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L^p-BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS OF CLASS $S_{0,0}$

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ABSTRACT. We study the L^p-boundedness of pseudo-differential operators with the support of their symbols being contained in $E \times \mathbf{R}^n$, where E is a compact subset of $\mathbf{R}^n$, and their symbols have derivatives with respect to x only up to order k, in the Hölder continuous sense, where $k > n/2$ (the case $1 < p \leq 2$) and $k > n/p$ (the case $2 < p < \infty$). We also give a new proof of the L^p-boundedness, $1 < p < \infty$, of pseudo-differential operators of class $S^m_{0,0}$, where $m = m(p) = -n|1/p - 1/2|$, and $a \in S^m_{0,0}$ satisfies $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^m$ for $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$, $|\alpha| \leq k$ and $|\beta| \leq k'$, in the Hölder continuous sense, where $k > n/2$, $k' > n/p$ (the case $1 < p \leq 2$) and $k > n/p$, $k' > n/2$ (the case $2 < p < \infty$).

1. INTRODUCTION

Let $a(x, \xi)$ be a sufficiently regular function defined on $\mathbf{R}^n \times \mathbf{R}^n$. The pseudo-differential operators considered in this paper are of the following form:

$$a(x, D)u(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} e^{ix\xi} a(x, \xi)\hat{u}(\xi)\, d\xi, \quad x \in \mathbf{R}^n,\ u \in C_0^\infty(\mathbf{R}^n),$$

where $\hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi} u(x)\, dx$ is the Fourier transform of u. The function $a(x, \xi)$ is called the symbol of the operator $a(x, D)$. A symbol $a(x, \xi)$ is said to be of class $S^m_{\rho,\delta}$, where $m \in \mathbf{R}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$, if it satisfies the inequalities

$$(1.1) \qquad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|}, \quad x, \xi \in \mathbf{R}^n,$$

for all multi-indices α and β, where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. For the L^p-boundedness of pseudo-differential operators, we only require the derivatives $\partial_x^\alpha a$ (or $\partial_x^\alpha \partial_\xi^\beta a$) up to finite order. So we give the following definitions.

Definition 1.1. Let $m \in \mathbf{R}$, $k > 0$, and $k \notin \mathbf{N}$. We define $\Lambda_k^m(\mathbf{R}^n \times \mathbf{R}^n)$ to be the collection of continuous functions $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ whose derivatives $\partial_x^\alpha a$ in the distribution sense satisfy the following conditions:

(1.2) There is a constant $C > 0$ such that for $\alpha \in \mathbf{N}^n$, $|\alpha| \leq [k]$ and $x, \xi, h \in \mathbf{R}^n$, we have

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(1) If $|\alpha| \le [k]$ then $|\partial_x^\alpha(x, \xi)| \le C\langle\xi\rangle^m$.

(2) If $|h| \le 1$ and $|\alpha| = [k]$ then $|\partial_x^\alpha a(x+h, \xi) - \partial_x^\alpha(x, \xi)| \le C\langle\xi\rangle^m |h|^{k-[k]}$.

We denote by $\|a\|_{m,k}$ the smallest C such that (1.2) holds. Throughout all sections, we denote a constant depending only on n and m (resp. n, m, s or n) by $C_{n,m}$ (resp. $C_{n,m,s}$ or C_n), which may vary from time to time.

Definition 1.2. Let $m \in \mathbf{R}$, $0 \le \delta < 1$, $k, k' > 0$, and $k, k' \notin \mathbf{N}$. We define $\Lambda^m_{\delta,k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$ to be the collection of continuous functions $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy the following conditions:

(1.3) There is a constant $C > 0$ such that for $\alpha, \beta \in \mathbf{N}^n$, $|\alpha| \le [k]$, $|\beta| \le [k']$ and $x, \xi, h, \eta \in \mathbf{R}^n$, we have

(1) If $|\alpha| \le [k]$ and $|\beta| \le [k']$ then

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C\langle\xi\rangle^{m+\delta(|\alpha|-|\beta|)}.$$

(2) If $|h| \le 1$, $|\alpha| = [k]$ and $|\beta| \le [k']$ then

$$|\partial_x^\alpha \partial_\xi^\beta a(x+h, \xi) - \partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C\langle\xi\rangle^{m+\delta(k-|\beta|)}|h|^{k-[k]}.$$

(3) If $|\eta| \le 1$, $|\alpha| \le [k]$ and $|\beta| = [k']$ then

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi+\eta) - \partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C\langle\xi\rangle^{m+\delta(|\alpha|-k')}|\eta|^{k'-[k']}.$$

(4) If $|h| \le 1$, $|\eta| \le 1$, $|\alpha| = [k]$ and $|\beta| = [k']$ then

$$|\partial_x^\alpha \partial_\xi^\beta a(x+h, \xi+\eta) - \partial_x^\alpha \partial_\xi^\beta a(x+h, \xi) - \partial_x^\alpha \partial_\xi^\beta a(x, \xi+\eta) + \partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C\langle\xi\rangle^{m+\delta(k-k')}|h|^{k-[k]}|\eta|^{k'-[k']}.$$

Remark 1.1. In the case of $\delta = 0$, we denote by $\Lambda^m_{k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$ the class $\Lambda^m_{0,k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$ and by $\|a\|_{m,k,k'}$ the smallest C such that (1.3) holds.

As to the boundedness of the pseudo-differential operators with symbols belonging to the class $S^m_{\rho,\delta}$ or $\Lambda^m_{\delta,k,k'}$, the following theorems are known.

Theorem A. *Let $1 < p < \infty$, $\delta = \rho = 0$ and $m = -n|1/p - 1/2|$. If k, k' are sufficiently large real numbers and $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy* (1.1) *with $|\alpha| \le k$ and $|\beta| \le k'$, then $a(x, D)$ is continuous from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$.*

Theorem A is due to Calderón-Vaillancourt [6] (the case $p = 2$) and Coifman-Meyer [8] (the case $1 < p < \infty$). Calderón-Vaillancourt proved it for $\alpha, \beta \in \{0, 1, 2, 3\}^n$. Coifman-Meyer proved it for $k, k' \ge 2n$. Cordes [9] proved it (the case $p = 2$) for $|\alpha|, |\beta| \le [n/2] + 1$.

Theorem B. *Let $1 < p < \infty$, $0 \le \delta = \rho < 1$ and $m = -n(1-\rho)|1/p - 1/2|$. If k, k' are sufficiently large real numbers and $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta a$ in the distribution sense satisfy* (1.1) *with $|\alpha| \le k$ and $|\beta| \le k'$, then $a(x, D)$ is continuous from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$.*

Theorem B is due to Calderón-Vaillancourt [7] (the case $p = 2$) and Fefferman [10] (the case $1 < p < \infty$; cf. Wang-Li [24, p. 194]). Calderón-Vaillancourt proved it for $|\beta| \le 2[n/2] + n$ and $|\alpha| \le 2m'$ with $m' \in \mathbf{N}$

and $m'(1-\rho) \geq 5n/4$. Coifman-Meyer [8] proved it (the case $p=2$) for $|\alpha|, |\beta| \leq m'$ with $m' \in \mathbf{N}$ and $m' \geq [n/2]+1$. Kato [13] proved it (the case $p=2$) by using the method of Cordes [9]. Beal [2] proved it (the case $p=2$ and $-\infty < \rho < 1$). Nagase [17] proved it (the case $2 \leq p \leq \infty$) for $k, k' = [n/2]+1$. I. L. Hwang [12] proved it (the case $p=2$ and $-\infty < \rho < 1$) for $\alpha, \beta \in \{0, 1\}^n$.

Miyachi [14], [15] proved the following theorem, which gives the sharpest results.

Theorem C. *Let* $0 \leq \delta < 1$ *and* $m = -n(1-\delta)|1/p - 1/2|$.

(1) *If* $0 < p \leq 1, \delta = 0, k > n/2, k' > n/p$ *and* $a \in \Lambda^m_{\delta,k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$, *then* $a(x, D)$ *is continuous from* $H^p(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$, *where* H^p *are the Hardy spaces.*

(2) *If* $0 < p < 1, k > n/2, k' > n/p$ *and* $a \in \Lambda^m_{\delta,k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$, *then* $a(x, D)$ *is continuous from* $h^p(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$, *where* h^p *are the local Hardy spaces.*

(3) *If* $1 < p \leq 2, k > n/2, k' > n/p$ *and* $a \in \Lambda^m_{\delta,k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$, *then* $a(x, D)$ *is continuous from* $L^p(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$.

(4) *If* $2 < p < \infty, k > n/p, k' > n/2$ *and* $a \in \Lambda^m_{\delta,k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$, *then* $a(x, D)$ *is continuous from* $L^p(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$.

Sugimoto [20, 21] proved L^p-boundedness results, $0 < p < \infty$, by means of Besov spaces, which are an improvement of Theorem C with $\delta = 0$. Muramatu [16] also obtained some L^2-boundedness results by means of Besov spaces, which also are an improvement of Theorem C with $0 \leq \delta < 1$. Bourdaud-Meyer [4] proved Theorem C with $p = 2$ and $\delta = 0$, and obtained the sharpest result.

The following theorem is Sugimoto's result ([22, Theorem 2.2]), which is closely related to our Theorem 3.1.

Theorem D. *Let* (1) $p = 2, \mathbf{q} = (q, q') \in [2, \infty)^2 \cup \{\infty\} \times \{2, \infty\}$ *or* (2) $p \in [1, 2), \mathbf{q} = (q, q') \in (2, \infty) \times [2, \infty) \cup \{\infty\} \times \{2, \infty\}$. *Then for* $a \in B^{(n/2-n/q', n/p-n/q)}_{\mathbf{q},(1,1),(0,n/p-n/2)}$ *and* $f \in \mathcal{S} \cap H^p$, *we have*

$$\|a(x, D)f\|_{L^p} \leq c\|a(x, \xi)\|_{B^{(n/2-n/q', n/p-n/q)}_{\mathbf{q},(1,1),(0,n/p-n/2)}}\|f\|_{H^p},$$

where c *is a constant independent of* a *and* f, $\mathcal{S}$ *is the collection of rapidly decreasing functions,* H^p *is the Hardy space, and* $B^{(n/2-n/q', n/p-n/q)}_{\mathbf{q},(1,1),(0,n/p-n/2)}$ *is a Besov space defined in* [22].

