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CLASSIFICATIONS OF BAIRE-1 FUNCTIONS AND c_0-SPREADING MODELS

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ABSTRACT. We prove that if for a bounded function f defined on a compact space K there exists a bounded sequence (f_n) of continuous functions, with spreading model of order ξ, $1 \leq \xi < \omega_1$, equivalent to the summing basis of c_0, converging pointwise to f, then $r_{\mathrm{ND}}(f) > \omega^\xi$ (the index r_{ND} as defined by A. Kechris and A. Louveau). As a corollary of this result we have that the Banach spaces $V_\xi(K)$, $1 \leq \xi < \omega_1$, which previously defined by the author, consist of functions with rank greater than ω^ξ. For the case $\xi = 1$ we have the equality of these classes. For every countable ordinal number ξ we construct a function S with $r_{\mathrm{ND}}(S) > \omega^\xi$. Defining the notion of null-coefficient sequences of order ξ, $1 \leq \xi < \omega_1$, we prove that every bounded sequence (f_n) of continuous functions converging pointwise to a function f with $r_{\mathrm{ND}}(f) \leq \omega^\xi$ is a null-coefficient sequence of order ξ. As a corollary to this we have the following c_0-spreading model theorem: Every nontrivial, weak-Cauchy sequence in a Banach space either has a convex block subsequence generating a spreading model equivalent to the summing basis of c_0 or is a null-coefficient sequence of order 1.

INTRODUCTION

In the last few years various classifications of the class $B_1(K)$ of bounded Baire-1 functions on a compact metric space K were given by many authors (see [1, 7, 8]). Recently in [5] the class $B_1(K)$ was classified into a transfinite, decreasing hierarchy $V_\xi(K)$, $1 \leq \xi < \omega_1$, of Banach spaces. The first space coincides with $B_{1/4}(K)$, which was first defined in [7]; and the intersection of all $V_\xi(K)$ is equal to the space $\mathrm{DBSC}(K)$ of differences of bounded semicontinuous functions on K. As proved in [7] and [5], $f \in B_{1/4}(K)$ if and only if there exists a sequence (f_n) of continuous functions on K converging pointwise to f and generating a spreading model equivalent to the summing basis of c_0. Extending the notion of spreading models in [5], it is proved that the functions in $V_\xi(K)$ have a stronger property, namely, that there exists a sequence of continuous functions on K with spreading model of order ξ equivalent to the summing basis of c_0, converging pointwise to f.

A. Kechris and A. Louveau in [8] defined a natural rank r_{ND} on every bounded function f defined on a compact metric space K not in $\mathrm{DBSC}(K)$, which has values of the form ω^ξ for countable ordinals ξ [6] (by [8] all such ordinals are obtained). With a different terminology but equivalent formulation

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this index is used by H. Rosenthal in [9] to prove the important result: that every bounded sequence (f_n) of continuous functions on K converging pointwise to a function f not in $\mathrm{DBSC}(K)$ has a strongly summing subsequence. From this result and the characterization of functions in $\mathrm{DBSC}(K)$ given by C. Bessaga and A. Pelczynski [4], there follows the c_0-theorem of Rosenthal, namely, that every nontrivial, weak-Cauchy sequence in a Banach space has either a strongly summing subsequence or a convex block basis equivalent to the summing basis of c_0.

In this paper we give a relation between the rank r_{ND} and the functions which are pointwise limits of sequences of continuous functions with spreading model of order ξ, $1 \leq \xi < \omega_1$, equivalent to the summing basis of c_0. Namely, we prove (Theorem 9) that if for a bounded function f defined on a compact metric space K there exists a bounded sequence (f_n) of continuous functions on K, with spreading model of order ξ $(1 \leq \xi < \omega_1)$, equivalent to the summing basis of c_0, converging pointwise to f, then $r_{\mathrm{ND}}(f) > \omega^\xi$. As a corollary of this result we have that for every $1 \leq \xi < \omega_1$

$$V_\xi(K) \subseteq \{f \in B_1(K) \colon r_{\mathrm{ND}}(f) > \omega^\xi\}.$$

For the case $\xi = 1$ we have the equality of these classes. Finally, for every countable ordinal number ξ we construct a linear, Baire-1 function S on a compact metric space K which is not in $\mathrm{DBSC}(K)$ and prove that $r_{\mathrm{ND}}(S) > \omega^\xi$ using Theorem 9.

Defining the notion of null-coefficient sequences of order ξ, $1 \leq \xi < \omega_1$, we prove a result similar to Rosenthal's for the case of functions with rank less or equal to ω^ξ. Namely, we prove that every bounded sequence (f_n) of continuous functions converging pointwise to a function f with $r_{\mathrm{ND}}(f) \leq \omega^\xi$ $(1 \leq \xi < \omega)$ is null-coefficient of order ξ (Theorem 14). In particular (case $\xi = 1$) it is proved that $f \notin B_{1/4}(K)$ if and only if every bounded sequence of continuous functions converging pointwise to f is null-coefficient of order 1. As a corollary to this and the characterization of functions in $B_{1/4}(K) \setminus C(K)$ given in [5] we have the following c_0-spreading model theorem: Every nontrivial, weak-Cauchy sequence in a Banach space either has a convex block subsequence generating a spreading model equivalent to the summing basis of c_0 or is a null-coefficient sequence of order 1 (Theorem 18).

We will use standard terminology and notation. For completeness we will give some definitions and notation which we will use in the following.

