

UNIVALENT FUNCTIONS AND THE POMPEIU PROBLEM

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ABSTRACT. In this paper we prove a result on the Pompeiu problem. If the Schwarz function Φ of the boundary of a simply-connected domain $\Omega \subset \mathbb{R}^2$ extends meromorphically into a certain portion E of Ω with a pole at some point $z_0 \in E$, then Ω has the Pompeiu property unless Φ is a Möbius transformation, in which case Ω is a disk.

1. INTRODUCTION

In 1929 the Rumanian mathematician D. Pompeiu formulated the following problem: “*To characterize those bounded domains $\Omega \subset \mathbb{R}^2$ for which $f \equiv 0$ is the only continuous function such that*

$$(1.1) \quad \int_{\sigma(D)} f \, dx = 0,$$

for every rigid motion σ of $\mathbb{R}^2$ ”.

One says that Ω has the Pompeiu property if $f \equiv 0$ is the only continuous function for which (1.1) holds. For a historical introduction to the problem we refer the reader to [GS1]. In that paper we conjectured that (modulo sets of measure zero) the disk is the only simply-connected domain that does not have the Pompeiu property. Chakalov [C] was the first one to realize that the disk fails to have the Pompeiu property. In fact, if one considers the function $f(x_1, x_2) = \sin(ax_1)$, then one has

$$\int_{B_r(x_0)} f(x) \, dx = \frac{2\pi r}{a} \sin(ax_{0,1}) J_1(ar),$$

where $x_0 = (x_{0,1}, x_{0,2})$ is fixed, $B_r(x_0) = \{x \mid |x - x_0| < r\}$, and J_1 is the Bessel function of order one. It is therefore enough to choose $a > 0$, such that $J_1(ar) = 0$, for (1.1) to hold.

This paper contains some progress toward the above conjecture. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain whose boundary $\partial\Omega$ is a piecewise C^1 Jordan curve. By the Riemann mapping theorem there exists a univalent function $h: D \rightarrow \Omega$, where $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Moreover, h can be extended in a one-to-one fashion to a continuous map of $\overline{D}$ onto $\overline{\Omega}$. In order

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to state the main result in this paper we need to introduce some definitions. We consider the Schwarz function of $\partial\Omega$ given by

$$\Phi(w) = h \left(\overline{\frac{1}{h^{-1}(w)}} \right).$$

A priori, Φ is well defined on $\partial\Omega$. Given a straight line $L \subset \mathbb{C}$, and a point $z_0 \notin L$, we denote by $\Lambda(L; z_0)$ the open half-plane lying on one side of L and containing z_0 . We also let

$$E(L; z_0) = \Lambda(L; z_0) \cap \Omega.$$

The main result in this paper is given by the following

Theorem 1. *Suppose that there exist $z_0 \in \Omega$ and a straight line $L \subset \mathbb{C}$ such that:*

(i) Φ can be extended to a holomorphic function in $E(L; z_0) \setminus \{z_0\}$ having a pole in z_0 ;

(ii) Φ is not a Möbius transformation.

Then, Ω has the Pompeiu property.

Figure 1 below illustrates the situation.

We now state two remarkable consequences of Theorem 1.

Corollary 2. *Suppose that h is univalent in D and meromorphic in $\mathbb{C}$, with at least one pole in $\overline{\mathbb{C}} \setminus \overline{D}$. If, moreover, h is not a Möbius transformation, then $\Omega = h(D)$ has the Pompeiu property.*

If we specialize Theorem 1 to the class of convex domains we obtain the following partial solution of the Pompeiu problem.

