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GENERALIZATION OF THE WHITNEY-MAHOWALD THEOREM

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ABSTRACT. The Whitney-Mahowald theorem gave normal Euler number (mod 4) for embeddings of a closed $2n$-manifold in Euclidean $4n$-space. We generalize this theorem to embeddings of closed $2n$-manifolds in an oriented $4n$-manifold with an approach in the framework of unoriented bordism groups of maps.

1. INTRODUCTION

Given a map $f: M \to N$, where M and N are smooth connected manifolds with dimensions n and $2n$ respectively, and M is closed, N is oriented, there is a question:

(∗) What is the set of the normal Euler classes of smooth embeddings in the homotopy class $[f]$?

An orientation of N and a map $f: M \to N$ determine an isomorphism $H^n(M, \widetilde{Z}) \to Z$ which sends the normal Euler class of an embedding g in $[f]$ to an integer $\chi(g)$ called the normal Euler number g, where $\widetilde{Z}$ is the local integer coefficients associated to the orientation line bundle of M. For odd n, the normal Euler classes are always zero. Hence Question (∗) is interesting only for even n.

H. Whitney [Wh1] first proved that any n manifold embeds in R^{2n}. By using Whitney's technique, J. Milnor proved in [Mi] that if N is simply connected and $n > 2$, then any $f: M \to N$ is homotopic to embeddings. The following more general result is due to Haefliger [Ha]:

Theorem 1 (Haefliger). *If $n > 2$ and $f: M^n \to N^{2n}$ is a map with $f_*: \pi_1(M) \to \pi_1(N)$ surjective, then f is homotopic to embeddings.*

If M is orientable, then the normal Euler numbers of embeddings are uniquely determined by their homotopy classes. However, if M is nonorientable, the situation changes. Whitney [Wh2] in the case $n = 2$, and Mahowald [Mah] in the case of n even, proved that if $f: M^n \to R^{2n}$ is an embedding, then

$$\chi(f) = 2\overline{w}_1(M)\overline{w}_{n-1}(M) \mod 4,$$

where $2\overline{w}_1(M)\overline{w}_{n-1}(M)$ is understood as the image of the dual Stiefel-Whitney number $\overline{w}_1(M)\overline{w}_{n-1}(M)$ under the natural inclusion $Z_2 \to Z_4$.

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Malyi [Mal] proved that if $n > 2$ is even and M^n is nonorientable, then for any integer x with $x = 2\overline{w}_1(M)\overline{w}_{n-1}(M) \bmod 4$, there is an embedding $f : M^n \to R^{2n}$ with $\chi(f) = x$.

W. S. Massey gave a new proof of Mahowald's theorem in [Mas], by using the following formula proved also by him:

$$\widetilde{P}(U_2) = (\tilde{\rho}_4(X_n) + \tilde{\theta}(w_1 w_{n-1})) \cdot U,$$

where U is the Thom class of an n-dimensional vector bundle ξ over B with n even, X_n the Euler class of ξ, both U and X_n take local integer coefficients $\widetilde{Z}$ determined by ξ, $U_2 = U \bmod 2$, w_i the ith Whitney class of ξ, $\widetilde{P}$ the Pontryagin square, and

$$\tilde{\rho} : H^q(B, \widetilde{Z}) \to H^q(B, \widetilde{Z}_4), \quad \tilde{\theta} : H^q(B, Z_2) \to H^q(B, \widetilde{Z}_4)$$

the natural homomorphisms.