In this paper, we prove the L^p-boundedness, $1 < p < \infty$, of pseudo-differential operators with the support of their symbols being contained in $E \times \mathbf{R}^n$, where E is a compact subset of $\mathbf{R}^n$. We also prove Theorem C with $\delta = 0$ and $1 < p < \infty$. The method originated in [12] and is different from Miyachi and Sugimoto (they used an interpolation theorem; see [18], [19] and [5, §3]). Roughly speaking, our proof is that for $u, v \in C_0^\infty(\mathbf{R}^n)$, $a \in \Lambda^m_k(\mathbf{R}^n \times \mathbf{R}^n)$ and $\operatorname{supp} a \subseteq E \times \mathbf{R}^n$, where E is a compact subset of $\mathbf{R}^n$, we can write $(a(x, D)u, v)$ in the following form:

$$(a(x, D)u, v) = \sum_{i=1}^{r_1} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} b_i(x, \xi)\hat{u}(\xi)h_i(x, \xi)\, d\xi\, dx,$$

where $r_1 \in \mathbf{N}$. Similarly, for $u, v \in C_0^\infty(\mathbf{R}^n)$ and $a \in \Lambda^m_{k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$, we also can write $(a(x, D)u, v)$ in the following form:

$$(a(x, D)u, v) = \sum_{r=1}^{r_2} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} b_i(x, \xi) g_i(x, \xi) h_i(x, \xi)\, d\xi\, dx,$$

where $r_2 \in \mathbf{N}$. Here, $b_i(x, \xi)$ are related to $a(x, \xi)$ and its derivatives, g_i, h_i are Wigner functions which have the following forms:

$$(1) \quad g_i(x, \xi) = \int_{\mathbf{R}^n} e^{-iy\xi} \varphi_i(x - y) u(y)\, dy,$$

$$(2) \quad h_i(x, \xi) = \int_{\mathbf{R}^n} e^{ix\lambda} \psi_i(\xi + \lambda) \bar{\hat{v}}(\lambda)\, d\lambda,$$

where $x, \xi \in \mathbf{R}^n$ and $\varphi_i \in L^p(\mathbf{R}^n)$, $\psi_i \in L^2(\mathbf{R}^n)$ (the case $1 < p \leq 2$), $\varphi_i \in L^2(\mathbf{R}^n)$, $\psi_i \in L^p(\mathbf{R}^n)$ (the case $2 < p < \infty$).

Then, by Paley's inequality, we can get

$$|(a(x, D)u, v)| \leq C\|u\|_{L^p}\|v\|_{L^q}, \quad 1 < p < \infty,\ 1/p + 1/q = 1,$$

where $C = C_{E,n,p,k}\|a\|_{m,k}$ or $C_{n,p,k,k'}\|a\|_{m,k,k'}$.

The contents of the following sections are as follows. In §2, we give some lemmas and corollaries. In §3, we state the main results. §4 is devoted to the proofs of our main results. In §5, we give two counterexamples which are related to the class of $\Lambda^m_k(\mathbf{R}^n \times \mathbf{R}^n)$.

2. Lemmas and Corollaries

In order to prove the L^p-boundedness of a pseudo-differential operators, we need to study the Fourier transform of its symbol. First, we have the following lemma. Its proof can be found in [1].

Lemma 2.1. *Let* $\varphi_s(\lambda) = (1 + |\lambda|^2)^{s/2}$ *with* $\lambda \in \mathbf{R}^n$ *and* $0 < s < 1$. *Then the Fourier transform of* φ_s *has the following properties*:

$$\widehat{\varphi_s} \in C^\infty(\mathbf{R}^n \backslash \{0\}). \tag{2.1}$$

There are constants $C_{n,s}$ *and* $C_{n,s,t}$ *such that*

$$|\widehat{\varphi_s}(x)| \leq C_{n,s,t}|x|^{-t-1} \quad \text{for } |x| > 1 \text{ and } t \in \mathbf{N}, \tag{2.2}$$

and

$$|\widehat{\varphi_s}(x)| \leq C_{n,s}|x|^{-n-s} \quad \text{for } 0 < |x| \leq 1. \tag{2.3}$$

Remark 2.1. In fact, if we define $\varphi_{s,\varepsilon}(\lambda) = \varphi_s(\lambda) e^{-\varepsilon|\lambda|^2}$, $\lambda \in \mathbf{R}^n$ and $0 < s, \varepsilon < 1$, then $\widehat{\varphi_{s,\varepsilon}}$ satisfies (2.1)–(2.3) with $C_{n,s}$ and $C_{n,s,t}$ independent of ε.

For $a \in \Lambda^m_{k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$, we define $\hat{a}^1$, $\hat{a}^2$ as follows:

$$(1) \quad \hat{a}^1(\lambda, \xi) = \int_{\mathbf{R}^n} e^{-ix\lambda} a(x, \xi)\, dx, \quad \lambda, \xi \in \mathbf{R}^n.$$

$$(2) \quad \hat{a}^2(x, y) = \int_{\mathbf{R}^n} e^{-i\xi y} a(x, \xi)\, d\xi, \quad x, y \in \mathbf{R}^n.$$

Then we have the following lemma.

Lemma 2.2. *Let* $m \in \mathbf{R}$, $0 < s < k$, $k' < 1$, $a \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n) \cap \Lambda^m_{k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$ *and* $\varphi_s(\lambda) = (1 + |\lambda|^2)^{s/2}$, $\lambda \in \mathbf{R}^n$. *Suppose* $\widehat{g_1}^1(\cdot, \xi) = \hat{a}^1(\cdot, \xi)\varphi_s(\cdot)$ *and* $\widehat{g_2}^2(x, \cdot) = \hat{a}^2(x, \cdot)\varphi_s(\cdot)$, $x, \xi \in \mathbf{R}^n$. *Then we have*

$$|g_i(x, \xi)| \le C_{n,m,s}\|a\|\langle\xi\rangle^m, \quad x, \xi \in \mathbf{R}^n, \tag{2.4}$$

where $i = 1, 2$, $\|a\| = \|a\|_{m,k,k'}$ *and* $C_{n,m,s}$ *depends only on* k *or* k'.

Proof. We shall prove the case $i = 2$ only, since the proof of the case $i = 1$ is similar. Without loss of generality, we may assume that $\varphi_s(\lambda) = \varphi_{s,\varepsilon}(\lambda) = (1 + |\lambda|^2)^{s/2}e^{-\varepsilon|\lambda|^2}$, $\lambda \in \mathbf{R}^n$ and $0 < \varepsilon < 1$. Then we have

$$\begin{aligned} g_2(x, \xi) &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} e^{i\xi y}\hat{a}^2(x, y)\varphi_{s,\varepsilon}(y)\, dy \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \widehat{\varphi_{s,\varepsilon}}(\eta)a(x, \eta + \xi)\, d\eta \\ &= I_1(x, \xi) + I_2(x, \xi), \quad x, \xi \in \mathbf{R}^n, \end{aligned} \tag{2.5}$$

where

$$I_1(x, \xi) = \left(\frac{1}{2\pi}\right)^n \int_{|\eta|\le 1} \widehat{\varphi_{s,\varepsilon}}(\eta)a(x, \eta + \xi)\, d\eta, \tag{2.6}$$

and

$$I_2(x, \xi) = \left(\frac{1}{2\pi}\right)^n \int_{|\eta|> 1} \widehat{\varphi_{s,\varepsilon}}(\eta)a(x, \eta + \xi)\, d\eta. \tag{2.7}$$

By (1.3) and (2.2), we obtain

$$|I_2(x, \xi)| \le C_{n,s,t}\|a\| \int_{|\eta|>1} |\eta|^{-1-t}(1 + |\eta + \xi|^2)^{m/2}\, d\eta, \quad t \in \mathbf{N}.$$

We have the following simple inequality (called Peetre's inequality; see [23, p. 17]):

(2.8) For all real numbers s' and for all vectors $\theta, \theta' \in \mathbf{R}^n$,

$$(1 + |\theta|^2)^{s'} \le 2^{|s'|}(1 + |\theta - \theta'|^2)^{|s'|}(1 + |\theta'|^2)^{s'}.$$

We get

$$|I_2(x, \xi)| \le C_{n,m,s}\|a\|\langle\xi\rangle^m.$$

We now estimate I_1. First, we write I_1 in the form

$$I_1(x, \xi) = I_{1,1}(x, \xi) + I_{1,2}(x, \xi), \quad x, \xi \in \mathbf{R}^n,$$

where

$$I_{1,1}(x, \xi) = \left(\frac{1}{2\pi}\right)^n \int_{|\eta|\le 1} \widehat{\varphi_{s,\varepsilon}}(\eta)(a(x, \eta + \xi) - a(x, \xi))\, d\eta,$$

and

$$I_{1,2}(x, \xi) = \left(\frac{1}{2\pi}\right)^n \int_{|\eta|\le 1} \widehat{\varphi_{s,\varepsilon}}(\eta)\, d\eta \cdot a(x, \xi).$$

By (1.3) and (2.3), we get

$$\begin{aligned} |I_{1,1}(x, \xi)| &\le C_{n,s}\|a\|\langle\xi\rangle^m \int_{|\eta|\le 1} \frac{1}{|\eta|^{n+s-k'}}\, d\eta \\ &\le C_{n,s}\|a\|\langle\xi\rangle^m, \quad \text{where } C_{n,s} \text{ depends on } k'. \end{aligned}$$

Since $\varphi_{s,\varepsilon}(0)=1$ and $\int_{|\eta|>1}|\widehat{\varphi_{s,\varepsilon}}(\eta)|\,d\eta \le C_{n,s}$, we obtain

$$|I_{1,2}(x,\xi)| \le C_{n,s}\|a\|\langle\xi\rangle^m, \quad \text{where } C_{n,s} \text{ depends on } k'. \quad \square$$

Corollary 2.1. *Let* $m\in\mathbf{R}$, $0<s<k<1$, $a\in C_0^\infty(\mathbf{R}^n\times\mathbf{R}^n)\cap\Lambda_k^m(\mathbf{R}^n\times\mathbf{R}^n)$ *and* $\varphi_s(\lambda)=(1+|\lambda|^2)^{s/2}$, $\lambda\in\mathbf{R}^n$. *Suppose* $\hat{g}^1(\cdot,\xi)=\hat{a}^1(\cdot,\xi)\varphi_s(\cdot)$, $\xi\in\mathbf{R}^n$. *Then*

$$|g(x,\xi)|\le C_{n,m,s}\|a\|\langle\xi\rangle^m, \quad x,\xi\in\mathbf{R}^n,$$

where $\|a\|=\|a\|_{m,k}$ *and* $C_{n,m,s}$ *depends only on* k.