Let K be a compact, metrizable space. The class of continuous functions on K is denoted by $C(K)$ and the class of Baire-1 functions on K (i.e., the pointwise limits of uniformly bounded sequences of continuous functions on K) by $B_1(K)$. $\mathrm{DBSC}(K)$ denotes the subclass of $B_1(K)$ consisting of differences of bounded semicontinuous functions. It is easy to see that

$$\mathrm{DBSC}(K) = \Big\{f \in B_1(K)\colon \text{ there exists } (f_n) \subseteq C(K) \text{ so that } f = \sum f_n \text{ and} \sum |f_n| \text{ is bounded}\Big\}.$$

The class $\mathrm{DBSC}(K)$ is a Banach space with respect to the norm

$$\|f\|_D = \inf\Big\{\Big\|\sum |f_n|\Big\|_\infty : (f_n) \subseteq C(K) \text{ and } \sum f_n = f\Big\}.$$

It is not hard to check that $\|f\|_\infty \le \|f\|_D$, but the two norms are not equivalent in general. The norm-closure of DBSC(K) is denoted by $B_{1/2}(K)$ in [7]. In the same paper the authors define the subclass $B_{1/4}(K)$ by

$$B_{1/4}(K) = \left\{ f \in B_1(K)\colon \text{ there exists } (f_n) \subseteq \mathrm{DBSC}(K) \right.$$
$$\left. \text{such that } \|f_n - f\|_\infty \to 0 \text{ and } \sup_n \|f_n\|_D < \infty \right\}.$$

The space $B_{1/4}(K)$ is complete with respect to the norm

$$\|f\|_{1/4} = \inf\left\{ \sup_n \|f_n\|_D\colon (f_n) \subseteq \mathrm{DBSC}(K) \quad \text{and} \quad \|f_n - f\|_\infty \to 0 \right\}.$$

In [5] this definition was extended in the transfinite as follows: Let

$$V_1(K) = B_{1/4}(K) \text{ and } \|\ \|_1 = \|\ \|_{1/4}.$$

If the normed space $(V_\xi(K), \|\ \|_\xi)$ has been defined, then

$$V_{\xi+1}(K) = \{ f \in B_1(K)\colon \text{ there exists } (f_n) \subseteq \mathrm{DBSC}(K)$$
$$\text{with } \|f_n - f\|_\xi \to 0 \text{ and } \sup \|f_n\|_D < \infty \}$$

and

$$\|f\|_{\xi+1} = \inf\left\{ \sup_n \|f_n\|_D\colon (f_n) \subseteq \mathrm{DBSC}(K) \text{ and } \|f_n - f\|_\xi \to 0 \right\}.$$

Finally, for a limit ordinal ξ

$$\|f\|_\xi = \sup\{\|f\|_\beta\colon 1 \le \beta < \xi\} \quad \text{for every } f \in \bigcap_{\beta<\xi} V_\beta(K)$$

and

$$V_\xi(K) = \{ f \in B_1(K)\colon \|f\|_\xi < \infty \}.$$

The spaces $(V_\xi(K), \|\ \|_\xi)$, $1 \le \xi < \omega_1$, are complete, and their intersection coincides with DBSC(K) [5]. It is easy to see that $V_\xi(K) \subseteq V_\beta(K)$ and $\|f\|_\infty \le \|f\|_\beta \le \|f\|_\xi$ for every $f \in V_\xi(K)$ and $\beta < \xi < \omega_1$. According to [7] and [5], the functions in $B_{1/4}(K)\backslash C(K)$ are characterized in terms of c_0-spreading models and the functions in $V_\xi(K)\backslash C(K)$ have an analogous stronger property. As we will need these results, we include a precise statement:

Let (x_n) be a seminormalized basic sequence in a Banach space X. A basic sequence (e_n) is said to be a spreading model of (x_n) if for every $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ so that if $m < n_1 < n_2 < \cdots < n_k$, then

$$\left| \left\| \sum_{i=1}^k \lambda_i x_{n_i} \right\| - \left\| \sum_{i=1}^k \lambda_i e_i \right\| \right| < \varepsilon \quad \text{for all scalars } \lambda_1, \ldots, \lambda_k \text{ with } \max_{1\le i<k} |\lambda_i| \le 1.$$

Every seminormalized basic sequence has a subsequence generating a spreading model.

If H, F are two finite subsets of $\mathbb{N}$, we denote $H < F$ iff $\max H < \min F$.

The summing basis (s_n) of c_0 is characterized by

$$\left\| \sum_{i=1}^\infty \lambda_i s_i \right\| = \sup_n \left| \sum_{i=1}^n \lambda_i \right|.$$

Definition 1 [1]. For every limit ordinal ξ, let (ξ_n) be a sequence of ordinal numbers strictly increasing to ξ. We define:

$\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\}$,

$\mathcal{F}_{\xi+1} = \{F \subseteq \mathbb{N} : F \subseteq F_1 \cup \cdots \cup F_n$ with $\{n\} \leq F_1 < \cdots < F_n$ and $F_i \in \mathcal{F}_\xi$, $i = 1, \ldots, . \}$,

and if ξ is a limit ordinal

$\mathcal{F}_\xi = \{F \subseteq \mathbb{N} : F \in \mathcal{F}_{\xi_n}$ and $n \leq \min F\}$.

Definition 2. Let X be a Banach space and (x_n) a sequence in X. We say that (x_n) has a spreading model of order ξ equivalent (or δ-equivalent) to the summing basis of c_0 if there exists $\delta > 0$ such that

$$(1/\delta)\left\|\sum_{i=1}^{k} \lambda_i s_i\right\|_\infty \leq \left\|\sum_{i=1}^{k} \lambda_i x_{n_i}\right\| \leq \delta \left\|\sum_{i=1}^{k} \lambda_i s_i\right\|_\infty,$$

for every $(n_1, \ldots, n_k) \in \mathcal{F}_\xi$ and scalars $\lambda_1, \ldots, \lambda_k$.

It is easy to see that a sequence (y_n) in X has a subsequence generating a spreading model equivalent to the summing basis of c_0 if and only if it has a subsequence with spreading model of order 1 equivalent to the summing basis of c_0.

Theorem 3 [5, 7]. *Let K be a compact metric space, f a real bounded function on K, and ξ a countable ordinal number. If $f \in V_\xi(K)\backslash C(K)$, then there exists a sequence $(f_n) \subseteq C(K)$, with spreading model of order ξ (for every choice of $(\mathcal{F}_\xi)$) equivalent to the summing basis of c_0, converging pointwise to f. Moreover, $f \in B_{1/4}(K)\backslash C(K)$ if and only if there exists $(f_n) \subseteq C(K)$, with spreading model (or order 1) equivalent to the summing basis of c_0, converging pointwise to f.*

In [8] the authors define a natural rank r_{ND} on every bounded function defined on a compact metric space K, as follows:

Let f be a bounded function on K. One defines the upper regularization of f, $\mathrm{ur}(f)$ (usually denoted by $\hat{f}$), by

$$\mathrm{ur}(f) = \inf\{g : g \in C(K) \text{ and } g \geq f\}.$$

The function $\mathrm{ur}(f)$ is upper semicontinuous, and one has

$$\mathrm{ur}(f)(x) = \overline{\lim_{y \to x}} f(y) = \max\{L \in [-\infty, \infty] : \exists x_n \to x, f(x_n) \to L\}$$
$$= \inf\left\{\sup_{y \in U} f(y) : U \text{ is a neighbourhood of } x\right\}.$$