Corollary 3. *Suppose that $\Omega = h(D)$ be a convex set. Assume that h has a pole on the boundary of the circle of convergence relative to its Taylor expansion at $z = 0$. If h is not a Möbius transformation, then Ω has the Pompeiu property.*

Remark. Corollary 2 contains the result in our paper [GS2] (see also [GS3]) concerned with the case

$$h(z) = \sum_{k=0}^N a_k z^k.$$

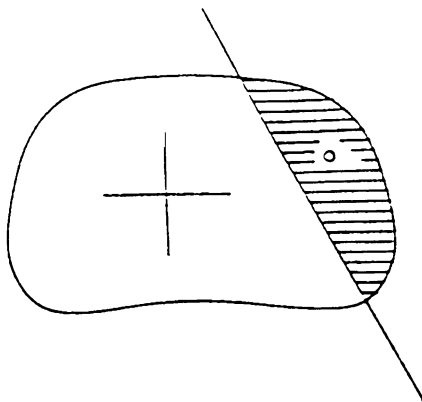


FIGURE 1

Furthermore, it contains a result in a recent paper by Ebenfelt [E]. The latter has proved that if h is a univalent function in D such that $h(z) = \frac{p(z)}{q(z)}$, with p and q polynomials, then $\Omega = h(D)$ has the Pompeiu property, unless h is a Möbius transformation.

Our strategy to prove Theorem 1 is to study, by Riemann's method of the steepest descent, the asymptotic behavior of the (complexified) Fourier transform of the characteristic function of Ω , $\hat{\chi}_\Omega$, along the algebraic variety of $\mathbb{C}^2$, $M_\alpha = \{\zeta_1^2 + \zeta_2^2 = \alpha\}$, $\alpha > 0$. This is due to an important characterization of the Pompeiu property established in 1973 by Brown, Schreiber, and Taylor [BST], see Theorem A in the next section. We mention that Berenstein [B] was the first one to use asymptotic expansions of $\hat{\chi}_\Omega$ in connection with the Pompeiu problem.

2. PRELIMINARY REDUCTIONS

We begin this section by recalling the above-mentioned characterization of the Pompeiu property due to Brown, Schreiber, and Taylor [BST].

Theorem A. *A bounded domain $\Omega \subset \mathbb{R}^2$ has the Pompeiu property if and only if there exists no $\alpha \in \mathbb{C} \setminus \{0\}$ such that the complexified Fourier transform of the characteristic function of Ω , $\hat{\chi}_\Omega$, vanishes identically on*

$$M_\alpha = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid \zeta_1^2 + \zeta_2^2 = \alpha\}.$$

It was observed by Berenstein [B] that, when Ω is simply connected, $\alpha \in \mathbb{C} \setminus \{0\}$ in the statement of Theorem A can be replaced by $\alpha > 0$. Furthermore, when $\partial\Omega$ is a rectifiable Jordan curve, then the divergence theorem allows to replace $\hat{\chi}_\Omega$ with $\hat{\chi}_{\partial\Omega}$. Note that for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$

$$(2.1) \quad \hat{\chi}_{\partial\Omega} = \int_{\partial\Omega} e^{i\langle \zeta, x \rangle} (dx_1 + idx_2)$$

where we have let $\langle \zeta, x \rangle = \zeta_1 x_1 + \zeta_2 x_2$. Changing ζ in $-i\zeta$ in (2.1) we are thus led to study the following oscillatory integral

$$(2.2) \quad \int_{\partial\Omega} e^{\langle \zeta, x \rangle} (dx_1 + idx_2)$$

for $\zeta \in M_{-\alpha}$, with $\alpha > 0$. We write ζ in the form

$$\zeta = r(\cos \theta, \sin \theta) + it(-\sin \theta, \cos \theta).$$

The condition $\zeta \in M_{-\alpha}$ becomes

$$(2.3) \quad t^2 = \alpha + r^2.$$

We have

$$(2.4) \quad \begin{aligned} \langle \zeta, x \rangle &= x_1(r \cos \theta - it \sin \theta) + x_2(r \sin \theta + it \cos \theta) \\ &= rx_1 e^{-i\theta} + irx_2 e^{-i\theta} - i(t-r)x_1 \sin \theta + i(t-r)x_2 \cos \theta \\ &= re^{-i\theta}(x_1 + ix_2) - i(t-r)(x_1 \sin \theta - x_2 \cos \theta). \end{aligned}$$

