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GEOMETRIC INVARIANTS FOR SEIFERT FIBRED 3-MANIFOLDS

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ABSTRACT. In this paper, we obtain a formula for the η-invariant of the signature operator for some circle bundles over Riemannian 2-orbifolds. We then apply it to Seifert fibred 3-manifolds endowed with one of the six Seifert geometries. By using a relation between the Chern-Simons invariant and the η-invariant, we also derive some elementary formulae for the Chern-Simons invariant of these manifolds. As applications, we show that some families of these manifolds cannot be conformally immersed into the Euclidean space $\mathbf{E}^4$.

1. INTRODUCTION

In dimension 3, the work of Thurston [15], [16] indicates that there are essentially eight relevant homogeneous geometries needed for geometric structures on 3-manifolds. Of these, six are the so-called Seifert geometries.

The purpose of this paper is to study two kinds of geometric invariants, the Chern-Simons invariant and the η-invariant, for a 3-manifold endowed with one of these six Seifert geometries.

The Chern-Simons invariant was first defined in [3] for a closed Riemannian 3-manifold by integrating a certain 3-form over the manifold. It is only defined mod $\mathbf{Z}$.

The η-invariant of a selfadjoint elliptic operator was originally introduced by Atiyah, Patodi, and Singer in [1] for odd-dimensional Riemannian manifolds in terms of the spectrum of the operator. The η-invariant we are interested in here is that of the signature operator. Such an invariant has a topological interpretation which allows us to compute it without using analytic tools. As shown in [1], it measures the extent to which the Hirzebruch signature formula fails for a nonclosed $4k$-dimensional Riemannian manifold whose metric is a product near its boundary. In dimension 3, it can be thought of as a real-valued generalization of the Chern-Simons invariant.

The remainder of this paper consists of three sections. In Section 2, we obtain a formula for the η-invariant of S^1-bundles over Riemannian 2-orbifolds. We then, in Section 3, apply our formula to the geometric Seifert fibred 3-manifolds to get explicit expressions of the η-invariant under these geometries. Finally, in Section 4, we derive some elementary formulae for the Chern-Simons invariant of geometric Seifert fibred 3-manifolds. As applications, we show that some

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families of these manifolds cannot be conformally immersed into the Euclidean space $\mathbf{E}^4$.

2. η-INVARIANT FOR S^1-BUNDLES OVER RIEMANNIAN 2-ORBIFOLDS

In [10], Komuro obtained a formula for the η-invariant of some circle bundles over Riemann surfaces. In this section, we extend his formula to the case of S^1-bundles over Riemannian 2-orbifolds by using the generalized Hirzebruch signature formula as developed by Kawasaki in [8] and [9]. We start with the following topological interpretation for the η-invariant due to Atiyah, Patodi, and Singer:

Theorem 2.0 [1]. *Let N be a $4k$-dimensional Riemannian manifold with $\partial N = M$. Assume that N has a product metric near the boundary. Then*

$$\eta(M) = \int_N L_k(P) - \operatorname{Sign}(N)$$

where L_k is the kth Hirzebruch L-polynomial, P is the Pontrjagin form of N and $\operatorname{Sign}(N)$ *is the signature of N.*

Kawasaki in [8] generalized the above theorem to the case where M^{4k-1} bounds a $4k$-dimensional Riemannian orbifold.

To state his result, we need some preliminaries from [8] and [9].

Let $(X, \mathscr{U})$ be a compact orbifold. Denote by ΣX the singular set of X. For every $x \in \Sigma X$, choose precisely one linear chart $(\tilde{U}_x, G_x, U_x, \pi_x)$ such that $\pi_x(0) = x$. For each $g \in G_x$, a local group of x, the centralizer $C_{G_x}(g)$ of g in G_x acts on $\tilde{U}_x^g = \{y \in \tilde{U}_x | g \cdot y = y\}$. Let $(1), (h_x^1), \cdots, (h_x^{\rho_x})$ be all the conjugacy classes in G_x. Suppose $y \in U_x \cap \Sigma X$. Then an overlap map ψ_{yx} between $\tilde{U}_y$ and $\tilde{U}_x$ induces a homomorphism $\psi_{yx}^\star : G_y \longrightarrow G_x$. The following is a bijection

$$\{(y, (h_y^j)) | y \in U_x \cap \Sigma X, \psi_{yx}^\star(h_y^j) = h_x^i\} \longrightarrow \tilde{U}_x^{h_x^i} / C_{G_x}(h_x^i),$$
$$(y, (h_y^j)) \longmapsto [\psi_{yx}(0)]$$

which defines an orbifold structure on

$$\tilde{\Sigma} X = \{(x, (h_x^j)) | x \in \Sigma X, j = 1, 2, \cdots, \rho_x\}.$$

The multiplicity of $\tilde{\Sigma} X$ in X at $(x, (h_x^j))$ is the order of the trivially acting subgroup of $C_{G_x}(h_x^j)$ on $\tilde{U}_x^{h_x^j}$.

Let $\tilde{\Sigma}_1 X, \cdots, \tilde{\Sigma}_c X$ be the connected components of $\tilde{\Sigma} X$. The multiplicity is locally constant on $\tilde{\Sigma} X$. Thus we can assign a number $m_s = m(\tilde{\Sigma}_s X)$ to each $\tilde{\Sigma}_s X$, called the multiplicity of $\tilde{\Sigma}_s X$ in X.

On each orbifold chart $\tilde{U}_x^h$ of $\tilde{\Sigma} X$, we have the normal bundle $\nu(\tilde{U}_x^h)$ in $\tilde{U}_x$ and the tangent bundle $\tau(\tilde{U}_x^h)$. h acts on $\nu(\tilde{U}_x^h)$. We have the decomposition

$$\nu(\tilde{U}_x^h) = \bigoplus_{0<\theta\leq\pi} \nu_\theta^h$$

where ν_θ^h is the bundles of eigenspace for h with eigenvalues $e^{\pm i\theta}$. Introduce complex structures on the bundles ν_θ^h such that $h \cdot v = e^{i\theta} v$ if $v \in \nu_\theta^h$. (For simplicity, we assume here that ν_π^h can be given a compatible complex structure so that $\theta = \pi$ need not play a special role). The collection of these $C_{G_x}(h)$ bundles form complex vector bundles over $\tilde{\Sigma}X$. Choose a $C_{G_x}(h)$-invariant connection for each bundle. Write the total Chern class

$$c(\nu_\theta^h) = \prod_j (1 + x_j),$$

i.e., $c_n(\nu_\theta^h) = n$th symmmetric polynomial in x_j's. Define

$$L_\theta(\nu_\theta^h) = \prod_j \coth(x_j + \frac{i\theta}{2}) = \prod_j \left(\frac{e^{i\theta}e^{2x_j} + 1}{e^{i\theta}e^{2x_j} - 1}\right).$$

The local characteristic form

$$L^{\tilde{\Sigma}}(\tilde{U}_x^h) = L(\tilde{U}_x^h) \cdot \prod_{0<\theta\leq\pi} L_\theta(\nu_\theta^h)$$

defines an L-class in $\tilde{\Sigma}X$.