Proof. This is an immediate consequence of Lemma (2.2). $\square$

Corollary 2.2. *Let* $m\in\mathbf{R}$, $0<s<k<1$, $0<s'<k'<1$, $a\in C_0^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ $\cap\,\Lambda^m_{k,k'}(\mathbf{R}^n\times\mathbf{R}^n)$ *and* $\varphi_{\tilde{s}}(\lambda)=(1+|\lambda|^2)^{\tilde{s}/2}$, $\lambda\in\mathbf{R}^n$ *and* $\tilde{s}=s,s'$. *Suppose* $\hat{g}(\lambda,y)=\hat{a}(\lambda,y)\varphi_s(\lambda)\varphi_{s'}(y)$, $y,\lambda\in\mathbf{R}^n$. *Then we have*

$$|g(x,\xi)|\le C_{n,m,s}\|a\|\langle\xi\rangle^m, \quad x,\xi\in\mathbf{R}^n,$$

where $\|a\|=\|a\|_{m,k,k'}$ *and* $C_{n,m,s}$ *depends only on* k, k'.

Proof. Without loss of generality, we may assume that $\varphi_{\tilde{s}}(\lambda)=\varphi_{\tilde{s},\varepsilon}(\lambda)=(1+|\lambda|^2)^{\tilde{s}/2}e^{-\varepsilon|\lambda|^2}$, $\lambda\in\mathbf{R}^n$, $\tilde{s}=s,s'$ and $0<\varepsilon<1$. First, we have

$$\begin{aligned} g(x,\xi) &= \left(\frac{1}{2\pi}\right)^{2n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)a(z+x,\eta+\xi)\,dz\,d\eta \\ &= \left(\frac{1}{2\pi}\right)^{2n}\sum_{j=1}^4 I_j(x,\xi), \quad x,\xi\in\mathbf{R}^n, \end{aligned}$$

where

$$\begin{aligned} I_1(x,\xi) &= \int_{|z|\le1}\int_{|\eta|\le1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)a(z+x,\eta+\xi)\,dz\,d\eta, \\ I_2(x,\xi) &= \int_{|z|\le1}\int_{|\eta|>1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)a(z+x,\eta+\xi)\,dz\,d\eta, \\ I_3(x,\xi) &= \int_{|z|>1}\int_{|\eta|\le1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)a(z+x,\eta+\xi)\,dz\,d\eta, \end{aligned}$$

and

$$I_4(x,\xi) = \int_{|z|>1}\int_{|\eta|>1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)a(z+x,\eta+\xi)\,dz\,d\eta.$$

By (1.3), (2.2), and (2.8), we get

$$|I_4(x,\xi)|\le C_{n,m,s}\|a\|\langle\xi\rangle^m, \quad \text{where } C_{n,m,s} \text{ depends on } k, k'.$$

We now estimate I_1 only and leave the estimates of I_2 and I_3 to the reader. We write I_1 in the form

$$I_1(x,\xi)=\sum_{t=1}^4 I_{1,t}(x,\xi), \quad x,\xi\in\mathbf{R}^n,$$

where

$$I_{1,1}(x,\xi)=\int_{|z|\le 1}\int_{|\eta|\le 1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)(a(z+x,\eta+\xi)-a(x+x,\xi)$$
$$-a(x,\eta+\xi)+a(x,\xi))\,dz\,d\eta,$$

$$I_{1,2}(x,\xi)=\int_{|z|\le 1}\int_{|\eta|\le 1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)(a(z+x,\xi)-a(x,\xi))\,dz\,d\eta,$$

$$I_{1,3}(x,\xi)=\int_{|z|\le 1}\int_{|\eta|\le 1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)(a(x,\eta+\xi)-a(x,\xi))\,dz\,d\eta,$$

and

$$I_{1,4}(x,\xi)=\int_{|z|\le 1}\int_{|\eta|\le 1}\widehat{\varphi_{s,\varepsilon}}(z)\widehat{\varphi_{s',\varepsilon}}(\eta)\,dz\,d\eta\cdot a(x,\xi).$$

By (1.3) and (2.3), we get

$$|I_{1,t}(x,\xi)|\le C_{n,m,s}\|a\|\langle\xi\rangle^m,$$

where $t=1,2,3,4$ and $C_{n,m,s}$ depends on k, k'. □

We shall state a crucial lemma which is related to the Winger function. Its proof can be found in [12, p. 62].

Lemma 2.3. *For* $u,\varphi\in C_0^\infty(\mathbf{R}^n)$, *we define*

$$g(x,\xi)=\int_{\mathbf{R}^n}e^{-iy\xi}\varphi(x-y)u(y)\,dy,$$

and

$$h(x,\xi)=\int_{\mathbf{R}^n}e^{ix\lambda}\varphi(\xi+\lambda)u(\lambda)\,d\lambda,\quad x,\xi\in\mathbf{R}^n.$$

Then we have

$$\|g\|_{L^2(\mathbf{R}^n\times\mathbf{R}^n)}=\|h\|_{L^2(\mathbf{R}^n\times\mathbf{R}^n)}=(2\pi)^{n/2}\|\varphi\|_{L^2(\mathbf{R}^n)}\|u\|_{L^2(\mathbf{R}^n)}.$$

To prove the L^p-boundedness of pseudo-differential operators, we also need the following lemma which is related to the Hausdorff-Young inequality and Paley's inequality (see [3, Chapter I] and [11, p. 105]). It can be found in [3, p. 17] and [11, p. 106].

Lemma 2.4. *If* $1<p\le 2$, $1/p+1/q=1$ *and* $p\le r\le q$, *then*

$$\left(\int_{\mathbf{R}^n}|\xi|^{-n(1-r/q)}|\hat f(\xi)|^r\,d\xi\right)^{1/r}\le C_p\|f\|_{L^p(\mathbf{R}^n)}.$$

Remark 2.2. In this paper, Lemma 2.4 is applied in the case of $r=2$.

We shall use the following partition of unity. We leave its construction to the reader.

Let $r>0$ and $s=1,\dots,n$. We define

$$\Gamma_{s,r}=\{\xi\in\mathbf{R}^n|\xi=(\xi_1,\dots,\xi_n),\ |\xi_t|\le r|\xi_s|\text{ if }t\ne s\}.$$

Then we can find $W_0\in C_0^\infty(\mathbf{R}^n)$ and $W_s\in C^\infty(\mathbf{R}^n)$, $s=1,\dots,n$, such that the following conditions hold:

(1) $0\le W_s\le 1$, $s=0,1,\dots,n$,

(2) $\operatorname{supp} W_0 \subseteq \{|\xi| \le 1\}$, $\operatorname{supp} W_s \subseteq \Gamma_{s,3/2} \cap \{|\xi| \ge 1/2\}$, $W_s(\xi) = W_s(\frac{\xi}{|\xi|})$ for $|\xi| \ge 1$, and $W_s(\xi) = 1$ for $\xi \in \Gamma_{s,1/2}$ and $|\xi| \ge 1$, $s = 1, \dots, n$,

(3) $\sum_{s=0}^n W_s \equiv 1$,

(4) for $\alpha \in \mathbf{N}^n$, there exists a constant $C_\alpha > 0$ such that

$$|\partial_\xi^\alpha W_s(\xi)| \le C_\alpha (1 + |\xi|)^{-|\alpha|}, \quad \xi \in \mathbf{R}^n \text{ and } s = 1, \dots, n.$$

To prove the L^p-boundedness, $2 < p < \infty$, of pseudo-differential operators, we need to study the Fourier transform of the following functions:

$$\psi_s(\xi) = W_s(\xi) \frac{1}{1 + i\xi_s^{[n/p]}} \frac{1}{(1 + |\xi|^2)^{\frac{1}{2}(n/p - [n/p] + \varepsilon/2)}}, \tag{2.9}$$

where $\xi \in \mathbf{R}^n$, $s = 1, \dots, n$, W_s are defined as above and $\varepsilon, \varepsilon > 0$, is so small that $n/p + \varepsilon/2 \notin \mathbf{N}$, $n/p - [n/p] + \varepsilon < 1$, $[n/p + \varepsilon] = [n/p]$ and $n/q - [n/q] \ne \varepsilon/2$ with $1/p + 1/q = 1$.

It is clear that $\psi_s \in L^p(\mathbf{R}^n)$, and we shall show that $\widehat{\psi_s} \in L^q(\mathbf{R}^n)$.

Lemma 2.5. *Let ψ_s be defined as in (2.9). Then we have*

$$|\widehat{\psi_s}(x)| \le C_{n,\varepsilon} |x|^{-n/q + \varepsilon/2} \quad \text{for } 0 < |x| \le 1, \tag{2.10}$$

$$|\widehat{\psi_s}(x)| \le C_{n,t} |x|^{-t} \quad \text{for } |x| > 1 \text{ and } t \in \mathbf{N}, \tag{2.11}$$

where $x \in \mathbf{R}^n$ and $1/p + 1/q = 1$ with $2 < p < \infty$.

