TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 346, Number 2, December 1994

ON SQUARE-PRESERVING ISOMETRIES OF CONVOLUTION ALGEBRAS

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ABSTRACT. Let S and S' be two semigroups, each contained in a locally compact group. Under certain conditions on S and S', we shall characterize those isometric additive surjections $T : M(S) \to M(S')$ which preserve convolution squares. Our results generalize the classical results of Wendel and of Johnson and also Patterson's characterization of isometric involutions on measure algebras.

Let G be a locally compact group, and let $M(G)$ be the convolution measure algebra consisting of all regular complex Borel measures on G (J. L. Taylor [13]). For each subsemigroup S of G, let $M(S)$ be the set of all $\mu \in M(G)$ that are concentrated on S. Then $M(S)$ forms a norm-closed subalgebra of $M(G)$. Under certain conditions on two semigroups S and S', each contained in a locally compact group, we shall characterize those isometric additive surjections $T : M(S) \to M(S')$ which preserve convolution squares: $T(\mu * \mu) = T\mu * T\mu$. Such a T is either an isomorphism or an anti-isomorphism (as a mapping between two rings), and also either linear or conjugately linear (Theorem 7). Our characterization of such mappings generalizes not only the classical results of J. G. Wendel [14] and of B. E. Johnson [4] about isometric isomorphisms on measure algebras, but also P. L. Patterson's characterization of isometric involutions on measure algebras [7]. Our main result (Theorem 4) appears to be new even for finite groups.

Recall that a semigroup S is said to satisfy the *cancellation law* if $x \neq y$ in S implies $ax \neq ay$ and $xa \neq ya$ for each $a \in S$. Let $f: S \to S'$ be a mapping from a semigroup (or a ring) into another. We say that f is a *semihomomorphism* if it is either a homomorphism or an anti-homomorphism ($f(xy) = f(y)f(x)$). If, in addition, f is a bijection, we call f a *semi-isomorphism.*

We begin with a purely algebraic result, which will play central role in our work.

Proposition 1. *Let S be a semigroup, let S' be a semigroup satisfying the cancellation law, and let $x \to x' : S \to S'$ be a mapping such that for each x and $y \in S$, either $(xy)' = x'y'$ or $(xy)' = y'x'$. Then $x \to x'$ is a semihomomorphism.*

Received by the editors November 9, 1993 and, in revised form, May 30, 1994.
1991 *Mathematics Subject Classification.* Primary 43A10; Secondary 20M15.
Key words and phrases. Measure algebras, semihomomorphisms, Jordan products, tight convergence.

Proof. We shall prove the result in five steps.

(I) $(xy)' = x'y'$ if and only if $(yx)' = y'x'$. Similarly, $(xy)' = y'x'$ if and only if $(yx)' = x'y'$.

To prove these, suppose $(xy)' = x'y'$. Then $(xy^2)' = x'(y')^2$. For, otherwise, $(xy^2)' = (y')^2x'$. Hence

$$(y')^2x' = (xy \cdot y)' = \begin{cases} (xy)' \cdot y' = x'y' \cdot y', & \text{or} \\ y' \cdot (xy)' = y' \cdot x'y'. \end{cases}$$

Since S' satisfies the cancellation law, it follows that $(y')^2x' = x'(y')^2 = (xy^2)'$ in either case. Therefore $(xy^2)' = x'(y')^2$ whenever $(xy)' = x'y'$.

Suppose to the contrary that $(xy)' = x'y'$ but $(yx)' \neq y'x'$. Then $(yx)' = x'y'$, and so

$$\begin{aligned} x'y' \cdot x'y' &= (xy \cdot yx)' = (xy^2 \cdot x)' \\ &= \begin{cases} (xy^2)' \cdot x' = x'(y')^2x', & \text{or} \\ x'(xy^2)' = x' \cdot x'(y')^2. \end{cases} \end{aligned}$$

Hence x' and y' commute in either case, which contradicts $x'y' = (yx)' \neq y'x'$. The second assertion follows from the first assertion.

(II) $(xyx)' = x'y'x'$ for $x, y \in S$.

First suppose $(xy)' = x'y'$. Then $(xy^2)' = x'(y')^2 = (xy)' \cdot y'$, as was shown in the above proof. Hence $(y \cdot xy)' = y' \cdot (xy)' = y'x'y'$ by (I) with (x, y) replaced by (xy, y). By (I), we may exchange x and y in this conclusion to get $(xyx)' = x'y'x'$.

