$ and

$$f \underset{u}{\sim} g \Leftrightarrow h(u) = 0, \quad h'|_{T_u \widetilde{\mathscr{F}}} = 0, \dots, h^{(k)}|_{T_u \widetilde{\mathscr{F}}} = 0 \quad \text{where } h = f - g,$$

and let $\mathrm{Met}(J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R})) \to P$ denote the bundle whose sections are smooth Riemannian metrics on $J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R})$. The following proposition holds:

**Proposition 4.2.2.** *There is a natural map from* $\mathrm{Met}(T\widetilde{\mathscr{F}})$ *into* $\mathrm{Met}(J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R}))$ *sending* $\widetilde{\xi}$ *to* $\widetilde{\xi}_k$ *with the following properties:*

> (1) *If* $\Phi$ *is an* $\mathscr{F}$*-preserving diffeomorphim of* $M$ *and* $\widetilde{\xi} \in \mathrm{Met}(T\widetilde{\mathscr{F}})$ *then* $(\widetilde{\Phi}^*(\widetilde{\xi}))_k = (\widetilde{\Phi})^*_k[\widetilde{\xi}_k].$
>
> (2) *For any* $l \geq k$ *the map is continuous where* $\mathrm{Met}(T\widetilde{\mathscr{F}})$ *has the topology of* $C^l$*-uniform convergence on compact sets and* $\mathrm{Met}(J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R}))$ *has the topology of* $C^{l-k}$*-uniform convergence on compact sets.*

*Proof.* Take inspiration from the proof of [2, Proposition 2.5, p. 162].

Let $\omega$ denote the smooth $a$-invariant Riemannian metric on $T\mathscr{F}$. By Proposition 4.2.1 $\omega$ yields a smooth Riemannian metric $\widetilde{\omega}$ on $T\widetilde{\mathscr{F}}$ which is both $GL(p)$- and $\widetilde{a}$-invariant, and which in turn, by Proposition 4.2.2, yields for each integer $k$ a smooth Riemannian metric $\widetilde{\omega}_k$ on the vector bundle $J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R})$ which is both $GL(p)$- and $\widetilde{a}$-invariant.

We now see that the existence of a measurable Riemannian metric on $T\mathscr{F}$ which is a.e. invariant under the perturbed action $a'$ implies the existence, for each integer $k$, of a measurable Riemannian metric $\widetilde{\eta}'_k$ on $J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R})$ which is both $GL(p)$-invariant and a.e. $\widetilde{a}'$-invariant.

We first give a new construction of $J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R})$ as a balanced product: let $G(p + p^2)$ be the group of germs at $0 \in \mathbf{R}^{p+p^2}$ of local diffeomorphisms of $\mathbf{R}^{p+p^2}$ fixing 0. For $k \geq 1$, let $G_k(p + p^2)$ be the normal subgroup consisting of diffeomorphisms that agree with the identity up to order $k$ and let $GL(p+p^2)^{(k)}$ be the quotient $G(p+p^2)/G_k(p+p^2)$. Let $J^k(\mathbf{R}^{p+p^2}, 0, \mathbf{R})$ be the vector space of $k$-jets at 0 of smooth $\mathbf{R}$-valued functions on $\mathbf{R}^{p+p^2}$. If $\phi \in GL(p + p^2)^{(k)}$ and $f \in J^k(\mathbf{R}^{p+p^2}, 0, \mathbf{R})$ then $(\phi, f) \mapsto f \circ \phi^{-1}$ defines a faithful left linear action of $GL(p + p^2)^{(k)}$ on $J^k(\mathbf{R}^{p+p^2}, 0, \mathbf{R})$. Let $P^{(k)}_{\widetilde{\mathscr{F}}} \to P$ be the bundle of $k$-jets along the leaves of $\widetilde{\mathscr{F}}$ of local diffeomorphisms of $P$ (an element of the fiber over $u \in P$ is a class of local diffeomorphisms of $P$ fixing $u$ which derivatives restricted to $\widetilde{D}_u = T_u\widetilde{\mathscr{F}}$ agree up to order $k$): it is a principal $GL(p + p^2)^{(k)}$ fiber bundle and we have

$$J^k_{\widetilde{\mathscr{F}}}(P, \mathbf{R}) = P^{(k)}_{\widetilde{\mathscr{F}}} \times_{GL(p+p^2)^{(k)}} J^k(\mathbf{R}^{p+p^2}, 0, \mathbf{R}).$$

**Proposition 4.2.3.**

> (1) *Let* $\alpha^{(k)}$ *be a cocycle corresponding to a measurable trivialization of* $P^{(k)}_{\widetilde{\mathscr{F}}} \to P$ *and the* $H$*-action on* $P^{(k)}_{\widetilde{\mathscr{F}}}$ *induced by* $\widetilde{a}'$, *and let* $\alpha$ *be the cocycle corresponding to* $\widetilde{a}'$ *and the associated trivialization of* $P \to M$.

*As $\alpha$ is equivalent to a cocycle taking values in a compact subgroup of $GL(p)$, $\alpha^{(k)}$ is also equivalent to a cocycle taking values in a compact subgroup of $GL(p + p^2)^{(k)}$: thus, for each integer $k$, there exists a measurable Riemannian metric $\widetilde{\Upsilon}'_k$ on $J^k_{\widetilde{\mathcal{F}}}(P, \mathbf{R})$ which is a.e. $\tilde{a}'$-invariant.*

(2) *If there exists such a measurable a.e. $\tilde{a}'$-invariant Riemannian metric $\widetilde{\Upsilon}'_k$ on $J^k_{\widetilde{\mathcal{F}}}(P, \mathbf{R})$, then there exists a measurable Riemannian metric $\widetilde{\eta}'_k$ on $J^k_{\widetilde{\mathcal{F}}}(P, \mathbf{R})$ which is both $GL(p)$-invariant and a.e. $\tilde{a}'$-invariant.*

*Proof.* (1) follows from [2, Proposition 4.7, p. 173] and (2) follows from [2, Corollary 4.13, p. 176].

### 4.3. The volume $\tilde{v}'$ and the Sobolev spaces $L^{s,k}_{\widetilde{\mathcal{F}}, \widetilde{\eta}'_k}(P, \mathbf{R})$.

The $\tilde{a}'$-invariant smooth volume $v'$ on $M$ together with Haar measure $\mu$ on $GL(p)$ (which is unimodular) yields a smooth volume $\tilde{v}'$ on $P$ which is both $GL(p)$- and $\tilde{a}'$-invariant.

Each $f \in C^\infty(P, \mathbf{R})$ defines a $k$-jet extension $j^k_{\widetilde{\mathcal{F}}}(f) \in C^\infty(P, J^k_{\widetilde{\mathcal{F}}}(P, \mathbf{R}))$. Let $\widetilde{\eta}'_k$ be the a.e. $\tilde{a}'$-invariant measurable Riemannian metric on the vector bundle $J^k_{\widetilde{\mathcal{F}}}(P, \mathbf{R}) \to P$ obtained in §4.2 and set

$$C^\infty(P, \mathbf{R})^{s,k}_{\widetilde{\mathcal{F}}, \widetilde{\eta}'_k} = \{ f \in C^\infty(P, \mathbf{R}) \mid \|j^k_{\widetilde{\mathcal{F}}}(f)\|_{\widetilde{\eta}'_k} \in L^s_{\tilde{v}'}(P) \}.$$

