

## DYNAMICS NEAR THE ESSENTIAL SINGULARITY OF A CLASS OF ENTIRE VECTOR FIELDS

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**ABSTRACT.** We investigate the dynamics near the essential singularity at infinity for a class of zero-free entire vector fields of finite order, i.e., those of the form  $f(z) = e^{P(z)}$  where  $P(z) = z^d$  or  $P(z) = az^2 + bz + c$ . We show that the flow generated by such a vector field has a “bouquet of flowers” attached to the point at infinity.

### 0. INTRODUCTION

In this paper we consider the dynamics in a neighborhood of infinity of the flow generated by an entire vector field of the form  $f(z) = \exp(P(z))$ , where  $z \in \mathbb{C}$  and  $P(z) = z^d$  or  $P(z) = az^2 + bz + c$ . Notice that such entire functions have an essential singularity at infinity. Our motivation for examining this rather special looking class of vector fields comes from three sources. First we are interested in extending to entire vector fields the work of Benzinger [1991]. There a fairly complete description of the global dynamics of the flow generated by a rational vector field  $f(z) = p(z)/q(z)$  is given. As we shall discover, however, the dynamics near the essential singularity for the class of entire vector fields we consider here is rather more complicated than is the case for a rational vector field.

The choice of entire vector field studied here is motivated by the theory of entire functions (see, e.g., Boas [1954]). An entire function  $f$  is of *finite order*  $\rho$  if it satisfies the following growth condition. Let  $M(r) = \max_{|z| \leq r} |f(z)|$  and define  $\rho = \inf\{s \mid \limsup_{r \rightarrow \infty} r^{-s} \log M(r) < \infty\}$ . The required condition is  $\rho < \infty$ . Entire functions of finite order have many nice properties not shared by entire functions not in this class. For example, such functions can be represented in the form

$$f(z) = z^k e^{P(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{z_n}\right)^p\right),$$

where  $z_n \neq 0$ ,  $f(z_n) = 0$ ,  $p \leq \rho$ , and  $P(z)$  is a polynomial of degree at most  $\rho$ . Thus in this paper we consider the “leading order term” in the class of zero-free entire vector fields of finite order. We view this as a first step in

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understanding the global dynamics of the flow generated by an arbitrary entire vector field of finite order. Note that most of the “standard” examples of entire functions ( $\sin z$ ,  $\cos z$ ,  $\sinh z$ , etc.) are of finite order.

Finally, we are also interested in issues related to numerical analysis. In Hockett [1992] the global dynamics of the forward and backward Euler algorithms are analyzed in the case where the vector field is a complex polynomial. There techniques from complex analytic dynamics are used to show how to modify the standard algorithms so as to obtain mappings which provide global approximations to the flow. The same techniques, with minor modifications, may be applied to rational vector fields as well. An important element of the analysis involves understanding the dynamics of the flow near the poles of the vector field. In the setting of one complex variable it remains, therefore, to understand how to deal with the case of a vector field with essential singularities.

Our main tool for analyzing the dynamics near infinity is the procedure of “blowing up” a singularity. Roughly speaking one introduces a singular change of coordinates (perhaps more than once) which replaces a singular point in the vector field by a boundary onto which the original vector field may be smoothly extended (after, perhaps, a reparametrization of time). After a finite number of such blowups one has created a flow-invariant boundary containing isolated equilibria. An analysis of these equilibria provides a way of understanding the structure of the flow near the singularity of the original system. We shall discover that the isolated equilibria found after blowing up are degenerate, i.e., have an eigenvalue with zero real part. Thus we will need to make use of center manifold theory in order to understand the dynamics near these equilibria.

In Section 1 we provide a synopsis of those results about blowing up and center manifold theory which we need later. Section 2 is the main part of the paper in which we show that the flow generated by  $f(z) = \exp(z^d)$  near the essential singularity at infinity contains  $d$  “flowers”. We also show that the general quadratic case  $f(z) = \exp(az^2 + bz + c)$  has essentially the same dynamics as the special case  $f(z) = \exp(z^2)$ . We end, in Section 3, with some speculations about the general zero-free finite order case  $f(z) = \exp(P(z))$  where  $P$  is a monic polynomial of degree  $d$ .

## 1. BACKGROUND ON BLOWING UP AND CENTER MANIFOLDS

As indicated in the introduction, our main techniques for analyzing the dynamics of the flow near the essential singularity at infinity involve blowing up the singularity, then using center manifold theory to analyze the degenerate equilibria which we find after blowing up. It will be useful to discuss these techniques in general before applying them to the cases of interest here.