Next suppose $(xy)' = y'x'$, but $(xy \cdot x)' \neq x'(xy)' = x'y'x'$. Then we must have $(xy \cdot x)' = (xy)'x'$, which equals $y'(x')^2$. Hence

$$\begin{aligned} y'x' \cdot y'x' &= (xy \cdot xy)' = (xyx \cdot y)' \\ &= \begin{cases} (xyx)' \cdot y' = y'(x')^2 \cdot y', & \text{or} \\ y' \cdot (xyx)' = y' \cdot y'(x')^2. \end{cases} \end{aligned}$$

In either case, x' and y' commute, which yields a contradiction.

(III) Suppose $x, y, z \in S$,

$$(1) \qquad (xy)' = x'y' \neq y'x' \quad \text{and} \quad (yz)' = y'z'.$$

Then we have $(xz)' = x'z'$.

In fact, suppose to the contrary that $(xz)' \neq x'z'$. Then

$$(2) \qquad (xz)' = z'x' \neq x'z' = (zx)',$$

where the second equality follows from (I). Moreover,

$$\begin{aligned} z' \cdot x'y' \cdot z' &= (z \cdot xy \cdot z)' \quad \text{by (II) and (1)} \\ &= (zx \cdot yz)' \\ &= \begin{cases} (zx)'(yz)' = x'z' \cdot y'z', & \text{or} \\ (yz)'(zx)' = y'z' \cdot x'z' \end{cases} \end{aligned}$$

by (1) and (2). In the first case, $z'x' = x'z'$, which contradicts (2). Therefore we must have

$$(3) \qquad z'x'y' = y'z'x' \quad \text{and} \quad y'z' \neq z'y'.$$

(If $y'z' = z'y'$, then the equality in (3) would yield $x'y' = y'x'$, which contradicts (1).) Moreover, writing $zxy = z \cdot xy$ and applying (1), we get

$$(4) \qquad (zxy)' = z'x'y' \quad \text{or} \quad x'y'z'.$$

From $zxy = zx \cdot y$ and (2), we also get

$$(5) \qquad (zxy)' = x'z'y' \text{ or } y'x'z'.$$

Suppose $(zxy)' = z'x'y'$. Then $z'x'y' = y'x'z'$ by (5) and (2); hence $y'z'x' = y'x'z'$ by (3), so $z'x' = x'z'$, which contradicts (2). Therefore we must have $(zxy)' = x'y'z'$ by (4). Hence (5) shows that either $x'y'z' = x'z'y'$ or $x'y'z' = y'x'z'$. In either case, we get a contradiction by either (3) or (1). Therefore (1) implies $(xz)' = x'z'$.

(IV) Let $x \in S$ be given. If there exists $y \in S$ such that

$$(6) \qquad (xy)' = x'y' \neq y'x',$$

then $(xz)' = x'z'$ for each $z \in S$.

To prove this, fix any $y \in S$ satisfying (6) and any $z \in S$. Suppose to the contrary that $(xz)' \neq x'z'$. Then

$$(7) \qquad (xz)' = z'x' \neq x'z' = (zx)'.$$

Therefore $(yz)' = y'z'$ *cannot* happen by (III). Hence

$$(8) \qquad (yz)' = z'y' \neq y'z'.$$

Note that

$$(xyz)' = (xy \cdot z)' = x'y'z' \text{ or } z'x'y'$$

by (6), and

$$(xyz)' = (x \cdot yz)' = x'z'y' \text{ or } z'y'x'$$

by (8). Therefore, in light of (7) and (8),

$$(9) \qquad (xyz)' = x'y'z' = z'y'x'$$

is the unique possibility. In particular, $(xy \cdot z)' = (xy)'z'$ by (6), and so (I) ensures that

$$(10) \qquad (zxy)' = z'(xy)' = z'x'y'.$$

Finally, consider the following three expansions of $(yzxy)'$:

$$(11) \qquad (y \cdot zx \cdot y)' = y'x'z'y' \quad \text{by (II) and (7)},$$

$$(12) \qquad (y \cdot zxy)' = y'z'x'y' \quad \text{or} \quad z'x'y'y' \quad \text{by (10)},$$

$$(13) \qquad (yz \cdot xy)' = z'y'x'y' \quad \text{or} \quad x'y'z'y' \quad \text{by (8) and (6)}.$$

Since $x'z' \neq z'x'$, we have $y'x'z' = z'x'y'$ by (11) and (12). Since $x'y' \neq y'x'$, we also have $y'x'z' = z'y'x'$ by (11) and (13). But then $x'y' = y'x'$, a contradiction.