Proof. (2.11) is obvious and we leave the proof to the reader. We shall estimate (2.10) in the case $s = 1$ only, since the estimates of the other cases are similar. First, we prove the case $n = 1$. Making the change of variables $x\xi \to \xi$, we have

$$|\hat{\psi}_1(x)| \le C|x|^{-1/q + \varepsilon/2} \left| \int_{\mathbf{R}} e^{-i\xi} \varphi(x, \xi)\, d\xi \right|, \quad x \in \mathbf{R}, 0 < |x| \le 1,$$

where

$$\varphi(x, \xi) = \frac{1}{(|x|^2 + |\xi|^2)^{\frac{1}{2}(1/p + \varepsilon/2)}}, \quad \xi \in \mathbf{R}.$$

Since $1/p + \varepsilon/2 < 1$, we have

$$\left| \int_0^1 e^{-i\xi} \varphi(x, \xi)\, d\xi \right| \le C_\varepsilon,$$

and, by integration by part, we have

$$\left| \int_1^\infty e^{-i\xi} \varphi(x, \xi)\, d\xi \right| \le C_\varepsilon.$$

Hence, we get

$$|\widehat{\psi_1}(x)| \le C_\varepsilon |x|^{-1/q + \varepsilon/2}, \quad x \in \mathbf{R}, 0 < |x| \le 1.$$

Now, we prove the case $n \ge 2$. First, we write $\widehat{\psi_1}$ in the form

$$\widehat{\psi_1}(x) = I_1(x) + (-i) I_2(x), \quad x \in \mathbf{R}^n, 0 < |x| \le 1,$$

where

$$I_k(x) = \int_{\mathbf{R}^n} e^{-ix\xi} \varphi_k(\xi)\, d\xi, \qquad k = 1, 2,$$

with

$$\varphi_1(\xi) = W_1(\xi)\frac{1}{1+\xi_1^{2[n/p]}}\frac{1}{(1+|\xi|^2)^{\frac{1}{2}(n/p-[n/p]+\varepsilon/2)}},$$

and

$$\varphi_2(\xi) = W_1(\xi)\frac{\xi_1^{[n/p]}}{1+\xi_1^{2[n/p]}}\frac{1}{(1+|\xi|^2)^{\frac{1}{2}(n/p-[n/p]+\varepsilon/2)}},$$

$$\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n.$$

We first estimate I_1, making use of the following identity:

$$\frac{1}{(-ix_j)^\alpha}(\partial_{\xi_j}^\alpha e^{-ix\xi}) = e^{-ix\xi},$$

where

$$x = (x_1, \dots, x_n) \in \mathbf{R}^n\backslash\{0\} \quad \text{and} \quad |x_j| = \max_{1\le t\le n}\{|x_t|\},$$

(2.12) $$\alpha = n - 2[n/p] \quad \text{if } [n/p] \ge 1,$$

(2.13) $$\alpha = [n/q] - 1 \quad \text{if } [n/p] = 0 \text{ and } n/q - [n/q] < \varepsilon/2,$$

and

$$\alpha = [n/q] \quad \text{if } [n/p] = 0 \text{ and } n/q - [n/q] > \varepsilon/2.$$

Without loss of generality, we may assume that $j = 1$ and we write I_1 in the form

$$I_1(x) = \frac{1}{(ix_1)^\alpha}\sum_{\alpha_1,\alpha_2,\alpha_3} C_{\alpha_1,\alpha_2,\alpha_3}\int_{\mathbf{R}^n} e^{-ix\xi}\varphi_{\alpha_1}(\xi)\varphi_{\alpha_2}(\xi)\varphi_{\alpha_3}(\xi)\,d\xi,$$

$$x = (x_1, \dots, x_n) \in \mathbf{R}^n,$$

where

$$\varphi_{\alpha_1}(\xi) = \partial_{\xi_1}^{\alpha_1}(W_1(\xi)),$$

$$\varphi_{\alpha_2}(\xi) = \partial_{\xi_1}^{\alpha_2}\left(\frac{1}{1+\xi_1^{2[n/p]}}\right),$$

$$\varphi_{\alpha_3}(\xi) = \partial_{\xi_1}^{\alpha_3}\left(\frac{1}{(1+|\xi|^2)^{\frac{1}{2}(n/p-[n/p]+\varepsilon/2)}}\right), \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n,$$

and

$$\alpha_1, \alpha_2, \alpha_3 \in \mathbf{N}, \quad \alpha_1 + \alpha_2 + \alpha_3 = \alpha.$$

Let

$$J(x) = \frac{1}{(ix_1)^\alpha}\int_{\mathbf{R}^n} e^{-ix\xi}\varphi_{\alpha_1}(\xi)\varphi_{\alpha_2}(\xi)\varphi_{\alpha_3}(\xi)\,d\xi, \quad x = (x_1, \dots, x_n) \in \mathbf{R}^n,$$

and we shall estimate the case $\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = 0$ only, since the estimates of the other cases are similar. First, we consider $[n/p] \ge 1$. By (2.12), we get

$$\begin{aligned}|J(x)| &\le C_{n,\varepsilon}|x_1|^{-\alpha}\int_{\mathbf{R}^n}\frac{1}{(1+|\xi|^2)^{\frac{1}{2}\alpha'}}\,d\xi \\ &\le C_{n,\varepsilon}|x|^{-\alpha},\end{aligned}$$

where $\alpha' = n + n/p - [n/p] + \varepsilon/2 > n$.

We now consider $[n/p]=0$ and we shall estimate the case $n/q-[n/q]<\varepsilon/2$ only, since the estimate of the other case is similar. By (2.13), we have

$$|J(x)| \le C_{n,\varepsilon}|x_1|^{-\alpha}\left|\int_{s_{n-1}}\int_0^{\infty} e^{-if(x,\xi)|x|r}\frac{r^{n-1}}{(1+r^2)^{\frac{1}{2}(n-1)}}\frac{1}{(1+r^2)^{\frac{1}{2}\alpha'}}\,dr\,ds\right|,$$

where $f(x,\xi)=\frac{x}{|x|}\frac{\xi}{|\xi|}$, $x,\xi\in\mathbf{R}^n$, $0<\alpha'=-(n/q-[n/q])+\varepsilon/2<1$ and s_{n-1} denotes the n-dimensional unit sphere.

Making the change of variables $(f(x,\xi)|x|)r\to r$, we have

$$|J(x)| \le C_{n,\varepsilon}|x_1|^{-n/q+\varepsilon/2}\left|\int_{s_{n-1}}\frac{1}{(f(x,\xi))^{1-\alpha'}}\right.$$
$$\left.\times\int_0^{\infty} e^{-ir}\frac{r^{n-1}}{((f(x,\xi)|x|^2)+r^2)^{\frac{1}{2}(n-1)}}\frac{1}{((f(x,\xi)|x|)^2+r^2)^{\frac{1}{2}\alpha'}}\,dr\,ds\right|.$$

By an argument similar to the proof of the case $n=1$, we obtain

$$\begin{aligned}|J(x)| &\le C_{n,\varepsilon}|x|^{-n/q+\varepsilon/2}\omega_{n-2}\left|\int_0^{\pi}\frac{1}{(\cos\theta)^{1-\alpha'}}(\sin\theta)^{n-2}\,d\theta\right|\\ &\le C_{n,\varepsilon}|x|^{-n/q+\varepsilon/2},\end{aligned}$$

where ω_{n-2} is the volume of the unit ball in $\mathbf{R}^{n-2}$.

Now, we estimate I_2, making use of the following identity:

$$\frac{1}{(-ix_j)^{\beta}}(\partial_{\xi_j}^{\beta}e^{-ix\xi})=e^{-ix\xi},$$

where

$$x=(x_1,\dots,x_n)\in\mathbf{R}^n\backslash\{0\}\quad\text{and}\quad |x_j|=\max_{1\le t\le n}\{|x_t|\},$$
$$\beta=[n/q]-1\quad\text{if } n/q-[n/q]<\varepsilon/2,$$

and

$$\beta=[n/q]\quad\text{if } n/q-[n/q]>\varepsilon/2.$$

By an argument similar to the proof of I_1 for the case $[n/p]=0$, we get

$$|I_2(x)|\le C_{n,\varepsilon}|x|^{-n/q+\varepsilon/2},\quad x\in\mathbf{R}^n,\ 0<|x|<1,$$

for either $n/q-[n/q]<\varepsilon/2$ or $n/q-[n/q]>\varepsilon/2$. □

Corollary 2.3. *For $\xi\in\mathbf{R}^n$ and $2<p<\infty$, we define*

$$\psi(\xi)=\frac{1}{(1+|\xi|^2)^{\frac{1}{2}(n/p+\varepsilon/2)}},$$

where ε, $\varepsilon>0$, is so small that $n/p+\varepsilon/2\notin\mathbf{N}$, $n/p-[n/p]+\varepsilon<1$, $[n/p+\varepsilon]=[n/p]$ and $n/q-[n/q]\ne\varepsilon/2$ with $1/p+1/q=1$.

Then we have

$$|\hat{\psi}(x)|\le C_{n,\varepsilon}|x|^{-n/q+\varepsilon/2}\quad\text{for } 0<|x|\le 1,$$

and

$$|\hat{\psi}(x)|\le C_{n,t}|x|^{-t}\quad\text{for } |x|>1 \text{ and } t\in\mathbf{N},$$

where $x\in\mathbf{R}^n$.

Proof. By an argument similar to the proof of Lemma 2.5, Corollary 2.3 is obtained. □

3. Main results

First, set $m = m(p) = -n|1/p - 1/2|$, $\|a\| = \|a\|_{m,k}$ if $a \in \Lambda_k^m(\mathbf{R}^n \times \mathbf{R}^n)$ and $\|a\| = \|a\|_{m,k,k'}$ if $a \in \Lambda_{k,k'}^m(\mathbf{R}^n \times \mathbf{R}^n)$. Then we have the following theorems.

Theorem 3.1. *Let* $1 < p \le 2$, $k > n/2$, $k \notin \mathbf{N}$, E *a compact subset of* $\mathbf{R}^n$ *and* $\Omega_1 = \{x \in \mathbf{R}^n | d(x, E) \le 1\}$. *If* $a \in \Lambda_k^m(\mathbf{R}^n \times \mathbf{R}^n)$ *and* $\operatorname{supp} a \subseteq E \times \mathbf{R}^n$, *then* $a(x, D)$ *is continuous from* $L^p(\mathbf{R}^n)$ *to* $L^p(E)$ *with its norm bounded by* $C_{E,n,p,k}|\Omega_1|^{1/p}\|a\|$, *where* $|\ |$ *denotes the Lebesgue measure.*

Theorem 3.2. *Let* $2 < p < \infty$, $k > n/p$, $k \notin \mathbf{N}$, *and* E *a compact subset of* $\mathbf{R}^n$. *If* $a \in \Lambda_k^m(\mathbf{R}^n \times \mathbf{R}^n)$ *and* $\operatorname{supp} a \subseteq E \times \mathbf{R}^n$, *then* $a(x, D)$ *is continuous from* $L^p_{\mathrm{loc}}(\mathbf{R}^n)$ *to* $L^p(E)$.

Theorem 3.3. *Let* $1 < p \le 2$, $k > n/2$, $k' > n/p$ *and* $k, k' \notin \mathbf{N}$. *If* $a \in \Lambda_{k,k'}^m(\mathbf{R}^n \times \mathbf{R}^n)$, *then* $a(x, D)$ *is continuous from* $L^p(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$ *with its norm bounded by* $C_{n,p,k,k'}\|a\|$.