In [8] the authors associate with each bounded function f an increasing sequence $(f_\xi)_{1 \leq \xi < \omega_1}$ of upper semicontinuous functions. In a different formulation (but equivalently) in [9] the author defines an increasing sequence $(u_\xi(f))_{1 \leq \xi < \omega_1}$ as

$$u_1(f) = \mathrm{ur}(\mathrm{ur}(f) - f).$$

If $u_\xi(f)$ is defined,

$$u_{\xi+1}(f) = \mathrm{ur}(\mathrm{ur}(u_\xi(f) + f) - f).$$

For a limit ξ, $u_\xi(f)$ is defined if and only if $u_\beta(f)$ is defined for all $\beta < \xi$ and $\sup_{\beta<\xi} u_\beta(f)$ is bounded, and then

$$u_\xi(f) = \mathrm{ur}\left(\sup_{\beta<\xi} u_\beta(f)\right).$$

According to [8], f is in DBSC(K) if and only if $u_\xi(f)$ is defined for all $\xi < \omega_1$ or, equivalently, if there exists a $\xi < \omega_1$ such that $u_\xi(f)$ is defined and $u_{\xi+1}(f) = u_\xi(f)$. Hence, to every bounded function f on K there corresponds a rank:

$$r_{\mathrm{ND}}(f) = \inf\{1 \le \xi < \omega_1 : u_\xi(f) \text{ is undefined}\}, \quad \text{if such a } \xi \text{ exists}$$

and $r_{\mathrm{ND}}(f) = \omega_1$ otherwise.

Note that the values of this rank are always limit ordinals. It is proved in [6] that if $f \notin \mathrm{DBSC}(K)$, then $r_{\mathrm{ND}}(f) = \omega^\xi$ for some $1 \le \xi < \omega_1$ (by [8] all such ordinals are obtained) according to the following lemma.

Lemma 4 [6]. *Let f be a bounded function on K, and suppose that $u_\xi(f)$ is defined. Then $u_{\xi\cdot n}(f)$ is defined and $\|u_{\xi\cdot n}(f)\|_\infty \le n\|u_\xi(f)\|_\infty$ for all $n \in \mathbb{N}$.*

Proof. Let $M = \|u_\xi(f)\|_\infty$. By induction $u_{\xi+\beta}(f)$ is defined and $M + u_\beta(f) \ge u_{\xi+\beta}(f)$ for every $\beta \le \xi$. Finally, $u_{\xi\cdot 2}(f)$ is defined and $\|u_{\xi\cdot 2}(f)\|_\infty \le 2\|u_\xi(f)\|_\infty$. The result then follows by induction on n.

In the proof of the main theorem we will use two lemmas which are proved in [9]. For completeness we give them below.

Lemma 5 [9]. *Let f be a bounded real function defined on a compact metric space K, ξ a countable ordinal number, and $x \in K$. Assume that $0 < u_\xi(f)(x) < u_{\xi+1}(f)(x) = M < \infty$. If U is an open neighborhood of x and $0 < \varepsilon < 1$, then there exist positive numbers λ, δ, and $x_1 \in U$ such that:*

(i) $(1-\varepsilon_1)M < \lambda + \delta < (1-\varepsilon_1)M$,

(**) (ii) $x_1 \in \mathrm{cl}(L)$, where $L = \{y \in K : \lambda \le u_\xi(f)(y) < (1-\varepsilon_1)M - \delta\}$,

(iii) $\overline{\lim}_{y\in L,\, y\to x_1} (f(y) - f(x_1)) = \delta.$

Lemma 6 [9]. *Let K be a compact metric space and $(f_n) \subseteq C(K)$ converging pointwise to a bounded function f. If $x_1 \in K$, L is a subset of K with $x_1 \in \mathrm{cl}(L)$, $\delta = \overline{\lim}_{y\in L,\, y\to x_1}(f(y) - f(x_1)) > 0$, $0 < \varepsilon < 1$, and U is an open neighborhood of x_1, then there exists a subsequence (f_{n_i}) of (f_n) such that given $t > 1$ there exists an $x_2 \in U \cap L$ satisfying:*

(i) $f(x_2) - f(x_1) > (1-\varepsilon)\delta$,

(*) (ii) $\sum_{1\le i<t} |f_{n_i}(x_2) - f(x_1)| < \varepsilon\delta$,

(iii) $\sum_{i\ge t} |f_{n_i}(x_2) - f(x_2)| < \varepsilon\delta.$

We will define for every countable ordinal number ξ a family $\mathscr{A}_\xi$ of finite subsets of $\mathbb{N}$ such that $\mathscr{A}_{\omega^\beta} = \mathscr{F}_\beta$ for every $1 \le \beta < \omega_1$.

Definition 7. Let $(\mathcal{F}_\xi)_{1\le\xi<\omega_1}$ be a family of finite subsets of $\mathbb{N}$ as described in Definition 1. We define:

$$\mathcal{A}_1 = \{F \subset \mathbb{N}: \#F = 2\},$$

$$\mathcal{A}_{\xi+1} = \{F \subseteq \mathbb{N}: F \subseteq F_1 \cup F_2 \text{ where } F_1 < F_2,\, F_1 \in \mathcal{A}_1, \text{ and } F_2 \in \mathcal{A}_\xi\}.$$

If ξ is a limit ordinal, then $\xi = \sum_{i=1}^{m} \rho_i\omega^{\beta_i}$, where $m, \rho_1, \dots, \rho_m \in \mathbb{N}$ and $\beta_1, \dots, \beta_m$ are ordinal numbers with $\beta_1 > \cdots > \beta_m > 0$. We define

$$\mathcal{A}_{\rho\omega^\beta} = \{F \subseteq \mathbb{N}: F \subseteq F_1 \cup \cdots \cup F_\rho \text{ with } F_1 < \cdots < F_\rho \text{ and } F_i \in \mathcal{F}_\beta \text{ for } i = 1, \dots, \rho\}$$

and in general

$$\mathcal{A}_\xi = \left\{F \subseteq \mathbb{N}: F \subseteq F_1 \cup F_2 \text{ with } F_1 < F_2,\, F_1 \in \mathcal{A}_\gamma, \text{ and } F_2 \in \mathcal{A}_\beta \text{ where } \gamma = \rho_m\omega^{\beta_m},\, \beta = \sum_{i=1}^{m-1} \rho_i\omega^{\beta_i}\right\}.$$

The following theorem is inspired by Theorem 4.1 of Rosenthal in [9].