Theorem 2.1 (Kawasaki [8]). *Let N be a $4k$-dimensional Riemannian orbifold with $\partial N = M$ a $(4k-1)$-dimensional Riemannian manifold. Assume that N has a product metric near the boundary. Then*

$$\eta(M) = \int_N L_k(P) - \mathrm{Sign}(N) + \sum_{s=1}^{c} \frac{1}{m_s} \langle L^{\tilde{\Sigma}_s}, [\tilde{\Sigma}_s N]\rangle.$$

In Theorems 2.0 and 2.1, it is required that the metric on N is a product near the boundary. But this may not be the case in practice. We need a so-called "boundary correction term". The following formulation is due to Gilkey [4].

Let $C = M \times [0, 1]$ and identify $M \times \{0\}$ with $M = \partial N$. Let g_0 be the product metric on C. Extend the metric g on N to a metric g_1 on $C \cup N$ such that g_1 is a product near $M \times \{1\}$ and agrees with g_0 near $M \times 1$. Let ∇_0 and ∇_1 be the Riemannian connections determined by g_0 and g_1 respectively. Denote ω_0, ω_1 and Ω_0, Ω_1 the corresponding connection forms and curvature forms of ∇_0 and ∇_1 respectively.

Let $\omega = \omega_1 - \omega_0$, $\omega_t = (1-t)\omega_0 + t\omega_1$, and Ω_t the curvature form of ω_t. Define

$$TL_k = 2k \int_0^1 L_k(\omega, \Omega_t, \cdots, \Omega_t) dt.$$

Then

$$dTL_k = L_k(\Omega_1) - L_k(\Omega_0).$$

Since $L_k(\Omega_0) = 0$, we get

$$dTL_k = L_k(\Omega_1) = L_k(\Omega_1, \cdots, \Omega_1).$$

Thus

$$\int_C L_k(\Omega_1) = \int_C dTL_k = \int_{\partial C} TL_k = \int_{M\times\{0\}} TL_k = -\int_M TL_k.$$

It follows from Theorem 2.1 that

$$\begin{aligned}\eta(M) &= \int_{C\cup N} L_k(P) - \operatorname{Sign}(C\cup N) + \sum_{s=1}^{c}\frac{1}{m_s}\langle L^{\tilde{\Sigma}_s}, [\tilde{\Sigma}_s N]\rangle \\ &= \int_C L_k(\Omega_1) + \int_N L_k(P) - \operatorname{Sign}(N) + \sum_{s=1}^{c}\frac{1}{m_s}\langle L^{\tilde{\Sigma}_s}, [\tilde{\Sigma}_s N]\rangle \\ &= -\int_M TL_k + \int_N L_k(P) - \operatorname{Sign}(N) + \sum_{s=1}^{c}\frac{1}{m_s}\langle L^{\tilde{\Sigma}_s}, [\tilde{\Sigma}_s N]\rangle.\end{aligned}$$

Now, let $S(E) \longrightarrow F$ be a principal S^1-bundle over an oriented, Riemannian 2-orbifold $(F, \hat{g})$. Let $p : E \longrightarrow F$ be the associated $\mathbf{C}^1$-bundle.

Choose a fiber metric $\tilde{g}$ and a $\tilde{g}$-preserving connection $\tilde{\nabla}$. Then we obtain a Riemannian orbifold (E, g) by assuming that

$$g|_{\text{horizontal}} = p^\star \hat{g} \quad \text{and} \quad g|_{\text{vertical}} = \tilde{g}.$$

We now determine the Riemannian connection of g on E.

For any $x \in F$, choose local orthonormal (horizontal) sections e_1, e_2 near x, Then they determine a coordinate system u^1, u^2 along the fiber with $e_\alpha = \partial/\partial u^\alpha$, $\alpha = 1, 2$.

Let x_3, x_4 be a local orthonormal frame of F near x and e_3, e_4 their horizontal lifts with respect to $\tilde{\nabla}$.

Let $\tilde{R}$ be the curvature tensor of the connection $\tilde{\nabla}$ and $\tilde{\Omega}_\alpha^\beta$ its components with respect to e_1, e_2. Then $\tilde{\Omega}_1^2 = \tilde{R}_{1234} x_3^\star \wedge x_4^\star$ where

$$\tilde{R}_{1234} = \tilde{g}(\tilde{R}(x_3, x_4)e_1, e_2).$$

We will denote $\tilde{R} = \tilde{R}_{1234}$ for brevity. For convenience, we use the polar coordinate system (r, θ) in the vertical space so that $u^1 = r\cos\theta$ and $u^2 = r\sin\theta$.

A straightforward calculation yields

Lemma 2.2. *With respect to the basis* $\{e_1 = \frac{1}{r}\frac{\partial}{\partial\theta}, e_2 = \frac{\partial}{\partial r}, e_3, e_4\}$, *the connection form* ω *of* ∇ *is given by*

$$\omega_2^1 = \widetilde{d\theta}, \quad \omega_3^1 = -\frac{1}{2} r\tilde{R} e_4^\star, \quad \omega_4^1 = \frac{1}{2} r\tilde{R} e_3^\star,$$

$$\omega_3^2 = \omega_4^2 = 0, \quad \omega_4^3 = -\frac{1}{2} r^2 \tilde{R}\widetilde{d\theta} + p^\star \hat{\omega}_4^3$$

where $\{r\widetilde{d\theta}, dr, e_3^\star, e_4^\star\}$ *is the dual basis of* $\{\frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial r}, e_3, e_4\}$ *and* $\hat{\omega}_4^3$ *is the component of the connection form of* $\hat{g}$ *with respect to* x_3, x_4.

From Lemma 2.2, we have the following

Lemma 2.3. *If $\tilde{R}$ is constant on F, then with respect to the same basis as in Lemma 2.2, the components of the curvature form Ω of ∇ are given by*

$$\begin{aligned}
\Omega_1^2 &= \tilde{R}e_3^\star \wedge e_4^\star, \\
\Omega_1^3 &= -\frac{1}{2}\tilde{R}e_4^\star \wedge dr + \frac{1}{4}r^2\tilde{R}^2 e_3^\star \wedge r\widetilde{d\theta}, \\
\Omega_1^4 &= \frac{1}{2}\tilde{R}e_3^\star \wedge dr + \frac{1}{4}r^2\tilde{R}^2 e_4^\star \wedge r\widetilde{d\theta}, \\
\Omega_2^3 &= \frac{1}{2}\tilde{R}e_4^\star \wedge r\widetilde{d\theta}, \\
\Omega_2^4 &= -\frac{1}{2}\tilde{R}e_3^\star \wedge r\widetilde{d\theta}, \\
\Omega_4^3 &= p^\star\hat{\Omega}_3^4 - \frac{3}{4}r^2\tilde{R}^2 p^\star\omega_F + \tilde{R}dr \wedge r\widetilde{d\theta},
\end{aligned} \tag{2.1}$$

where $\hat{\Omega}_3^4$ is the curvature form of $\hat{\omega}_3^4$ and $\omega_F = x_3^\star \wedge x_4^\star$ is the volume form of $(F, \hat{g})$. Equation (2) is valid in general.