The set $C^\infty(P, \mathbf{R})^{s,k}_{\widetilde{\mathcal{F}}, \widetilde{\eta}'_k}$ with the norm

$$\|f\|^{s,k}_{\widetilde{\mathcal{F}}, \widetilde{\eta}'_k} = \left( \int_P \|j^k_{\widetilde{\mathcal{F}}}(f)\|^s_{\widetilde{\eta}'_k} \, d\tilde{v}' \right)^{1/s}$$

is a normed linear vector space whose completion we denote by $L^{s,k}_{\widetilde{\mathcal{F}}, \widetilde{\eta}'_k}(P, \mathbf{R})$. If the Riemannian metric $\widetilde{\eta}'_k$ is only measurable we might have $L^{s,k}_{\widetilde{\mathcal{F}}, \widetilde{\eta}'_k}(P, \mathbf{R}) = 0$. But if $\widetilde{\eta}'_k$ is smooth we clearly have $L^{s,k}_{\widetilde{\mathcal{F}}, \widetilde{\eta}'_k}(P, \mathbf{R}) \supset C^\infty_c(P, \mathbf{R})$, the smooth compactly supported functions on $P$. We are going to see that, provided $\widetilde{\eta}'_k$ satisfies some integrability conditions with respect to the smooth metric $\tilde{\omega}_k$, this inclusion also holds, together with a Sobolev-embedding type theorem. We first introduce the following definitions:

**Definition 4.3.1.** Let $V$ be a real vector space and let $\eta, \zeta \in \mathrm{Inn}(V)$. Set $M(\eta/\zeta) = \max\{\|x\|_\eta \mid \|x\|_\zeta = 1\}$. If $E \to P$ is a vector bundle and $\eta, \zeta$ are measurable metrics on $E$, then let $M(\eta/\zeta) :\to \mathbf{R}$ be the measurable function defined by $M(\eta/\zeta)(u) = M(\eta_u/\zeta_u)$.

We know that there exists an atlas $\mathcal{A} = (h_\alpha)_{\alpha \in \{1, \dots, i\}}$ on $M$ whose charts $h_\alpha : U_\alpha \to M$ trivialize $\mathcal{F}$. To this atlas corresponds the atlas

$$\widetilde{\mathcal{A}} = (\widetilde{h_\alpha})_{\alpha \in \{1, \dots, i\}}$$

of $P$ whose charts $\widetilde{h_\alpha} : U_\alpha \times GL(p) \to P$ trivialize $\widetilde{\mathcal{F}}$.

**Definition 4.3.2.** Let $f : P \to \mathbf{R}$ be a function on $P$. We will say that $f$ is $C^r$-*for-almost-every-leaf-of-*$\widetilde{\mathscr{F}}$ if in each chart of the atlas $\widetilde{\mathscr{A}}$ and for almost every $x \in \pi_2(U_\alpha) \subset \mathbf{R}^q$ (where $q = d - p$ is the codimension of the foliation $\mathscr{F}$) the function $f \circ \widetilde{h}_\alpha(., x, .)$ is $C^r$.

We then have the following proposition:

**Proposition 4.3.3.**

(1) *If* $M(\widetilde{\eta}_k'/\widetilde{\omega}_k) \in L^2_{\mathrm{loc}}(P)$, *then* $C^\infty_c(P, \mathbf{R}) \subset L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}_k'}(P, \mathbf{R})$.

(2) *If* $M(\widetilde{\omega}_k/\widetilde{\eta}_k') \in L^2_{\mathrm{loc}}(P)$, *then* $L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}_k'}(P, \mathbf{R}) \subset L^{1,k}_{\widetilde{\mathscr{F}},\widetilde{\omega}_k,\mathrm{loc}}(P, \mathbf{R})$.

(3) *If* $f \in L^{1,k}_{\widetilde{\mathscr{F}},\widetilde{\omega}_k,\mathrm{loc}}(P, \mathbf{R})$ *and* $k > r + \dim(\widetilde{\mathscr{F}})$, *then* $f$ *is* $C^r$-*for-almost-every-leaf-of-*$\widetilde{\mathscr{F}}$.

(4) *Assume that* $f : P \to \mathbf{R}$ *is* $C^r$-*for-almost-every-leaf-of-*$\widetilde{\mathscr{F}}$. *If* $f$ *is moreover* $\widetilde{a}'$-*invariant then* $f \in C^r(P)$.

*Proof.* For (1) and (2) take inspiration from [2, Lemma 2.8, p. 164]. To prove (3) apply Fubini's Theorem and the classical Sobolev Embedding Theorem in the charts of the atlas $\widetilde{\mathscr{A}}$, and to prove (4) use the $\widetilde{a}'$-invariance of $f$ together with the transversality of the $H$-action $\widetilde{a}'$ on $P$ to the foliation $\widetilde{\mathscr{F}}$.

We now show that the integrability conditions of Proposition 4.3.3 (namely that $M(\widetilde{\eta}_k'/\widetilde{\omega}_k)$ and $M(\widetilde{\omega}_k/\widetilde{\eta}_k')$ both belong to $L^2_{\mathrm{loc}}(P)$) are realized provided that the perturbed action $a'$ is $C^\infty$-close to $a$:

**Proposition 4.3.4.** *If* $a'$ *is a sufficiently* $C^\infty$-*small perturbation of* $a$ *then for each integer* $k$, $M(\widetilde{\eta}_k'/\widetilde{\omega}_k)$ *and* $M(\widetilde{\omega}_k/\widetilde{\eta}_k')$ *both belong to* $L^2_{\mathrm{loc}}(P)$.

*Proof.* In order to insure that $M(\widetilde{\eta}_k'/\widetilde{\omega}_k)$ belongs to $L^2_{\mathrm{loc}}(P)$ we will examine the growth of this function along $\widetilde{a}'$-orbits and show that this can be converted into the desired integrability property by an application of Kazhdan's property. We first make the following remark: if $\widetilde{\eta}_k'$ and $\widetilde{\omega}_k$ are both $GL(p)$-invariant, then $M(\widetilde{\eta}_k'/\widetilde{\omega}_k)$ can be considered as a function on $M$ so that in order to show that this function, considered as a function on $P$, belongs to $L^2_{\mathrm{loc}}(P)$, it is enough to show that, considered as a function on $M$, it is in $L^2_{v'}(M)$. Furthermore, as $\widetilde{\eta}_k'$ is $\widetilde{a}'$-invariant, then for each $h \in H$ and a.e. $m \in M$ we have

$$
\begin{aligned}
M(\widetilde{\eta}_k'/\widetilde{\omega}_k)(hm) &= M(h^*\widetilde{\eta}_k'/h^*\widetilde{\omega}_k)(m) \\
&\leq M(h^*\widetilde{\eta}_k'/\widetilde{\omega}_k)(m) M(\widetilde{\omega}_k/h^*\widetilde{\omega}_k)(m) \\
&= M(\widetilde{\eta}_k'/\widetilde{\omega}_k)(m) M(\widetilde{\omega}_k/h^*\widetilde{\omega}_k)(m).
\end{aligned}
$$

Thus the growth of $M(\widetilde{\eta}_k'/\widetilde{\omega}_k)(m)$ along the $a'$-orbit $Hm$ is governed by the growth of $\|M(\widetilde{\omega}_k/h^*\widetilde{\omega}_k)\|_\infty$ (we recall that $\omega$ is smooth) along the same orbit. The following proposition is then clearly germane:

**Proposition 4.3.5** [2, Theorem 8.1, p. 187]. *Assume that* $H$ *is a Kazhdan group. Then there exist a compact subset* $K_0 \subset H$ *and a constant* $C_0 > 1$ *(depending only on the group* $H$) *satisfying the following property: whenever* $H$ *acts ergod-*

*ically on a probability space* $(X, \mu)$ *preserving* $\mu$, *any measurable* $f : X \to \mathbb{R}$ *which satisfies, for all* $h \in K$, $|f(hx)| \leq C_0 |f(x)|$ *for a.e.* $x \in X$, *belongs to* $L^2_\mu(X)$.

Thus by Proposition 4.3.5 to insure that $M(\widetilde{\eta}'_k / \widetilde{\omega}_k) \in L^2_{\widetilde{v}'}(M)$ it is enough to insure that for all $h \in K_0$, $\|M(\widetilde{\omega}_k / h^* \widetilde{\omega}_k)\|_\infty \leq C_0$. But by (2) of Proposition 4.2.1 and (2) of Proposition 4.2.2 $\|M(\widetilde{\omega}_k / h^* \widetilde{\omega}_k)\|_\infty$ will be uniformly close to 1 for all $h \in K_0$ provided that $a'$ is a $C^{k+1}$-small perturbation of $a$.