Theorem 8. *Let f be a real function defined on a compact metric space K and (f_n) a uniformly bounded sequence of continuous functions converging pointwise to f. Let also ξ be a countable ordinal and $x \in K$ with $0 < u_\xi(f)(x) < \infty$. For every open neighborhood U of x and $0 < \varepsilon < 1$ there exists a subsequence (f_{n_t}) of (f_n) with the following properties: Given an infinite sequence of integers $1 \le t_1 < t_2 < \cdots$ there exists $F \in \mathcal{A}_\xi$, where $F = \{n_{t_1} < \cdots < n_{t_{2k}}\}$ $(k \in \mathbb{N})$, and $y \in U$ such that:*

(i) $f_{n_{t_{2i}}} - f_{n_{t_{2i-1}}}(y) > 0$ *for* $i = 1, \dots, k$ *and*

(ii) $\sum_{i=1}^{k} f_{n_{t_{2i}}} - f_{n_{t_{2i-1}}}(y) > (1-\varepsilon)u_\xi(f)(x)$.

Proof. The argument is similar to the proof of Theorem 4.1 in [9], except that additional work is required to locate F in $\mathcal{A}_\xi$.

Let $1 < \varepsilon < 0$ and U be an open neighborhood of x.

Case $\xi = 1$. Let $0 < \varepsilon_1 < 1$ with $(1-\varepsilon_1)(1-3\varepsilon_1) > 1-\varepsilon$ and $M = u_1(f)(x)$. According to the definition there exists $x_1 \in U$ with

$$(1-\varepsilon_1)M < \mathrm{ur}(f)(x_1) - f(x_1) = \delta < (1-\varepsilon_1)M.$$

From Lemma 6 there exists a subsequence (f_{n_t}) of (f_n) such that given $t > 1$ there exists $x_2 \in U$ satisfying $(*)$ (i)–(iii):

$$(*)\quad \begin{aligned} &\text{(i) } f(x_2) - f(x_1) > (1-\varepsilon_1)\delta, \\ &\text{(ii) } \sum_{1\le i<t} |f_{n_i}(x_2) - f(x_1)| < \varepsilon_1\delta, \\ &\text{(iii) } \sum_{i\ge t} |f_{n_i}(x_2) - f(x_2)| < \varepsilon_1\delta. \end{aligned}$$

Then given $1 \le t_1 < t_2$ there exists $x_2 \in U$ satisfying $(*)$ for $t = t_2$. Thus $F = \{n_{t_1}, n_{t_2}\} \in \mathcal{A}_1$ and

$$\begin{aligned} f_{n_{t_2}}(x_2) - f_{n_{t_1}}(x_2) &> f(x_2) - f(x_1) - 2\varepsilon_1\delta \\ &> (1-\varepsilon_1)\delta - 2\varepsilon_1\delta > (1-3\varepsilon_1)(1-\varepsilon_1)M > (1-\varepsilon)M. \end{aligned}$$

Case $\xi + 1$. Suppose the result is established for ξ. Let $0 < u_{\xi+1}(f)(x) = M < \infty$ and $0 < \varepsilon_1 < 1$ with $(1-\varepsilon_1)(1-3\varepsilon_1) > 1-\varepsilon$. We may assume that $0 < u_\xi(f)(x) < u_{\xi+1}(f)(x)$. Otherwise, if $0 < u_\xi(f)(x) = u_{\xi+1}(f)(x)$, the result follows by hypothesis and $u_\xi(f)(x) = 0$ is impossible.

According to Lemma 5 there exist $\lambda > 0$, $\delta > 0$, and $x_1 \in U$ satisfying $(**)$ (i)–(iii):

$$(**)\quad \begin{array}{l} \text{(i) } (1-\varepsilon_1)M < \lambda + \delta < (1-\varepsilon_1)M, \\ \text{(ii) } x_1 \in \operatorname{cl}(L), \text{ where } L = \{y \in K : \lambda \le u_\xi(f)(y) < (1-\varepsilon_1)M - \delta\}, \\ \text{(iii) } \overline{\lim}_{y\in L,\, y\to x_1} (f(y) - f(x_1)) = \delta. \end{array}$$

From Lemma 6 there exists a subsequence (f_{n_i}) of (f_n) such that given $t > 1$ there exists $x_2 \in U \cap L$ satisfying $(*)$ (i)–(iii). Without loss of generality we may assume that (f_n) itself has this property.

We will construct positive integers n_s, $s \in \mathbb{N}$, and infinite subsets M_s, $s \in \mathbb{N}$, of $\mathbb{N}$ satisfying $(***)$ (i)–(viii):

(i) $n_1 < \cdots < n_s < M_s$,
(ii) $M_s \subseteq M_{s-1}$,
(iii) $n_s = \min M_{s-1}$.

Given $r \in \mathbb{N}$ with $1 < r \le s$ there exist an open set $V \subseteq U$ and $x_2 \in V$ so that:

$(***)$

(iv) $f(x_2) - f(x_1) > (1-\varepsilon_1)\delta$,
(v) $\sum_{1\le i<r} |f_{n_i}(y) - f(x_1)| < \varepsilon_1\delta$ for every $y \in V$,
(vi) $\sum_{r\le i\le s} |f_{n_i}(y) - f(x_2)| < \varepsilon_1\delta$ for every $y \in V$,
(vii) $\lambda \le u_\xi(f)(x_2) < (1+\varepsilon_1)M - \delta$,
(viii) given $\{m_1, m_2, \dots\} \subseteq M_s$ with $1 \le m_1 < m_2 < \cdots$ there exists $y \in V$ and $F = \{m_1, m_2, \dots, m_{2k}\} \in \mathscr{A}_\xi$ $(k \in \mathbb{N})$ such that $f_{m_{2i}} - f_{m_{2i-1}}(y) > 0$, $i = 1, \dots, k$, and

$$\sum_{i=1}^{k} f_{m_{2i}} - f_{m_{2i-1}}(y) > (1-\varepsilon_1)u_\xi(f)(x_2).$$