To generalize Mahowald's theorem to embeddings of M^n in an oriented N^{2n}, we define first $P : H_n(N, Z_2) \to Z_4$ as follows:

For any $x \in H_n(N, Z_2)$, take a compact submanifold N_x^{2n} of N so that $x = i_* y$, where

$$i_* : H_n(N_x, Z_2) \to H_n(N, Z_2)$$

is the natural homomorphism and $y \in H_n(N_x, Z_2)$. Let

$$Dy \in H^n(N_x, \partial N_x, Z_2) \cong H^n(N_x/\partial N_x, Z_2)$$

be the Lefschetz dual of y. Then

$$P(x) = \langle \widetilde{P}(Dy), [N_x/\partial N_x] \rangle,$$

where $[N_x/\partial N_x]$ is the fundamental class of

$$H_{2n}(N_x/\partial N_x, Z_4) \cong Z_4$$

determined by the orientation of N_x inherited from that of N, and $\langle\ ,\ \rangle$ stands for the Kronecker product.

It is easy to see that P is well defined and if n is even, then

$$P(x+y) = P(x) + P(y) + 2x \cdot y,$$

where $x \cdot y$ is the intersection number (the proof depends on a formula for the Pontryagin square; cf. [MT, p. 21]).

Now we are in a position to state

Theorem 2. *Let n be even.*

(1) *If $f : M^n \to N^{2n}$ is an embedding, then*

$$\chi(f) = P(f_*[M]) + 2w_1(f)w_{n-1}(f) \bmod 4$$

where $[M]$ is the generator of $H_n(M, Z_2) \cong Z_2$, and $w_i(f)$ the ith normal Whitney class of f.

(2) *If $n > 2$, M is nonorientable, and f is a map with $f_* : \tilde{\pi}_1(M) \to \pi_1(N)$ surjective, where $\tilde{\pi}_1(M)$ is the subgroup of $\pi_1(M)$ consisting of orientation-preserving elements, then normal Euler numbers of embeddings in $[f]$ fill the* $\bmod 4$ *residue class $P(f_*[M]) + 2w_1(f)w_{n-1}(f)$.*

Corollary 1. *If n is even, and $f, g : M^n \to N^{2n}$ are homotopic embeddings, then $\chi(f) = \chi(g) \bmod 4$.*

Corollary 2 (The generalized Whitney congruence of Rohlin [Ro]). *Let $f: M^2 \to N^4$ be an embedding, where N is oriented closed and $f_*[M]$ is characteristic; then*

$$\sigma(N) = \chi(f) + 2\chi(M) \mod 4,$$

where $\sigma(N)$ is the signature of N, and $\chi(M)$ stands for the Euler characteristic number of M.

Theorem 3. *Let $\mathcal{N}_n(N)$ be the bordism group of maps from closed (possibly non-orientable.) n-manifolds into the oriented 2n-manifold N. Then $P(f_*[M]) + 2w_1(f)w_{n-1}(f)$ gives a map $q: \mathcal{N}_n(N) \to Z_4$ with the following properties:*

(1) *Any element $x \in \mathcal{N}_n(N)$ includes embeddings with normal Euler numbers* mod 4 *equal to $q(x)$.*

(2) *A self-transversal immersion with only double points in x has* mod 4 *normal Euler number equal to $q(x)$ (or $q(x)+2$) if and only if its number of self-intersection points is even (or odd).*

(3) *Let $x, y \in \mathcal{N}_n(N)$ be represented by $f: M_1 \to N$ and $g: M_2 \to N$, and define $x \cdot y$ as $f_*[M_1] \cdot g_*[M_2] \in Z_2$. Then $x \cdot y$ gives a bilinear form on $\mathcal{N}_n(N)$ and*

$$q(x+y) = q(x) + q(y) + 2x \cdot y.$$

In general, it is difficult to calculate Pontryagin squares. We shall give another formula for $q(x)$ in a special case that N is the total space of the orientation line bundle of a manifold K^{2n-1}, e.g. $N = R^{2n-1} \times R = R^{2n}$.