A similar argument holds for $\|M(\widetilde{\omega}_k / \widetilde{\eta}'_k)\|_\infty$. $\square$

## 4.4. The $\widetilde{a}'$-invariant functions $F_k \in L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})$.

From now on we always assume that $a'$ is a sufficiently $C^\infty$-small perturbation of $a$ so that if $k$ and $r$ are integers such that $k > r + \dim(\widetilde{\mathscr{F}})$, we have $C^\infty_c(P, \mathbb{R}) \subset L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})$ and $L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R}) \subset L^{1,k}_{\widetilde{\mathscr{F}}, \widetilde{\omega}_k, \text{loc}}(P, \mathbb{R}) \subset C^r(P)$. We can moreover assume that the Riemannian metric $\widetilde{\eta}'_k$ agrees with the standard metric on the naturally split trivial bundle $J^0_{\widetilde{\mathscr{F}}}(P, \mathbb{R}) \subset J^k_{\widetilde{\mathscr{F}}}(P, \mathbb{R})$ so that $\|f\|^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k} \leq \|f\|_{2, \widetilde{v}'}$ for all $f \in C^\infty_c(P, \mathbb{R})$. By extending the identity on the subspace of smooth compactly supported functions we obtain canonical continuous injections $i_k : L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R}) \hookrightarrow L^2_{\widetilde{v}'}(P) \cap C^r(P)$ such that $\|i_k\|_{\text{op}} \leq 1$.

**Proposition 4.4.1.** *For each integer* $k$, *the existence of a measurable Riemannian metric* $\widetilde{\eta}'_k$ *on the vector bundle* $J^k_{\widetilde{\mathscr{F}}}(P, \mathbb{R})$ *which is both* $GL(p)$- *and* $\widetilde{a}'$-*invariant and which moreover satisfies* $M(\widetilde{\eta}'_k / \widetilde{\omega}_k) \in L^2_{\text{loc}}(P)$ *implies the existence of a nonzero* $\widetilde{a}'$-*invariant* $F_k \in L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})$.

*Proof.* We first show that there exists a nonzero $\widetilde{a}'$-invariant $f \in L^2_{\widetilde{v}'}(P)$: let $\Phi : M \times GL(p) \to P$ be a measurable trivialisation of $P$ such that the corresponding cocycle $\alpha : H \times M \to GL(p)$ satisfies $\alpha(H \times M) \subset K$ where $K \subset GL(p)$ is a compact subgroup (such a trivialization exists as $a'$ leaves a measurable metric on $T\mathscr{F}$ a.e. invariant). It therefore suffices to see that there exists a nonzero $f \in L^2(M \times GL(p))$ which is $H$-invariant under the action $h.(m, g) = (h.m, \alpha(h, m)g)$. But clearly, if $\phi \in L^2(GL(p))$ is a nonzero left-$K$-invariant function, then $f(m, g) = \phi(g)$ is such a function.

Let $i_k : L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R}) \to L^2_{\widetilde{v}'}(P)$ denote the canonical inclusion. Since $\widetilde{\eta}'_k$ and $\widetilde{v}'$ are $\widetilde{a}'$-invariant the $H$-action $\widetilde{a}'$ on $P$ induces unitary representations of $H$ on $L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})$ and $L^2_{\widetilde{v}'}(P)$ respectively. Clearly $i_k(h.f) = h.i_k(f)$, i.e., $i_k$ is an intertwining operator between these two unitary representations. Let $(i_k)^* : \left(L^2_{\widetilde{v}'}(P)\right)^* \to \left(L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})\right)^*$ denote as usual the dual map of $i_k$, and let $d_k : \left(L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})\right)^* \to L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})$ and $d : \left(L^2_{\widetilde{v}'}(P)\right)^* \to L^2_{\widetilde{v}'}(P)$ denote respectively the canonical maps identifying $L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbb{R})$ and $L^2_{\widetilde{v}'}(P)$ with their duals.

$$L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}'_k}(P,\mathbb{R}) \xrightarrow{\ i_k\ } L^2_{\widetilde{v}'}(P)$$

$$\uparrow{\scriptstyle d_k} \qquad\qquad\qquad d\uparrow$$

$$\big(L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}'_k}(P,\mathbb{R})\big)^* \xleftarrow{\ (i_k)^*\ } (L^2_{\widetilde{v}'}(P))^*$$

The map $\Psi_k = d_k \circ i_k^* \circ d^{-1}$ also intertwines the two unitary representations of $H$ so that if $f \in L^2_{\widetilde{v}'}(P)$ is $H$-invariant, $\Psi_k(f) \in L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}'_k}(P,\mathbb{R})$ will also be $H$-invariant. Since $i_k(L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}'_k}(P,\mathbb{R}))$ contains $C^\infty_c(P,\mathbb{R})$ (by (1) of Proposition 4.3.3) it is dense in $L^2_{\widetilde{v}'}(P)$ and thus $(i_k)^*$ is injective. Thus the function $F_k = \Psi_k(f) \in L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}'_k}(P,\mathbb{R})$ (where $f \in L^2_{\widetilde{v}'}(P)$ is the nonzero $H$-invariant function obtained before) is nonzero and $H$-invariant.  □

**4.5. Proof of the Main Theorem.** Let $a'$ be a sufficiently $C^\infty$-small perturbation of $a$ and fix an integer $k$. In §4.2 we have proved the existence of a measurable Riemannian metric $\widetilde{\eta}'_k$ on the vector bundle $J^k_{\widetilde{\mathscr{F}}}(P,\mathbb{R}) \to P$ which is both $GL(p)$- and $\widetilde{a}'$-invariant, and the existence of a smooth Riemannian metric $\widetilde{\omega}_k$ on the same vector bundle which is both $GL(p)$- and $\widetilde{a}$-invariant. By Proposition 4.3.4 the metric $\widetilde{\eta}'_k$ necessarily satisfies the integrability conditions (1) and (2) of Proposition 4.3.3 (with respect to the smooth metric $\widetilde{\omega}_k$).

From now on we fix such a perturbation $a'$ and two integers $k$ and $r$ such that $r \geq 1$ and $k > r + \dim(\widetilde{\mathscr{F}})$.

By Proposition 4.4.1 there exists a nonzero $\widetilde{a}'$-invariant $F_k \in L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}'_k}(P,\mathbb{R})$ which, by Proposition 4.3.3, belongs to $C^r(P)$. Our Main Theorem 4.1.1 thus clearly follows from the following proposition:

**Proposition 4.5.1.** *The existence of such an* $F_k \in L^{2,k}_{\widetilde{\mathscr{F}},\widetilde{\eta}'_k}(P,\mathbb{R}) \cap C^r(P)$ *implies the existence of an* $a'$-*invariant* $C^r$-*Riemannian metric on* $T\mathscr{F}$.

We recall the latter is equivalent to the existence of a compact subgroup $K \subset GL(p)$ and an $a'$-invariant $C^r$ cross-section $\sigma' : M \to P/K$ of the natural projection $P/K \to M$.

**Lemma 4.5.2.** *Assume that* $f \in C^r(P) \cap L^2_{\widetilde{v}'}(P)$ *(respectively* $f \in C^r(P) \cap L^1_{\widetilde{v}'}(P)$*) is nonzero and* $\widetilde{a}'$-*invariant. Set* $P_m = \pi^{-1}(m)$ *and* $f_m = f|_{P_m}$. *Then*

(1) $f_m \in L^2(P_m)$ *(respectively* $L^1(P_m)$*) for all* $m \in M$.

(2) *Let* $\Pi$ *denote the left regular representation of* $GL(p)$ *on* $L^2(GL(p))$ *(respectively on* $L^1(GL(p))$*) and set* $W = \{m \in M \mid f_m \neq 0\}$. *Then* $W$ *is an open* $H$-*invariant conull set and there exists* $\lambda \in C^r(GL(p)) \cap L^2(GL(p))$ *(respectively* $C^r(GL(p)) \cap L^1(GL(p))$*) such that in any trivialization of* $P$ *and for all* $m \in W$ *we have* $f_m \in \Pi(GL(p))(\lambda)$.