Sometimes the process of blowing up is described by saying that one introduces a singular change of coordinates which expands a singular point into a boundary onto which one extends the original vector field. This is somewhat inaccurate and a little confusing if one regards the coordinate change as taking place on a given fixed phase space. A more accurate description can be given in the following terms. For simplicity we discuss only the case of a planar vector field with a singularity at the origin.

Consider a dynamical system  $\dot{x} = F(x)$ ,  $x \in \mathbb{R}^2 - \{0\}$  where the vector field  $F$  is smooth except at the origin, where  $F$  has a singularity. The basic

idea behind blowing up is to find a diffeomorphism  $\Phi$  from the interior of a (possibly noncompact) manifold *with boundary*  $M$  into a neighborhood of the origin in  $\mathbb{R}^2$  such that  $\Phi(\partial M) = 0$ . One may pull back the vector field  $F$  via  $\Phi$  to define a vector field  $\Phi_*(F)$  in a neighborhood of  $\partial M$ . With an appropriate choice of the mapping  $\Phi$  it is often the case that the pullback  $\Phi_*(F)$  may be desingularized by reparametrizing time, so that the desingularized vector field then extends smoothly to  $\partial M$ . An analysis of the dynamics in a neighborhood of  $\partial M$  can be used to understand the dynamics near the origin in the plane.

As a simple example consider the system  $\dot{z} = 1/z^4$ ,  $z \in \mathbb{C} - \{0\}$ . Let  $(\theta, r)$  denote coordinates on the open cylinder  $C = S^1 \times (0, \infty)$  and consider the mapping from this cylinder to the punctured plane defined by  $z = \Phi(\theta, r) = re^{i\theta}$ . Notice that  $\Phi^{-1}$  is not well-defined at the origin, i.e., the point  $z = 0$  has been "blown up" into  $\partial C$ . The pullback by this mapping determines the following dynamical system on the open cylinder

$$\dot{\theta} = -\frac{1}{r^4} \sin 4\theta, \quad \dot{r} = \frac{1}{r^3} \cos 4\theta.$$

We may remove the singularity on  $\partial C$  by a reparametrization of time (this does not change the geometry of the trajectories of the flow) to obtain

$$\dot{\theta} = -\sin 4\theta, \quad \dot{r} = r \cos 4\theta.$$

This latter system on  $C$  extends smoothly to  $\partial C$  and in fact  $\partial C$  is invariant under the flow. There are eight hyperbolic equilibria at  $(\theta, r) = (k\pi/4, 0)$ ,  $k = 0, 1, 2, \dots, 7$ , with eigenvalues  $4(-1)^{k+1}$  and  $(-1)^k$ . Thus these equilibria on  $\partial C$  are all of saddle type. This analysis allows us to conclude that the origin is an 8-pronged (singular) saddle point for the original system.

The technique of blowing up a singularity appears in a variety of contexts in dynamical systems. See, for example, Devaney [1981, 1982], Chapter 7 of Guckenheimer and Holmes [1983] and Hockett [1992].

The other main tool in our analysis is center manifold theory. We collect here the main results which we will need later. Since our analysis takes place exclusively in two dimensions we shall state results in that setting only. For general background on invariant manifold theory and center manifolds see, e.g., Guckenheimer and Holmes [1983] or Carr [1981]. Consider the planar system

$$(1.1) \quad \dot{x} = f(x, y), \quad \dot{y} = \lambda y + g(x, y),$$

where  $\lambda \neq 0$  and  $f$  and  $g$  are smooth functions satisfying  $f(0, 0) = g(0, 0) = 0$  and  $Df(0, 0) = Dg(0, 0) = (0, 0)$ . Notice that the origin is an equilibrium point for (1.1) with eigenvalues 0 and  $\lambda$ . The first result concerns the existence of a local center manifold for this system.

**Theorem CM1.** *The system (1.1) has a local center manifold  $y = h(x)$ ,  $h(0) = h'(0) = 0$  defined on  $|x| < \delta$  for  $\delta$  sufficiently small. Moreover,  $h$  is  $C^2$  and the flow in the center manifold is governed by the equation*

$$(1.2) \quad \dot{u} = f(u, h(u)).$$

The next result shows how one may use properties of (1.2) to deduce properties of (1.1) near the origin.