Let $f: M^n \to K^{2n-1}$ be a map, K' a compact $(2n-1)$-dimensional submanifold of K containing $f(M)$, and $D: H_n(K', Z_2) \to H^{n-1}(K', \partial K'; Z_2)$ the Lefschetz dual. Then

$$u(f) = f^* D f_*[M] \in H^{n-1}(M, Z_2)$$

is well defined. We have

Theorem 4. *If N is the total space of the orientation line bundle of K^{2n-1} with n even, and $f: M \to N$ is an embedding, then*

$$\chi(f) = 2\tilde{w}_1(f)(u(f) + \tilde{w}_{n-1}(f)) \mod 4,$$

where $\tilde{w}_i(f)$ is the ith stable normal Whitney class of f regarded as a map $M \to K$, and $u(f)$ is understood similarly.

Remark 1. The proof of Theorem 4 is geometric, hence different from those of Mahowald and of Massey for $N = R^{2n}$.

Remark 2. By Theorem 3 and the proof of Theorem 4, we have

$$q(x) = 2\tilde{w}_1(f)(u(f) + \tilde{w}_{n-1}(f)),$$

where $f \in x \in \mathcal{N}_n(K) = \mathcal{N}_n(N)$, and a formula expressing $P(f_*[M])$ in terms of $u(f)$, $w_1(K)$, and $w_i(f)$, $i \leq n-1$, can be gotten. From this we see that if K^{2n-1} is orientable with n even and $N = K \times R$, $y \in H_n(N, Z_2)$ with $P(y) \neq 0$, then there is no map $f: M^n \to N$ with M orientable and $f_*[M] = y$.

Example. Let $g : M = RP^n \# RP^n \to RP^n$ be the map collapsing the second copy to a point, and f be the composition

$$M \xrightarrow{g} RP^n \subset RP^{2n-1} \subset RP^{2n-1} \times R.$$

Then

$$f_* : \tilde{\pi}_1(M) \to \pi_1(RP^{2n-1} \times R)$$

is surjective, and it follows from (2) of Theorem 2 that if $n > 2$ is even, then the normal Euler numbers of the embeddings homotopic to f fill a mod 4 residue class. Now Theorem 4 tells us that this class is $4Z$. And we see that for the generator x of $H_n(RP_{2n-1} \times R)$, $P(x) = 2$, and hence x is not represented by maps from orientable n-manifolds.

Remark 3. The earliest preprint of this paper was typed in 1989, under the title "Embedding n-manifolds in $2n$-manifolds" and was used and quoted in [Li3]. The main content of the paper was given in a talk in a conference held at Tokyo University, September, 1990.

Remark 4. Recently, Yamada [Ya] found the formula in part (1) of Theorem 2 for the case $n = 2$ with $H_1(M, Z) = 0$ independently, using a geometric method.

2. Proof of Theorem 2

Let $f : M \to N$ be an embedding, N_f a compact tubular neighbourhood of $f(M)$. Regard N_f as the disk bundle of the normal bundle of f; we have by Massey's formula

$$\widetilde{P}(U_2) = (\tilde{\rho}_4(X(f)) + \theta(w_1(f)w_{n-1}(f))) \cdot U$$

where $X(f)$ is the normal Euler class.

Let $[N_f/\partial N_f]$ be the fundamental class of

$$H^{2n}(N_f/\partial N_f, Z_4) \cong Z_4$$

corresponding to the orientation of N_f inherited from that of N. Since $U_2 \in H^n(N_f, \partial N_f; Z_2)$ is the Lefschetz dual of $f_*[M] \in H_n(N_f, Z_2)$, we have

$$\widetilde{P}(U_2) = P(f_*[M])[N_f/\partial N_f].$$

Let $\chi(f)$ be the normal Euler number determined by the orientation of N. Then the Thom isomorphism

$$H^n(M, \widetilde{Z}_4) \cong H^{2n}(N_f/\partial N_f, Z_4)$$

given by $x \to x \cdot U$ sends

$$(\tilde{\rho}_4(X(f)) + \tilde{\theta}(w_1(f)w_{n-1}(f))) \cdot U$$

to

$$(\chi(f) + 2w_1(f)w_{n-1}(f))[N_f/\partial N_f];$$

hence

$$\chi(f) = P(f_*[M]) + 2w_1(f)w_{n-1}(f) \bmod 4$$

and (1) is proved.