Let $M_1 = \mathbb{N}\backslash\{1\}$, $n_1 = 1$, and $n_2 = 2$. We set $s = 2 = r$. As we assumed previously, there exists $x_2 \in U \cap L$ such that

$$f(x_2) - f(x_1) > (1-\varepsilon_1)\delta,\ |f_1(x_2) - f(x_1)| < \varepsilon_1\delta,\ \sum_{i\ge 2} |f_i(x_2) - f(x_2)| < \varepsilon_1\delta.$$

Using the continuity of f_1 and f_2 we can choose an open subset V of U with $x_2 \in V$ such that $|f_1(y) - f(x_1)| < \varepsilon_1\delta$ and $|f_2(y) - f(x_2)| < \varepsilon_1\delta$ for every $y \in V$. Finally, using the induction hypothesis we choose an infinite subset M_2 of $\mathbb{N}$ with $2 < M_2$ satisfying the conclusion of the theorem for the case ξ, $\varepsilon = \varepsilon_1$, $U = V$, and $x = x_2$. The proof for $s = 2 = r$ is complete.

Let $s \ge 2$, and suppose that $n_1, \dots, n_s, M_1, \dots, M_s$ have been constructed. Then $n_{s+1} = \min M_s$. We will construct infinite subsets $M^1, M^2, \dots, M^{s+1}$ of $\mathbb{N}$ such that $M_s\backslash\{n_{s+1}\} = M^1 \supseteq M^2 \supseteq \cdots \supseteq M^{s+1}$ and for every $1 < r \le s+1$ there is an open subset V of U and $x_2 \in V$ satisfying $(***)$ (iv)–(viii), where we replace "s" by "$s+1$" in (vi) and "M_s" by "M^r" in (viii). Once this is done we set $M_{s+1} = M^{s+1}$.

Let $1 < r \le s+1$, and suppose M^{r-1} is defined. Using the property of (f_n) we can find $x_2 \in U \cap L$ satisfying $(*)$ (i)–(iii) for $t = n_r$. Hence we have

$$f(x_2) - f(x_1) > (1-\varepsilon_1)\delta,$$
$$\sum_{1 \le i < r} |f_{n_i}(x_2) - f(x_1)| < \varepsilon_1 \delta, \qquad \sum_{r \le i \le s+1} |f_{n_i}(x_2) - f(x_2)| < \varepsilon_1 \delta.$$

Using the continuity of $f_{n_1}, \dots, f_{n_{s+1}}$ we can find an open subset V of U with $x_2 \in V$ satisfying $(***)$ (v) and $(***)$ (vi) with "s" replaced by "$s+1$". At last by the induction hypothesis we choose $M^r \subseteq M^{r-1}$ so that $(***)$ (viii) holds with "M_s" replaced by "M^r".

The sequence (f_{n_t}) satisfies the conclusion of the theorem for the case $\xi+1$. Indeed, let $1 \le r_1 < r_2 < t_1 < t_2 < \cdots$ be an infinite sequence of integers. We set $m_i = n_{t_i}$ for every $i \in \mathbb{N}$. Then $m_1 < m_2 < \cdots$ and $\{m_1, m_2, \dots\} \subseteq M_{t_1 - 1}$. Hence from $(***)$ there exist an open subset V of U and $x_2 \in V$ such that

$f(x_2) - f(x_1) > (1-\varepsilon_1)\delta$,
$|f_{n_{r_1}}(y) - f(x_1)| < \varepsilon_1\delta$, $|f_{n_{r_2}}(y) - f(x_2)| < \varepsilon_1\delta$ for every $y \in V$,
$\lambda \le u_\xi(f)(x_2) < (1+\varepsilon_1)M - \delta$.

Also there exist $y \in V$ and $F_2 = \{m_1, m_2, \dots, m_{2k}\} \in \mathscr{A}_\xi$ such that

$$f_{m_{2i}} - f_{m_{2i-1}}(y) > 0 \text{ for all } 1 \le i \le k \text{ and } \sum_{i=1}^{k} f_{m_{2i}} - f_{m_{2i-1}}(y) > (1-\varepsilon_1)u_\xi(f)(x_2).$$

Set $F = \{n_{r_1}, n_{r_2}\} \cup F_2 \in \mathscr{A}_{\xi+1}$. Then

$$f_{n_{r_1}} - f_{n_{r_2}}(y) > f(x_2) - f(x_1) - 2\varepsilon_1\delta > (1-\varepsilon_1)\delta - 2\varepsilon_1\delta > (1-3\varepsilon_1)\delta > 0$$

and

$$\begin{aligned} f_{n_{r_1}} - f_{n_{r_2}}(y) + \sum_{i=1}^{k} f_{n_{t_{2i}}} - f_{n_{t_{2i-1}}}(y) &> (1-3\varepsilon_1)\delta + (1-\varepsilon_1)u_\xi(f)(x_2) \\ &\ge (1-3\varepsilon_1)\delta + (1-\varepsilon_1)\lambda > (1-3\varepsilon_1)(\delta+\lambda) \\ &> (1-3\varepsilon_1)(1-\varepsilon_1)M > (1-\varepsilon)M. \end{aligned}$$

This finishes the proof of the theorem for the case $\xi + 1$.

Case ξ: limit ordinal. Suppose the theorem is proved for all ordinal numbers a with $a < \xi$. By the definition of $u_\xi(f)(x)$ there exist $x_1 \in U$ and $a < \xi$ such that:

$$(1-\varepsilon/2)u_\xi(f)(x) < u_a(f)(x_1) < (1+\varepsilon/2)u_\xi(f)(x).$$

In particular, if $\xi = \sum_{i=1}^{m} \rho_i \omega^{\beta_i}$, where $m, \rho_1, \dots, \rho_m$ are positive natural numbers and $\beta_1 > \beta_2 > \cdots > \beta_m > 0$ are countable ordinals numbers, then we can choose $\mu \in \mathbb{N}$ such that $a = \beta + \gamma$, where $\beta = \sum_{i=1}^{m-1} \rho_i \omega^{\beta_i}$ ($\beta = 0$ if $m = 1$) and $\gamma = (\rho_m - 1)\omega^{\beta_m} + \mu\omega^{\zeta}$ if $\beta_m = \zeta + 1$ or $\gamma = (\rho_m - 1)\omega^{\beta_m} + \omega^{\zeta_\mu}$ if β_m is a limit ordinal and (ζ_n) is the sequence of ordinal numbers strictly increasing to β_m.