Since from our assumptions in the introduction $\partial\Omega = h(\partial D)$, where h is univalent in $D = \{w \in \mathbb{C} \mid |w| < 1\}$, we have for $s \in [0, 2\pi]$

$$(2.5) \quad x_1(s) = \frac{1}{2}h(e^{is}) + \frac{1}{2}k(e^{is}).$$

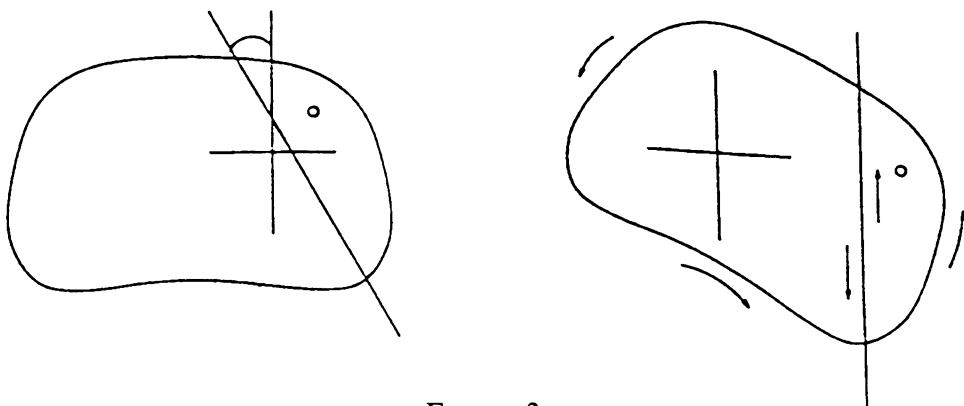


FIGURE 2

Here, we have let

$$(2.6) \quad k(w) = \overline{h\left(\frac{1}{\overline{w}}\right)}.$$

Analogously, we have

$$(2.7) \quad x_2(s) = \frac{1}{2i}h(e^{is}) - \frac{1}{2i}k(e^{is}).$$

Inserting (2.5), (2.7) in (2.4) we obtain

$$\langle \zeta, x \rangle = re^{-i\theta}h - i\frac{(t-r)}{2}[(h+k)\sin\theta + i(h-k)\cos\theta],$$

which, after some easy reductions, gives

$$(2.8) \quad \langle \zeta, x \rangle = \frac{t+r}{2}e^{-i\theta}h(e^{is}) - \frac{t-r}{2}e^{i\theta}k(e^{is}).$$

Taking (2.8) into account, we see that (up to a factor of i) the integral in (2.2) becomes

$$(2.9) \quad \begin{aligned} I(r) &= \int_{\partial\Omega} \exp\left[\frac{t+r}{2}e^{-i\theta}w - \frac{t-r}{2}e^{i\theta}\Phi(w)\right] dw \\ &= e^{i\theta} \int_{\partial\Sigma} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] dw \end{aligned}$$

where $\Sigma = e^{-i\theta}\Omega$, $\Psi(w) = e^{i\theta}\Phi(e^{i\theta}w)$. At this point we choose $\theta \in [0, 2\pi]$ in such a way that the straight line $e^{-i\theta}L$, where L is as in the statement of Theorem 1, becomes parallel to the imaginary axis. We let $w_0 = e^{-i\theta}z_0$, $M = e^{-i\theta}L$, where $z_0 \in \Omega$ is as in the assumption of Theorem 1, see Figure 2.

We now have from (2.9)

$$(2.10) \quad e^{-i\theta}I(r) = \left(\int_{\partial E(M; w_0)} + \int_{\partial[\Sigma \setminus E(M; w_0)]} \right) \exp\left[\left(\frac{t+r}{2}\right)w - \frac{t-r}{2}\Psi(w)\right] dw.$$