Suppose $n > 2$ and $f_* : \tilde{\pi}_1(M) \to \pi_1(N)$ is surjective. Then by Theorem 1, f is homotopic to embeddings and we may assume f is an embedding.

For any $m \in Z$, take a self-transversal immersion $h : S^n \to S^{2n}$ with $2|m|$ self-intersection points and $\chi(h) = 4m$. Making a suitable connected sum of f and h, we get a self-transversal immersion $g = f\#h : M \to N$ homotopic to f with $2|m|$ self-intersection points and $\chi(g) = \chi(f) + 4m$. Assume $a_1, b_1, a_2, b_2, \ldots, a_{2|m|}, b_{2|m|}$ are distinct points in M such that $g(a_i) = g(b_i)$.

Choose simple curves I_i, $i = 1, 2$, connecting a_i and b_i such that $I_1 \cap I_2 = \varnothing$ and $I_i \cap \{a_3, b_3, \ldots, a_{2|m|}, b_{2|m|}\} = \varnothing$. Then $g(I_1)$ and $g(I_2)$ form a simple closed curve γ in N. Given orientations of $T(M)$ on I_1 and I_2, the signs of self-intersections of g at $g(a_1)$ and $g(a_2)$ are determined. The surjectivity of $f_* : \tilde{\pi}_1(M) \to \pi_1(N)$ together with its implicate that there exist elements in $\pi(M)\backslash\tilde{\pi}(M)$ which are in the kernel of f_* allows us to choose I_2 so that γ is nullhomotopic and the signs at $g(a_1)$ and $g(a_2)$ are opposite. Thus, by Whitney's technique, we can get an immersion regularly homotopic to g with $2|m| - 2$ self-intersection points. Continuing in this way, we will get at last an embedding regularly homotopic to g. This proves (2).

Proof of Corollary 2. By a formula of Wu (see [Wu] or [Th]),

$$P(f_*[M]) = \widetilde{P}(w_2(N)) = p_1(N) \bmod 4 + 2w_4(N),$$

where p_1 is the first Pontryagin class of N. $w_4(N) + \sigma(N) \bmod 2$ is an $\mathcal{N}_4^{\mathrm{so}}$-invariant, and $w_4(CP^2) + \sigma(CP^2) \bmod 2 = 0$; hence $w_4(N) + \sigma(N) \bmod 2 = 0$ for any N. This fact together with $p_1(N) = 3\sigma(N)$ implies $P(f_*[M]) = \sigma(N) \bmod 4$. Now, Corollary 2 follows from (1) of Theorem 2.

3. Proof of Theorem 3

First, since $f_*([M])$ and $w_1(f)w_{n-1}(f)$ are bordism invariants, q is well defined.

Now, let $f : M \to N$ be a self-transversal immersion, and $b \in N$ be a self-intersection point of f such that $b = f(a_1) = f(a_2)$, but $a_1 \neq a_2$. Take a neighbourhood of b which corresponds to R^{2n} diffeomorphically such that the image of f in this neighbourhood corresponds to $R^n \times 0 \cup 0 \times R^n$. For $x \in R^n$ with $|x| < 2$, let $v(x, 0) = (0, x)$, $v(0, x) = (x, 0)$. Then v extends to a normal vector field of f denoted by v also. We may assume v is transversal to the zero section since it is already so at a_1 and a_2. The normal Euler number of f is the algebraic sum of the zeros of v, and the total contribution of a_1 and a_2 is ± 2.

Let γ be a curve in the plane as shown in Figure 1.