Theorem 3.4. *Let* $2 < p < \infty$, $k > n/p$, $k' > n/2$ *and* $k, k' \notin \mathbf{N}$. *If* $a \in \Lambda_{k,k'}^m(\mathbf{R}^n \times \mathbf{R}^n)$, *then* $a(x, D)$ *is continuous from* $L^p(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$ *with its norm bounded by* $C_{n,p,k,k'}\|a\|$.

4. Proofs of the main results

Proof of Theorem 3.1. Without loss of generality, we may assume that

$$a \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n) \cap \Lambda_k^m(\mathbf{R}^n \times \mathbf{R}^n).$$

Let $k = n/2 + \varepsilon$ and $\varphi_2(\lambda) = (1 + |\lambda|^2)^{\frac{1}{2}(n/2 - [n/2] + \varepsilon/2)}$, where $\lambda \in \mathbf{R}^n$ and ε, $\varepsilon > 0$, is so small that $n/2 - [n/2] - \varepsilon < 1$ and $[n/2 + \varepsilon] = [n/2]$. Then for $u, v \in C_0^\infty(\mathbf{R}^n)$, we have

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{ix\xi} a(x, \xi)\hat{u}(\xi)\overline{v}(x)\, d\xi\, dx. \tag{4.1}$$

We write (4.1) in the form

$$\begin{aligned}(a(x, D)u, v) &= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{a}^1(\lambda, \xi)\hat{u}(\xi)\overline{\hat{v}}(\lambda + \xi)\, d\lambda\, d\xi \\ &= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{a}^1(\lambda, \xi)\varphi_2(\lambda)\varphi_2^{-1}(\lambda)\hat{u}(\xi)\overline{\hat{v}}(\lambda + \xi)\, d\lambda\, d\xi \\ &= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-ix\lambda} b(x, \xi)\hat{u}(\xi)\varphi_2^{-1}(\lambda)\overline{\hat{v}}(\lambda + \xi)\, d\lambda\, d\xi\, dx,\end{aligned} \tag{4.2}$$

where

$$\hat{b}^1(\cdot, \xi) = \hat{a}^1(\cdot, \xi)\varphi_2(\cdot), \qquad \xi \in \mathbf{R}^n.$$

Making use of the partition of unity W_s, $s = 0, 1, \dots, n$, we write (4.2) in the form

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^{2n} \sum_{s=0}^{n} I_s,$$

where

$$I_s = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-ix\lambda} b(x, \xi)\hat{u}(\xi) W_s(\lambda)\varphi_2^{-1}(\lambda)\overline{\hat{v}}(\lambda+\xi)\, d\lambda\, d\xi\, dx,$$
$$s = 0, 1, \dots, n.$$

We shall estimate I_1 only, since the other cases are similar. Integrating the above integral with respect to x first and making the use of the identity

$$\frac{1}{1 + i\lambda_1^{[n/2]}}(1 - (-i)^{1-[n/2]}\partial_{x_1}^{[n/2]})(e^{-ix\lambda}) = e^{ix\lambda},$$

we write I_1 in the form

$$I_1 = I_{1,1} + (i)^{1-[n/2]} I_{1,2},$$

where

$$I_{1,1} = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(x, \xi)\hat{u}(\xi) h(x, \xi)\, d\xi\, dx,$$

and

$$I_{1,2} = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \tilde{b}(x, \xi)\hat{u}(\xi) h(x, \xi)\, d\xi\, dx, \tag{4.3}$$

with

$$\begin{aligned} \tilde{b}(x, \xi) &= \partial_{x_1}^{[n/2]}(b(x, \xi)), \\ h(x, \xi) &= \int_{\mathbf{R}^n} e^{-ix\lambda}\psi(\lambda)\overline{\hat{v}}(\lambda+\xi)\, d\lambda, \qquad x, \xi \in \mathbf{R}^n, \end{aligned} \tag{4.4}$$

and

$$\psi(\lambda) = W_1(\lambda)\varphi_2^{-1}(\lambda)\frac{1}{1 + i\lambda_1^{[n/2]}}, \qquad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n.$$

We shall estimate $I_{1,2}$ only, since the estimate of $I_{1,1}$ is similar. First, we write (4.4) in the form

$$h(x, \xi) = \int_{\mathbf{R}^n} e^{i(x-z)\xi}\hat{\psi}(z)\overline{v}(x-z)\, dz. \tag{4.5}$$

Substituting (4.5) into (4.3), we write $I_{1,2}$ in the form

$$I_{1,2} = J_1 + J_2, \tag{4.6}$$

where

$$J_1 = \int_{\Omega_1}\int_{\mathbf{R}^n} \tilde{b}(x, \xi)\hat{u}(\xi) h(x, \xi)\, d\xi\, dx,$$

and

$$J_2 = \int_{\mathbf{R}\setminus\Omega_1}\int_{\mathbf{R}^n} \tilde{b}(x, \xi)\hat{u}(\xi) h(x, \xi)\, d\xi\, dx.$$

We first estimate J_1. By Hölder's inequality, Corollary 2.1, Lemma 2.4 and Parseval's formula, we obtain

$$|J_1| \le C \int_{\Omega_1} \left(\int_{\mathbf{R}^n} |\langle \xi \rangle^m \hat{u}(\xi)|^2 \, d\xi \right)^{1/2} \left(\int_{\mathbf{R}^n} |h(x, \xi)|^2 \, d\xi \right)^{1/2} dx$$
$$\le C \|u\|_{L^p(\mathbf{R}^n)} \int_{\Omega_1} \left(\int_{\mathbf{R}^n} |\hat{\psi}(z) \overline{v}(x - z)|^2 \, dz \right)^{1/2} dx,$$

where $C = C_{E,n,p,k} \|a\|$ and $1 < p \le 2$.

By Hölder's inequality and Minkowski's inequality, we get

$$|J_1| \le C |\Omega_1|^{1/p} \|u\|_{L^p(\mathbf{R}^n)} \|\psi\|_{L^2(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)}, \qquad 1/p + 1/q = 1.$$

Now, we estimate J_2. We have

$$\tilde{b}(x, \xi) = \left(\frac{1}{2\pi} \right)^n \int_{\mathbf{R}^n} \partial_{x_1}^{[n/2]} (a(x + y, \xi)) \hat{\varphi}(y) \, dy$$
$$= \left(\frac{1}{2\pi} \right)^n \int_{\mathbf{R}^n} (\partial_{x_1}^{[n/2]} a)(y, \xi) \hat{\varphi}(y - x) \, dy, \qquad x, \xi \in \mathbf{R}^n,$$

and $|x - y| > 1$ for $x \in \mathbf{R}^n \backslash \Omega_1$ and $y \in E$. Hence, by Hölder's inequality, Corollary 2.1, Lemma 2.4, and Parseval's formula, we have

$$|J_2| \le C \|u\|_{L^p(\mathbf{R}^n)} \int_{\mathbf{R}^n \backslash \Omega_1} A(x) \left(\int_{\mathbf{R}^n} |\hat{\psi}(z) \overline{v}(x - z)|^2 \, dz \right)^{1/2} dx,$$

where $C = C_{E,n,p,k} \|a\|$, $1 < p \le 2$, and

$$A(x) = \int_{\mathbf{R}^n} \frac{\chi_E(y)}{|x - y|^{n+1}} \, dy, \qquad x \in \mathbf{R}^n,$$

with

$$\chi_E(y) = \begin{cases} 1 & \text{if } y \in E. \\ 0 & \text{if } y \notin E. \end{cases}$$

By Hölder's inequality and Minkowski's inequality, we obtain

$$|J_2| \le C |E|^{1/p} \|u\|_{L^p(\mathbf{R}^n)} \|\psi\|_{L^2(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)}, \qquad 1/p + 1/q = 1. \quad \square$$

Proof of Theorem 3.2. Without loss of generality, we may assume that

$$a \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n) \cap \Lambda_k^m(\mathbf{R}^n \times \mathbf{R}^n).$$

Let $k = n/p + \varepsilon$ and $\varphi_p(\lambda) = (1 + |\lambda|^2)^{\frac{1}{2}(n/p - [n/p] + \varepsilon/2)}$, where $\lambda \in \mathbf{R}^n$ and ε, $\varepsilon > 0$, is so small that $n/p + \varepsilon/2 \notin \mathbf{N}$, $n/p - [n/p] + \varepsilon < 1$, $[n/p + \varepsilon] = [n/p]$ and $n/q - [n/q] \neq \varepsilon/2$ with $1/p + 1/q = 1$. It is enough to show that the conclusion holds in every open ball. So fix a ball, say B. Then for $u, v \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and $\operatorname{supp} u \subseteq B$, we have

$$(a(x, D)u, v) = \left(\frac{1}{2\pi} \right)^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\xi} a(x, \xi) \hat{u}(\xi) \overline{v}(x) \, d\xi \, dx.$$

Since the rest of the arguments are similar to the proof of Theorem 3.1, we shall only study the following lemma.

Lemma 4.1. *For* $u, v \in C_0^\infty(\mathbf{R}^n)$ *and* $\operatorname{supp} u \subseteq B$, *we define*

$$J = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \tilde{b}(x, \xi)\hat{u}(\xi)h(x, \xi)\, d\xi\, dx, \tag{4.7}$$

where

$$\tilde{b}(x, \xi) = \partial_{x_1}^{[n/p]}(b(x, \xi)), \qquad x = (x_1, \dots, x_n), \xi \in \mathbf{R}^n, 2 < p < \infty,$$

with

$$\begin{aligned} &\hat{b}^1(\cdot, \xi) = \hat{a}^1(\cdot, \xi)\varphi_p(\cdot), \\ &h(x, \xi) = \int_{\mathbf{R}^n} e^{-ix\lambda}\psi_p(\lambda)\overline{v}(\lambda + \xi)\, d\lambda, \qquad x, \xi \in \mathbf{R}^n, \end{aligned} \tag{4.8}$$

with

$$\psi_p(\lambda) = W_1(\lambda)\varphi_p^{-1}(\lambda)\frac{1}{1 + i\lambda_1^{[n/p]}}, \qquad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n,$$

and W_1 *is defined as in the proof of Theorem* 3.1. *Then we have*

$$|J| \le C_{E,n,p,k}\|a\|\,|B|^{\frac{p-2}{2p}}|\Omega_1|^{1/p}\|u\|_{L^p(B)}\|v\|_{L^q(\mathbf{R}^n)},$$

where $|B|$, $|\Omega_1|$ *denote the Lebesque measure of* B, Ω_1, *respectively, with* $\Omega_1 = \{x \in \mathbf{R}^n | d(x, E) \le 1\}$, *and* $1/p + 1/q = 1$ *with* $2 < p < \infty$.