3. ASYMPTOTIC EXPANSION OF $\hat{\chi}_{\partial\Omega}$ AND THE POMPEIU PROPERTY

The aim of this section is to establish the asymptotic behavior as $r \rightarrow \infty$ of the integral in the right-hand side of (2.10). We begin by analyzing that

part of the integral on the set $\partial(\Sigma \setminus E(M; w_0))$. We let $A = \max |\Psi|$ on $\partial(\Sigma \setminus E(M; w_0))$. Then

$$\begin{aligned} & \left| \int_{\partial(\Sigma \setminus E(M; w_0))} \exp \left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w) \right] dw \right| \\ & \leq \int_{\partial(\Sigma \setminus E(M; w_0))} \exp \left[\frac{t+r}{2} \Re w + \frac{t-r}{2} A \right] ds. \end{aligned}$$

We now choose $\beta > 0$ such that on the set $\partial(\Sigma \setminus E(M; w_0))$ we have (see Figure 2)

$$\Re w \leq \Re w_0 - \beta.$$

Noting that (2.3) gives

$$(3.1) \quad \frac{t+r}{2} = r \left(1 + O \left(\frac{1}{r^2} \right) \right), \quad t-r = \frac{\alpha}{2r} \left(1 + O \left(\frac{1}{r} \right) \right)$$

as $r \rightarrow \infty$, it follows that on the set $\partial(\Sigma \setminus E(M; w_0))$ we have uniformly as $r \rightarrow \infty$

$$\frac{t+r}{2} \Re w + \frac{t-r}{2} A \leq \frac{t+r}{2} (\Re w_0 - \beta) + \frac{t-r}{2} A = r (\Re w_0 - \beta) (1 + o(1)).$$

From this we derive the estimate for $r \rightarrow \infty$

$$(3.2) \quad \left| \int_{\partial(\Sigma \setminus E(M; w_0))} \exp \left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w) \right] dw \right| \leq C \exp \left[r \left(\Re w_0 - \frac{\beta}{2} \right) \right].$$

We will now analyze the first integral in the right-hand side of (2.10). We have for $\delta > 0$ small by Cauchy's theorem

$$\begin{aligned} (3.3) \quad & \int_{\partial E(M; w_0)} \exp \left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w) \right] dw \\ & = \exp \left(\frac{t+r}{2} w_0 \right) \int_{|w-w_0|=\delta} \exp \left[\frac{t+r}{2} (w-w_0) - \frac{t-r}{2} \Psi(w) \right] dw \\ & = \exp \left(\frac{t+r}{2} w_0 \right) \int_{|w|=\delta} \exp \left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w_0+w) \right] dw. \end{aligned}$$

By the assumptions on Φ in Theorem 1, there exists $m \in \mathbb{N}$ such that

$$(3.4) \quad \Psi(w_0+w) = \sum_{k=-m}^{\infty} a_k w^k$$

for $|w| \leq \delta$, with $a_{-m} \neq 0$. We now distinguish two cases.

First case. $m \geq 2$.

Using (2.3) we can write

$$(3.5) \quad \frac{t-r}{2} = \frac{\alpha}{2(t+r)}.$$

By (3.4), (3.5) we have on the circle $\{w \in \delta e^{i\tau} \mid -\pi \leq \tau \leq \pi\}$

$$\begin{aligned} (3.6) \quad & \frac{t+r}{2} w - \frac{t-r}{2} \Psi(w_0+w) \\ & = \frac{t+r}{2} \delta e^{i\tau} - \frac{\alpha a_{-m}}{2(t+r)} \delta^{-m} e^{-im\tau} - \frac{\alpha}{2(t+r)} \sum_{k=-m+1}^{\infty} a_k \delta^k e^{ik\tau}. \end{aligned}$$

We now choose

$$\delta = \left(\frac{t+r}{2} \right)^{-2/(m+1)}.$$

Then, (3.6) becomes as $r \rightarrow \infty$

$$\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0 + w) = \left(\frac{t+r}{2} \right)^{(m-1)/(m+1)} \left[e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau} + o(1) \right].$$

From the first equality in (3.1) we conclude that

$$(3.7) \quad \frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0 + w) = r^{(m-1)/(m+1)} q(r) \left\{ e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau} + o(1) \right\}$$

with $q(r) \rightarrow 1$ as $r \rightarrow \infty$, uniformly on the circle $\{w = \delta e^{i\tau} \mid -\pi \leq \tau \leq \pi\}$.