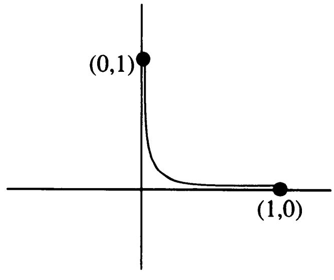

FIGURE 1

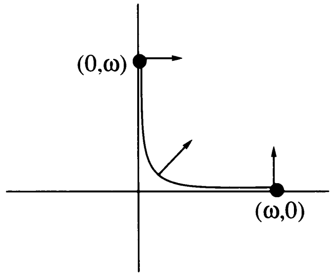

FIGURE 2

We assume $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $0 \le t \le 1$, such that $\gamma(0) = (0, 1)$, $\gamma(1) = (1, 0)$, and γ contacts with the x-axis and y-axis smoothly.

Let

$$D^n = \{x \in R^n \mid |x| < 1\},$$
$$S^{n-1} = \{x \in R^n \mid |x| = 1\}.$$

Take off the neighbourhoods of a_1 and a_2 in M which correspond to $D^n \times 0$ and $0 \times D^n$, and then add $S^{2n-1} \times I$ naturally; we get a new manifold M'. Mapping $S^{n-1} \times I$ to R^{2n} by

$$(\omega, t) \to (\gamma_1(t)\omega, \gamma_2(t)\omega),$$

we have an immersion f' of M' in N and a normal vector field ν' of f' which are identical with f and v respectively outside the neighbourhoods of a_1 and a_2. On $S^{n-1} \times I$, we may assume ν' has no zeros, as can be seen from Figure 2.

Now it is clear that the numbers of self-intersection points of f and f' differ by 1 and their normal Euler numbers differ by 2.

Since f and f' are obviously bordant, and any bordism class in $\mathcal{N}_n(N)$ is represented by a self-transversal immersion, we prove property (1) by repeating the process from f to f' and Theorem 2, (1).

If f has even (odd) number of self-intersection points, then f is bordant to an embedding g with $\chi(f) - \chi(g) = 2 \times$ even number ($2\times$ odd number). This proves (2).

Property (3) is straightforward since

$$P(x+y) = P(x) + P(y) + 2x \cdot y$$

and

$$w_1(x+y)w_{n-1}(x+y) = w_1(x)w_{n-1}(x) + w_1(y)w_{n-1}(y).$$

The proof is complete.

4. PROOF OF THEOREM 4

We divide the proof into three steps.

Step 1. First, we notice that the homotopy class $[f] \in [M, N]$ is represented by a self-transversal immersion $g: M \to K$ (cf. [LP]). Regard g as an immersion of M in N; then its normal bundle includes a line bundle, hence has an orientation-reversing automorphism. Thus $2\chi(g) = 0$ (cf. [Li1]), and $\chi(g) = 0$. We will prove in Steps 2 and 3 that g is regularly homotopic to a self-transversal immersion f_1 of M in N with mod 2 number of self-intersection points $w_1(g)(u(g)+w_{n-1}(g))$. Since an immersion regularly homotopic to embeddings must have an even number of self-intersections, and $\chi(g)=0$, Theorem 3 follows immediately from the classification theorem of immersions of M in N (cf. [Li2]).

The aim of Step 2 (for $n>2$) and Step 3 (for $n=2$) is to construct an immersion $f_1 : M \to N$ which is self-transversal and regularly homotopic to g, and calculate the number of self-intersection points of f_1.

Step 2. Suppose $n>2$. Then the multiple points of g consist only of double points whose set $X \subset M$ is the disjoint union of some circles $S_1, \dots, S_k$ such that $g(S_{2i-1}) = g(S_{2i})$, $i=1,\dots,j$, and $S_i \to g(S_i)$ is a nontrivial 2-sheet covering if $i>2j$. By formula (1) and the introduction of [He], we see that the homology class $[X]$ in $H_1(M, Z_2)$ represented by X is the Poincaré dual of $u(g)+w_{n-1}(g)$.