(V) Set

$$A = \{x \in S : (xz)' = x'z' \ \forall z \in S\},$$
$$B = \{x \in S : (xz)' = z'x' \ \forall z \in S\}.$$

If $x \in S$ and $(xy)' = x'y' \neq y'x'$ for some $y \in S$, then $x \in A$ by (IV). Hence $x \notin A$ implies that, for each $y \in S$, either $(xy)' = y'x'$ or $x'y' = y'x'$, so $(xy)' = y'x'$ in either case; hence $x \in B$. Thus we have proved that $S = A \cup B$.

Suppose to the contrary that neither $A \subset B$ nor $B \subset A$. Then there exist $x, y \in S$ such that $x \in A \backslash B$ and $y \in B \backslash A$. Note that $(xy)' = x'y'$. Since $S = A \cup B$, we have either $xy \in A$ or $xy \in B$. First suppose that $xy \in A$. Then $z \in S$ implies

$$\begin{aligned} x'y'z' &= (xy)'z' = (xyz)' \\ &= x'(yz)' \quad \text{since } x \in A \\ &= x'z'y' \quad \text{since } y \in B. \end{aligned}$$

Hence $(yz)' = z'y' = y'z'$ for every $z \in S$; that is, $y \in A$, which contradicts our choice of y. Therefore we must have $xy \in B$. But then $z \in S$ implies

$$\begin{aligned} z'x'y' &= z'(xy)' = (xyz)' \quad \text{since } xy \in B \\ &= x'(yz)' \quad \text{since } x \in A \\ &= x'z'y' \quad \text{since } y \in B. \end{aligned}$$

Hence $z'x' = x'z'$, so $(xz)' = z'x'$ for every $z \in S$. By the definition of B, this means $x \in B$, which contradicts our choice of x.

Thus we have proved that either $A \subset B$ or $B \subset A$. Since $S = A \cup B$, we conclude that either $S = A$ or $S = B$, which is nothing but the desired conclusion.

Remarks. Proposition 1 (Hua's theorem) is proved in E. Artin's book [1, pp. 39–40] for additive mappings $x \to x' : S \to S'$ from a field into another. However, the proof there is not applicable to our case because the additivity of $x \to x'$ plays an essential role in that proof. For generalizations of Hua's theorem in some other settings, see N. Jacobson and C. E. Rickart [3], W. R. Scott [10], and L. N. Ševrin [11] and [12].

The following result characterizes the mappings $\beta : S \to S'$ of the form $\beta(x) = zh(x)$ $(x \in S)$, where $h : S \to S'$ is a semihomomorphism and $z \in S'$ is an element which commutes with every element of $h(S)$.

Corollary 2. *Suppose S is a semigroup with an identity e, and S' is a semigroup satisfying the cancellation law. Then a mapping $\alpha : S \to S'$ is a semihomomorphism if and only if there exists a mapping $\beta : S \to S'$ such that*

(i) $$\{\beta(xy), \beta(yx)\} = \{\beta(x)\alpha(y), \alpha(y)\beta(x)\} \ \forall x \quad \text{and} \quad y \in S.$$

Moreover, if β is a mapping satisfying (i), *then $\beta(x) = \beta(e)\alpha(x) = \alpha(x)\beta(e)$ for each $x \in S$.*

Proof. One direction is obvious. To prove the nontrivial direction, let α and $\beta : S \to S'$ be two mappings satisfying (i). Taking $x = e$ in (i), we get $\beta(y) = \beta(e)\alpha(y) = \alpha(y)\beta(e)$ for each $y \in S$. Replacing y by xy in the last representation of β, we infer from (i) that $\{\alpha(xy), \alpha(yx)\} = \{\alpha(x)\alpha(y), \alpha(y)\alpha(x)\}$ because S' satisfies the cancellation law. Hence α is a semihomomorphism by Proposition 1, which completes the proof.