Taking (3.7) into account, we obtain for (3.3) with some $p(r) \rightarrow 1$ as $r \rightarrow \infty$

$$(3.8) \quad \int_{\partial E(M; w_0)} \exp \left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w) \right] dw = ir^{-2/(m+1)} p(r) \exp \left(\frac{t+r}{2}w_0 \right) \\ \cdot \int_{-\pi}^{\pi} \exp \left\{ r^{(m-1)/(m+1)} q(r) \left[e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau} + o(1) \right] \right\} e^{i\tau} d\tau.$$

At this point we observe that the integral on the right-hand side of (3.8) is of the type studied in the paper [GS2]. By virtue of the work done in [GS2] we can conclude that the asymptotic behavior, as $r \rightarrow \infty$, of the above-mentioned integral is as follows

$$(3.9) \quad \int_{-\pi}^{\pi} \exp \left\{ r^{(m-1)/(m+1)} q(r) \left[e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau} + o(1) \right] \right\} e^{i\tau} d\tau \\ = r^{-(m-1)/2(m+1)} A(r) \exp[r^{(m-1)/(m+1)} B(r)],$$

where, having let $\varphi(\tau) = e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau}$ for $\tau \in \mathbb{C}$, one has for $r \rightarrow \infty$

$$A(r) \rightarrow \frac{e^{i\tau_0}}{\sqrt{2\varphi''(\tau_0)}} = A_0 \neq 0, \quad B(r) \rightarrow \varphi(\tau_0).$$

Here, τ_0 is a suitable simple critical point of the function φ . Inserting (3.9) in (3.8) and recalling (3.1), we obtain

$$(3.10) \quad \int_{\partial E(M; w_0)} \exp \left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w) \right] dw \\ = r^{-(m+3)/2(m+1)} A_1(r) \exp[rw_0 + r^{(m-1)/(m+1)} B(r)],$$

where $A_1(r) \rightarrow iA_0$, as $r \rightarrow \infty$.

Using (3.2), (3.10) in (2.10) we finally conclude for $r \rightarrow \infty$

$$(3.11) \quad e^{-i\theta} I(r) = r^{-(m+3)/2(m+1)} A_1(r) \exp[rw_0 + r^{(m-1)/(m+1)} B(r)] \\ \cdot \left\{ 1 + O \left(r^{(m+3)/2(m+1)} \exp \left[-\frac{\beta}{2}r + Cr^{(m-1)/(m+1)} \right] \right) \right\},$$

for some number $C > 0$. Observing now that $0 < \frac{m-1}{m+1} < 1$, we infer that

$$O \left(r^{(m+3)/2(m+1)} \exp \left[-\frac{\beta}{2}r + Cr^{(m-1)/(m+1)} \right] \right) = o(1)$$

as $r \rightarrow \infty$. In conclusion, we obtain from (3.11)

$$(3.12) \quad e^{-i\theta} I(r) = r^{-(m+3)/2(m+1)} A_2(r) \exp[rw_0 + r^{(m-1)/(m+1)} B(r)],$$

where $A_2(r) \rightarrow iA_0$ as $r \rightarrow \infty$.

To conclude the study of the asymptotic behavior of the integral $e^{-i\theta} I(r)$ in (2.10) we need to analyze the case in which Ψ has as simple pole in w_0 , i.e., the case in which $m = 1$ in (3.4).

Second case. $m = 1$.