**Theorem CM2.** *Suppose that the origin is stable (asymptotically stable) (unstable) for (1.2). Then the same stability property holds for the origin in (1.1). Moreover, if the origin is stable and  $(x(t), y(t))$  is a solution of (1.1) with  $(x(t_0), y(t_0))$  sufficiently small, then there exists a solution  $u(t)$  of (1.2) such that*

$$\begin{aligned}x(t) &= u(t - t_0) + O(e^{-\gamma(t-t_0)}), \\y(t) &= h(u(t - t_0)) + O(e^{-\gamma(t-t_0)})\end{aligned}$$

as  $t \rightarrow \infty$ .

Thus, in the stable case, solutions of the full system which start near the origin approach solutions in the center manifold exponentially fast. Notice that there is an analogous result which holds as  $t \rightarrow -\infty$  when the origin is a source.

The last result we need shows us how to approximate the local center manifold to any degree of accuracy. If we put  $y(t) = h(x(t))$  into the second of equations (1.1) we find that  $h$  must satisfy the equation  $(\mathcal{N}h)(x) = \lambda h(x) + g(x, h(x)) - h'(x)f(x, h(x)) = 0$ . This equation, together with the conditions  $h(0) = h'(0) = 0$  determine the local center manifold. Of course we cannot expect to solve this equation exactly since that is tantamount to solving the original system. On the other hand, we have the following result.

**Theorem CM3.** *Let  $\phi$  be a  $C^1$  real-valued mapping of a neighborhood of the origin in  $\mathbb{R}$  with  $\phi(0) = \phi'(0) = 0$  and satisfying  $(\mathcal{N}\phi)(x) = O(|x|^k)$ ,  $k > 1$ , as  $x \rightarrow 0$ . Then  $|h(x) - \phi(x)| = O(|x|^k)$  as  $x \rightarrow 0$ .*

## 2. FLOWERS NEAR INFINITY

Let us now consider the dynamics near infinity of the flow generated by the system

$$(2.1) \quad \dot{z} = \exp(z^d).$$

Later we will show that the dynamics near infinity for  $\dot{z} = \exp(az^2 + bz + c)$  are essentially the same as those of (2.1) when  $d = 2$ .

We begin by introducing some terminology. Let  $\gamma(t) = x(t) + iy(t)$ ,  $t \in (t_0, t_1)$ , be a smooth parametrized curve in the plane without self-intersections such that  $\lim_{t \rightarrow t_0^+} \gamma(t) = \lim_{t \rightarrow t_1^-} \gamma(t) = z_0$ . We call  $\gamma$  a *petal* (with root  $z_0$  and direction  $\theta$ ) if in addition

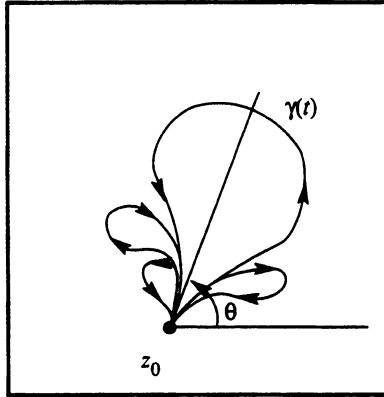
$$\theta = \lim_{t \rightarrow t_0^+} \arg \left( \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} \right) = \pi \pm \lim_{t \rightarrow t_1^-} \arg \left( \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} \right),$$

where all angles are chosen to lie in the interval  $(-\pi, \pi]$ . Thus the trace of  $\gamma$  has a cusp at  $z_0$ . See Figure 0. The *interior* of a petal is the open bounded region in the plane determined by the Jordan curve  $\gamma((t_0, t_1)) \cup \{z_0\}$ . A *flower* is a countable collection of petals with pairwise disjoint interiors and a common root and direction.

**Theorem 2.1.** *The flow generated by (2.1) has a "bouquet" of  $d$  flowers with root the point at infinity and directions  $k\pi/d$ ,  $k$  odd.*

*Proof.* First we change coordinates to a neighborhood of infinity via  $w = 1/z$  to obtain the system

$$(2.2) \quad \dot{w} = -w^2 \exp(1/w^d)$$

FIGURE 0. A flower with root  $z_0$  and direction  $\theta$ .