Proof. First, we write (4.8) in the form

$$h(x, \xi) = \int_{\mathbf{R}^n} e^{i(x-z)\xi}\hat{\psi}(z)\overline{v}(x - z)\, dz. \tag{4.9}$$

Substituting (4.9) into (4.7), we write J in the form $J = J_1 + J_2$, where

$$J_1 = \int_{\Omega_1}\int_{\mathbf{R}^n} \tilde{b}(x, \xi)\hat{u}(\xi)h(x, \xi)\, d\xi\, dx,$$

and

$$J_2 = \int_{\mathbf{R}^n\setminus\Omega_1}\int_{\mathbf{R}^n} \tilde{b}(x, \xi)\hat{u}(\xi)h(x, \xi)\, d\xi\, dx.$$

By an argument similar to the proof of Theorem 3.1, we have

$$\begin{aligned} |J_1| &\le C\int_{\Omega_1}\left(\int_{\mathbf{R}^n} |\hat{u}(\xi)|^2\, d\xi\right)^{1/2}\left(\int_{\mathbf{R}^n} |\langle\xi\rangle^m h(x, \xi)|^2\, d\xi\right)^{1/2} dx \\ &\le C\|u\|_{L^2(B)}\int_{\Omega_1}\left(\int_{\mathbf{R}^n} |\hat{\psi}(z)\overline{v}(x - z)|^q\, dz\right)^{1/q} dx, \end{aligned}$$

and

$$|J_2| \le C\|u\|_{L^2(B)}\int_{\mathbf{R}^n\setminus\Omega_1} A(x)\left(\int_{\mathbf{R}^n} |\hat{\psi}(z)\overline{v}(x - z)|^q\, dz\right)^{1/q} dx,$$

where $C = C_{E,n,p,k}\|a\|$, A is defined as in the proof of Theorem 3.1 and $1/p + 1/q - 1$ with $2 < p < \infty$.

By Hölder's inequality and Fubini's theorem, we obtain

$$|J_1| \le C|B|^{\frac{p-2}{2p}}\|u\|_{L^p(B)}|\Omega_1|^{1/p}\|\hat{\psi}(z)\|_{L^q(\mathbf{R}^n)}\|v\|_{L^q(\mathbf{R}^n)},$$

and

$$|J_2| \le C|B|^{\frac{p-2}{2p}}\|u\|_{L^p(B)}|E|^{1/p}\|\hat{\psi}(z)\|_{L^q(\mathbf{R}^n)}\|v\|_{L^q(\mathbf{R}^n)}. \quad \square$$

Then Theorem 3.2 follows by applying Lemma 2.5. □

Proof of Theorem 3.3. Without loss of generality, we may assume that $a \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n) \cap \Lambda^m_{k,k'}(\mathbf{R}^n \times \mathbf{R}^n)$. Let $k = n/2 + \varepsilon$, $k' = n/p + \varepsilon$ and $\varphi_p(\lambda) = (1 + |\lambda|^2)^{\frac{1}{2}(n/p-[n/p]+\varepsilon/2)}$, where $\lambda \in \mathbf{R}^n$ and $\varepsilon, \varepsilon > 0$, is so small that $n/p - [n/p] + \varepsilon < 1$ and $[n/p + \varepsilon] = [n/p]$. Then for $u, v \in C_0^\infty(\mathbf{R}^n)$, we have

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{ix\lambda} a(x, \xi)\hat{u}(\xi)\overline{v}(x)\, d\xi\, dx. \tag{4.10}$$

We write (4.10) in the form

$$\begin{aligned}(a(x, D)u, v) &= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{a}(\lambda, y) f(\lambda, y)\, dy\, d\lambda \\ &= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{a}(\lambda, y)\varphi_p(y)\varphi_2(\lambda)\varphi_p^{-1}(y)\varphi_1^{-1}(\lambda) f(\lambda, y)\, dy\, d\lambda \\ &= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(x, \xi) e^{-i\xi y} e^{-ix\lambda} \varphi_p^{-1}(y)\varphi_2^{-1}(\lambda) \\ &\qquad \times f(\lambda, y)\, dy\, d\lambda\, d\xi\, dx,\end{aligned} \tag{4.11}$$

where

$$\hat{b}(\lambda, y) = \hat{a}(\lambda, y)\varphi_p(y)\varphi_2(\lambda), \qquad 1 < p \le 2, \tag{4.12}$$

and

$$f(\lambda, y) = \int_{\mathbf{R}^n} e^{iw\lambda} u(w + y)\overline{v}(w)\, dw, \qquad \lambda, y \in \mathbf{R}^n. \tag{4.13}$$

Making use of the partition of unity W_s, $s = 0, 1, \dots, n$, we write (4.11) in the form

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^{2n} \sum_{s,t=0}^{n} I_{s,t},$$

where

$$I_{s,t} = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(x, \xi) e^{-i\xi y} e^{-ix\lambda} W_s(y)\varphi_p^{-1}(y) W_t(\lambda)\varphi_2^{-1}(\lambda) \times f(\lambda, y)\, dy\, d\lambda\, d\xi\, dx, \quad s, t = 0, 1, \dots, n.$$

We shall estimate $I_{1,1}$ only, since the estimates of other cases are similar. By an argument similar to the proof of Theorem 3.1, we use the following method:

(1) We integrate the above integral with respect to ξ first and make use of the identity

$$\frac{1}{1 + iy_1^{[n/p]}}(1 - (-i)^{1-[n/p]}\partial_{\xi_1}^{[n/p]})(e^{-i\xi y}) = e^{-i\xi y}.$$

(2) We integrate the result of (1) with respect to x first and use the identity

$$\frac{1}{1 + i\lambda_1^{[n/2]}}(1 - (-i)^{1-[n/2]}\partial_{x_1}^{[n/2]})(e^{-ix\lambda}) = e^{-ix\lambda}.$$

Then we obtain

$$I_{1,1} = J_1 + (i)^{1-[n/2]} J_2 + (i)^{1-[n/p]} J_3 + (i)^{1-[n/2]} (i)^{1-[n/p]} J_4 ,$$

where

$$J_k = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} b_k(x, \xi) \Delta(x, \xi) \, d\xi \, dx , \qquad k = 1, 2, 3, 4,$$

with

$$\begin{aligned} b_1(x, \xi) &= b(x, \xi), \\ b_2(x, \xi) &= \partial_{x_1}^{[n/2]}(b(x, \xi)), \\ b_3(x, \xi) &= \partial_{\xi_1}^{[n/p]}(b(x, \xi)), \\ b_4(x, \xi) &= \partial_{x_1}^{[n/2]} \partial_{\xi_1}^{[n/p]}(b(x, \xi)), \end{aligned}$$

(4.14) $$\Delta(x, \xi) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{-i\xi y} e^{-ix\lambda} \psi_p(y) \psi_2(\lambda) f(\lambda, y) \, dy \, d\lambda , \quad x, \xi \in \mathbf{R}^n ,$$

and

$$\psi_p(y) = W_1(y) \frac{1}{1 + iy_1^{[n/p]}} \varphi_p^{-1}(y) , \qquad y = (y_1, \ldots, y_n) \in \mathbf{R}^n , \; 1 < p \le 2 .$$

We shall estimate J_4 only, since the other cases are similar. First, we estimate the following integral:

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle \xi \rangle^m \Delta(x, \xi)| \, d\xi \, dx .$$

By Lemma 2.3 and $\psi_p \in L^2(\mathbf{R}^n)$, we see that the integral in (4.14) is in $L^1(\mathbf{R}^n \times \mathbf{R}^n)$. Therefore, without loss of generality, we may consider the following integral:

(4.15) $$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle \xi \rangle^m \Delta_\delta(x, \xi)| \, d\xi \, dx ,$$

where

(4.16) $$\Delta_\delta(x, \xi) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{-i\xi y} e^{-ix\lambda} \psi_{p,\delta}(y) \psi_{2,\delta}(\lambda) f(\lambda, y) \, dy \, d\lambda , \qquad x, \xi \in \mathbf{R}^n ,$$

with

$$\psi_{p,\delta}(y) = \psi_p(y) e^{-\delta |y|^2} , \quad y \in \mathbf{R}^n , \; 1 < p \le 2 \text{ and } 0 < \delta < 1 .$$

We now give a proposition that will help us to study (4.15).