Now let G be a locally compact group, and let $M(G)$ be the convolution measure algebra on G (cf. E. Hewitt–K. A. Ross [2] and Taylor [13]). Recall that an L-subspace of $M(G)$ is a closed subspace X of $M(G)$ such that $\mu \in M(G)$, $\nu \in X$, and $\mu \ll |\nu|$ implies $\mu \in X$ [13]. For each $E \subset G$, let E^0 be the interior of E and let $M(E)$ be the set of all $\mu \in M(G)$ that are concentrated on σ-compact subsets of E (depending on μ). Thus $M(E)$ is an L-subspace of $M(G)$, and if E is a subsemigroup of G, then $M(E)$ is also a subalgebra of $M(G)$. Let λ_G be a left Haar measure on G, and let

$$M_a(E) = \{\mu \in M(E) : \mu \ll \lambda_G\}.$$

Thus $M_a(G)$ is a two-sided L-ideal in $M(G)$.

In the sequel, let G and G' be two locally compact groups, and let S and S' be (not necessarily closed) subsemigroups of G and of G', respectively.

Definition 3. (a) A net (μ_n) in $M(G)$ is said to be *tight* if it is norm-bounded and if for each $\varepsilon > 0$, there exists a compact set $K \subset G$ such that eventually $|\mu_n|(G\backslash K) < \varepsilon$ (M. Loève [6]).

(b) Let $X \subset M(G)$ and $\mu \in M(G)$. A *tight net for* μ *in* X means a net (μ_n) in X which is tight as a net in $M(G)$ and converges to μ in the weak-$*$ topology of $M(G)$.

(c) Let $X \subset M(G)$. A mapping $A : X \to M(G')$ is said to be *tightly continuous* if whenever $\mu \in X$ and (μ_n) is a tight net for μ in X, then $A\mu_n \to A\mu$ in the weak-$*$ topology of $M(G')$.

Let $C(G)$ be the space of all bounded continuous functions on G. It is a well-known fact that if $\mu \in M(G)$ and (μ_n) is a tight net for μ in $M(G)$, then $\mu_n \to \mu$ in the weak topology of $M(G)$ induced by $C(G)$ (cf. [6]).

In order to give a motivation for our next result, let $A : \mathscr{R} \to \mathscr{R}'$ be an additive mapping from a ring into another which preserves squares. Then we may apply A to both sides of $xy + yx = (x+y)^2 - x^2 - y^2$ to obtain

$$\text{(I)} \qquad A(xy + yx) = (Ax)Ay + (Ay)Ax.$$

In other words, A preserves *Jordan products.* Now fix any $z' \in \mathscr{R}'$ which commutes with every Ax $(x \in \mathscr{R})$, and define $Bx = z'(Ax)$ for $x \in \mathscr{R}$. Then (I) ensures that

$$\text{(J}'\text{)} \qquad B(xy + yx) = (Bx)Ay + (Ay)Bx.$$

Our next result shows that, under certain strong assumptions on A and B, (J′) implies (I) and B is obtained in the above fashion (and much more).

Theorem 4. *Suppose* (i) $S \subset G$ *is a subsemigroup with* $S^0 \neq \varnothing$ *and* $e \in \overline{S}$, (ii) *each of* $A, B : M(S) \to M(G')$ *is an additive isometry onto a weak-$*$ closed L-subspace of* $M(G')$, *and* (iii) A *and* B *satisfy*

$$\text{(J)} \qquad B(\mu * \nu + \nu * \mu) = B\mu * A\nu + A\nu * B\mu \ \forall \mu, \nu \in M(S).$$

Then S is necessarily closed, and there exist a unique continuous homomorphism $\gamma : S \to \mathbb{T}$ *and a unique continuous semi-isomorphism* α *from* S *onto a closed subsemigroup of* G' *such that either*

(a) $A\mu = (\gamma\mu) \circ \alpha^{-1}$ $\forall \mu \in M(S)$, *or*

(b) $A\mu = (\gamma\bar{\mu}) \circ \alpha^{-1}$ $\forall \mu \in M(S)$.

In particular, A is a semihomomorphism which is either linear or conjugately linear. Moreover, there exists a unique $c \in \mathbb{T}$ *and a unique* $z \in G'$ *such that* $B\mu = (c\delta_z) * A\mu = (A\mu) * (c\delta_z)$ *for each* $\mu \in M(S)$.

To prove this result, we need three lemmas. The first one of them is well known and so we shall only give a sketchy proof to it.

Lemma 4.1. (i) *For each* $\mu \in M(G)$, *the mappings* $\nu \to \mu * \nu$ *and* $\nu \to \nu * \mu$ *are weak-*$*$ *continuous on* $M(G)$.

(ii) *If* (μ_n) [*resp.* (ν_n)] *is a tight net for* μ [*resp.* ν] *in* $M(G)$, *then* $(\mu_n * \nu_n)$ *is a tight net for* $\mu * \nu$ *in* $M(G)$.