Let $\varepsilon_i = \pm 1$ ($\delta = \pm 1$) so that $\varepsilon_i = 1$ ($\delta_i = 1$) iff S_i ($g(S_i)$) is orientation-preserving in M (in K). Then it is easy to see that

$$\delta_{2i-1} = \delta_{2i} = \varepsilon_{2i-1}\varepsilon_{2i} \quad \text{if } 1 \le i \le j,$$
$$\delta_i = -\varepsilon_i \quad \text{if } 2j < i \le k.$$

Denoting by ξ the orientation bundle of K, and ξ_0 the bundle of nonzero vectors of ξ, we define f_1 on X as follows:

(1) If $1 \le i \le j$, then $f_1 = g$ on S_{2i-1} and $f_1 = u \circ g$ on S_{2i}, where u is a smooth section of ξ over $g(S_{2i})$ transversal to the zero section.

(2) If $i > 2j$ and $\delta_i = -1$, then there is a smooth map $f_1 : S_i \to \xi_0$ such that $p \circ f_1 = g$, where p is the projection of ξ.

(3) If $i>2j$ and $\delta_i = 1$, then there is a smooth map $f_1 : S_i \to \xi$ with $p \circ f_1 = g$ such that f_1 has only one transversal self-intersection point.

Extend f_1 to an immersion of M in N so that $f_1 = g$ outside a tubular neighbourhood of X, and $p \circ f_1 = g$ on this neighbourhood. Then f_1 is regularly homotopic to g in N and has only transversal self-intersection points in $f_1(X)$.

Now we calculate

$$\langle w_1(g), [X]\rangle = \sum_{i=1}^{k} \langle w_1(g), [S_i]\rangle .$$

Letting $s(1)=0$ and $s(-1)=1$, we have

$$\langle w_1(M), [S_i]\rangle = s(\varepsilon_i),$$
$$\langle g^* w_1(K), [S_i]\rangle = \begin{cases} s(\delta_i), & 1 \le i \le 2j, \\ 0, & 2j < i \le k. \end{cases}$$

It follows then from

$$w_1(M) + w_1(g) = g^* w_1(K)$$

that

(1) if $1 \le i \le j$,

$$\langle w_1(g), [S_{2i-1}] + [S_{2i}]\rangle = \begin{cases} 0, & \text{if } \delta_{2i} = 1, \\ 1, & \text{if } \delta_{2i} = -1; \end{cases}$$

(2) if $2j < i \le k$ and $\delta_i = -1$,

$$\langle w_1(g), [S_i]\rangle = 0;$$

(3) if $2j < i \leq k$ and $\delta_i = 1$,

$$\langle w_1(g), [S_i] \rangle = 1.$$

This shows that $\langle w_1(g), [X] \rangle$ is equal to mod 2 number of self-intersection points of f_1. Since

$$\langle w_1(g), [X] \rangle = w_1(g)(u(g) + w_{n-1}(g)),$$

the proof for the case $n > 2$ is complete.

Step 3. Suppose $n = 2$. Now, the multiple points of g consist of double points and triple points whose set $X \subset M$ is the image of a self-transversal immersion h of the disjoint union of some l copies of the circle S. Denote by h_i the restriction of h on the ith copy of S.

Let

$$X_1 = \bigcup_{i=1}^{k} h_i(S), \qquad X_2 = \bigcup_{i=k+1}^{l} h_i(S)$$

such that each $h_i(S)$ in X_1 includes triple points, while X_2 does not. Then $X_1 \cap X_2 = \varnothing$ and $h_\alpha(S) \cap h_\beta(S) = \varnothing$, if $k < \alpha < \beta \leq l$, and $X = X_1 \cup X_2$. Moreover, we have

$$g(X_1) \cap g(X_2) = \varnothing.$$

We are able to cope with X_2 exactly as in Step 2. To cope with X_1, we may assume first that

$$g(h_{2i-1}(S)) = g(h_{2i}(S)), \quad \text{if } 1 \leq i \leq j,$$
$$g(h_i(S)) \neq g(h_\alpha(S)), \quad \text{if } 2j < i \leq k, \alpha \neq i, 1 \leq \alpha \leq k.$$