Proposition 4.1. *For* $u, v \in C_0^\infty(\mathbf{R}^n)$, $1 < p \le 2$ *and* $0 < \delta < 1$, *let* Δ_δ *be defined as in* (4.16). *Then we have*

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle \xi \rangle^m \Delta_\delta(x, \xi)| \, d\xi \, dx \le C_{n,p,k,k'} \|u\|_{L^p(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)} , \qquad 1/p + 1/q = 1 .$$

Proof. Substituting (4.13) into (4.16), writing $\overline{v}(w)$ in the form

$$\overline{v}(w) = \left(\frac{1}{2\pi} \right)^n \int_{\mathbf{R}^n} e^{-iw\eta} \hat{\overline{v}}(\eta) \, d\eta , \qquad w \in \mathbf{R}^n ,$$

and making the change of variables $w + y \to w$, we write Δ_δ in the form

(4.17)
$$\Delta_\delta(x, \xi) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix(\xi-\eta)} e^{-iw\xi} \times \left(\int_{\mathbf{R}^n} e^{i(w-x)\lambda} \widehat{\psi_{p,\delta}}(\lambda) \psi_{2,\delta}(\lambda + \eta - \xi)\, d\lambda\right) u(w)\overline{v}(\eta)\, dw\, d\eta,$$

$x, \xi \in \mathbf{R}^n$. By Taylor's expansion formula, we write $\psi_{2,\delta}$ in the form

(4.18)
$$\psi_{2,\delta}(\lambda + \eta - \xi) = \sum_{|\alpha| \le 4n} \frac{\lambda^\alpha}{\alpha!} \psi_{2,\delta}^{(\alpha)}(\eta - \xi) + (4n+1) \sum_{|\alpha| = 4n+1} \frac{\lambda^\alpha}{\alpha!} \int_0^1 (1-\theta)^{4n+1} \psi_{2,\delta}^{(\alpha)}(\eta - \xi + \theta\lambda)\, d\theta, \quad \lambda, \eta, \xi \in \mathbf{R}^n.$$

Substituting (4.18) into (4.17), we have

$$\Delta_\delta(x, \xi) = \left(\frac{1}{2\pi}\right)^n \sum_{|\alpha| \le 4n} \frac{1}{\alpha!} g_{\alpha,p,\delta}(x, \xi) h_{\alpha,2,\delta}(x, \xi) + \left(\frac{1}{2\pi}\right)^n (4n+1) \Box_\delta(x, \xi), \qquad s, \xi \in \mathbf{R}^n,$$

where

(4.19)
$$g_{\alpha,p,\delta}(x, \xi) = (i)^{-|\alpha|} \int_{\mathbf{R}^n} e^{-iw\xi} \psi_{p,\delta}^{(\alpha)}(w) u(x + w)\, dw,$$

(4.20)
$$h_{\alpha,2,\delta}(x, \xi) = \int_{\mathbf{R}^n} e^{-ix\eta} \psi_{2,\delta}^{(\alpha)}(\eta - \xi) \overline{v}(\eta)\, d\eta,$$

and

(4.21)
$$\Box_\delta(x, \xi) = \sum_{|\alpha| = 4n+1} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{4n+1} \times \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} e^{i(w-x)\lambda} \psi_{2,\delta}^{(\alpha)}(\eta - \xi + \theta\lambda) \lambda^\alpha \widehat{\psi_{p,\delta}}(\lambda)\, d\lambda\right) \times e^{ix(\xi-\eta)} e^{-iw\xi} u(w) \overline{v}(\eta)\, dw\, d\eta\, d\theta.$$

We now give a lemma to help us study the following integral:

(4.22)
$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle\xi\rangle^m g_{\alpha,p,\delta}(x, \xi) h_{\alpha,2,\delta}(x, \xi)|\, d\xi\, dx,$$

for $|\alpha| \le 4n$, $1 < p \le 2$ and $0 < \delta < 1$.

Lemma 4.2. *For $u, v \in C_0^\infty(\mathbf{R}^n)$, $|\alpha| \le 4n$, $1 < p \le 2$ and $0 < \delta < 1$, let $g_{\alpha,p,\delta}$ and $h_{\alpha,2,\delta}$ be defined as in* (4.19) *and* (4.20), *respectively. Then we have*

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle\xi\rangle^m g_{\alpha,p,\delta}(x, \xi) h_{\alpha,2,\delta}(x, \xi)|\, d\xi\, dx \le C_{n,p,k,k'} \|u\|_{L^p(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)}, \quad 1/p + 1/q = 1.$$

Proof. First, we write (4.20) in the form

$$h_{\alpha,2,\delta}(x,\xi)=\int_{\mathbf{R}^n} e^{-i\xi z}\widehat{\psi_{2,\delta}^{(\alpha)}}(z)\overline{v}(x-z)\,dz. \tag{4.23}$$

Substituting (4.23) into (4.22), by Hölder's inequality, Lemma 2.4 and Parseval's formula, we obtain

$$\begin{aligned}&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\langle\xi\rangle^m g_{\alpha,p,\delta}(x,\xi)h_{\alpha,2,\delta}(x,\xi)|\,d\xi\,dx\\ &\le\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}|\langle\xi\rangle^m g_{\alpha,p,\delta}(x,\xi)|^2\,d\xi\right)^{1/2}\left(\int_{\mathbf{R}^n}|h_{\alpha,2,\delta}(x,\xi)|^2\,d\xi\right)^{1/2}dx\\ &\le C\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}|\psi_{p,\delta}^{(\alpha)}(w)u(x+w)|^p\,dw\right)^{1/p}\left(\int_{\mathbf{R}^n}|\widehat{\psi_{2,\delta}^{(\alpha)}}(z)\overline{v}(x-z)|^2\,dz\right)^{1/2}dx,\end{aligned}$$

where $C=C_{n,p,k,k'}$.

By Hölder's inequality, Fubini's theorem and Minkowski's inequality, we have

$$\begin{aligned}&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\langle\xi\rangle^m g_{\alpha,p,\delta}(x,\xi)h_{\alpha,2,\delta}(x,\xi)|\,d\xi\,dx\\ &\le C_{n,p,k,k'}\|\psi_{p,\delta}^{(\alpha)}\|_{L^p(\mathbf{R}^n)}\|u\|_{L^p(\mathbf{R}^n)}\|\psi_{2,\delta}^{(\alpha)}\|_{L^2(\mathbf{R}^n)}\|v\|_{L^q(\mathbf{R}^n)},\quad 1/p+1/q=1.\quad\square\end{aligned}$$

We now give a lemma to help us study $\square_\delta$.

Lemma 4.3. *For $u, v\in C_0^\infty$, $1<p\le 2$ and $0<\delta<1$, let $\square_\delta$ be defined as in (4.21). Then we have*

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\langle\xi\rangle^m\square_\delta(x,\xi)|\,d\xi\,dx\le C_{n,p,k,k'}\|u\|_{L^p(\mathbf{R}^n)}\|v\|_{L^q(\mathbf{R}^n)},\qquad 1/p+1/q=1.$$

Proof. First, we study the following integral:

$$\int_{\mathbf{R}^n} e^{i(w-x)\lambda}\psi_{2,\delta}^{(\alpha)}(\eta-\xi+\theta\lambda)(\lambda^\alpha\widehat{\psi_{p,\delta}}(\lambda))\,d\lambda, \tag{4.25}$$

where $w, x, \eta, \xi\in\mathbf{R}^n$, $0\le\theta\le 1$ and $|\alpha|=4n+1$.

Making use of the following identity:

$$\left(\prod_{s=1}^n\frac{1}{1+i(w_s-x_s)}\right)\left(\prod_{s=1}^n(1+\partial_{\lambda_s})\right)(e^{i(w-x)\lambda})=e^{i(w-x)\lambda},$$

we write (4.25) in the form

$$\begin{aligned}&\int_{\mathbf{R}^n} e^{i(w-x)\lambda}\psi_{2,\delta}^{(\alpha)}(\eta-\xi+\theta\lambda)(\lambda^\alpha\widehat{\psi_{p,\delta}}(\lambda))\,d\lambda\\ &=\left(\prod_{s=1}^n\frac{1}{1+i(w_s-x_s)}\right)\sum_{\beta\in T}(-1)^{|\beta|}\sum_{\gamma\le\beta}\int_{\mathbf{R}^n}e^{i(w-x)\lambda}\\ &\qquad\times\partial_\lambda^\gamma(\psi_{2,\delta}^{(\alpha)}(\eta-\xi+\theta\lambda))\partial_\lambda^{\beta-\gamma}(\lambda^\alpha\widehat{\psi_{p,\delta}}(\lambda))\,d\lambda,\end{aligned} \tag{4.26}$$

with

$$T=\{(\beta_1,\beta_2,\dots,\beta_n)\in\mathbf{N}^n\,|\,\beta_t=0\text{ or }1,\ t=1,\dots,n\}.$$

Substituting (4.26) into (4.21), we get

$$\begin{aligned}(4.27)\qquad &\square_\delta(x, \xi)\\ &= \sum_{|\alpha|=4n+1} \frac{1}{\alpha!} \sum_{\beta\in T} (-1)^{|\beta|} \sum_{\gamma\leq\beta} \int_0^1 \int_{\mathbf{R}^n} (1-\theta)^{4n+1} \partial_\lambda^{\beta-\gamma}(\lambda^\alpha \widehat{\psi_{p,\delta}}(\lambda))\\ &\quad \times e^{-ix(\lambda-\xi)} \tilde{g}(x, \xi, \lambda) \tilde{h}_{\alpha,\gamma}(x, \xi, \theta\lambda)\, d\lambda\, d\theta, \quad x, \xi \in \mathbf{R}^n,\end{aligned}$$

where

$$\tilde{g}(x, \xi, \lambda) = \int_{\mathbf{R}^n} e^{iw(\lambda-\xi)} \left(\prod_{s=1}^n \frac{1}{1 + i(w_s - x_s)} \right) u(w)\, dw,$$

and

$$(4.29)\qquad \tilde{h}_{\alpha,\gamma}(x, \xi, \theta\lambda) = \int_{\mathbf{R}^n} e^{-ix\eta} \partial_\lambda^\gamma(\psi_{2,\delta}^{(\alpha)}(\eta - \xi + \theta\lambda)) \overline{v}(\eta)\, d\eta, \quad \lambda \in \mathbf{R}^n.$$

By an argument similar to the proof of Lemma 4.2, we have

$$\begin{aligned}(4.30)\qquad &\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle\xi\rangle^m \tilde{g}(x, \xi, \lambda) \tilde{h}_{\alpha,\gamma}(x, \xi, \theta\lambda)|\, d\xi\, dx\\ &\leq C_{n,p} \|l\|_{L^p(\mathbf{R}^n)} \|u\|_{L^p(\mathbf{R}^n)} \theta^{|r|} \|\psi_{2,\delta}^{(\alpha+\gamma)}\|_{L^2(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)},\end{aligned}$$

where

$$l(x) = \prod_{s=1}^n \frac{1}{1 + ix_s} \quad \text{and} \quad 1/p + 1/q = 1.$$

Also, we have

$$(4.31)\qquad \int_{\mathbf{R}^n} |\partial_\lambda^\beta (\lambda^\alpha \widehat{\psi_{p,\delta}}(\lambda))|\, d\lambda \leq C_n,$$

with $|\alpha| = 4n + 1$ and $\beta \in T$. Therefore, (4.25)–(4.31) imply

$$\begin{aligned}&\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle\xi\rangle^m \square_\delta(x, \xi)|\, d\xi\, dx\\ &\leq \sum_{|\alpha|=4n+1} \frac{1}{\alpha!} \sum_{\beta\in T} \sum_{\gamma\leq\beta} \int_0^1 \int_{\mathbf{R}^n} (1-\theta)^{4n+1} |\partial_\lambda^{\beta-\gamma}(\lambda^\alpha \widehat{\psi_{p,\delta}}(\lambda))|\\ &\quad \times \left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle\xi\rangle^m \tilde{g}(x, \xi, \lambda) \tilde{h}_{\alpha,\gamma}(x, \xi, \theta\lambda)|\, d\xi\, dx \right) d\lambda\, d\theta\\ &\leq C_{n,p,k,k'} \|u\|_{L^p(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)}, \quad 1/p + 1/q = 1,\ 1 < p \leq 2. \quad \square\end{aligned}$$