(iii) *If* (μ_n) *is a tight net for* μ *in* $M(G)$ *and if* $\nu \in M_a(G)$, *then* $\mu_n * \nu \to \mu * \nu$ *and* $\nu * \mu_n \to \nu * \mu$ *both in norm.*

Proof. (i) is obvious.

(ii) For $\mu \in M(G)$, $f \in C(G)$, and $y \in G$, set $(\mu \cdot f)(y) = \int f(xy)\, d\mu(x)$. Then $\mu \cdot f \in C(G)$ and $|(\mu \cdot f)(y)| \le \|\mu\|_M \|f\|_u$ for each $y \in G$. Now let (μ_n), (ν_n), μ, and ν be as in (ii). Plainly $(\mu_n * \nu_n)$ is a tight net in $M(G)$. If $f \in C(G)$, then $(\mu_n \cdot f)$ is a uniformly bounded net in $C(G)$ and converges to $\mu \cdot f$ uniformly on each compact subset of G. Therefore $\int f d(\mu_n * \nu_n) = \int (\mu_n \cdot f)\, d\nu_n \to \int (\mu \cdot f)\, d\nu$.

(iii) Replacing each μ_n by $\mu_n - \mu$, we may suppose $\mu = 0$. If $f \in C_c(G)$, then $\mu_n * f \to 0$ uniformly on G and the measures $\mu_n * (f\lambda_G) = (\mu_n * f)\lambda_G$ form a tight net in $M(G)$ by (ii). Hence $\|\mu_n * f\|_1 \to 0$. Since $C_c(G)$ is dense in $L_1(G)$, it follows that $\|\mu_n * \nu\| \to 0$ for each $\nu \in M_a(G)$. Similarly $\|\nu * \mu_n\| \to 0$ for each $\nu \in M_a(G)$, as desired.

Lemma 4.2. *Let* $\mu \in M(G)$ *be given. If there exists* $x \in G$ *such that*

$$(*) \qquad \mu * \delta_y + \delta_y * \mu = 0$$

for $y = x$ *and* x^2, *then* $\mu = 0$.

Proof. Let D be the set of all $y \in G$ that satisfy $(*)$. Then D is a symmetric closed subset of G. Moreover, $y \in G$ belongs to D if and only if $\delta_{y^{-1}} * \mu * \delta_y = -\mu$. Hence $D^3 = D$. Accordingly the set D^2 is a subgroup of G (unless $D = \varnothing$) and $yD^2 = D$ whenever $y \in D$. If both x and x^2 belong to D, then $e \in D^2 = x^2 D^2 = D$ and therefore $2\mu = 0$ by $(*)$ with $y = e$, as desired.

Lemma 4.3. *Let* $S \subset G$ *be a subsemigroup with nonempty interior, and let* $A, B, C, D : M(S) \to M(G')$ *be (not necessarily additive) mappings such that*

$$(*) \qquad A(\mu * \nu + \nu * \mu) = B\mu * C\nu + C\nu * D\mu \;\forall \mu, \nu \in M(S).$$

Suppose A *is a norm-continuous injection,* C *is a bounded mapping, and* $C(M(S))$ *is weak-*$*$ *closed in* $M(G')$. *Then* S *is necessarily closed in* G *and* C *is tightly continuous.*

Proof. Given $\nu \in M(\overline{S})$, $M(S)$ has a tight net (ν_n) for ν. Since C is a bounded mapping, $(C\nu_n)$ is a norm-bounded net in $M(G')$, so it has a weak-$*$ cluster point $\nu' \in M(G')$. To prove $\nu \in M(S)$ and $C\nu = \nu'$, fix any $\mu \in M_a(S)$. Then $\mu * \nu_n + \nu_n * \mu \to \mu * \nu + \nu * \mu$ in norm by Lemma 4.1(iii); in particular, $\mu * \nu + \nu * \mu \in M(S)$. It follows from the continuity of A that

$$(1) \qquad \begin{aligned} A(\mu * \nu + \nu * \mu) &= \lim A(\mu * \nu_n + \nu_n * \mu) \\ &= \lim\{B\mu * C\nu_n + C\nu_n * D\mu\} \quad \text{by } (*) \\ &= B\mu * \nu' + \nu' * D\mu, \end{aligned}$$

where the last equality follows from Lemma 4.1(i).