*Proof.* We will prove Lemma 4.5.2 for $f \in L^2_{\widetilde{v}'}(P)$, the proof for $f \in L^1_{\widetilde{v}'}(P)$ being identical. Fix $\Phi : M \times GL(p) \to P$ a measurable trivialization of $P$ and

let $\alpha : H \times M \to GL(p)$ be the cocycle corresponding to the $H$-action $\tilde{a}'$ on $P$ and the trivialization $\Phi$. We can then look upon $f$ as a measurable function on $M \times GL(p)$, and $f_m$ as a $C^r$-function on $GL(p)$. As $f$ is $H$-invariant, for all $m \in M$, $g \in GL(p)$ and $h \in H$ we have

$$f(\Phi(m, g)) = f(h\Phi(m, g)) = f(\Phi(hm, \alpha(h, m)g))$$

and thus

$(*)$ $$f(\Phi(hm, .)) = \Pi(\alpha(h, m))[f(\Phi(m, .))].$$

Set $W_1 = \{ m \in M \mid f_m \in L^2(P_m) \}$ and let $\beta : W_1 \to L^2(GL(p))$ be the map $m \mapsto f(\Phi(m, .))$. Let $q : L^2(GL(p)) \to L^2(GL(p))/GL(p)$ be the canonical projection associated to the left regular representation of $GL(p)$ on $L^2(GL(p))$ and define $\overline{f} : M \to L^2(GL(p))/GL(p)$ by $\overline{f} = q \circ \beta$. By Fubini $W_1$ is conull. By $(*)$ $W_1$ is $H$-invariant and for all $h \in H$ we have $\overline{f(hm)} = \overline{f(m)}$ a.e. in $W_1$. As the $H$-action $a'$ on $M$ is assumed to be ergodic, by [1, Proposition 2.1.11, p. 11] the function is constant a.e. on $W_1$ provided that the space $L^2(GL(p))/GL(p)$ is countably separated:

*Claim* 1. The space $L^2(GL(p))/GL(p)$ is countably separated.

*Proof.* By [1, Proposition 2.1.14, p. 12], Claim 1 is equivalent to saying that each $GL(p)$-orbit in $L^2(GL(p))$ is locally closed. In fact, for any lcsc group $G$, any $G$-orbit in $L^2(G)$ is closed.

Thus there exists $\lambda \in L^2(GL(p))$ and $W_2 \subset W_1$ conull and $H$-invariant such that

$$f(\Phi(m, .)) \in \Pi(GL(p))\lambda \qquad \text{for all } m \in W_2.$$

This implies that $\lambda \in C^r(GL(p))$ and $\lambda \neq 0$.

*Claim* 2. $W_1 = M$ (this proves (1)).

*Proof.* Fix $m_0 \in M$. As $W_2$ is conull there exists a sequence $(m_n)_{n \in \mathbf{N}}$ of points of $W_2$ such that $m_n \xrightarrow[n \to \infty]{} m_0$. Fix some continuous trivialization of $P$ centered at $m_0$, say $\Psi : U(m_0) \times GL(p) \to \pi^{-1}(U(m_0))$. Then by definition of $W_2$ there exists a sequence $(g_n)_{n \in \mathbf{N}}$ of elements of $GL(p)$ such that

$$f(\Psi(m_n, .)) = \Pi(g_n)\lambda = f_n.$$

By continuity of $\Psi$ and $f$, $\underline{\lim}_{n \to \infty} |f_n| = |f(\Psi(m_0, .))|$, and by Fatou's Lemma,

$$\int_{GL(p)} \underline{\lim}_{n \to \infty} |f_n|^2 \, dg = \int_{GL(p)} |f(\Psi(m_0, .))|^2 \, dg$$

$$\leq \underline{\lim}_{n \to \infty} \int_{GL(p)} |f_n|^2 \, dg \leq \underline{\lim}_{n \to \infty} \int_{GL(p)} |\lambda(g_n g)|^2 \, dg = \|\lambda\|_2^2$$

and thus $f_{m_0} \in L^2(P_{m_0})$, i.e., $m_0 \in W_1$.

*Claim* 3. Set $W = \{ m \in M \mid f_m \neq 0 \}$. Then $W = W_2$. (This proves (2).)

*Proof.* Clearly $W_2 \subset W$ so we just have to show that $W \subset W_2$, i.e., that if $m_0 \in W_2^c$ then $f_{m_0} = 0$. Fix $m_0 \in W_2^c$. As $W_2$ is conull there exists a sequence $(m_n)_{n \in \mathbf{N}}$ of points of $W_2$ such that $m_n \xrightarrow[n \to \infty]{} m_0$. As in the proof of Claim 2, fix some continuous trivialization centered at $m_0$, $\Psi : U(m_0) \times GL(p) \to$

$\pi^{-1}(U(m_0))$. By definition of $W_2$ there exists a sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $GL(p)$ such that

$$f(\Psi(m_n, .)) = \Pi(g_n)\lambda = f_n.$$

Assume that $(g_n)$ has a convergent subsequence $g_{n_k} \xrightarrow[k \to \infty]{} g_0$. Then $\Pi(g_{n_k})$ $\xrightarrow[k \to \infty]{} \Pi(g_0)\lambda$ pointwise in $GL(p)$ and by continuity of $\Psi$ and $f$,

$$\Pi(g_{n_k})\lambda = f(\Psi(m_{n_k}, .)) \xrightarrow[k \to \infty]{} f(\Psi(m_0, .)) \qquad \text{pointwise in } GL(p),$$

and thus $f(\Psi(m_0, .)) = \Pi(g_0)\lambda$, which contradicts the assumption that $m_0 \in W_2^c$. We thus necessarily have $g_n \xrightarrow[n \to \infty]{} \infty$ so that

$$\Pi(g_n)(\lambda) = f(\Psi(m_n, .)) \xrightarrow[n \to \infty]{L^2_{\text{loc}}(GL(p))} 0.$$

It is therefore enough to see that

$$f(\Psi(m_n, .)) \xrightarrow[n \to \infty]{L^2_{\text{loc}}(GL(p))} f(\Psi(m_0, .)).$$

Fix $K \subset GL(p)$ a compact. By continuity of $\Psi$,

$$f(\Psi(m_n, .)) \xrightarrow[n \to \infty]{} f(\Psi(m_0, .))$$

pointwise in $GL(p)$. Without loss of generality we can assume that $U(m_0)$ is relatively compact and thus that for all integers $N$ and all $g \in K$

$$|f(\Psi(m_n, g)) - f(\Psi(m_0, g))|^2 \leq C.$$

As Haar measure is finite on compact sets $C \in L^2_K(GL(p))$, so that we are done by Lebesgue's Dominated Convergence Theorem.  □

**Lemma 4.5.3.** *Let* $F_k \in L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}_k'}(P, \mathbb{R}) \subset C^r(P) \cap L^2_{v'}(P)$ *be the nonzero* $\widetilde{a}'$-*invariant function on* $P$ *obtained at the beginning of* §4.5, *and let* $W = \{m \in M \mid (F_k)_m \neq 0\}$ *be the* $a'$-*invariant open conull subset of* $M$ *introduced in* (2) *of Lemma* 4.5.2. *Fix* $m_0 \in W^c$. *Then there exists a nonzero* $\widetilde{a}'$-*invariant function* $f^{m_0} \in L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}_k'}(P, \mathbb{R})$ *such that* $(f^{m_0})_{m_0} \neq 0$.