Before presenting our main theorem, we need some facts about the local structure of a 2-orbifold and a complex line bundle over it.

Let F be an oriented, closed 2-orbifold. Then, by the classification theorem of 2-orbifolds (see [15], for example), the only possible orbifold charts have the form $\mathbf{R}^2/(\mathbf{Z}/\alpha)$ with $\mathbf{Z}/\alpha$ acting on $\mathbf{R}^2$ by multiplication by $e^{2\pi\beta i/\alpha}$ for some β prime to α. ΣF consists of a set of isolated points $\{x_1, \cdots, x_n\}$ in F. Thus, F is determined by the data $(g; \alpha_1, \cdots, \alpha_n)$ where g is the genus of F and the α_s's are such that $G_{x_s} = \mathbf{Z}/\alpha_s$. We assume thereafter, without loss of generality, that $\alpha_s > 0$.

Let $E \longrightarrow F$ be a complex line bundle over F such that E is a 4-orbifold with orbifold chart $(\mathbf{R}^2 \times \mathbf{R}^2)/\mathbf{Z}/\alpha$ where $\mathbf{Z}/\alpha$ acts on the first coordinate by $e^{2\pi i\beta/\alpha}$ and on the second by $e^{2\pi\gamma i/\alpha}$ for some β and γ prime to α. Thus $\Sigma E = \{(x_1, 0), \cdots, (x_n, 0)\}$. We call

$$((\alpha_1; \beta_1, \gamma_1), \cdots, (\alpha_n; \beta_n, \gamma_n))$$

the orbifold data of E.

Theorem 2.4. *Let $p: E \longrightarrow F$ be a complex line bundle over an oriented, closed Riemannian 2-orbifold $(F, \hat{g})$. Suppose that the total space E has orbifold data $((\alpha_1; \beta_1, \gamma_1), \cdots, (\alpha_n; \beta_n, \gamma_n))$. Choose a fiber metric $\tilde{g}$ in E and let $\tilde{\nabla}$ be a $\tilde{g}$-preserving connection in E. Then (E, g) becomes a Riemannian orbifold. Assume that $\tilde{R}$ is constant on F. Then the η-invariant of the circle bundle of radius r is given by*

$$\eta(S_rE) = \frac{2}{3}c_1\left\{\frac{\pi r^2}{\mathrm{Vol}(F)}\chi - \left(\frac{\pi r^2}{\mathrm{Vol}(F)}\right)^2 c_1^2\right\} + \frac{1}{3}c_1 - \varepsilon + \sum_{j=1}^n 4s(\beta_j, \gamma_j; \alpha_j)$$

where c_1 is the (rational) Euler number of the bundle $E \longrightarrow F$. χ is the (rational) Euler characteristic of the base orbifold F. $s(\beta_j, \gamma_j; \alpha_j)$ is the following *generalized Dedekind sum as in* [7]:

$$s(\beta, \gamma; \alpha) = \frac{1}{4\alpha}\sum_{k=1}^{\alpha-1} \cot\left(\frac{k\beta\pi}{\alpha}\right)\cot\left(\frac{k\gamma\pi}{\alpha}\right).$$

ε *is defined by*

$$\varepsilon = \begin{cases} 1 & \text{if } c_1 > 0, \\ 0 & \text{if } c_1 = 0, \\ -1 & \text{if } c_1 < 0. \end{cases}$$

Proof. Denote by $D_r(E)$ the disk-bundle of radius r of E. Let $g_0 = g\,|_{S_r(E)} \times dt^2$ be the product metric on $D_{r+1}(E) - D_r(E) = S_r(E) \times [0, 1]$. Let ∇_0 be the Riemannian connection determined by g_0 and ω_0, Ω_0 the connection form and curvature form of ∇_0 respectively. Choose a metric g_1 on $D_{r+1}(E) - D_r(E)$ such that $g_1 = g$ on $D_{r+\frac{1}{4}}(E) - D_r(E)$ and $g_1 = g_0$ on $D_{r+1}(E) - D_{r+\frac{3}{4}}(E)$. Let ∇_1 be the Riemannian connection determined by the metric g_1 and ω_1, Ω_1 the connection form and curvature form of ∇_1 respectively. Write $\omega = \omega_1 - \omega_0$ and $\omega_t = (1-t)\omega_0 + t\omega_1$. Then from Lemma 2.2, with respect to the basis $\{\frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial r}, e_3, e_4\}$, the components of ω_t on $D_{r+\frac{1}{4}}(E) - D_r(E)$ are given by

$$(\omega_t)_2^1 = t\widetilde{d\theta}, \quad (\omega_t)_3^1 = -\frac{1}{2}r\tilde{R}e_4^\star, \quad (\omega_t)_4^1 = \frac{1}{2}r\tilde{R}e_3^\star,$$
$$(\omega_t)_3^2 = (\omega_t)_4^2 = 0, \quad (\omega_t)_4^3 = -\frac{1}{2}r^2\tilde{R}\widetilde{d\theta} + p^\star\hat{\omega}_4^3.$$

It follows that $(\Omega_t)_1^2 = tp^\star\tilde{\Omega}_1^2$. Also, $\omega_1^2 = -\widetilde{d\theta}$ and $\omega_s^q = 0$ others. From Lemma 2.3 we get

$$(\Omega_1^2)^2 = (\Omega_2^3)^2 = (\Omega_2^4)^2 = 0,$$
$$(\Omega_1^3)^2 + (\Omega_1^4)^2 = -\frac{1}{2}r^2\tilde{R}^3 e_3^\star \wedge e_4^\star \wedge rdr \wedge \widetilde{d\theta},$$
$$(\Omega_3^4)^2 = -\frac{3}{2}r^2\tilde{R}^3 e_3^\star \wedge e_4^\star \wedge rdr \wedge \widetilde{d\theta} + 2p^\star\hat{\Omega}_3^4 \wedge \tilde{R}dr \wedge r\widetilde{d\theta}.$$

Thus,

$$L_1(g) = \frac{1}{3}P_1(g) = \frac{1}{12\pi^2}\sum_{s<q}(\Omega_s^q)^2$$
$$= \frac{1}{6\pi^2}(\tilde{R}p^\star\hat{\Omega}_3^4 \wedge rdr \wedge \widetilde{d\theta} - r^2\tilde{R}^3 p^\star\omega_F \wedge rdr \wedge \widetilde{d\theta}),$$

$$TL_1 = 2\int_0^1 L_1(\omega, \Omega_t)dt$$
$$= 2\cdot\frac{1}{12\pi^2}\int_0^1 tp^\star\tilde{\Omega}_1^2 \wedge (-\widetilde{d\theta})dt$$
$$= -\frac{1}{12\pi^2}p^\star\tilde{\Omega}_1^2 \wedge \widetilde{d\theta}.$$