Now $C(M(S))$ is weak-$*$ closed in $M(G')$; hence there exists $\sigma \in M(S)$ such that $C\sigma = \nu'$. Since A is injective, it follows from $(*)$ and (1) that

$$\mu * (\nu - \sigma) + (\nu - \sigma) * \mu = 0 \ \forall \mu \in M_a(S). \tag{2}$$

For each $x \in S^0$, $M_a(S)$ has a tight net for δ_x. Therefore (2) combined with Lemma 4.1(i) ensures that $\delta_x * (\nu - \sigma) + (\nu - \sigma) * \delta_x = 0$ for each $x \in S^0$. But S^0 is obviously a semigroup and $S^0 \neq \varnothing$ by hypothesis. It follows from Lemma 4.2 that $\nu - \sigma = 0$; hence $\nu = \sigma \in M(S)$ and $C\nu = \nu'$. As ν' was an arbitrary weak-$*$ cluster point of the norm-bounded net $(C\nu_n)$ in $M(G')$, we conclude that $C\nu_n \to C\nu$ weak-$*$.

Thus we have confirmed that $M(\overline{S}) = M(S)$ and C is tightly continuous. In particular, S is closed in G, which completes the proof.

Proof of Theorem 4. (I) Let $x \in S$ be given. Then there exists a unique $\gamma(x) \in \mathbb{T}$ and a unique $\alpha(x) \in G'$ such that

$$A(\delta_x) = \gamma(x)\delta_{\alpha(x)}. \tag{1}$$

Similarly, there exists a unique $\chi(x) \in \mathbb{T}$ and a unique $\beta(x) \in G'$ such that

$$B(\delta_x) = \chi(x)\delta_{\beta(x)}. \tag{2}$$

To confirm these facts, note that two bounded measures μ and ν are mutually singular if and only if $\|\mu \pm \nu\| = \|\mu\| + \|\nu\|$ (S. Kakutani [5]). Since A is an additive isometry, it follows that A preserves the mutual singularity of measures. Moreover, $A(M(S))$ is an L-subspace of $M(G')$ by (ii). Therefore the support of $A(\delta_x)$ is a singleton and $\|A(\delta_x)\| = \|\delta_x\| = 1$. Hence there exist a unique $\gamma(x) \in \mathbb{T}$ and a unique $\alpha(x) \in G'$ satisfying (1). The proof for B is the same.

(II) The function $\gamma : S \to \mathbb{T}$ is a homomorphism and $\alpha : S \to G'$ is a semihomomorphism. Moreover, $\chi(x) = \chi(e)\gamma(x)$ and $\beta(x) = \beta(e)\alpha(x) = \alpha(x)\beta(e)$ for each $x \in S$. (Later, we shall show $e \in S$.)

In fact, let $x, y \in S$ be given. Then

$$\begin{aligned} \chi(xy)\delta_{\beta(xy)} + \chi(yx)\delta_{\beta(yx)} &= B(\delta_{xy} + \delta_{yx}) \quad \text{by (2)} \\ &= (B\delta_x) * (A\delta_y) + (A\delta_y) * (B\delta_x) \quad \text{by (J)} \\ &= \chi(x)\gamma(y)\{\delta_{\beta(x)\alpha(y)} + \delta_{\alpha(y)\beta(x)}\} \quad \text{by (1) and (2).} \end{aligned} \tag{3}$$

Hence $\chi(xy) = \chi(yx) = \chi(x)\gamma(y)$ and

$$\{\beta(xy), \beta(yx)\} = \{\beta(x)\alpha(y), \alpha(y)\beta(x)\}.$$

This, combined with Corollary 2, establishes the results.

(III) We have

$$A(i\delta_x) = i\gamma(x)\delta_{\alpha(x)} \ \forall x \in S, \text{ or} \tag{4.A}$$

$$A(i\delta_x) = -i\gamma(x)\delta_{\alpha(x)} \ \forall x \in S. \tag{5.A}$$

Moreover, A satisfies (4.A) or (5.A) if and only if B satisfies

$$B(i\delta_x) = i\chi(x)\delta_{\beta(x)} \ \forall x \in S, \text{ or} \tag{4.B}$$

$$B(i\delta_x) = -i\chi(x)\delta_{\beta(x)} \ \forall x \in S, \tag{5.B}$$

respectively.