on the punctured plane  $\mathbb{C} - \{0\}$ . We now perform the first of our blowups. Let  $(\theta, r)$  be coordinates on the (open) cylinder  $C = S^1 \times (0, \infty)$ , so that  $w = re^{i\theta}$  defines a conformal diffeomorphism between  $C$  and the punctured plane  $\mathbb{C} - \{0\}$ . We now pull back the vector field of (2.2) to  $C$  via this diffeomorphism, i.e.,

$$(\dot{r} + i r \dot{\theta})e^{i\theta} = -r^2 e^{2i\theta} \exp\left(\frac{1}{r^d} e^{-i d \theta}\right)$$

which may be written in the form

$$\begin{aligned} (2.3) \quad \dot{r} + i r \dot{\theta} &= -r^2 e^{i\theta} \exp\left(\frac{1}{r^d} e^{-i d \theta}\right) \\ &= -r^2 e^{r^{-d} \cos(d\theta)} e^{i(\theta - r^{-d} \sin(d\theta))}. \end{aligned}$$

We now observe that the term  $e^{r^{-d} \cos(d\theta)}$  is a positive real-valued function which we can remove by a reparametrization of time. Thus we consider instead the following system whose trajectories are the same as those of (2.3) (but whose parametrization by time is different):

$$(2.4) \quad \dot{r} + i r \dot{\theta} = -r^2 e^{i(\theta - r^{-d} \sin(d\theta))}.$$

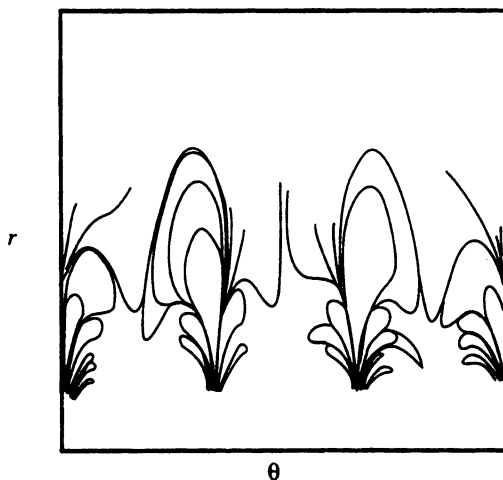
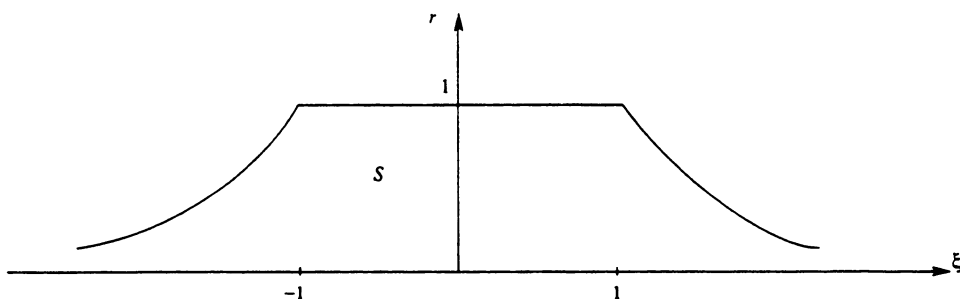
Taking the real and imaginary parts of (2.4), we obtain on the cylinder  $C$  the system

$$\begin{aligned} (2.5) \quad \dot{\theta} &= -r \sin(\theta - r^{-d} \sin(d\theta)), \\ \dot{r} &= -r^2 \cos(\theta - r^{-d} \sin(d\theta)). \end{aligned}$$

Notice that the vector field of (2.5) can be extended to the boundary  $\partial C = S^1 \times \{0\}$ , although the extended vector field is not smooth. This lack of smoothness will not be of concern. Figure 1 shows trajectories of (2.5) for the case  $d = 3$ .

We shall be interested in solutions in a neighborhood of the points  $(\theta, r) = (k\pi/d, 0)$ ,  $k = 0, \pm 1, \dots, \pm d$ . We will perform a second blowing up in a neighborhood of each of these points by means of a mapping from an infinite strip into a neighborhood of the point  $(k\pi/d, 0)$  in  $\partial C$ .

**Lemma 2.2.** *Let  $\xi$  be any real number. The equation  $r^{-d} \sin(d\theta) = \xi$  has a unique solution  $\theta(r)$  in a neighborhood  $U_k$  of each of the points  $(\theta, r) =$*

FIGURE 1. The flow for (2.5) in the case  $d = 3$ .FIGURE 2. The infinite strip  $S$ .

$(k\pi/d, 0)$ ,  $k = 0, \pm 1, \dots, \pm d$ , such that  $\lim_{r \rightarrow 0^+} \theta(r) = k\pi/d$  and  $\lim_{r \rightarrow 0^+} \theta'(r) = 0$ .