*Proof.* As the $H$-action $a'$ on $M$ is transverse to the foliation $\mathscr{F}$, there exist vectors $H_1, \ldots, H_q \in \mathfrak{h}$ (where $\mathfrak{h}$ denotes the Lie algebra of $H$ and $q = d - p$ is the codimension of $\mathscr{F}$) such that $T_{m_0}M = T_{m_0}\mathscr{F} \oplus \langle k^{H_1}(m_0), \ldots, k^{H_q}(m_0)\rangle$ (where the second factor denotes the subspace spanned at $m_0$ by the the the Killing fields $k^{H_i}$). Let $\phi : V(0) \to V_{L_{m_0}}(m_0)$ be a local chart of the leaf $L_{m_0}$ centered at $m_0$ (where $V(0) \subset \mathbb{R}^p$ is an open convex relatively compact neighbourhood of the origin, so that $V_{L_{m_0}}(m_0) \subset L_{m_0}$ is an open connected relatively compact neighbourhood of $m_0$ in the leaf $L_{m_0}$), fix $\epsilon > 0$ and define the map $\Phi :$ $V(0) \times (-\epsilon, \epsilon)^q \to M$ by

$$\Phi(l, t_1, \ldots, t_q) = \exp\left(\sum_{i=1}^{i=q} t_i H^i\right) . \phi(l).$$

For each $(l, 0) \in V(0) \times (-\epsilon, \epsilon)^q$ the map $\Phi$ is a local diffeomorphism at $(l, 0)$ so that by picking $\epsilon$ small enough we can assume that $\Phi$ is a diffeomorphism:

$$\Phi : V(0) \times (-\epsilon, \epsilon)^q \to U(m_0)$$

where $U(m_0) \subset M$ is an open connected relatively compact neighbourhood of $m_0 \in W^c$. As the $H$-action $a'$ on $M$ permutes the leaves of the foliation $\mathscr{F}$ this diffeomorphism is moreover a local trivialisation of $\mathscr{F}$ which, by construction, is centered at $m_0$ and satisfies

$$\Phi(V(0) \times \{0\}) = \phi(V(0)) = V_{L_{m_0}}(m_0) \subset L_{m_0}.$$

We will thus refer to the coordinates $l$ and $t$ on $U(m_0)$ as, respectively, the leafwise and transverse coordinates.

As the map $\Phi$ is a local trivialisation of $\mathscr{F}$, it maps the canonical frame $(e_1, \ldots, e_p)$ at $(l, t) \in V(0) \times (-\epsilon, \epsilon)^q \subset \mathbf{R}^p \times \mathbf{R}^q$ onto a frame of $\mathscr{F}$ at $\Phi(l, t) \in U(m_0)$ which we denote $u(l, t) \in P$. Set $u(0, 0) = u_0$ and let $\pi : P \to M$ be the natural projection. By construction $\pi(u_0) = m_0$ and $\pi^{-1}(U(m_0))$ is an open neighbourhood of $u_0$. We can naturally define a diffeomorphism

$$\tilde{\Phi} : V(0) \times (-\epsilon, \epsilon)^q \times GL(p) \to \pi^{-1}(U(m_0))$$

by the formula

$$\tilde{\Phi}(l, t, g) = u(l, t) \cdot g$$

which is both a local trivialization of the $GL(p)$-bundle $\pi^{-1}(U(m_0)) \to U(m_0)$ and a local trivialization of the foliation $\widetilde{\mathscr{F}}$ on $P$ centered at $u_0$, which satisfies

$$\tilde{\Phi}(V(0) \times \{0\} \times GL(p)) \subset \tilde{L}_{u_0}.$$

The coordinates $(l, t, g)$ on $U(u_0)$ will be respectively referred to as the leafwise, transverse and group coordinates (note that leafwise is with respect to the foliation $\mathscr{F}$ and not $\widetilde{\mathscr{F}}$).

As $W \subset M$ is $H$-invariant and, by hypothesis, $m_0 \in W^c$, necessarily $\Phi(\{0\} \times \mathbf{R}^q) \subset W^c$ and $\Phi^{-1}(W \cap U(m_0)) \subset V(0) \times (-\epsilon, \epsilon)^q$ is invariant under small transverse translations (by this we mean that if $(l, t) \in V(0) \times (-\epsilon, \epsilon)^q$, $t_0 \in (-\epsilon, \epsilon)^q$, and $\Phi(l, t) \in W$, then $\Phi(l, t_0) \in W$). Therefore $W^c \cap V_{L_{m_0}}(m_0) \subset L_{m_0}$ must also have measure zero in $V(m_0) \subset L_{m_0}$ (otherwise $W^c \subset M$ would contain a subset of strictly positive measure). We can therefore assert that there exists a sequence $(m_j)_{j \in \mathbf{N}}$ such that

$$m_j \in W \cap V_{L_{m_0}}(m_0) \subset L_{m_0} \quad \text{and} \quad m_j \xrightarrow[j \to \infty]{} m_0.$$

By (2) of Lemma 4.5.2 there exist a nonzero $\lambda \in C^r(GL(p)) \cap L^2(GL(p))$ and a sequence $(g_j)_{j \in \mathbf{N}}$ of elements of $GL(p)$ such that $(F_k)_{m_j} = \Pi(g_j)\lambda$ where $(F_k)_{m_j}(g) = F_k \circ \tilde{\Phi}(\Phi^{-1}(m_j), g)$. Set $u_j = \tilde{\Phi}(\Phi^{-1}(m_j), e) \in \tilde{L}_{u_0}$. By construction $\pi(u_j) = m_j \in L_{m_0}$. By continuity of $\Phi$ and $\tilde{\Phi}$, we have $u_j = \tilde{\Phi}(\Phi^{-1}(m_j), e) \xrightarrow[j \to \infty]{} \tilde{\Phi}(\Phi^{-1}(m_0), e) = \tilde{\Phi}(0, 0, e) = u_0$ and as $\lambda \neq 0$ there exists $g_0 \in GL(p)$ such that $\lambda(g_0) \neq 0$. Set

$$f_j = \Pi_R(g_j g_0)(F_k)$$

where $\Pi_R$ denotes the right regular representation of $GL(p)$ on $L^{2,k}_{\widetilde{\mathscr{F}}, \tilde{\eta}'_k}(P, \mathbf{R})$ (which is unitary as both the volume $\tilde{v}'$ and the metric $\tilde{\eta}'_k$ are $GL(p)$-invariant).

Thus $f_j \in L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbf{R})$, $\|f_j\|^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k} = \|F_k\|^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}$, and as $H$ acts on the left of $P$ and $F_k$ is $\widetilde{a}'$-invariant (and nonzero), $f_j$ is also $\widetilde{a}'$-invariant (and nonzero): in order to prove Lemma 4.5.3 it is therefore enough to show that there exists an integer $j_0$ such that $f_{j_0}(u_0) \neq 0$ (we are then done by setting $f^{m_0} = f_{j_0}$).

*Claim.* There exists an integer $j_0$ such that $f_{j_0}(u_0) \neq 0$.

We prove this claim by contradiction: assume that for all $j \in \mathbf{N}$ we have $f_j(u_0) = 0$. We will use the following fact:

$$f_j(u_j) = f_j(\widetilde{\Phi}(\Phi^{-1}(m_j), e)) = [\Pi_R(g_j g_0)\Pi(g_j)\lambda](e) = \lambda(g_j^{-1} e g_j g_0) = \lambda(g_0).$$

Let $V(e) \subset GL(p) \subset \mathbf{R}^{p^2}$ be an open convex relatively compact neighbourhood of the identity and set $U(u_0) = \widetilde{\Phi}(V(0) \times (-\epsilon, \epsilon)^q \times V(e))$. Clearly $U(u_0)$ is an open connected relatively compact neighbourhood of $u_0 \in P$ and by passing to a subsequence we can assume that the $u_j$ lie in $U(u_0)$. For each integer $j$ define a $C^r$-function

$$G_j = f_j \circ \widetilde{\Phi} : V(0) \times (-\epsilon, \epsilon)^q \times V(e) \to \mathbf{R}$$

to which we can apply the Mean Value Theorem ($V(0) \times (-\epsilon, \epsilon)^q \times V(e) \subset \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{p^2}$ being convex and open) between the points $\widetilde{\Phi}^{-1}(u_j)$ and $0$: there exists $c_j \in [0, \widetilde{\Phi}^{-1}(u_j)] \subset \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{p^2}$ such that

$$dG_j(c_j)(\widetilde{\Phi}^{-1}(u_j) - 0) = G_j(\widetilde{\Phi}^{-1}(u_j)) - G_j(0) = f_j(u_j) - f_j(u_0)$$
$$= f_j(u_j) = \lambda(g_0).$$