To see these, we first argue as in (I) to get $A(i\delta_x) = \gamma'(x)\delta_{\alpha'(x)}$ for some $\gamma'(x) \in \mathbb{T}$ and some $\alpha'(x) \in G'$. By comparing the norms of both sides of $A(\delta_x + i\delta_x) = \gamma(x)\delta_{\alpha(x)} + \gamma'(x)\delta_{\alpha'(x)}$, we see that $\alpha(x) = \alpha'(x)$ and $\gamma'(x) = \pm i\gamma(x)$. Hence, for each fixed $x \in S$, either (4.A) or (5.A) holds. Similarly, for each fixed $x \in S$, either (4.B) or (5.B) holds.

Suppose to the contrary that there exist $x, y \in S$ such that x satisfies (4.B) but y satisfies (5.A) with x replaced by y. Then

$$\begin{aligned}\chi(x)\gamma(y)\{\delta_{\beta(xy)} + \delta_{\beta(xy)}\} &= B(\delta_{xy} + \delta_{yx}) \quad \text{by (3)} \\ &= -B(i\delta_x * i\delta_y + i\delta_y * i\delta_x) \\ &= -\{B(i\delta_x) * A(i\delta_y) + A(i\delta_y) * B(i\delta_x)\} \\ &= -\chi(x)\gamma(y)\{\delta_{\beta(x)\alpha(y)} + \delta_{\alpha(y)\beta(x)}\},\end{aligned}$$

which is of course absurd. Therefore, if (4.B) holds for some $x \in S$, then (4.A) holds for every $x \in S$. Similarly, if (4.A) holds for some $x \in S$, then (4.B) holds for every $x \in S$. Hence A satisfies either (4.A) or (5.A), (4.A) implies (4.B), and (5.A) implies (5.B).

(IV) S is closed (hence $e \in S$ by (i)), each of α, β, γ, χ is continuous, and α is a semi-isomorphism onto a closed subsemigroup of G'.

In fact, S is closed by Lemma 4.3. To prove the continuity of α and γ, suppose (x_n) is a convergent net in S with limit x. Then (δ_{x_n}) is a tight net for δ_x in $M(S)$. Since A is tightly continuous by Lemma 4.3, it follows that $A(\delta_{x_n}) \to A(\delta_x)$ weak-$*$. In light of (1), this means that $\gamma(x_n) \to \gamma(x)$ and $\alpha(x_n) \to \alpha(x)$. Hence both γ and α are continuous, and so both χ and β are also continuous by (II).

Now α is a semihomomorphism; hence $\alpha(S)$ is a subsemigroup of G'. Moreover, α is an injection by (1) since A is an additive injection. To prove that $\alpha(S)$ is closed in G', let $x' \in G'$ be a cluster point of $\alpha(S)$. Then there exists a net (x_n) in S such that $\alpha(x_n) \to x'$. By (1), this means

$$\overline{\gamma(x_n)}A(\delta_{x_n}) = \delta_{\alpha(x_n)} \to \delta_{x'} \text{ weak-}*.$$

Since $A(M(S))$ is a weak-$*$ closed subspace of $M(G')$, it follows that $\delta_{x'} \in A(M(S))$. It is now obvious from the proof of (I) that $x' = \alpha(x)$ for some $x \in S$. Hence $\alpha(S)$ is closed in G'.

(V) Suppose that A satisfies (4.A). Then (1) ensures that A is linear on $M_d(S)$ (the discrete measures) and so $A\mu = (\gamma\mu) \circ \alpha^{-1}$ for each $\mu \in M_d(S)$. Given $\mu \in M(S)$, $M_d(S)$ has a tight net (μ_n) for μ. Moreover, A is tightly continuous by Lemma 4.3 and $\alpha : S \to G'$ is continuous by (IV). It follows that

$$A\mu = *\text{-}\lim A\mu_n = *\text{-}\lim(\gamma\mu_n) \circ \alpha^{-1} = (\gamma\mu) \circ \alpha^{-1}.$$

Hence A is linear and satisfies condition (a) in Theorem 4. It is easy to check that A is a homomorphism (resp. an antihomomorphism) if and only if α is a homomorphism (resp. an antihomomorphism). Moreover, writing $c = \chi(e)$ and $z = \beta(e)$, we have

$$\begin{aligned}B(\delta_x) &= \chi(x)\delta_{\beta(x)} \quad \text{by (2)} \\ &= c\gamma(x)\delta_{z\alpha(x)} \quad \text{by (II)} \\ &= c\delta_z * \gamma(x)\delta_{\alpha(x)} \\ &= c\delta_z * A(\delta_x) = A(\delta_x) * c\delta_z \quad \text{by (1) and (II).}\end{aligned}$$

Since A satisfies (4.A), B must satisfy (4.B) by (III). Hence an argument similar to the above one shows that

$$B\mu = c\delta_z * A\mu = (A\mu) * c\delta_z \ \forall \mu \in M(S). \tag{6}$$

If A satisfies (5.A), then we can similarly show that A is conjugately linear and satisfies (b), and (6) holds. This completes the proof of Theorem 4.