Now, from the inductive hypothesis there exists a subsequence (f_{n_t}) of (f_n) such that $2\mu < n_1$ and given $t_1 < t_2 < \cdots$ an infinite sequence of integers there exists $k \in \mathbb{N}$ and $y \in U$ such that $F = \{n_{t_1}, \dots, n_{t_{2k}}\} \in \mathscr{A}_a$,

$$f_{n_{t_{2i}}} - f_{n_{t_{2i-1}}}(y) > 0 \quad \text{for } i = 1, \dots, k$$

and

$$\sum_{i=1}^{k} f_{n_{t_{2i}}} - f_{n_{t_{2i-1}}}(y) > (1-\varepsilon/2)u_a(f)(x_1) > (1-\varepsilon)u_\xi(f)(x).$$

We claim that $F \in \mathscr{A}_\xi$. Indeed, we have that $2\mu < F$. If $\xi = \omega$, then $F \in \mathscr{A}_\mu$ and since $\#F \leq 2\mu$ we have that $F \in \mathscr{F}_1 = \mathscr{A}_\omega$. If $\xi = \omega^{\zeta+1}$, then $F \in \mathscr{A}_{\mu\omega^\zeta}$ and since $F \subseteq F_1 \cup \cdots \cup F_\mu$, where $F_1 < \cdots < F_\mu$ and $F_i \in \mathscr{F}_\xi$ for all $i = 1, \dots, \mu$, we have that $F \in \mathscr{F}_{\zeta+1} = \mathscr{A}_\xi$. If $\xi = \omega^\beta$ and β is a limit ordinal, then if (β_n) is the sequence or ordinals increasing to β, we have $F \in \mathscr{F}_{\beta_\mu}$ and finally $F \in \mathscr{F}_\beta = \mathscr{A}_\xi$. Let $\xi = \rho\omega^\beta$, where $\rho \in \mathbb{N}$, $\rho > 1$, and $1 \leq \beta < \omega_1$. Then $F \in \mathscr{A}_\gamma$, where $\gamma = (\rho-1)\omega^\beta + \gamma_\mu$ with $\gamma_\mu = \mu\omega^\zeta$ if $\beta = \zeta+1$ or $\gamma_\mu = \omega^{\beta_\mu}$ if β is a limit ordinal. Since $F \subseteq F_1 \cup \cdots \cup F_\rho$, where $F_1 \in \mathscr{A}_{\gamma_\mu}$ and $F_2 < \cdots < F_\rho \in \mathscr{F}_\beta$, it follows, analogously to the previous cases, that $F_1 \in \mathscr{F}_\beta$ and finally that $F \in \mathscr{A}_\xi$. In general, if $\xi = \sum_{i=1}^m \rho_i\omega^{\beta_i}$ with $m > 1$, $\rho_1, \dots, \rho_m > 0$, and $\beta_1 > \cdots > \beta_m > 0$, then $F \in \mathscr{A}_{\beta+\gamma}$ and since $F \subseteq F_1 \cup F_2$, where $F_1 \in \mathscr{A}_\gamma$, $F_2 \in \mathscr{A}_\beta$, and $F_1 < F_2$, we have, analogously to the previous cases, that $F_1 \in \mathscr{A}_\zeta$, where $\zeta \in \rho_m\omega^{\beta_m}$ and finally that $F \in \mathscr{A}_\xi$. This completes the proof of the theorem.

From the previous theorem we have the main theorem:

Theorem 9. *Let f be a bounded function defined on a compact metric space K, let (f_n) be a uniformly bounded sequence of continuous functions converging pointwise to f, and let ξ be a countable ordinal number. If (f_n) has spreading model of order ξ equivalent to the summing basis of c_0, then $u_{\omega^\xi}(f)$ is defined, equivalently $r_{\mathrm{ND}}(f) > \omega^\xi$.*

Proof. Let (f_n) have spreading model of order ξ δ-equivalent (for some $\delta > 0$) to the summing basis of c_0, and suppose $u_{\omega^\xi}(f)$ is undefined. Let $r_{\mathrm{ND}}(f) = \omega^\zeta$, with $\zeta \leq \xi$, according to Lemma 4. Hence there exist $x \in K$ and a countable ordinal number a, with $a < \omega^\zeta$, such that $2\delta < u_a(f)(x) < \infty$. We can choose $\mu \in \mathbb{N}$ such that $a = \mu\omega^\beta$ if $\zeta = \beta+1$ or $a = \omega^{\zeta_\mu}$ if ζ is a limit ordinal and (ζ_n) is the sequence of ordinal numbers strictly increasing to ζ.

From the definition of the families $\mathscr{F}_\xi$, $1 \leq \xi < \omega_1$, it is easy to see that for every $\zeta < \xi < \omega_1$ there exists $v(\zeta, \xi) \in \mathbb{N}$ such that if $F \in \mathscr{F}_\zeta$ and $v(\zeta, \xi) < F$, then $F \in \mathscr{F}_\xi$ (see [2]).

Let $v = \max(v(\zeta, \xi), \mu)$. According to Theorem 8 there exist $F \in \mathscr{A}_a$ with $2v < F = \{n_1, \dots, n_{2k}\} (k \in \mathbb{N})$ and $y \in K$ such that

$$\sum_{i=1}^{k} f_{n_{2i}} - f_{n_{2i-1}}(y) > (1/2)u_a(f)(x) > \delta.$$

Since $2\mu < F$, we have that $F \in \mathscr{F}_\zeta$ (see the proof of Theorem 8, case ξ: limit ordinal). Consequently, since $v(\zeta, \xi) < F$, we have that $F \in \mathscr{F}_\xi$. This is a contradiction, because (f_n) has spreading model of order ξ δ-equivalent to the summing basis of c_0. Hence $u_{\omega^\xi}(f)$ is defined.

The following two corollaries are already proved in [6]. Here we give a proof using the previous theorem.

Corollary 10. *For every compact metric space K and countable ordinal number ξ we have $V_\xi(K) \subseteq \{f \in B_1(K) : r_{\mathrm{ND}}(f) > \omega^\xi\}$.*

Proof. This is true according to the previous theorem and Theorem 3.