As $c_j \xrightarrow[j \to \infty]{} 0$ (as $c_j \in [0, \widetilde{\Phi}^{-1}(u_j)]$), and $\widetilde{\Phi}^{-1}(u_j) \xrightarrow[j \to \infty]{} 0$, we have

$$\|dG_j(c_j)\| \geq \frac{|\lambda(g_0)|}{\|\widetilde{\Phi}^{-1}(u_j)\|} \xrightarrow[j \to \infty]{} \infty$$

where $\|dG_j(c_j)\|$ denotes the norm of the derivative $dG_j(c_j) : \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{p^2} \to \mathbf{R}$. We moreover point out that as $u_j \in \widetilde{L}_{u_0}$, by construction of $\widetilde{\Phi}$ both $\widetilde{\Phi}^{-1}(u_j)$ and $c_j$ lie in $V(0) \times \{0\} \times V(e)$ so that if $\overline{G}_j$ denotes the restriction of $G_j$ to $V(0) \times \{0\} \times V(e) \subset V(0) \times (-\epsilon, \epsilon)^q \times V(e)$ then

$$(*) \qquad\qquad\qquad \|d\overline{G}_j(c_j)\| \xrightarrow[j \to \infty]{} \infty$$

where $\|d\overline{G}_j(c_j)\|$ denotes this time the norm of the derivative $d\overline{G}_j(c_j) : \mathbf{R}^p \times \mathbf{R}^{p^2} \to \mathbf{R}$.

*We claim that this contradicts* $\|f_j\|^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k} = \|F_k\|^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}$ *for all integers* $j$.

Indeed as $M(\widetilde{\omega}_k / \widetilde{\eta}'_k) \in L^2_{\text{loc}}(P)$ and $U(u_0)$ is relatively compact, for each integer $j$ we have on one hand:

$$\int_{U(u_0)} \|j^k_{\widetilde{\mathscr{F}}}(f_j)\|_{\widetilde{\omega}_k} d\widetilde{v}' \leq \int_{U(u_0)} \|j^k_{\widetilde{\mathscr{F}}}(f_j)\|_{\widetilde{\eta}'_k} M(\widetilde{\omega}_k / \widetilde{\eta}'_k) d\widetilde{v}'$$

$$\leq \left( \int_{U(u_0)} \|j^k_{\widetilde{\mathscr{F}}}(f_j)\|^2_{\widetilde{\eta}'_k} d\widetilde{v}' \right)^{1/2} \left( \int_{U(u_0)} M^2(\widetilde{\omega}_k / \widetilde{\eta}'_k) d\widetilde{v}' \right)^{1/2}$$

$$\leq \|f_j\|^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k} C_1 = \|F_k\|^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k} C_1 \leq C_2$$

for some constants $C_1$, $C_2 > 0$, so that the quantities $\int_{U(u_0)} \|j^k_{\widetilde{\mathscr{F}}}(f_j)\|_{\widetilde{\omega}_k} d\widetilde{v}'$ are uniformly bounded. On the other hand, by construction of $\widetilde{\Phi}$, the formula of change of variables, the relative compactness of $U(u_0) \subset P$, Fubini's Theorem and the $\widetilde{a}'$-invariance of the functions $f_j$ (which implies in the chart $\widetilde{\Phi}$ the invariance of $G_j$ by transverse translations), there exist constants $C_3 > 0$ and $C_4 > 0$ such that

$$\int_{U(u_0)} \|j^k_{\widetilde{\mathscr{F}}}(f_j)\|_{\widetilde{\omega}_k} d\widetilde{v}'$$

$$= \int_{V(0) \times (-\epsilon, \epsilon)^q \times V(e)} \|j^k_{\widetilde{\mathscr{F}_0}}(G_j)\|_{\widetilde{\Phi}^*(\widetilde{\omega}_k)} \frac{d[\widetilde{\Phi}^*(\widetilde{v}')]}{dl \times dt \times dg} dl \times dt \times dg$$

$$\geq C_3 \int_{V(0) \times (-\epsilon, \epsilon)^q \times V(e)} \|j^k_{\widetilde{\mathscr{F}_0}}(G_j)\|_0 (l, t, g) dl \times dt \times dg$$

$$= (2\epsilon)^q C_3 \int_{V(0) \times V(e)} \|j^k_{\mathscr{F}_0}(\overline{G}_j)\|_0 (l, g) dl \times dg \geq C_4 \|\overline{G}_j\|_{1,k}^{V(0) \times V(e)}$$

where $\widetilde{\mathscr{F}_0}$ denotes the canonical foliation on $\mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{p^2}$ obtained by translating $\mathbf{R}^p \times \{0\} \times \mathbf{R}^{p^2}$, $j^k_{\widetilde{\mathscr{F}_0}}(G_j)$ denotes the $k$-jet of the map $G_j$ in the direction of $\widetilde{\mathscr{F}_0}$, $\| \|_0$ in $\|j^k_{\widetilde{\mathscr{F}_0}}(G_j)\|_0$ denotes the canonical norm on the vector bundle of the $k$-jets of functions on $\mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{p^2}$ in the direction of $\widetilde{\mathscr{F}_0}$, $\mathscr{F}_0$ denotes the canonical foliation on $\mathbf{R}^p \times \mathbf{R}^{p^2}$ obtained by translating $\mathbf{R}^p \times \{0\}$, $j^k_{\mathscr{F}_0}(\overline{G}_j)$ denotes the $k$-jet of the map $\overline{G}_j$ in the direction of $\mathscr{F}_0$, and $\| \|_0$ in $\|j^k_{\mathscr{F}_0}(\overline{G}_j)\|_0$ denotes the canonical norm on the vector bundle of the $k$-jets of functions on $\mathbf{R}^p \times \mathbf{R}^{p^2}$ in the direction of $\mathscr{F}_0$. Thus the $\|\overline{G}_j\|_{1,k}^{V(0) \times V(e)}$ are uniformly bounded. We get the desired contradiction by applying the Sobolev Embedding Theorem: since we assumed $r \geq 1$ we have the following bounded injection:

$$L^{1,k}(V(0) \times V(e)) \hookrightarrow BC^1(V(0) \times V(e)).$$

Therefore the $\|d\overline{G}_j(c_j)\|$ should remain bounded, which contradicts ($*$). $\square$

**Lemma 4.5.4.** *There exists an* $\widetilde{a}'$*-invariant* $h \in C^r(P, \mathbf{R}) \cap L^1_{v'}(P)$ *such that* $h_m \neq 0$ *for all* $m \in M$.

*Proof.* By Lemma 4.5.3, for each $m_0 \in M$ there exists an $f^{m_0} \in L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbf{R})$ which is $\widetilde{a}'$-invariant and satisfies $(f^{m_0})_{m_0} \neq 0$. As $f^{m_0} \in L^{2,k}_{\widetilde{\mathscr{F}}, \widetilde{\eta}'_k}(P, \mathbf{R}) \subset C^r(P, \mathbf{R})$, necessarily $(f^{m_0})_m \neq 0$ for all $m \in U(m_0)$ where $U(m_0) \subset M$ is an open neighbourhood of $m_0$. As $M$ is compact there exists a finite covering $U(m_1), \ldots, U(m_j)$ of $M$ by such open neighbourhoods associated to $\widetilde{a}'$-invariant functions $f^{m_1}, \ldots, f^{m_j} \in C^r(P, \mathbf{R}) \cap L^2_{v'}(P)$, and clearly $h = (f^{m_1})^2 + \cdots + (f^{m_j})^2$ is a solution. $\square$

We now resume the proof of Proposition 4.5.1. We first apply Lemma 4.5.2 to the $\widetilde{a}'$-invariant function $h \in C^r(P, \mathbf{R}) \cap L^1_{v'}(P)$ yielded by Lemma