Also, note that $\hat{\Omega}_3^4$ and $\tilde{\Omega}_1^2$ represent $2\pi\chi(F)$ and $2\pi c_1(E)$ respectively. Thus, since $\tilde{R}$ is constant on F, we have

$$\tilde{R} = \frac{2\pi c_1}{\mathrm{Vol}(F)}.$$

Choose a partition of unity $\{f_\alpha\}_{\alpha\in\Lambda}$ subordinate to the open cover $\mathscr{U}$ of F. Then we get

$$\begin{aligned}
\int_{D_r(E)} L_1(g) &= \sum_{\alpha\in\Lambda} \frac{1}{|G_\alpha|} \int_{\tilde{E}_\alpha} (f_\alpha \circ p) \frac{1}{6\pi^2} (\tilde{R} p^\star \hat{\Omega}_3^4 \wedge r\,dr \wedge \widetilde{d\theta} \\
&\qquad\qquad - r^2 \tilde{R}^3 p^\star \omega_F \wedge r\,dr \wedge \widetilde{d\theta}) \\
&= \sum_{\alpha\in\Lambda} \frac{1}{|G_\alpha|} \int_{\tilde{U}_\alpha} f_\alpha \frac{1}{6\pi^2} \left(\pi r^2 \tilde{R} \hat{\Omega}_3^4 - \frac{1}{2}\pi r^4 \tilde{R}^2 \tilde{\Omega}_1^2\right) \\
&= \int_F \frac{1}{6\pi^2} \left(\pi r^2 \tilde{R} \hat{\Omega}_3^4 - \frac{1}{2}\pi r^4 \tilde{R}^2 \tilde{\Omega}_1^2\right) \\
&= \frac{1}{6\pi^2} \left(\pi r^2 \tilde{R} \hat{\Omega}_3^4 - \frac{1}{2}\pi r^4 \tilde{R}^2 \tilde{\Omega}_1^2\right) [F] \\
&= \frac{2}{3} c_1 \left(\frac{\pi r^2}{\mathrm{Vol}(F)} \chi - \left(\frac{\pi r^2}{\mathrm{Vol}(F)}\right)^2 c_1^2\right),
\end{aligned}$$

$$\begin{aligned}
\int_{S_r(E)} TL_1(g) &= -\sum_{\alpha\in\Lambda} \frac{1}{|G_\alpha|} \int_{\tilde{U}_\alpha \times S^1} (f_\alpha \circ p) \frac{1}{12\pi^2} (p^\star \tilde{\Omega}_1^2 \wedge \widetilde{d\theta}) \\
&= -\sum_{\alpha\in\Lambda} \frac{2\pi}{|G_\alpha|} \int_{\tilde{U}_\alpha} f_\alpha \frac{1}{12\pi^2} \tilde{\Omega}_1^2 = -\frac{1}{6\pi} \int_F \tilde{\Omega}_1^2 \\
&= -\frac{1}{6\pi} 2\pi c_1(E)[F] = -\frac{1}{3} c_1.
\end{aligned}$$

To determine $\mathrm{Sign}(D_r(E))$, we examine the following diagram:

$$\begin{array}{ccccc}
H^2(D_r(E), S_r(E)) & \otimes & H^2(D_r(E), S_r(E) & \longrightarrow & H^4(D_r(E), S_r(E)) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H_2(D_r(E)) & \otimes & H_2(D_r(E)) & \longrightarrow & H_0(D_r(E)) \\
\| & & \| & & \| \\
\mathbf{Z}(a) & \otimes & \mathbf{Z}(a) & \longrightarrow & \mathbf{Z}(1) \\
 & & & & \\
a & \bullet & a & \longrightarrow & c_1
\end{array}$$

where the vertical arrows are Poincaré-Lefschetz duality with $\mathbf{Q}$ coefficients and the last row is an orbifold version of the Euler characteristic of E (see

[11], for instance). Thus, we get

$$\operatorname{Sign}(D_r(E)) = \varepsilon = \begin{cases} 1 & \text{if } c_1 > 0, \\ 0 & \text{if } c_1 = 0, \\ -1 & \text{if } c_1 < 0. \end{cases}$$

Next, we determine the term

$$\sum \frac{1}{m_s} \langle L^{\Sigma_s}, [\tilde{\Sigma}_s D_r(E)] \rangle.$$

By the orbifold data, we have

$$\tilde{\Sigma} D_r(E) = \{(x_j, (g_j^k)) | j = 1, 2, \cdots, n, \ k = 1, 2, \cdots, \alpha_j - 1\}$$

and

$$m_j^k = \alpha_j \quad \text{for } k = 1, 2, \cdots, \alpha_j - 2.$$

Then,

$$\nu_{(x_j, (g_j^k))} = T_{(x_j, (g_j^k))} E = \mathbf{C}_1 \oplus \mathbf{C}_2$$

with g_j^k acting on $\mathbf{C}_1$ and $\mathbf{C}_2$ by multiplication by $e^{2\pi\beta_j k i/\alpha_j}$ and $e^{2\pi\gamma_j k i/\alpha_j}$ respectively. It follows that

$$\begin{aligned} \langle L^{\Sigma}, [\tilde{\Sigma}_j^k D_r(E)] \rangle &= \prod_{0<\theta\leq\pi} L_\theta(\nu^{g_j^k}_{(x_j, (g_j^k))})[(x_j, (g_j^k))] \\ &= \frac{e^{2\pi\beta_j k i/\alpha_j} + 1}{e^{2\pi\beta_j k i/\alpha_j} - 1} \cdot \frac{e^{2\pi\gamma_j k i/\alpha_j} + 1}{e^{2\pi\gamma_j k i/\alpha_j} - 1} \\ &= -\cot \frac{k\beta_j \pi}{\alpha_j} \cot \frac{k\gamma_j \pi}{\alpha_j}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum \frac{1}{m_s} \langle L^{\Sigma_s}, [\tilde{\Sigma}_s D_r(E)] \rangle &= -\sum_{j=1}^{n} \frac{1}{\alpha_j} \sum_{k=1}^{\alpha_j - 1} \cot \frac{k\beta_j \pi}{\alpha_j} \cot \frac{k\gamma_j \pi}{\alpha_j} \\ &= \sum_{j=1}^{n} \frac{1}{\alpha_j} (4\alpha_j s(\beta_j, \gamma_j; \alpha_j)) \\ &= 4 \sum_{j=1}^{n} s(\beta_j, \gamma_j; \alpha_j). \end{aligned}$$

Finally, from formula (2.1) we get

$$\begin{aligned} \eta(S_r E) &= \int_{D_r(E)} L_1 - \int_{S_r(E)} T L_1 - \operatorname{Sign}(D_r(E)) + \sum \frac{1}{m_s} \langle L^{\tilde{\Sigma}_s}, [\tilde{\Sigma}_s D_r(E)] \rangle \\ &= \frac{2}{3} c_1 \left\{ \frac{\pi r^2}{\operatorname{Vol}(F)} \chi - \left(\frac{\pi r^2}{\operatorname{Vol}(F)} \right)^2 c_1^2 \right\} + \frac{1}{3} c_1 - \varepsilon + 4 \sum_{j=1}^{n} s(\beta_j, \gamma_j; \alpha_j). \quad \square \end{aligned}$$

3. Application to geometric Seifert fibred 3-manifolds

This section is devoted to providing some explicit formulae for the η-invariant of geometric Seifert fibred 3-manifolds. We refer to [11] and [13] for basic material on Seifert fibred 3-manifolds and their relevant geometries.