Thus, Proposition 4.1 gives

$$|J_4| \leq C_{n,p,k,k'} \|a\|\, \|u\|_{L^p(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)}, \quad 1/p + 1/q = 1,\ 1 < p \leq 2. \quad \square$$

Proof of Theorem 3.4. Without loss of generality, we may assume that $a \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n) \cap \Lambda_{k,k'}^m(\mathbf{R}^n \times \mathbf{R}^n)$. Let $k = n/p + \varepsilon$, $k' = n/2 + \varepsilon$, and $\varphi_{p'}(\lambda) = (1 + |\lambda|^2)^{\frac{1}{2}(n/p' - [n/p'] + \varepsilon/2)}$, where $\lambda \in \mathbf{R}^n$, $2 \leq p' < \infty$ and $\varepsilon, \varepsilon > 0$, is so small that $n/p' + \varepsilon/2 \notin \mathbf{N}$, $n/p' - [n/p'] + \varepsilon < 1$, $[n/p' + \varepsilon] = [n/p']$ and $n/q - [n/q] \neq \varepsilon/2$ with $1/p + 1/q = 1$. Then for $u, v \in C_0^\infty$, we have

$$(a(x, D)u, v) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\xi} a(x, \xi) \hat{u}(\xi) \overline{v}(x)\, d\xi\, dx.$$

Since the following arguments are similar to the proof of Theorem 3.3, we shall only study the following lemma, which is similar to Lemma 4.2.

Lemma 4.4. *For* $u, v \in C_0^\infty$, $|\alpha| \le 4n$, $2 < p < \infty$ *and* $0 < \delta < 1$, *we define* $g_{\alpha,2,\delta}$ *and* $h_{\alpha,p,\delta}$ *as follows*:

$$g_{\alpha,2,\delta}(x, \xi) = (i)^{-|\alpha|} \int_{\mathbf{R}^n} e^{-iw\xi} \psi_{2,\delta}^{(\alpha)}(w) u(x+w)\, dw, \tag{4.32}$$

and

$$h_{\alpha,p,\delta}(x, \xi) = \int_{\mathbf{R}^n} e^{-ix\eta} \psi_{p,\delta}^{(\alpha)}(\eta - \xi) \overline{v}(\eta)\, d\eta, \quad x, \xi \in \mathbf{R}^n, \tag{4.33}$$

with

$$\psi_{p',\delta}(y) = \psi_{p'}(y) e^{-\delta|y|^2}, \qquad \psi_{p'}(y) = W_1(y) \frac{1}{1 + iy_1^{[n/p']}} \varphi_{p'}^{-1}(y),$$
$$y = (y_1, \dots, y_n) \in \mathbf{R}^n, \qquad 2 \le p' < \infty,$$

and W_1 *is defined as in the proof of Theorem* 3.3. *Then we have*

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle \xi \rangle^m g_{\alpha,2,\delta}(x, \xi) h_{\alpha,p,\delta}(x, \xi)|\, d\xi\, dx$$
$$\le C_{n,p,k,k'} \|u\|_{L^p(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)}, \quad 1/p + 1/q = 1.$$

Proof. First, we write (4.33) in the form

$$h_{\alpha,p,\delta}(x, \xi) = \int_{\mathbf{R}^n} e^{-i\xi z} \widehat{\psi_{p,\delta}^{(\alpha)}}(z) \overline{v}(x - z)\, dz.$$

By an argument similar to that in the proof of Lemma 4.2, we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle \xi \rangle^m g_{\alpha,2,\delta}(x, \xi) h_{\alpha,p,\delta}(x, \xi)|\, d\xi\, dx \\
&\le \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |g_{\alpha,2,\delta}(x, \xi)|^2\, d\xi \right)^{1/2} \left(\int_{\mathbf{R}^n} \langle \xi \rangle^m |h_{\alpha,p,\delta}(x, \xi)|^2\, d\xi \right)^{1/2} dx \\
&\le C \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |\psi_{2,\delta}^{(\alpha)}(w) u(x+w)|^2\, dw \right)^{1/2} \left(\int_{\mathbf{R}^n} |\widehat{\psi_{p,\delta}^{(\alpha)}}(z) \overline{v}(x-z)|^q\, dz \right)^{1/q} dx \\
&\le C \|\psi_{2,\delta}^{(\alpha)}\|_{L^2(\mathbf{R}^n)} \|u\|_{L^p(\mathbf{R}^n)} \|\widehat{\psi_{p,\delta}^{(\alpha)}}\|_{L^q(\mathbf{R}^n)} \|v\|_{L^q(\mathbf{R}^n)}, \quad 1/p + 1/q = 1,
\end{aligned}$$

where $C = C_{n,p,k,k'}$. □

Thus, by an argument similar to the proof of Theorem 3.3, Lemma 4.4, and Lemma 2.5, Theorem 3.4 is obtained. □

5. Counterexamples

In this section, we give two examples to show that the conditions on k given in Theorem 3.1 are the sharpest (the case $p = 2$) and are sufficiently sharp (the case $1 < p < 2$). They are modifications of the counterexamples of [4] and [14, p. 151], respectively.

Example 5.1. Let $p = 2$ and $0 < k \le n/2$ (n is the dimension). Define

$$a_k(x, \xi) = \psi(x) e^{-ix\xi} A(\xi), \qquad x, \xi \in \mathbf{R}^n,$$

where $\psi \in C_0^\infty(B(0, 1))$, $0 \le \psi \le 1$, $\psi(x) \equiv 1$ if $|x| \le 1/2$ and A is a positive continuous but not differential function such that $\langle \xi \rangle^{-k} \le A(\xi) \le 2\langle \xi \rangle^{-k}$.

It is obvious that $a_k \in \Lambda_k^0(\mathbf{R}^n \times \mathbf{R}^n)$, and we have

$$a_k(x, D)u(x) = \left(\frac{1}{2\pi}\right)^n \psi(x) \int_{\mathbf{R}^n} A(\xi)\hat{u}(\xi)\, d\xi .$$

Since $A \notin L^2(\mathbf{R}^n)$, we can find a function u, $u \in L^2(\mathbf{R}^n)$, such that

$$\int_{\mathbf{R}^n} A(\xi)\hat{u}(\xi)\, d\xi = \infty .$$

Hence, $a_k(x, D)$ is not continuous from $L^2(\mathbf{R}^n)$ to $L^2(B(0, 1))$. Thus $a(x, D)$ is not continuous from $L^2(\mathbf{R}^n)$ to $L^2(B(0, 1))$ if $0 < k \leq n/2$ and $a \in \Lambda_k^0(\mathbf{R}^n \times \mathbf{R}^n)$.

Example 5.2. If $1 < p < 2$, let

$$a_\varepsilon(x, \xi) = \psi(x)e^{-ix\xi}A(\xi), \qquad x, \xi \in \mathbf{R}^n,$$

where ψ is defined as in Example 5.1 and A is a positive continuous but not differential function such that $\langle\xi\rangle^{-n/p+\varepsilon} \leq A(\xi) \leq 2\langle\xi\rangle^{-n/p+\varepsilon}$ with $0 < \varepsilon < n/2$. It is obvious that $a_\varepsilon \in \Lambda^m_{n/2-\varepsilon}(\mathbf{R}^n \times \mathbf{R}^n)$, where $m = m(p) = -n(1/p - 1/2)$, and we have

$$a_\varepsilon(x, D)u(x) = \left(\frac{1}{2\pi}\right)^n \psi(x) \int_{\mathbf{R}^n} A(\xi)\hat{u}(\xi)\, d\xi .$$

Let u be the convolution of φ and $\overline{\varphi}$, where $\varphi \in C_0^\infty(B(0, 1/4))$ and $\varphi(x) \equiv 1$ if $|x| \leq 1/8$. Then $\hat{u} = \hat{\varphi} \cdot \overline{\hat{\varphi}} \geq 0$ and $\hat{u}$ cannot be identical equal to zero on any open subset of $\mathbf{R}^n$. Now, we define $u_r(x) = r^{-n/p}u(x/r)$ for $0 < r < 1$. We can see that $\|u_r\|_{L^p} = C$, where C is a constant that does not depend on r. Since

$$\begin{aligned}
\int_{\mathbf{R}^n} A(\xi)\widehat{u_r}(\xi)\, d\xi &\geq \int_{\mathbf{R}^n} \langle\xi\rangle^{-n/p+\varepsilon}\widehat{u_r}(\xi)\, d\xi \\
&= r^{n(1-1/p)} \int_{\mathbf{R}^n} \langle\xi\rangle^{-n/p+\varepsilon}\hat{u}(r\xi)\, d\xi \\
&= r^{-\varepsilon} \int_{\mathbf{R}^n} (r^2 + |\xi|^2)^{\frac{1}{2}(-n/p+\varepsilon)}\hat{u}(\xi)\, d\xi \\
&\geq r^{-\varepsilon} \int_{|\xi|<1} (1 + |\xi|^2)^{\frac{1}{2}(-n/p+\varepsilon)}\hat{u}(\xi)\, d\xi \\
&\geq C' r^{-\varepsilon} \to \infty \quad \text{as } r \to 0,
\end{aligned}$$

where C' is a positive constant, we have

$$\|a_\varepsilon(x, D)u_r\|_{L^p(B(0,1))} \geq Cr^{-\varepsilon}\|\psi\|_{L^p(B(0,1))} \to \infty \quad \text{as } r \to 0 .$$

So $a_\varepsilon(x, D)$ is not continuous from $L^p(B(0, 1))$ to $L^p(B(0, 1))$. Thus $a(x, D)$ is not continuous from $L^p(B(0, 1))$ to $L^p(B(0, 1))$ if $1 < p < 2$, $0 < k < n/2$ and $a \in \Lambda_k^m(\mathbf{R}^n \times \mathbf{R}^n)$, where $m = m(p) = -n(1/p - 1/2)$.

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