4.5.4: there exists $\lambda \in C^r(GL(p)) \cap L^1(GL(p))$ such that in any trivialisation of $P$ we have $h_m \in \Pi(GL(p))(\lambda)$ *for all* $m \in M$. Let $G_\lambda = \{g_0 \in GL(p) \mid \lambda(g_0 g) = \lambda(g)$ for all $g \in GL(p)\}$ be the stabilizer of $\lambda$ for $\Pi$. As $GL(p)/G_\lambda$ is a left $GL(p)$-space we can introduce the $GL(p)/G_\lambda$ fiber-bundle $P/G_\lambda \cong P \times_{GL(p)} (GL(p)/G_\lambda)$. An element $\bar{u}$ of $P/G_\lambda$ can be identified with a function on the fiber $P_m$ (by setting $f_m(g) = \lambda(g)$) where $m = \pi(u)$) which, in any trivialisation of the fiber $P_m \cong GL(p)$, belongs to $\Pi(GL(p))\lambda$. We can thus assert:

*The $\tilde{a}'$-invariant function $h$ on $P$ defines pointwise an $a'$-invariant section $\sigma' : M \to P/G_\lambda$. As $G_\lambda$ is compact, we will have proved the existence of an $a'$-invariant $C^r$-Riemannian metric on $T\mathscr{F}$ provided that we can show that $\sigma'$ is $C^r$ on $M$.*

Fix $m_0 \in M$. Let $\Phi : U(m_0) \times GL(p) \to \pi^{-1}(U(m_0))$ be a smooth trivialization of $\pi^{-1}(U(m_0))$ where $U(m_0)$ is an open neighbourhood of $m_0$ and let $\bar{\pi} : P/G_\lambda \to M$ denote the canonical projection of the fiber bundle $P/G_\lambda$. To the trivialization $\Phi$ corresponds the smooth quotient trivialization $\overline{\Phi} : U(m_0) \times GL(p)/G_\lambda \to \bar{\pi}^{-1}(U(m_0))$ defined by $\overline{\Phi}(m, \bar{g}) = \overline{\Phi(m, g)}$. We have just seen that for each $m \in U(m_0)$ there exists an $s(m) \in GL(p)$ such that

 (1)  $h(\Phi(m, g)) = \lambda(s(m)g)$,
 (2)  $h \circ \Phi \in C^r(U(m_0) \times GL(p))$,
 (3)  $(\overline{\Phi})^{-1} \circ \sigma'(m) = (m, \overline{s(m)})$.

We now want to use those three items to prove that the function $\bar{s} : U(m_0) \to GL(p)/G_\lambda$ defined by $\bar{s}(m) = \overline{s(m)}$ is $C^r$ on $U(m_0)$.

**Lemma 4.5.5.** $\bar{s}$ *is continuous on* $U(m_0)$.

*Proof.* Let $(m_n)_{n \in \mathbb{N}}$ be a sequence of points of $U(m_0)$ which converges to $m_0$. We want to show that $\bar{s}(m_n) \xrightarrow[n \to \infty]{} \bar{s}(m_0)$ (in $G/G_\lambda$). Assume that there exists a subsequence $(m_{n_p})_{p \in \mathbb{N}}$ such that $s(m_{n_p}) \xrightarrow[p \to \infty]{} \infty$. Then on one hand $\Pi(s(m_{n_p}))\lambda \xrightarrow[p \to \infty]{L^1_{loc}(GL(p))} 0$. On the other hand, fix $K \subset GL(p)$ a compact subset: by continuity of $h \circ \Phi$, we have $h(\Phi(m_{n_p}, g)) \xrightarrow[p \to \infty]{} f(\Phi(m_0, g))$ pointwise and as $|h(\Phi(m_{n_p}, g))| \leq C \in L^1(K)$ for all $p$ and all $g \in K$, by Lebesgue's Dominated Convergence Theorem $h(\Phi(m_{n_p}, g)) \xrightarrow[p \to \infty]{L^1_{loc}(GL(p))} h(\Phi(m_0, g))$, so that $h(\Phi(m_0, .)) = 0$ which contradicts the assumption $h_{m_0} \neq 0$. We can therefore assume without loss of generality that the $s(m_n)$ lie in a compact set (so that the $\overline{s(m_n)}$ are also contained in a compact set). It is then enough to show that any convergent subsequence $\overline{s(m_{n_p})}$ converges to $\overline{s(m_0)}$. Assume thus that $\overline{s(m_n)} \xrightarrow[n \to \infty]{} \bar{g}_0$. Then there exist $g_n \in GL(p)$, $g_n \xrightarrow[n \to \infty]{} g$ such that $\overline{g_n} = \overline{s(m_n)}$ and $\bar{g} = \bar{g}_0$ (take a continuous section $j$ of the canonical projection $\bar{\pi} : GL(p) \to GL(p)/G_\lambda$ in a neighbourhood of $\overline{s(m_0)}$ and set $g_n = j(\overline{s(m_n)})$). Then $\Pi(g_n)\lambda \xrightarrow[n \to \infty]{pointwise} \Pi(g)\lambda$. But by continuity of $h \circ \Phi$, we also have $\Pi(g_n)\lambda = \Pi(s(m_n))\lambda = h(\Phi(m_n, .)) \xrightarrow[n \to \infty]{pointwise} h(\Phi(m_{n_0}, .)) = \Pi(s(m_0))\lambda = \Pi(g)\lambda$, and thus $\bar{g} = \overline{s(m_0)} = \bar{g}_0$. $\square$

It remains to show that $\bar{s}$ is $C^r$ on $U(m_0)$. This follows from the following lemma:

**Lemma 4.5.6.** *Let $G$ be a connected Lie group, $\lambda \in C^r(G)$ $(1 \le r \le \infty)$ and $s : U \to G$ a map from an open subset $U \subset \mathbf{R}^n$ into $G$. Let $G_\lambda$ denote the closed Lie subgroup of $G$ defined by $G_\lambda = \{g_0 \in G \mid \forall g \in G, \lambda(g_0 g) = \lambda(g)\}$. Assume that*

    (1) *the quotient map $\bar{s} : U \to G/G_\lambda$ given by $m \to \overline{s(m)}$ is continuous,*

    (2) *the function $U \times G \to \mathbf{R}$ given by $(m, g) \to \lambda(s(m)g)$ is $C^r$.*

*Then the quotient map $\bar{s} : U \to G/G_\lambda$ given by $m \to \overline{s(m)}$ is also $C^r$.*

*Proof.* $\lambda$ being constant on the right cosets of $G_\lambda$ we can define $\bar{\lambda} : G\backslash G_\lambda \to \mathbf{R}$ by $\bar{g} \to \bar{\lambda}(\bar{g}) = \lambda(g)$. $\bar{\lambda}$ is also $C^k$ on $U$ (as we can look upon $G$ as a left $G_\lambda$-principal fiber bundle with base $G\backslash G_\lambda$ so that locally $\bar{\lambda}(\bar{g}) = \lambda(\Phi(\bar{g}))$ where $\Phi : W \subset G\backslash G_\lambda \to G$ is smooth). Set $d = \dim(G\backslash G_\lambda) = \dim(G) - \dim(G_\lambda)$. Given $g_1, g_2, \dots, g_d$ in $G$ we define a new $C^r$-function $\bar{\lambda}_{g_1 g_2 \dots g_d} : G\backslash G_\lambda \to \mathbf{R}^d$ by

$$\bar{\lambda}_{g_1 g_2 \dots g_d}(\bar{g}) = (\bar{\lambda}(\bar{g}g_1), \dots, \bar{\lambda}(\bar{g}g_d)).$$

**Sublemma.** *We can pick $g_1, g_2, \dots, g_d$ in $G$ in such a way that $\bar{\lambda}_{g_1 g_2 \dots g_d}$ is invertible in a neighbourhood of $\bar{e}$.*