A Seifert fibred 3-manifold can be viewed as an S^1-fibration $M \longrightarrow F$ over a closed 2-orbifold F.

Associated to M is the Seifert invariant $(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n))$.

The Euler number of the Seifert fibration is

$$e(M \longrightarrow F) = -\sum_{j=1}^{n} \frac{\beta_j}{\alpha_j}.$$

The Euler characteristic of the base orbifold F is

$$\chi = 2 - 2g - \sum_{j=1}^{n} \frac{\alpha_j - 1}{\alpha_j}.$$

The relevant geometry of a Seifert fibred 3-manifold is determined by e and χ as follows.

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$\mathbf{S}^2 \times \mathbf{E}^1$	$\mathbf{E}^3$	$\mathbf{H}^2 \times \mathbf{E}^1$
$e \neq 0$	$\mathbf{S}^3$	**Nil**	$\widetilde{\mathbf{PSL}}$

In what follows, we assume that the base space F is oriented.

Let $M = M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)) \longrightarrow F$ be a geometric Seifert fibred 3-manifold and $E \longrightarrow F$ be the associated $\mathbf{C}^1$-bundle. Then the set of singular points of the orbifold E is given by $\Sigma = \{(x_1, 0), \cdots, (x_n, 0)\}$. The local group $G_{x_j} = \mathbf{Z}/\alpha_j$ acts on the first and the second coordinate by multiplication by $e^{2\pi\beta_j i/\alpha_j}$ and $e^{2\pi i/\alpha_j}$ respectively.

Thus, in the expression of $\eta(M)$ in Theorem 2.4,

$$s(\beta_j, \gamma_j; \alpha_j) = s(\beta_j, 1; \alpha_j) = s(\beta_j, \alpha_j).$$

Choose a fiber metric $\tilde{g}$ on E such that the induced metric on M is the one from the corresponding Seifert geometry. Let $\tilde{\nabla}$ be a $\tilde{g}$-preserving connection on E.

Lemma 3.1. *If M is locally symmetric, then under the above assumption, $\tilde{R}$ is constant on F.*

Proof. Choose the same local basis $\{e_1 = \frac{\partial}{r\partial\theta}, e_2 = \frac{\partial}{\partial r}, e_3, e_4\}$ as in Section 2. For every $x \in M$, denote I_x the local reflection about x. Since $M = S_r(E)$ is locally symmetric, I_x is an isometry. Hence $dI_x = -Id$ commutes with ∇R. Thus

$$\begin{aligned} -\nabla_{e_\alpha} R(e_3, e_4)e_1 &= dI_x(\nabla_{e_\alpha} R(e_3, e_4)e_1) \\ &= \nabla_{-e_\alpha} R(-e_3, -e_4)(-e_1) = \nabla_{e_\alpha} R(e_3, e_4)e_1 \quad \text{for } \alpha = 3, 4. \end{aligned}$$

Therefore,

$$\nabla_{e_\alpha} R(e_3, e_4)e_1 = 0 \quad \text{for} \quad \alpha = 3, 4.$$

From Lemma 2.2, we have $\omega_3^2 = \omega_4^2 = 0$. Thus,

$$\nabla_{e_\alpha} e_2 = 0 \quad \text{for} \quad \alpha = 3, 4.$$

It follows that

$$\begin{aligned} e_\alpha(R_{1234}) &= e_\alpha(g(R(e_3, e_4)e_1, e_2)) \\ &= g(\nabla_{e_\alpha} R(e_3, e_4)e_1, e_2) + g(R(e_3, e_4)e_1, \nabla_{e_\alpha} e_2) \\ &= 0 \quad \text{for} \quad \alpha = 3, 4. \end{aligned}$$

Also, from formula (2) in Section 2, we have

$$\tilde{R} \circ \pi = \tilde{R}_{1234} \circ \pi = R_{1234}.$$

Thus we get $x_\alpha(\tilde{R}) = e_\alpha(R) = 0$ for $\alpha = 3, 4$. Hence, $\tilde{R}$ is constant on F. □

Now, we are in position to apply Theorem 2.4 to the geometric Seifert fibred 3-manifolds.

(a) M is modeled on $\mathbf{S}^2 \times \mathbf{E}^1$–, $\mathbf{E}^3$- or $\mathbf{H}^2 \times \mathbf{E}^1$-geometry.

Any Seifert manifold $M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n))$ with one of these geometries is locally symmetric and $c_1 = e = 0$. Thus we have by Theorem 2.4,

$$\eta(M) = 4\sum_{j=1}^{n} s(\beta_j, \alpha_j).$$

(b) M is modeled on $\mathbf{S}^3$-geometry.

Clearly, every Seifert manifold with this geometry is locally symmetric.

(1) $n \geq 3$. F has an $\mathbf{S}^2$-geometry. We have

$$2\pi\chi \cdot 2\pi r = \mathrm{Vol}(M) = \frac{\mathrm{Vol}(S^3(2))}{|\pi_1(M)|} = \frac{16\pi^2}{4|\frac{e}{\chi^2}|} = 4\pi^2 \frac{\chi^2}{|e|}.$$

Hence, $r = |\chi/e|$. Thus Theorem 2.4 yields

$$\eta(M) = \frac{1}{6}\frac{\chi^2}{e} + \frac{1}{3}e - \mathrm{sgn}(e) + 4\sum_{j=1}^{n} s(\beta_j, \alpha_j).$$

(2) $n \leq 2$. We have $M((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = L(p, q)$ where

$$\epsilon p = \alpha_1\beta_2 + \alpha_2\beta_1, \quad \epsilon q = \alpha_1\beta_2' + \alpha_2'\beta_1, \quad 1 = \alpha_2\beta_2' - \alpha_2'\beta_2,$$

and $\epsilon = \pm 1$.

In order to use Theorem 2.4, we need a "geometric fibration" $M \longrightarrow F$ such that F possesses a geometry from $\mathbf{S}^2$ as a 2-orbifold. This is equivalent to requiring that $\alpha_1 = \alpha_2$.

Lemma 3.2. *Any lens space $L(p, q)$ possesses exactly two geometric Seifert fibrations*

$$L(p, q) = M((\alpha, \gamma_1), (\alpha, \gamma_2))$$

where

$$\alpha = \frac{p}{\gcd(p, q-1)}, \quad \gamma_1 + \gamma_2 = \gcd(p, q-1), \quad \textit{and}$$

$$\gamma_2 \frac{q-1}{\gcd(p, q-1)} \equiv -1 \pmod{\alpha}$$

or

$$L(p, q) = M((\alpha', \gamma_1'), (\alpha', \gamma_2'))$$

where

$$\alpha' = \frac{p}{\gcd(p, q+1)}, \quad \gamma_1' + \gamma_2' = -\gcd(p, q+1), \quad \textit{and}$$

$$\gamma_2' \frac{q+1}{\gcd(p, q+1)} \equiv -1 \pmod{\alpha'}.$$

Proof. We will show it in one case. The other case is similar.