*Proof of Lemma 2.2.* Let  $\arcsin_k$  denote that branch of  $\arcsin$  such that  $\arcsin 0 = k\pi$ . Then  $\theta(\xi, r) = \arcsin_k(r^d \xi)/d$  has the desired properties.  $\square$

Let  $S = \{(\xi, r) \mid \xi \in \mathbb{R} \text{ and } 0 < r < 1 \text{ if } |\xi| \leq 1, 0 < r < 1/|\xi|^{1/d} \text{ otherwise}\}$  (see Figure 2). The conditions on  $r$  insure that  $r^d \xi$  lies in the domain of  $\arcsin_k$ . Then the mapping  $\Phi: S \rightarrow U_k$  given by  $\Phi(\xi, r) = (\arcsin_k(r^d \xi)/d, r)$  is a diffeomorphism from  $S$  into  $U_k$ . Note that Lemma 2.2 shows that the image under  $\Phi$  of the curves  $\xi = \text{const.}$  meet at  $(k\pi/d, 0)$  and are normal to  $\partial C$  at those points. See Figure 3. Thus the point  $(k\pi/d, 0)$  is blown up into  $\partial S$  and  $\Phi(S \cup \partial S) = U_k \cup \{(k\pi/d, 0)\}$ . Recall that  $k$  is a fixed integer between  $-d$  and  $d$ .

*Remark.* One may well wonder why we do not attempt to perform our blowup near  $(k\pi/d, 0)$  using polar coordinates as we just did above. It will be a consequence of the remainder of the proof of Theorem 2.1 that all of the trajectories which impinge on points of  $\partial C$  in either forward or backward time meet  $\partial C$

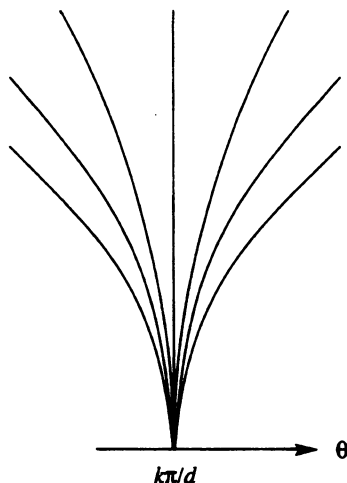


FIGURE 3. The image  $\Phi(c, r) \in U_k$  of coordinate curves  $\xi = c$  in  $S$ .

at right angles. Thus the use of polar coordinates is not helpful.

**Lemma 2.3.** *Let  $\gamma(t) = (\xi(t), r(t))$ ,  $t \in (t_0, t_1)$  be a smooth curve in  $S$  such that  $\lim_{t \rightarrow t_1^-} \gamma(t) \in \partial S$  and  $\lim_{t \rightarrow t_1^-} \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = (\alpha, \beta)$  with  $\beta \neq 0$ . Then the curve  $\eta(t) = \Phi \circ \gamma(t)$  satisfies  $\lim_{t \rightarrow t_1^-} \eta(t) = k\pi/d$  and  $\lim_{t \rightarrow t_1^-} \frac{\dot{\eta}(t)}{|\dot{\eta}(t)|} = (0, \pm 1)$ , i.e.,  $\eta(t)$  meets  $\partial C$  at  $(k\pi/d, 0)$  and is normal to  $\partial C$  at that point.*

*Proof of Lemma 2.3.* The proof is a straightforward computation.  $\square$

Now, pulling back the vector field in (2.5) by  $\Phi$  we obtain the system

$$\begin{aligned}\dot{\xi} &= dr^{-(d-1)} \sin \left( \frac{d-1}{d} \arcsin_k(\xi r^d) + \xi \right), \\ \dot{r} &= -r^2 \cos \left( \frac{1}{d} \arcsin_k(\xi r^d) - \xi \right)\end{aligned}$$

on  $S$ . Reparametrizing time we remove the singularity at  $r = 0$  in the first of these equations to arrive at the system

$$\begin{aligned}(2.6) \quad \dot{\xi} &= d \sin \left( \frac{d-1}{d} \arcsin_k(\xi r^d) + \xi \right), \\ \dot{r} &= -r^{d+1} \cos \left( \frac{1}{d} \arcsin_k(\xi r^d) - \xi \right).\end{aligned}$$

Now (2.6) defines a smooth vector field on the *half-closed* infinite strip  $S \cup \partial S$ . Thus we have “blown up” the point  $(k\pi/d, 0)$  on  $\partial C$  into  $\partial S$  (a copy of  $\mathbb{R}^1$ ) onto which the vector field of (2.6) extends *smoothly*. Setting  $r = 0$  in (2.6) we find there are equilibria at  $\xi_{kj} = (j - \frac{d-1}{d}k)\pi$ ,  $j \in \mathbb{Z}$  (recall that  $k$  is fixed). The dynamics within the boundary are easily seen to be as in Figure 4.