We have also

$$\delta_{2i-1} = \delta_{2j} = \varepsilon_{2i-1}\varepsilon_{2i}, \quad \text{if } 1 \leq i \leq j,$$
$$\delta_i = -\varepsilon_i, \quad \text{if } 2j < i \leq k,$$

where $\varepsilon_j = \pm 1$, $\delta_i = \pm 1$, and $\varepsilon_i = 1$ $(\delta_i = 1)$ iff $h_i : S \to M$ $(g \circ h_i : S \to K)$ is an orientation-preserving loop.

Let $X_3 = \{d_1, d_2, \dots, d_{3s}\} \subset X_1$ be the set of triple points of g which is the set of self-intersection points on h such that

$$g(d_{3i-2}) = g(d_{3i-1}) = g(d_{3i}), \qquad 1 \leq i \leq s,$$

and let u be a nonzero section of ξ over $g(X_3)$. Let $t : X_3 \to R$ be given by

$$t(d_{3i-2}) = -1, \quad t(d_{3i-1}) = 0, \quad t(d_{3i}) = 1.$$

Step 3(a). Suppose $j \geq 1$ and

$$h_1^{-1}(X_3) = \{a_1, a_2, \dots, a_\alpha\}.$$

Then

$$h_2^{-1}(X_3) = \{b_1, b_2, \dots, b_\alpha\}$$

has the following properties.

(1) $g(h_1(a_i)) = g(h_2(b_i))$,

(2) $h_1(a_i) \neq h_2(b_i)$.

Regarding S as $[0, 1]/\{0, 1\}$, we may assume that

$$a_1 = 0 < a_2 < \cdots < a_\alpha < 1 = a_{\alpha+1}\,,$$
$$a_i = b_i\,, \qquad i = 1, 2, \ldots, \alpha+1\,,$$
$$g(h_1(q)) = g(h_2(q))\,, \quad \text{for } q \in [0, 1]\,.$$

Define f_1 on X_3 by

$$f_1(x) = t(x)u(g(x))\,;$$

then extend f_1 to a smooth map on $h_1(S)$ so that

$$f_1(h_1(q)) = t_i(h_1(q))u_i(g(h_1(q)))\,, \quad \text{if } q \in [a_i, a_{i+1}]\,,$$

where u_i is a nonzero smooth section of ξ over $g(h_1([a_i, a_{i+1}]))$ with $(u_i \circ g \circ h_1)(q) = (u \circ g \circ h_1)(q)$ for $q = a_i, a_{i+1}$, and $t_i \circ h_1$ is a smooth function on $[a_i, a_{i+1}]$ such that

$$t_i(h_1(q))u_1(g(h_1(q))) = t(h_1(q))u_1(g(h_1(q)))\,, \text{ for } q = a_i, a_{i+1}\,,$$

and

$$\frac{d}{dq}t_i(h_1(q))\begin{cases} = 0 & \text{on } [a_i, a_i+\varepsilon] \cup [a_{i+1}-\varepsilon, a_{i+1}]\,, \\ \text{is either identically zero or nonzero} & \text{on } (a_i+\varepsilon, a_{i+1}-\varepsilon)\,, \end{cases}$$

where $0 < \varepsilon < \frac{1}{2}(a_{i+1} - a_i)$ for $i = 1, 2, \ldots, \alpha$.