Suppose that $\alpha, \gamma_1, \gamma_2$ satisfies

$$p = \alpha\gamma_2 + \alpha\gamma_1, \quad q = \alpha\beta_2' + \alpha_2'\gamma_1, \quad 1 = \alpha\beta_2' - \alpha_2'\gamma_2.$$

Then we have

$$1 = \alpha\beta_2' - \frac{q-1}{\gamma_1+\gamma_2}\gamma_2 = \frac{p}{\gamma_1+\gamma_2}\beta_2' - \frac{q-1}{\gamma_1+\gamma_2}\gamma_2.$$

It follows that

$$\gamma_1 + \gamma_2 = \gcd(p, q-1), \quad \gamma_2\frac{q-1}{\gcd(p, q-1)} \equiv -1 \pmod{\alpha}, \quad \text{and}$$

$$\alpha = \frac{p}{\gcd(p, q-1)}.$$

The converse is straightforward. □

For the geometric fibration $L(p, q) = M((\alpha, \gamma_1), (\alpha, \gamma_2))$, we have

$$e = -\frac{\gamma_1+\gamma_2}{\alpha} = -\frac{(\gcd(p, q-1))^2}{p} \quad \text{and} \quad \chi = 2 - 2\frac{\alpha-1}{\alpha} = 2\frac{\gcd(p, q-1)}{p}.$$

Thus, from Theorem 2.4, we get

$$\begin{aligned}
\eta(L(p, q)) &= \eta(M((\alpha, \gamma_1), (\alpha, \gamma_2))) \\
&= \frac{1}{6}\frac{\chi^2}{e} + \frac{1}{3}e + \operatorname{sgn}(p) + 4\sum_{j=1}^{2} s(\gamma_j, \alpha) \\
&= -\frac{2}{3p} - \frac{1}{3}\frac{(\gcd(p, q-1))^2}{p} + \operatorname{sgn}(p) + 4\sum_{j=1}^{2} s(\gamma_j, \alpha).
\end{aligned}$$

where

$$\alpha = \frac{p}{\gcd(p, q-1)}, \quad \gamma_1 + \gamma_2 = \gcd(p, q-1), \quad \text{and}$$

$$\gamma_2 \frac{q-1}{\gcd(p, q-1)} \equiv -1 \pmod{\alpha}.$$

Similar computation yields

$$\begin{aligned}\eta(L(p, q)) &= \eta(M((\alpha', \gamma_1'), (\alpha', \gamma_2'))) \\ &= \frac{2}{3p} + \frac{1}{3}\frac{(\gcd(p, q+1))^2}{p} - \operatorname{sgn}(p) + 4\sum_{j=1}^{2} s(\gamma_j', \alpha').\end{aligned}$$

where

$$\alpha' = \frac{p}{\gcd(p, q+1)}, \quad \gamma_1' + \gamma_2' = -\gcd(p, q+1), \quad \text{and}$$

$$\gamma_2' \frac{q+1}{\gcd(p, q+1)} \equiv -1 \pmod{\alpha'}.$$

On the other hand, as computed by Atiyah-Patodi-Singer in [2]

$$\eta(L(p, q)) = -4s(q, p).$$

Thus, by equating the above three formulae, we get the following interesting identities about the Dedekind sums:

$$\begin{aligned} s(q, p) &= -\sum_{j=1}^{2} s(\gamma_j, \alpha) + \frac{1}{6p} + \frac{1}{12}\frac{(\gcd(p, q-1))^2}{p} - \frac{1}{4}\operatorname{sgn}(p) \\ &= -\sum_{j=1}^{2} s(\gamma_j', \alpha') - \frac{1}{6p} - \frac{1}{12}\frac{(\gcd(p, q+1))^2}{p} + \frac{1}{4}\operatorname{sgn}(p) \end{aligned}$$

where

$$\alpha = \frac{p}{\gcd(p, q-1)}, \quad \gamma_1 + \gamma_2 = \gcd(p, q-1), \quad \text{and}$$

$$\gamma_2 \frac{q-1}{\gcd(p, q-1)} \equiv -1 \pmod{\alpha}$$

and

$$\alpha' = \frac{p}{\gcd(p, q+1)}, \quad \gamma_1' + \gamma_2' = -\gcd(p, q+1), \quad \text{and}$$

$$\gamma_2' \frac{q+1}{\gcd(p, q+1)} \equiv -1 \pmod{\alpha'}.$$

In particular, let $q = 1$ and $p > 0$ in the first equality or $q = -1$ and $p < 0$ in the second equation, we get

$$s(1, p) = \frac{1}{12p}(p - \operatorname{sgn}(p))(p - 2\operatorname{sgn}(p)). \tag{3.1}$$

Thus, we have

$$\sum_{k=1}^{|p|-1} \left(\cot\left(\frac{\pi k}{p}\right)\right)^2 = \frac{1}{3}(p - \operatorname{sgn}(p))(p - 2\operatorname{sgn}(p)).$$

(c) M is modeled on $\widetilde{PSL}$-geometry. Equip $\mathbf{H}^2$ with the standard hyperbolic metric. Then we have a natural metric on $T(\mathbf{H}^2)$. The identification between $PSL(2, \mathbf{R})$ and the unit tangent bundle $T^1(\mathbf{H}^2)$ gives rise to a (left-invariant) metric on $PSL(2, \mathbf{R})$ which induces a metric on $\widetilde{PSL}$.

For a Seifert manifold $M \longrightarrow F$ with this geometry and the given metric on $\widetilde{PSL}$, we have $\tilde{R} = -1$ on F. Thus, Theorem 2.4 implies.

From a homomorphism $\pi_1(M) \longrightarrow \mathrm{Isom}(\widetilde{PSL})$ giving a geometric structure on M, we get $r = |\chi/e|$ (see [11]). It follows from Theorem 2.4 that

$$\eta(M) = -\frac{1}{2}\frac{\chi^2}{e} + \frac{1}{3}e - \mathrm{sgn}(e) + 4\sum_{j=1}^{n} s(\beta_j, \alpha_j). \tag{3.2}$$

From the above discussion, we have the following

Corollary 3.3. *Under the above five geometries, if we fix the metric in each universal cover as above, then the η-invariant depends only on the topology.*

Finally, under the **Nil**-geometry, the volume of the base orbifold is indeterminate, so the Seifert invariant alone is not sufficient to express $\eta(M)$.