For the case $\xi = 1$ the two classes are equal, according to the following:

Corollary 11. *Let K be a compact metric space and f a function on K which is not continuous. The following are equivalent*:

(i) $f \in B_{1/4}(K)$,
(ii) $r_{\mathrm{ND}}(f) > \omega$,
(iii) *there exists a bounded sequence $(f_n) \subseteq C(K)$ converging pointwise to f and generating a spreading model equivalent to the summing basis of c_0.*

Proof. The equivalence of (i) and (iii) is proved in [7] and [5]. According to the previous corollary (i) implies (ii). That (ii) implies (i) is proved in [6].

After these results the following interesting problem remains:

Problem. Is it true that for every compact metric space K and every ordinal number $\xi < \omega_1$ we have $V_\xi(K) = \{f \in B_1(K) : r_{\mathrm{ND}}(f) > \omega^\xi\}$?

For every countable ordinal number ξ we will construct a Baire-1 function which is not a difference of bounded semicontinuous functions and has rank greater than ω^ξ.

Example 12. For every countable ordinal ξ, let T_ξ be the Tsirelson-like space which is defined by S. Argyros in [2]. For completeness we recall the definition of T_ξ.

Let $x : \mathbb{N} \to \mathbb{R}$ be a finitely supported function. For every $m \in \mathbb{N}$ set

$$\|x\|_0^\xi = \sup\{|x(p)| : p \in \mathbb{N}\} \text{ and}$$

$$\|x\|_{m+1}^\xi = \max\left\{\|x\|_m^\xi, \frac{1}{2}\sup\sum_{i=1}^{k-1} \|x|p_i, p_{i+1}-1|\,\|_m^\xi \text{ for all } (p_1, \dots, p_k) \in \mathscr{B}_\xi\right\},$$

where $x|p, q|$ $(p \le q)$ denotes the restriction of x on the set $\{p, p+1, \dots, q\}$ and $\mathscr{B}_\xi = \mathscr{F}_\xi U\{(n, p) : 2 \le n < p\}U\{\varnothing\}$ for all $1 \le \xi < \omega_1$. Finally, define

$$\begin{aligned}\|x\|^\xi &= \varliminf_{m\to\infty} \|x\|_m^\xi \\ &= \max\left\{\|x\|_0^\xi, \sup\frac{1}{2}\sum_{i=1}^{k-1} \|x|p_i, p_{i+1}-1|\,\|^\xi \text{ for } \{p_1, \dots, p_k\} \in \mathscr{B}_\xi\right\}.\end{aligned}$$

The space T_ξ is the completion of the linear space of all finitely supported functions with the norm $\|\ \|^\xi$. The usual basis (e_n) is an unconditional basis of T_ξ and, as proved in [2], T_ξ is reflexive.

Let X_ξ be the "Jamesification" of T_ξ [3]. Let us recall the definition.

For every finitely supported function $x : \mathbb{N} \to \mathbb{R}$ define:

$$\|x\|_\xi = \sup\left\{\left\|\sum_{j=1}^m (S_{n_j} - S_{p_j - 1})(x)e_{p_j}\right\|^\xi : 1 \le p_1 \le n_1 \le \dots \le p_m \le n_m\right\},$$

where $S_n(x) = \sum_{i=1}^n x(i)$ for every $n \in \mathbb{N}$, and $S_0(x) = 0$. The space X_ξ is the completion of the linear space of all finitely supported functions with the norm $\|\ \|_\xi$.

As shown in [3] the unit vectors e_n, $n \in \mathbb{N}$, form a boundedly complete normalized basis for X_ξ. Thus, X_ξ is isometric to the space Y_ξ^*, where $Y_\xi = [e_n^*]_{n=1}^\infty$ and (e_n^*) is the sequence of biorthogonal functionals of (e_n). Furthermore it was shown in [3] that Y_ξ is quasi-reflexive (of order one) and Y_ξ^{**} has a basis given by $\{S, e_1^*, e_2^*, \dots\}$, where $S(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^\infty a_i$. Of course $S_n = \sum_{n=1}^n e_i^*$ for every $n \in \mathbb{N}$ and (S_n) converges to S in the w^*-topology. Hence S is a Baire-1 function restricted on $K = (S_{Y_\xi^*}, w^*)$.

Since c_0 is not isomorphically embedding into Y_ξ [3] we have that $S \notin \mathrm{DBSC}(K)$. We will prove that $r_{\mathrm{ND}}(S) > \omega^\xi$. Let $x \in K$ and $F = (n_1, \dots, n_{2k}) \in \mathcal{F}_\xi$ $(k \in \mathbb{N})$. From the definition of the norms and since $(n_1 + 1, \dots, n_{2k-1} + 1, r) \in \mathcal{F}_\xi$ for $r \in \mathbb{N}$ with $r > n_{2k}$ we have

$$1 \geq \|x\|_\xi \geq \left\| \sum_{i=1}^k (S_{n_{2i}} - S_{n_{2i-1}})(x) e_{n_{2i-1}+1} \right\|^\xi \geq \frac{1}{2} \sum_{i=1}^k \left| S_{n_{2i}}(x) - S_{n_{2i-1}}(x) \right|.$$

If $r_{\mathrm{ND}}(S) \leq \omega^\xi$, then we can find, analogously to the proof of Theorem 9 $(\delta = 2)$, $y \in K$ and $F = \{n_1, \dots, n_{2k}\} \in \mathcal{F}_\xi$ such that

$$\sum_{i=1}^k \left| S_{n_{2i}}(y) - S_{n_{2i-1}}(y) \right| > 2.$$

This is a contradiction; hence, $r_{\mathrm{ND}}(S) > \omega^\xi$.

In [9] H. Rosenthal proved the fundamental result that if $f \notin \mathrm{DBSC}(K)$, then every bounded sequence (f_n) in $C(K)$ converging pointwise to f has a strongly summing subsequence. In this article we obtain a result, in the same spirit as the above, concerning the classes:

$$\{f \in B_1(K) : r_{\mathrm{ND}}(f) \leq \omega^\xi\} \subseteq B_1(K) \setminus \mathrm{DBSC}(K), \qquad 1 \leq \xi < \omega_1.$$

This result requires the following new concept:

Definition 13. A sequence (x_n) in a Banach space is called null-coefficient (n.c.) of order ξ, where ξ is a countable ordinal number, if whenever the scalars (c_n) satisfy

$$\sup \left\{ \left\| \sum_{i=1}^k c_{n_{2i}} (x_{n_{2i}} - x_{n_{2i-1}}) \right\| : (n_1, \dots, n_{2k}) \in \mathcal{F}_\xi \right\} < \infty,$$

the sequence (c_n) converges to 0.