We consider again the integral on the circle $\{w \in \delta e^{i\tau} | -\pi \leq \tau \leq \pi\}$ in the right-hand side of (3.3). By the assumptions in Theorem 1, the function Ψ is not a Möbius transformation. If $w = \sigma e^{i\tau}$, with $|\sigma| = \delta$, then we have from (3.5)

$$(3.13) \quad \begin{aligned} \frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) &= \frac{1}{2} \left[(t+r)w - \frac{\alpha}{t+r}\Psi(w_0+w) \right] \\ &= \frac{1}{2} \left\{ (t+r)\sigma e^{i\tau} - \frac{\alpha}{t+r} \left[\frac{a_{-1}}{\sigma} e^{-i\tau} + a_0 + \sum_{k=q}^{\infty} a_k \sigma^k e^{ik\tau} \right] \right\} \end{aligned}$$

for some $q \in \mathbb{N}$, with $a_q \neq 0$. At this point we choose

$$(3.14) \quad \sigma = \frac{i\sqrt{\alpha a_{-1}}}{t+r}.$$

It follows from (3.13), (3.14) that for $w = \sigma e^{i\tau}$ one has

$$(3.15) \quad \begin{aligned} \frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) &= -\frac{\alpha a_0}{t+r} + i\sqrt{\alpha a_{-1}} \cos \tau \\ &+ \frac{C}{(t+r)^{q+1}} e^{iq\tau} + O\left(\frac{1}{(t+r)^{q+2}}\right), \end{aligned}$$

for some $C \neq 0$. We conclude

$$(3.16) \quad \begin{aligned} \int_{|w|=\delta} \exp \left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) \right] dw \\ = i\sigma e^{-\alpha a_0/(t+r)} \int_{-\pi}^{\pi} e^{i\sqrt{\alpha a_{-1}} \cos \tau} \left\{ 1 + \frac{C}{(t+r)^{q+1}} e^{iq\tau} + O\left(\frac{1}{(t+r)^{q+2}}\right) \right\} e^{i\tau} d\tau. \end{aligned}$$

We now recall the integral representation of the Bessel function J_n (see [L])

$$J_n(z) = \frac{i^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{iz \cos \tau} e^{in\tau} d\tau, \quad n \in \mathbb{Z}.$$

Using this we can rewrite (3.16) as follows

$$(3.17) \quad \begin{aligned} \int_{|w|=\delta} \exp \left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) \right] dw \\ = -\frac{\sqrt{\alpha a_{-1}}}{t+r} e^{-\alpha a_0/(t+r)} \left\{ 2\pi i J_1(\sqrt{\alpha a_{-1}}) \right. \\ \left. + \frac{2\pi C i^q J_q(\sqrt{\alpha a_{-1}})}{(t+r)^{q+1}} + O\left(\frac{1}{(t+r)^{q+2}}\right) \right\}. \end{aligned}$$

Since, from (3.1), $t + r(1 + o(1))$ as $r \rightarrow \infty$, and by a theorem of Siegel, J_1 and J_q have no common zeros (see [W, p. 485]). (3.17) implies

$$(3.18) \quad \int_{|w|=\delta} \exp \left[\frac{t+r}{2} w - \frac{t-r}{2} \Psi(w_0 + w) \right] dw \\ = \frac{E_1(r)}{r} \left[J_1(\sqrt{\alpha a_{-1}}) + \frac{E_2}{r^{q+1}} J_q(\sqrt{\alpha a_{-1}}) \right],$$

where $E_1(r) \rightarrow E_0 \neq 0$ as $r \rightarrow \infty$, and $E_2 \neq 0$. From (2.10), (3.2) and (3.18) we finally obtain

$$(3.19) \quad \varepsilon^{-i\theta} I(r) = \varepsilon^{rw_0} \frac{E_3(r)}{r} \left[J_1(\sqrt{\alpha q_{-1}}) + \frac{E_2}{r^{q+1}} J_q(\sqrt{\alpha a_{-1}}) \right]$$

with $E_3(r) \rightarrow E_0$ as $r \rightarrow \infty$ (of course, in this estimate we have used again (3.1)).

We are now ready to conclude the proof of Theorem 1. We recall that from the reductions in §2, the oscillatory integral $\hat{\chi}_{\partial\Omega}(\zeta)$, with ζ moving out to infinity along a special path of $M_{-\alpha}$, was shown to equal $e^{-i\theta} I(r)$ in (2.10) (up to a factor of i).