We conclude this section with the following

Corollary 3.4. *Equip $PSL(2, \mathbf{R})$ with the above metric. Let $\Gamma \subseteq PSL(2, \mathbf{R})$ be a co-compact Fuchsian group of signature $\{g; \alpha_1, \cdots, \alpha_n\}$. Then*

$$\eta(PSL(2, \mathbf{R})/\Gamma) = \frac{1}{6}(2g + 4 + 7n) - \sum_{j=1}^{n}\left(\frac{1}{3}\alpha_j + \frac{5}{6\alpha_j}\right).$$

Proof. As shown in [11],

$$PSL(2, \mathbf{R})/\Gamma = M(g; (1, 2g-2), (\alpha_1, \alpha_1 - 1), \cdots, (\alpha_n, \alpha_n - 1)).$$

Thus,

$$\chi = e = 2 - 2g - \sum_{j=1}^{n}\frac{\alpha_j - 1}{\alpha_j}.$$

Also, from (3.1), we get

$$s(\alpha_j - 1; \alpha_j) = -s(1, \alpha_j) = -\frac{1}{12\alpha_j}(\alpha_j - 1)(\alpha_j - 2).$$

It follows from (3.2) that

$$\begin{aligned}\eta(PSL(2, \mathbf{R})/\Gamma) &= -\frac{1}{2}\chi + \frac{1}{3}\chi + 1 + 4\sum_{j=1}^{n} s(\alpha_j - 1, \alpha_j)\\ &= -\frac{1}{6}\chi + 1 + 4\sum_{j=1}^{n}\left(-\frac{1}{12\alpha_j}(\alpha_j - 1)(\alpha_j - 2)\right)\\ &= \frac{1}{6}(2g + 4 + 7n) - \sum_{j=1}^{n}\left(\frac{1}{3}\alpha_j + \frac{5}{6\alpha_j}\right). \quad \square\end{aligned}$$

Remark 1. A similar formula for the the η-invariant of $PSL(2, \mathbf{R})/\Gamma$ associated to the Dirac operator was obtained in [14].

4. Chern-Simons invariant and conformal immersions

Let M be a closed, oriented Riemannian 3-manifold. Chern and Simons defined a mod 1 invariant of M in [3], now commonly denoted by $CS(M)$, and showed that $CS(M) \equiv 0 \pmod 1$ if M conformally immerses into $\mathbf{E}^4$.

A surprising relation between $CS(M)$ and $\eta(M)$ is demonstrated by the following

Theorem 4.1 (Atiyah-Patodi-Singer [2]). *Let M be a closed, oriented Riemannian 3-manifold. Then*

$$CS(M) \equiv \frac{3}{2}\eta(M) + \frac{1}{2}\sigma(H_1(M;\mathbf{Z})) \pmod 1$$

where $\sigma(H_1(M;\mathbf{Z})) = \#$ of 2-primary summands in $H_1(M;\mathbf{Z})$.

In this section, we derive some elementary formulae for the Chern-Simons invariant of the geometric Seifert fibred 3-manifolds and show that some families of them cannot be conformally immersed into $\mathbf{E}^4$. We begin with the following

Lemma 4.2. *Let p, q be a pair of coprime positive integers. Choose r, s such that $ps + qr = 1$, $q + r$ is even and s is odd if p is even. Then*

$$6s(q, p) \equiv \frac{q+r}{2p} \pmod 1.$$

Proof. Case 1. p is odd. We have

$$6ps(q, p) = \frac{q+r_0}{2} + \frac{1}{2}pI(p, q) \in \mathbf{Z}$$

where r_0 is such that $qr_0 \equiv 1 \pmod p$, $-1 < r_0/p \leq 0$, and $I(p, q) \in \mathbf{Z}$ (see [7], for instance). Since p is odd, $q + r_0$ and $I(p, q)$ have the same parity. It follows that

$$6s(q, p) \equiv \frac{q + r_0 + \delta p}{2p} \pmod 1$$

with

$$\delta = \begin{cases} 0 & \text{if } q + r_0 \text{ is even}, \\ 1 & \text{if } q + r_0 \text{ is odd}. \end{cases}$$

Therefore we have

$$6s(q, p) \equiv \frac{q+r}{2p} \pmod 1$$

with $qr \equiv 1 \pmod p$ and $q + r$ even.

Case 2. p is even. Then q is odd.
From the Dedekind reciprocity law and Case 1, we get

$$\begin{aligned} 6s(q, p) &= -6s(p, q) + \frac{p^2+q^2+1-3pq}{2pq} \\ &\equiv \frac{-p-t}{2q} + \frac{p^2+q^2+1-3pq}{2pq} \pmod 1, \qquad pt \equiv 1 \pmod q,\ t \text{ even} \\ &\equiv \frac{q^2+1-ps}{2pq} \pmod 1, \qquad ps \equiv 1 \pmod q,\ s \text{ odd} \\ &\equiv \frac{q+r}{2p} \pmod 1, \qquad ps+qr=1,\ s \text{ odd}. \ \square \end{aligned}$$

Lemma 4.3. *Let*

$$M = M(g; (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n))$$

be a Seifert fibred 3*-manifold and* l *the number of even* α_j*'s. Then*

$$\sigma(H_1(M; \mathbf{Z})) = \begin{cases} l-1 & \text{if } l \geq 1, \\ 1 & \text{if } l = 0 \text{ and } \sum \beta_j \text{ is even}, \\ 0 & \text{if } l = 0 \text{ and } \sum \beta_j \text{ is odd}. \end{cases}$$

Proof. Arrange the Seifert invariant such that

$$M = M(g; (1, a), (\alpha_1, \beta_1), \cdots, (\alpha_l, \beta_l), (\alpha_{l+1}, \beta'_{l+1}), \cdots, (\alpha_n, \beta'_n))$$

where α_j is even for $j \leq l$, α_j is odd for $j > l$, β'_j is even for $j > l$, and a is the number of odd β_j's for $j > l$. Then we have

$$H_1(M; \mathbf{Z}) = \mathbf{Z}^{2g} \oplus \operatorname{Cok}\left(\begin{matrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a \\ & \alpha_1 & & & & & & \beta_1 \\ & & \ddots & & & & & \vdots \\ & & & \alpha_l & & & & \beta_l \\ & & & & \alpha_{l+1} & & & \beta_{l+1} \\ & & & & & \ddots & & \vdots \\ & & & & & & \alpha_n & \beta_n \end{matrix}\right).$$

Therefore

$$\sigma(H_1(M;\mathbf{Z})) = \sigma(\mathrm{Cok}(\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a \\ & \alpha_1 & & & & & & \beta_1 \\ & & \ddots & & & & & \vdots \\ & & & \alpha_l & & & & \beta_l \\ & & & & \alpha_{l+1} & & & \beta_{l+1} \\ & & & & & \ddots & & \vdots \\ & & & & & & \alpha_n & \beta_n \end{pmatrix}))$$

$$= \sigma(\mathrm{Cok}(\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a \\ & 0 & & & & & & 1 \\ & & \ddots & & & & & \vdots \\ & & & 0 & & & & 1 \\ & & & & 1 & & & 0 \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 & 0 \end{pmatrix})).$$

It follows that

$$\sigma(H_1(M;\mathbf{Z})) = \begin{cases} l-1 & \text{if } l \geq 1\,, \\ 1 & \text{if } l=0 \text{ and } a \text{ is even}, \\ 0 & \text{if } l=0 \text{ and } a \text{ is odd}, \end{cases}$$
$$= \begin{cases} l-1 & \text{if } l \geq 1\,, \\ 1 & \text{if } l=0 \text{ and } \sum \beta_j \text{ is even}, \\ 0 & \text{if } l=0 \text{ and } \sum \beta_j \text{ is odd}. \quad \square \end{cases}$$

By virtue of the formulae for $\eta(M)$ in Section 3, Theorem 4.1, Lemma 4.2 and Lemma 4.3, we derive the following elementary formulae for the Chern-Simons invariant of the geometric Seifert fibred 3-manifolds.