Let $v(q)$ be a vector in the fiber of ξ over $g(h_1(q))$ such that

$$v(a_i) = (u_1 \circ g \circ h_1)(a_1)\,,$$
$$v(q) = \pm(u_i \circ g \circ h_1)(q)\,, \quad \text{if } q \in [a_i, a_{i+1}]\,,$$

and v as a map $[a_1, a_{\alpha+1}] = [0, 1] \to N$ is continuous. Then v is smooth,

$$v(a_{\alpha+1}) = \delta_1 v(a_1)$$

and

$$f_1(h_1(a)) = s_1(q)v(q)\,,$$

where s_1 is a smooth real-valued function on $[0, 1]$ such that

$$\frac{ds_1}{dq}\begin{cases} = 0 & \text{if } |q - a_i| \le \varepsilon \text{ for some } i\,, \\ \text{is either identically zero or nonzero} & \text{on } (a_i+\varepsilon, a_{i+1}-\varepsilon)\,. \end{cases}$$

Similarly, we can define

$$f_1(h_2(q)) = s_2(q)v(q)\,, \qquad q \in [0, 1]\,,$$

with the same v, and s_2 sharing the same properties of s_1 stated above. Since

$$f_1(h_i(q)) = g(h_i(q))\,, \quad \text{for } q = 0, 1 \text{ and } i = 1, 2\,,$$

we have

$$s_i(0) = \delta_i s_i(1)\,, \quad i = 1, 2\,.$$

Moreover, $s_1(a_i)$ and $s_2(a_i)$ belong to the set $\{0, \pm 1\}$ and

$$s_1(a_i) \ne s_2(a_i)\,, \quad i = 1, 2, \ldots, \alpha+1\,,$$

because $f_1(h_1(a_i)) \ne f_1(h_2(a_i))$. Therefore, the graphs of s_1 and s_2 intersect transversally, and the number of their intersections is even iff $\delta_1 = 1$. Using the same method, we define f_1 on $h_{2i-1}(S) \cup h_{2i}(S)$ for any $i \in [2, j]$.

Step 3(b). Assume $k > 2j$ and

$$h_k^{-1}(X_3) = \{a_1, \dots, a_\alpha, b_1, \dots, b_\alpha\}$$

so that $g(h_k(a_i)) = g(h_k(b_i))$, and $a_2, \dots, a_\alpha$ are located in a half-circle bounded by a_1 and b_1. Let p_1 and p_2 be diffeomorphisms of $[0, 1]$ onto the half-circles containing a_2 and b_2 respectively such that

$$g \circ h_k \circ p_1 = g \circ h_k \circ p_2$$

and $f \circ h_k \circ p_i$ is smooth as a map defined on $[0, 1]/\{0, 1\}$. Then there are a map $v : [0, 1] \to \xi$ and real-valued functions s_1 and s_2 on $[0, 1]$ as in step 3(a), and

$$\begin{aligned}
&v(1) = \delta_k v(0), \\
&(f_1 \circ h_k \circ p_i)(q) = s_i(q)v(q), \quad \text{for } q \in [0, 1] \text{ and } i = 1, 2, \\
&(f_1 \circ h_k \circ p_1)(q) \neq (f_1 \circ h_k \circ p_2)(q), \qquad q \in \{0, 1\}, \\
&(f_1 \circ h_k \circ p_i)(0) \neq (f_1 \circ h_k \circ p_i)(1), \qquad i = 1, 2, \\
&(f_1 \circ h_k \circ p_1)(0) = (f_1 \circ h_k \circ p_2)(1), \\
&(f_1 \circ h_k \circ p_1)(1) = (f_1 \circ h_k \circ p_2)(0).
\end{aligned}$$

Therefore, the graphs of s_1 and s_2 are transversal and the number of their intersections is even iff $\delta_k = -1$. Do the same for $h_i(S)$ with $2j < i \leq k$.

Combining Steps 2, 3(a), and 3(b), we have defined f_1 on $X = X_1 \cup X_2$. f_1 can be extended to an immersion of M in N regularly homotopic to g with transversal self-intersection points whose $\bmod 2$ number contributed by X_1 is

$$\sum_{i=1}^{j} s(\delta_{2i}) + \sum_{i=2j+1}^{k} s(-\delta_i) = \langle w_1(g), [X_1] \rangle.$$

This together with Steps 1 and 2 proves Theorem 4.

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