From (3.12), (3.19) we see that, under the assumptions in Theorem 1, the latter cannot vanish identically on $M_{-\alpha}$. From Theorem A we conclude that Ω has the Pompeiu property. $\square$

4. PROOFS OF COROLLARIES 2 AND 3

The proof of Corollary 2 follows immediately from Theorem 1 by observing that if Φ is a Möbius transformation, then so is h . Moreover, if h has at least one pole, then Φ has at least one pole and at most one essential singularity ($w = 0$).

As for the proof of Corollary 3 we observe that if $\Omega = h(D)$ is convex, then by Study's theorem [S, Theorem 2.4] so is $h(D_r)$ for $0 < r \leq 1$, where $D_r = \{z \in \mathbb{C} \mid |z| \leq r\}$. Set $S = \{x \in \mathbb{C} \mid |z| < R\}$ with $R > 1$, and denote $S^{-1} = \{\frac{1}{z} \mid z \in S\}$. Assume that h is holomorphic in S with a pole z_0 on ∂S . Then Φ is holomorphic on $h(S^{-1})$ and has a pole in $h(\frac{1}{z_0})$. Since $h(S^{-1})$ is convex we are in a position to apply Theorem 1, see Figure 3.

We close this paper with an example of a one-parameter family of domains which fall within the scope of Theorem 1, but are not included in any previous result on the Pompeiu problem.

Example. Consider for $0 < \lambda < 2$ the map $h_\lambda : D \rightarrow \mathbb{C}$ given by $h_\lambda(x) = \frac{e^\lambda}{\lambda x - 2}$. In Figure 4, we represent $\Omega_\lambda = h_\lambda(D)$ for some values of λ . There exists $\lambda_0 \in (0, 2)$ such that for $0 < \lambda < \lambda_0$ the domain Ω_λ is convex. Furthermore, one verifies that for $\lambda \in (0, \lambda_0)$

$$\min \text{diam } \Omega_\lambda > \frac{1}{2} \max \text{diam } \Omega_\lambda$$

so that the result in [BK] cannot be invoked.

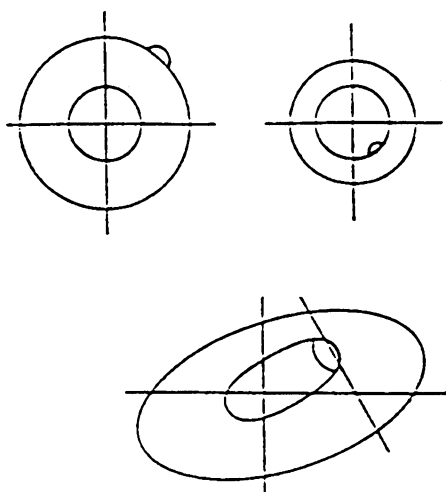


FIGURE 3

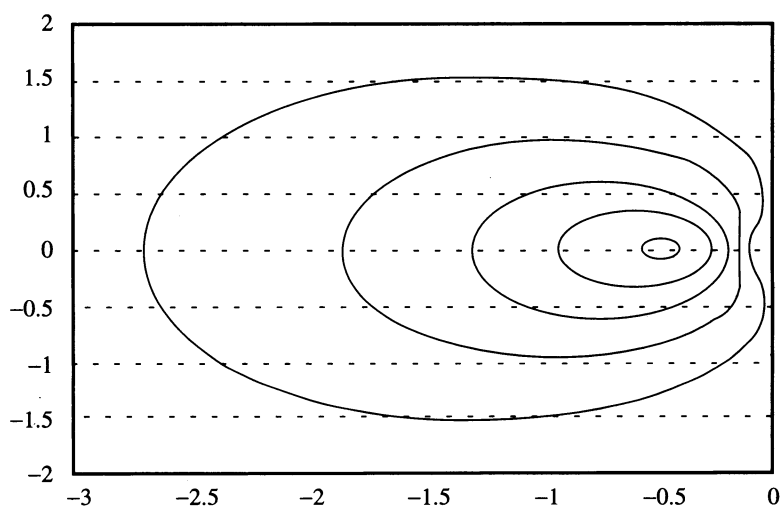


FIGURE 4

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