The linearization of (2.6) at the equilibrium point  $(\xi_{kj}, 0)$  is

$$\begin{bmatrix} \dot{\xi} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} d(-1)^j & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ r \end{bmatrix}$$

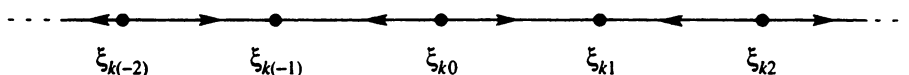


FIGURE 4. Dynamics in the invariant boundary at  $(\theta, r) = (k\pi/d, 0)$ .

with eigenvalues  $d(-1)^j$  and 0. Theorem CM1 of Section 1 guarantees that there is a smooth center manifold tangent to the center eigenspace  $E^c = \text{span}\{(0, 1)\}$  at the point  $(\xi_{kj}, 0)$  in  $\partial S$ . We seek to approximate the center manifold near  $(\xi_{kj}, 0)$ . We begin by shifting coordinates in (2.6) via  $\xi = u + \xi_{kj}$  to obtain

$$(2.7) \quad \begin{aligned} \dot{u} &= d \sin \left( \frac{d-1}{d} \arcsin_k \left( (u + \xi_{kj})r^d \right) + u + \xi_{kj} \right), \\ \dot{r} &= -r^{d+1} \cos \left( \frac{1}{d} \arcsin_k \left( (u + \xi_{kj})r^d \right) - u - \xi_{kj} \right). \end{aligned}$$

If we let  $u = h(r)$  be the equation of the (local) center manifold near the origin (with the boundary conditions  $h(0) = h'(0) = 0$ ) and substitute this into (2.7), we find that this local center manifold must satisfy the equation

$$\begin{aligned} (\mathcal{N}h)(r) &= r^{d+1}h'(r) \cos \left( \frac{1}{d} \arcsin_k \left( (h(r) + \xi_{kj})r^d \right) - h(r) - \xi_{kj} \right) \\ &\quad + d \sin \left( \frac{d-1}{d} \arcsin_k \left( (h(r) + \xi_{kj})r^d \right) + h(r) + \xi_{kj} \right) = 0. \end{aligned}$$

We now approximate the center manifold near the origin by  $h(r) = ar^d + O(r^{d+1})$ , yielding  $(\mathcal{N}h)(r) = d(\frac{d-1}{d}\xi_{kj} + a)r^d + O(r^{d+1}) = 0$ . Thus  $h(r) = -\frac{d-1}{d}\xi_{kj}r^d + O(r^{d+1})$ . By Theorem CM3,  $h(r)$  differs from the actual center manifold by an amount  $O(r^{d+1})$  as  $r \rightarrow 0$ .

*Remark.* It is very important in computing the above expansion of  $(\mathcal{N}h)(r)$  to remember that  $\arcsin_k(0) = k\pi$ . Thus, to leading order in  $r$ ,

$$\arcsin_k((h(r) + \xi_{kj})r^d) = k\pi + \xi_{kj}r^d + O(r^{d+1}).$$

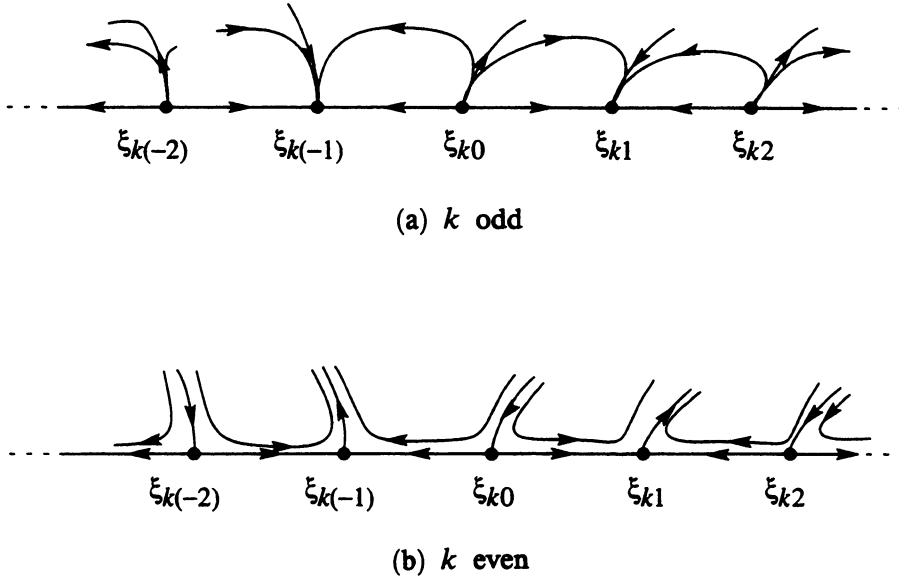
To obtain (approximately) the dynamics in the center manifold we plug our approximation to  $h(r)$  into the second of equations (2.7) to obtain