Remark. If a sequence (x_n) has spreading model of order ξ equivalent to the summing basis of c_0, then it is not null-coefficient. Indeed, take $c_n = 1$ for every $n \in \mathbb{N}$.

Theorem 14. *Let K be a compact metric space, f a bounded function on K, (f_n) a bounded sequence of continuous functions on K converging pointwise to f, and ξ a countable ordinal number. If $r_{\mathrm{ND}}(f) \leq \omega^\xi$, then (f_n) is null-coefficient of order ξ.*

Proof. Let $r_{\mathrm{ND}}(f) \leq \omega^\xi$. Then $r_{\mathrm{ND}}(f) = \omega^\zeta$ for some ordinal ζ with $\zeta \leq \xi$, according to Lemma 4. We assume that (f_n) is not a null-coefficient sequence of order ξ. Then there exists a sequence of scalars (c_n) and $\varepsilon > 0$ such that

$$\sup\left\{\left\|\sum_{i=1}^{k} c_{n_{2i}}(f_{n_{2i}} - f_{n_{2i-1}})\right\|_\infty : (n_1, \dots, n_{2k}) \in \mathcal{F}_\xi\right\} \leq 1$$

and $|c_n| > \varepsilon$ for infinite many n. Let (g_t) be a subsequence of (f_n) with $g_t = f_{n_t}$ and $c_{n_t} > \varepsilon$ for every $t \in \mathbb{N}$ (otherwise set $-c_n$ instead of c_n).

Since $r_{\mathrm{ND}}(f) = \omega^\zeta$, there exist $x \in K$ and $a < \omega^\zeta$ such that $2/\varepsilon < u_a(f)(x) < \infty$. We can choose $\mu \in \mathbb{N}$ such that $a = \mu\omega^\beta$ if $\zeta = \beta + 1$ or $a = \omega^{\zeta_\mu}$ if ζ is a limit ordinal and (ζ_n) is the sequence of ordinal numbers strictly increasing to ζ (according to Definition 1).

Let $v = \max(\mu, v(\zeta, \xi))$ (if $F \in \mathcal{F}_\zeta$ and $v(\zeta, \xi) < F$, then $F \in \mathcal{F}_\xi$). From Theorem 8, there exist $F \in \mathcal{A}_a$ with $2v < F = \{n_{t_1}, \dots, n_{t_{2k}}\}$ $(k \in \mathbb{N})$ and $y \in K$ such that $g_{t_{2i}} - g_{t_{2i-1}}(y) > 0$ for all $i = 1, \dots, k$ and

$$\sum_{i=1}^{k} g_{t_{2i}} - g_{t_{2i-1}}(y) > (1/2)u_a(f)(x) > 1/\varepsilon.$$

Then $F \in \mathcal{F}_\zeta$ (see the proof of Theorem 8, case ξ: limit ordinal) and consequently $F \in \mathcal{F}_\xi$. Also,

$$\sum_{i=1}^{k} c_{n_{t_{2i}}}(f_{n_{t_{2i}}} - f_{n_{t_{2i-1}}})(y) > 1.$$

This is a contradiction, since $(n_{t_1}, \dots, n_{t_{2k}}) \in \mathcal{F}_\xi$. Thus, (f_n) is null-coefficient of order ξ.

For the case $\xi = 1$, after Corollary 11, we have the following characterization of functions not in $B_{1/4}(K)$:

Theorem 15. *Let K be a compact metric space and $f \in B_1(K)\backslash C(K)$. Then f is not in $B_{1/4}(K)$ if and only if every uniformly bounded sequence of continuous functions on K converging pointwise to f is null-coefficient of order* 1.

Proof. If $f \in B_1(K)\backslash B_{1/4}(K)$, then $r_{\mathrm{ND}}(f) = \omega$ according to Corollary 11. From Theorem 14 we have that every bounded sequence $(f_n) \subseteq C(K)$ converging pointwise to f is null-coefficient of order 1. On the other hand, if every bounded sequence of continuous functions on K converging pointwise to f is null-coefficient of order 1, then according to the remark there is no bounded sequence (f_n) in $C(K)$ converging pointwise to f with spreading model (of order 1) equivalent to the summing basis of c_0. From Corollary 11, it follows that $f \notin B_{1/4}(K)$.

As a consequence of Theorems 3 and 15 we have the following dichotomy:

Theorem 16. *Let K be a compact metric space and $f \in B_1(K)\backslash C(K)$. Then, either there exists a bounded sequence $(f_n) \subseteq C(K)$ converging pointwise to f and generating a spreading model equivalent to the summing basis of c_0 or every uniformly bounded sequence of continuous functions converging pointwise to f is null-coefficient of order* 1.

Corollary 17. *Let K be a compact metric space, $f \in B_1(K)\backslash C(K)$, and (f_n) a bounded sequence in $C(K)$ converging pointwise to f. Then either there exists a* *convex block subsequence of* (f_n) *generating a spreading model equivalent to the summing basis of* c_0 *or every convex block subsequence of* (f_n) *is null-coefficient of order* 1.

Proof. If $f \in B_{1/4}(K)\backslash C(K)$, then, according to [7] and [5], (f_n) has a convex block subsequence generating a spreading model equivalent to the summing basis of c_0. If $f \notin B_{1/4}(K)$, then Theorem 15 finishes the proof.

Now we will give the c_0-spreading model theorem:

Theorem 18. *Every weak-Cauchy and non-weakly convergent sequence in a separable Banach space either has a convex block subsequence generating a spreading model equivalent to the summing basis of* c_0 *or is null-coefficient of order* 1 *(in fact, every convex block subsequence is null-coefficient of order* 1*)*.

Proof. Let X be a separable Banach space, and let K denote the unit ball of the dual space X^* endowed with the weak*-topology. If (x_n) is a weak-Cauchy and nonweakly convergent sequence in x, then let $x^{**} \in X^{**}\backslash X$ be the weak*-limit of (x_n). The restriction of x^{**} to K is in $B_1(K)\backslash C(K)$. Theorem 17 finishes the proof.

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