Without loss of generality, we assume in what follows that our Seifert invariant is in a normal form $((1\,,\,b)\,,(\alpha_1\,,\,\beta_1)\,,\,\cdots\,,(\alpha_n\,,\,\beta_n))$. For every pair of $(\alpha_j\,,\,\beta_j)$, choose $r_j\,,\ s_j$ such that $\alpha_j s_j + \beta_j r_j = 1\,,\ \beta_j + r_j$ is even and s_j is odd if α_j is even.

(a) M is modeled on $\mathbf{E}^3$, $\mathbf{S}^2\times\mathbf{E}^1$, or $\mathbf{H}^2\times\mathbf{E}^1$-geometry. We have

$$CS(M) \equiv 6\sum_{j=1}^{n} s(\beta_j\,,\,\alpha_j) + \frac{1}{2}\sigma(H_1(M;\mathbf{Z})) \pmod 1$$
$$\equiv \begin{cases} \sum_{j=1}^n r_j/(2\alpha_j) + (l-1)/2 \pmod 1 \text{ if } l\geq 1\,, \\ \sum_{j=1}^n r_j/(2\alpha_j) + 1/2 \pmod 1 \text{ if } l=0 \text{ and } b+\sum\beta_j \text{ is even}, \\ \sum_{j=1}^n r_j/(2\alpha_j) \pmod 1 \text{ if } l=0 \text{ and } b+\sum\beta_j \text{ is odd}. \end{cases}$$

(b) M is modeled on $\mathbf{S}^3$-geometry.

(i) $n \geq 3$. We have

$$CS(M) \equiv \frac{1}{4}\frac{\chi^2}{e} + \frac{1}{2}e + \frac{1}{2} + 6\sum_{j=1}^{n} s(\beta_j, \alpha_j) + \frac{1}{2}\sigma(H_1(M; \mathbf{Z})) \pmod 1.$$

Thus,

$$\begin{aligned} CS(M) &\equiv \frac{1}{4}\frac{\chi^2}{e} + \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} + \frac{1}{2}l \pmod 1 \quad \text{if } l \geq 1, \\ &\equiv \frac{\chi^2}{4e} + \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} \pmod 1 \ \text{ if } l = 0 \text{ and } b + \textstyle\sum \beta_j \text{ is even}, \\ &\equiv \frac{\chi^2}{4e} + \frac{1}{2} + \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} \pmod 1 \ \text{ if } l = 0 \text{ and } b + \textstyle\sum \beta_j \text{ is odd.} \end{aligned}$$

(ii) $n \leq 2$. We have $M((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = L(p, q)$ and

$$\begin{aligned} CS(L(p, q)) &\equiv -6s(q, p) + \frac{1}{2}\delta(L(p, q)) \pmod 1 \\ &\equiv -\frac{q + r}{2p} \pmod 1 \end{aligned} \tag{4.1}$$

where $ps + qr = 1$ with $q + r$ even and s even if p is even.

(c) M is modeled on $\widetilde{\mathbf{PSL}}$-geometry.

$$CS(M) \equiv -\frac{3}{4}\frac{\chi^2}{e} + \frac{1}{2}e + \frac{1}{2} + 6\sum_{j=1}^{n} s(\beta_j, \alpha_j) + \frac{1}{2}\sigma(H_1(M; \mathbf{Z})) \pmod 1.$$

Thus,

$$\begin{aligned} CS(M) &\equiv -\frac{3}{4}\frac{\chi^2}{e} + \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} + \frac{1}{2}l \pmod 1 \quad \text{if } l \geq 1, \\ &\equiv -\frac{3\chi^2}{4e} + \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} \pmod 1 \ \text{ if } l = 0 \text{ and } b + \textstyle\sum \beta_j \text{ is even}, \\ &\equiv -\frac{3\chi^2}{4e} + \frac{1}{2} + \sum_{j=1}^{n} \frac{r_j}{2\alpha_j} \pmod 1 \ \text{ if } l = 0 \text{ and } b + \textstyle\sum \beta_j \text{ is odd.} \end{aligned} \tag{4.2}$$

As Hirsch showed in [6], all compact 3-manifolds immerse in $\mathbf{R}^4$. We will show that some families of the geometric Seifert fibred 3-manifolds cannot be conformally immersed into $\mathbf{E}^4$.

Corollary 4.4. *Let F_g be the surface of genus $g > 1$ with a hyperbolic geometry. If g is even, then $T^1(F_g) = M(g, (1, 2g-2))$ with the induced metric from $\mathbf{H}^2$ doesn't conformally immerse into $\mathbf{E}^4$.*

Proof. We have $e = \chi = 2 - 2g$. It follows from (4.2) that

$$CS(T^1(F_g)) \equiv \frac{g}{2} - \frac{1}{2} \pmod 1 \neq 0 \text{ if } g \text{ is even.} \qquad \square$$

Corollary 4.5. *Equip $SO(3)$ with a bi-invariant metric. Let Γ be a finite subgroup of $SO(3)$. Then $SO(3)/\Gamma$ doesn't conformally immerse into $\mathbf{E}^4$.*

Proof. By the conformal invariance of the Chern-Simons invariant, we can assume that $SO(3)$ possesses the standard metric from $\mathbf{S}^3(1)$.

From (5) we have

$$CS(SO(3)) = CS(L(2, 1)) \equiv -\frac{1}{2} \pmod 1.$$

Thus, $SO(3)$ cannot be conformally immersed into $\mathbf{E}^4$. Hence $SO(3)/\Gamma$ cannot be conformally immersed into $\mathbf{E}^4$. $\square$

Remark. Heitsch and Lawson in [5] showed that a similar result holds in general for $SO(2k+1)/\Gamma$ where Γ is a discrete subgroup of $SO(2k+1)$ and $SO(2k+1)$ is equipped with a bi-invariant metric.

Corollary 4.6. *$L(p, q)$ with the standard metric cannot be conformally immersed into $\mathbf{E}^4$ except possibly when $q^2 + 1 \equiv 0 \pmod p$ and p is odd.*

Proof. If $p = 2k$ is even, then

$$L(p, q) = \mathbf{S}^3/(\mathbf{Z}/2k) = SO(3)/(\mathbf{Z}/k).$$

Thus, Corollary 4.5 implies that $L(p, q)$ cannot be conformally immersed into $\mathbf{E}^4$;

If p is odd, then from (4.1) we get

$$CS(L(p, q)) \equiv -\frac{q+r}{2p} \pmod 1$$

with $qr \equiv 1 \pmod p$ and $q + r$ even. Thus, $CS(L(p, q)) \equiv 0 \pmod 1$ implies $q + r \equiv 0 \pmod p$. Hence $q^2 + 1 \equiv 0 \pmod p$. $\square$

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