$$(2.8) \quad \begin{aligned} \dot{r} &= -r^{d+1} \cos \left( \frac{1}{d} \arcsin_k \left( (h(r) + \xi_{kj})r^d \right) - h(r) - \xi_{kj} \right) \\ &= -r^{d+1} \cos((k-j)\pi) + O(r^{d+2}) \\ &= (-1)^{1+k-j} r^{d+1} + O(r^{d+2}) \end{aligned}$$

Recall now the phase portrait in the boundary  $r = 0$  of Figure 4. Equation (2.8) shows that the normal dynamics at the points  $\xi_{kj}$  depend on whether  $k$  is even or odd. See Figure 5.

Thus we see that the equilibria in  $\partial S$  are alternately sources and sinks when  $k$  is odd, and saddle points when  $k$  is even.



FIGURE 5. Dynamics in the center manifold near  $\xi_{kj}$ .

Let us examine the case when  $k$  is odd more closely. First observe that since each equilibrium point in this case is either a source or a sink, there are trajectories which connect adjacent equilibria as in Figure 5(a). Notice also that  $(\xi_{kj}, 0)$  is a source if  $j$  is even, a sink if  $j$  is odd. Let  $\gamma_l(t) = (\xi(t), r(t))$  be such a connecting trajectory, say connecting  $(\xi_{k(2l)}, 0)$  to  $(\xi_{k(2l+1)}, 0)$ . That is

$$\lim_{t \rightarrow -\infty} \gamma_l(t) = (\xi_{k(2l)}, 0) \quad \text{and} \quad \lim_{t \rightarrow +\infty} \gamma_l(t) = (\xi_{k(2l+1)}, 0).$$

By Theorem CM2 there exists a neighborhood  $V$  of  $(\xi_{k(2l+1)}, 0)$  such that any trajectory entering  $V$  approaches a trajectory in the center manifold exponentially fast. Let  $t_0 > 0$  be chosen so that  $\gamma_l(t) \in V$  for  $t > t_0$ . Hence there exists  $u_l(t)$  such that

$$(2.9) \quad \gamma_l(t) = (u_l(t - t_0), h(u_l(t - t_0))) + O(e^{-\gamma(t-t_0)})$$

for  $t > t_0$ . Let  $c_l(t) = (u_l(t - t_0), h(u_l(t - t_0)))$ ,  $t > t_0$ , denote the trajectory in the center manifold at  $(\xi_{k(2l+1)}, 0)$ . Then  $c_l(t)$  is normal to  $\partial S$  at that point, i.e.,  $\lim_{t \rightarrow \infty} \frac{\dot{c}_l(t)}{|c_l(t)|} = (0, 1)$ . Hence by (2.9) we also have  $\lim_{t \rightarrow \infty} \frac{\dot{\gamma}_l(t)}{|\gamma_l(t)|} = (0, 1)$  as well. A similar argument may be used, letting  $t \rightarrow -\infty$ , to conclude that  $\gamma_l(t)$  is normal to  $\partial S$  at  $(\xi_{k(2l)}, 0)$  as well. Now the curve  $\eta_l(t) = \Phi \circ \gamma_l(t)$  is, except for its parametrization by time, a trajectory of the system (2.5). Since  $\gamma_l(t)$  is normal to  $\partial S$  at  $(\xi_{k(2l)}, 0)$  and  $(\xi_{k(2l+1)}, 0)$ , we have by Lemma 2.3 that  $\eta_l(t)$  satisfies the following:

(a)  $\eta_l(t)$ ,  $t \in \mathbb{R}$ , is a smooth curve without self-intersections such that  $\lim_{t \rightarrow \pm\infty} \eta_l(t) = k\pi/d$ .