*Proof of the sublemma.* Let $\mathfrak{g}_\lambda \subset \mathfrak{g}$ be the Lie algebra of $G_\lambda \subset G$. We can assume without loss of generality that $\lambda$ is nonconstant, i.e., $G_\lambda \ne G$. Fix an inner product $\langle\ \rangle$ on $\mathfrak{g}$, let $\langle\ \rangle$ also denote the corresponding right-invariant metric on $G$, and introduce the function $V : G \to \mathfrak{g}$ defined by

$$V(g) = R_g^*(\overrightarrow{\mathrm{grad}}\,\lambda(g))$$

(where $R_g$ denotes the right multiplication on $G$ and $\overrightarrow{\mathrm{grad}}$ is considered with respect to the right-invariant metric $\langle\ \rangle$ on $G$). We claim that

$$V(g) \in \mathfrak{g}_\lambda \iff \overrightarrow{\mathrm{grad}}\,\lambda(g) = 0 \iff V(g) = 0.$$

Indeed assume that $V(g) \in \mathfrak{g}_\lambda$. Then $\mathrm{Exp}(tV(g)) \in G_\lambda$ for all $t \in \mathbf{R}$. Thus

$$0 = \frac{d}{dt}\bigg|_{t=0} \lambda(\mathrm{Exp}(tV(g)).g) = d\lambda(g)\left[\frac{d}{dt}\bigg|_{t=0} \mathrm{Exp}(tV(g)).g\right]$$

$$= \langle \overrightarrow{\mathrm{grad}}\,\lambda(g), (R_g)_*(V(g))\rangle_{T_g G} = \|\overrightarrow{\mathrm{grad}}\,\lambda(g)\|^2.$$

We now show that there exist $g_1, g_2, \dots, g_d$ in $G$ such that $V(g_1), \dots, V(g_d)$ are linearly independent and satisfy $\mathfrak{g} = \mathfrak{g}_\lambda \oplus (V(g_1), \dots, V(g_d))$ where the second factor denotes the linear subspace spanned by $V(g_1), \dots, V(g_d)$: assume that $\{V(g) \in \mathfrak{g} \mid \overrightarrow{\mathrm{grad}}\,\lambda(g) \ne 0\}$ generates a subspace $\mathcal{H} = (V(g_1), \dots, V(g_k))$ with $k < d$. Then there exists a nontrivial $W \in (\mathfrak{g}_\lambda \oplus \mathcal{H})^\perp = \mathcal{H}^\perp \cap \mathfrak{g}_\lambda^\perp$. Let $W$ denotes the corresponding right-invariant vector field on $G$ and fix $g \in G$. We have

$$\frac{d}{dt}\bigg|_{t=t_0} \lambda(\mathrm{Exp}(tW).g) = \langle \overrightarrow{\mathrm{grad}}\,\lambda(\mathrm{Exp}(t_0 W)g), W(\mathrm{Exp}(t_0)g)\rangle_{T_{\mathrm{Exp}(t_0 W)g}G}$$

$$= \langle V(\mathrm{Exp}(t_0 W)g), W\rangle_{\mathfrak{g}} = 0$$

(by right-invariance of the metric $\langle \ \rangle$ and by definition of $W$). Since at $t = 0$ we have $\lambda(\exp(0W)g) = \lambda(g)$, for all $t \in \mathbb{R}$ we have $\mathrm{Exp}(tW) \in G_\lambda$ and thus $W \in \mathfrak{g}_\lambda$, which is a contradiction.

Now we see that $\bar{\lambda}_{g_1...g_d}$ is invertible in a neighbourhood of $\bar{e}$. Let $k^{V(g_i)}$ be the killing vector field on $G\backslash G_\lambda$ associated to $V(g_i)$ and the canonical right-action of $G$ on $G\backslash G_\lambda$. As $V(g_1), ..., V(g_d)$ is a basis of $\mathfrak{g}_\lambda^\perp \subset \mathfrak{g}$ and $dq|_{\mathfrak{g}_\lambda^\perp} : \mathfrak{g}_\lambda^\perp \to T_{\bar{e}}(G\backslash G_\lambda)$ is a linear isomorphism, $(k^{V(g_1)}(\bar{e}), ..., k^{V(g_d)}(\bar{e}))$ is a basis of $T_{\bar{e}}(G\backslash G_\lambda)$, and we have:

$$d(R_{g_i}\bar{\lambda})(k^{V(g_j)}(\bar{e})) = \frac{d}{dt}\Big|_{t=0} \bar{\lambda}(\bar{e}.[\mathrm{Exp}(tV(g_j))]g_i)$$

$$= \frac{d}{dt}\Big|_{t=0} \lambda([\mathrm{Exp}(tV(g_j))]g_i)$$

$$= \langle \overrightarrow{\mathrm{grad}\lambda}(g_i), [V(g_j)].g_i \rangle_{T_{g_i}G} = \langle V(g_i), V(g_j)\rangle_{\mathfrak{g}}$$

(by right-invariance of the metric $\langle \ \rangle$). As $\bar{\lambda}_{g_1...g_d} = (R_{g_1}\bar{\lambda}, ..., R_{g_d}\bar{\lambda})$, the matrix of $d\bar{\lambda}_{g_1...g_d}(\bar{e})$ expressed on the basis $k^{V(g_1)}(\bar{e}), ..., k^{V(g_d)}(\bar{e})$ of $T_{\bar{e}}(G\backslash G_\lambda)$ and the canonical basis of $\mathbb{R}^d$ is $(a_{ij}) = \langle V(g_i), V(g_j)\rangle_{\mathfrak{g}}$, which is invertible (as $V(g_i) \notin \mathfrak{g}_\lambda$ for each $i$ and $\mathfrak{g} = g_\lambda \oplus (V(g_1), ..., V(g_d)))$. $\square$

*Proof of Lemma* 4.5.6. Fix $m_0 \in U$ and let $W(\bar{e}) \subset G\backslash G_\lambda$ be a neighbourhood of $\bar{e}$ on which $\bar{\lambda}_{g_1...g_d}$ is invertible. As $G$ acts transitively on the left cosets of $G/G_\lambda$ there exists $g_0 \in G$ such that $g_0\overline{s(m_0)} = \beta(\bar{e})$ where $\beta : G\backslash G_\lambda \to G/G_\lambda$ is the canonical diffeomorphism defined by $\beta(G_\lambda g) = gG_\lambda$. As $\bar{s}$ is continuous on $U$ there exists $U'(m_0) \subset U$ such that $\beta^{-1}[g_0\bar{s}(U'(m_0))] \subset W(\bar{e}) \subset G\backslash G_\lambda$. By construction

$$\bar{\lambda}_{g_1...g_d}[\beta^{-1}(g_0\overline{s(m)})] = (\lambda(g_0s(m)g_1), ..., \lambda(g_0s(m)g_d))$$

is a $C^r$-function of $U(m_0)$. By the Inverse Function Theorem we can invert $\bar{\lambda}_{g_1...g_d}$ on $W(\bar{e})$ which implies that $m \mapsto \beta^{-1}(g_0\overline{s(m)}) \in C^r(U'(m_0))$ and thus that $m \mapsto \overline{s(m)} \in C^r(U(m_0))$. $\square$

## REFERENCES

1. Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monographs Math., vol. 81, Birkhäuser, Boston, 1984.

2. _____, *Lattices in semisimple groups and invariant geometric structures on compact manifolds*, Discrete Groups in Geometry and Analysis, Papers in Honor of G.D. Mostow on His Sixtieth Birthday, Prog. Math., vol. 67, Birkhäuser, Boston, 1987, pp. 152–210.

3. _____, *Ergodic theory and the automorphism group of a G-structure*, Group Representations, Ergodic Theory and Mathematical Physics (C. C. Moore, ed.), Springer-Verlag, Berlin and New York, 1987, pp. 247–278.

4. _____, *On the algebraic hull of an automorphism group of a principal bundle*, Comment Math. Helv. **65** (1990), 375–387.

5. Pierre de la Harpe et Alain Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts*, 175, Astérisque, Soc. Math. France, 1989.

6. A. Katok and J. Lewis, *Local rigidity for certain groups of toral automorphisms*, Preprint.

UNIVERSITÉ DE METZ, D. M. I., ILE DU SAULCY, 57045 METZ CEDEX 01, FRANCE
*E-mail address*: seydoux@dmi.univ-metz.fr