(b)  $\lim_{t \rightarrow \pm\infty} \frac{\dot{\eta}_l(t)}{|\dot{\eta}_l(t)|} = (0, \mp 1)$ .

(c) Since  $\Phi$  is a diffeomorphism the curves  $\eta_l(t)$ ,  $l \in \mathbb{Z}$ , have disjoint interiors.

Thus (2.5) has flowers with root at each of the points  $(k\pi/d, 0)$  and direction  $\pi/2$ . The image of each flower under the conformal map  $(\theta, r) \mapsto w = re^{i\theta}$  is a flower with root 0 and direction  $k\pi/d$ . Thus (2.2) has a “bouquet” of flowers with root  $w = 0$  (the point at infinity) and directions  $k\pi/d$ .  $\square$

Let us briefly discuss the system  $\dot{z} = \exp(az^2 + bz + c)$  where  $a, b, c \in \mathbb{C}$ ,  $a \neq 0$ . We begin by showing that this can be reduced, by a sequence of coordinate changes, to the system  $\dot{z} = \exp(z^2 + C)$ . First we change coordinates via  $z = w/\sqrt{a}$  to obtain  $\dot{w} = \exp(w^2 + Bw + c_1)$  where  $B = b/\sqrt{a}$  and  $c_1 = c + \log(a)/2$ . Next we let  $w = Z - B/2$  to obtain  $\dot{Z} = \exp(Z^2 + C)$ , where  $C = c_1 - B^2/4$ . Thus we consider instead the system

$$(2.10) \quad \dot{z} = \exp(z^2 + C), \quad C \in \mathbb{C}.$$

We now change coordinates to a neighborhood of infinity via  $w = 1/z$  to obtain

$$(2.11) \quad \dot{w} = -w^2 \exp(1/w^2 + C)$$

on  $\mathbb{C} - \{0\}$ . Using polar coordinates to pull this system back to the cylinder, yields (after an appropriate rescaling of time)

$$(2.12) \quad \begin{aligned} \dot{\theta} &= -r \sin \left( \theta + \beta - \frac{1}{r^2} \sin(2\theta) \right), \\ \dot{r} &= -r^2 \cos \left( \theta + \beta - \frac{1}{r^2} \sin(2\theta) \right), \end{aligned}$$

where  $\beta = \text{Im}(C)$ . Finally, in a neighborhood of the point  $(\theta, r) = (k\pi/2, 0)$  we use the mapping  $\Phi: S \rightarrow U_k$  to pull this system back to the strip  $S$ , so that (after an appropriate reparametrization of time) we have

$$(2.13) \quad \begin{aligned} \dot{\xi} &= 2 \sin(\arcsin_k(\xi r^2)/2 + \xi - \beta), \\ \dot{r} &= -r^3 \cos(\arcsin_k(\xi r^2)/2 - \xi + \beta). \end{aligned}$$

This system has equilibria at the points  $\xi_{kj} = (j - \frac{k}{2})\pi + \beta$ , and a center manifold calculation similar to that in the proof of Theorem 2.1 establishes that the dynamics near these equilibria is as in Figure 5. Thus (2.10) has flowers with roots at the point at infinity with directions  $\pm\pi/2$ .

### 3. CONCLUSION

There are several global questions that should ultimately be addressed in light of our local analysis. First, it is interesting to consider the role of the center manifolds in the above proof. Notice that the petals in the flowers are not made up of the center manifolds themselves but, rather, trajectories which connect the equilibria associated with adjacent center manifolds. Notice also that any given petal gives rise to an uncountable collection of “subpetals” lying in the interior of the given petal. We conjecture that (some) of the *global* center manifolds project under  $\Phi$  to the *envelope* of a given group of petals. See Figure 1. On the other hand, numerical experimentation suggests that the projection of some of these center manifolds are the envelope of trajectories which connect distinct flowers in the bouquet. For example, in Figure 1 there are (numerical)

trajectories which connect the flower with direction  $\pi/3$  to the flower with direction  $\pi$ .

It is interesting that the general quadratic case exhibits (up to a conformal change of coordinates in the punctured plane) essentially the same dynamics as the special case  $P(z) = z^2$ . We conjecture, on the basis of numerical experiments, that this is true of the general polynomial case as well. However, the techniques of this paper appear to be a rather cumbersome way to approach this.

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