Corollary 5. *Let S, A, B, α, γ be as in Theorem 4. Suppose that for each $x \in S$, there exists $\mu \in M(S)$ such that*

$$\limsup_n \|\delta_{\alpha(x_n)} * A\mu - \delta_{\alpha(x)} * A\mu\| < 2\|\mu\| \tag{$*$}$$

whenever (x_n) is a net in S such that $\alpha(x_n) \to \alpha(x)$. Then α is an (injective) homeomorphism.

Proof. We shall only confirm the result for the case that α is a homomorphism. Replacing A by $\mu \to A(\bar{\gamma}\mu)$, we may suppose $\gamma = 1$. Thus $x \in S$ and $\mu \in M(S)$ implies $A(\delta_x * \mu) = A\delta_x * A\mu = \delta_{\alpha(x)} * A\mu$.

Now suppose that the continuous bijection $\alpha : S \to \alpha(S)$ is not a homeomorphism. Then there exists $x \in S$ such that α^{-1} is not continuous at $\alpha(x)$. Since S is a closed subset of the locally compact space G, it follows that for each neighborhood V of $\alpha(x)$, $\alpha^{-1}(V)$ has noncompact closure in G. Therefore S has a net (x_n) such that $\alpha(x_n) \to \alpha(x)$ but (x_n) recedes to the point at infinity in the one-point compactification of G. Fix any $\mu \in M(S)$. Then $\delta_{x_n} * \mu$ and $\delta_x * \mu$ are eventually concentrated on "almost disjoint" sets; hence $\|\delta_{x_n} * \mu - \delta_x * \mu\| \to 2\|\mu\|$. Since A is an additive isometry, it follows that

$$\begin{aligned} 2\|\mu\| &= \lim_n \|A(\delta_{x_n} * \mu - \delta_x * \mu)\| \\ &= \lim_n \|\delta_{\alpha(x_n)} * A\mu - \delta_{\alpha(x)} * A\mu\|. \end{aligned}$$

Since $\mu \in M(S)$ was arbitrary, this contradicts our additional assumption, which completes the proof.

Corollary 6. *Suppose, in addition to the hypotheses of Theorem 4, that there exists $\mu \in M(S)$ such that $A\mu$ has a nonzero absolutely continuous component. Then the mapping α in Theorem 4 is a homeomorphism.*

Proof. If $\mu \in M(S)$ is as above, then it satisfies condition $(*)$ in Corollary 5 at each $x \in S$. Hence α is a homeomorphism.

Theorem 7. *Suppose that $S \subset G$ is a subsemigroup with nonempty interior, and that $T : M(S) \to M(G')$ is an additive isometry such that $T(M(S))$ is a weak-$*$ closed L-subspace of $M(G')$ and $T(\mu * \mu) = T\mu * T\mu$ for each $\mu \in M(S)$. Then S is necessarily closed in G, and there exist a unique continuous homomorphism $\gamma : S \to \mathbb{T}$ and a unique continuous semi-isomorphism α from S onto a closed subsemigroup of G' such that either*

(a) $T\mu = (\gamma\mu) \circ \alpha^{-1} \ \forall \mu \in M(S)$, *or*

(b) $T\mu = (\gamma\bar{\mu}) \circ \alpha^{-1} \ \forall \mu \in M(S)$.

In particular, T is a semi-homomorphism which is either linear or conjugately linear.

Proof. In the proof of Theorem 4, take $A = B = T$ and invoke Proposition 1 instead of Corollary 2.

Corollary 8. *Let $S \subset G$ be a closed subsemigroup with nonempty interior, let $T : M(S) \to M(S)$ be a linear surjective isometry which preserves convolution squares, and let $n \geq 2$ be a natural number. Then $T^n = I$ if and only if there exist a continuous homomorphism $\chi : S \to \mathbb{T}$ and a homeomorphic semi-isomorphism α on S such that* (i) $T\mu = \chi \cdot (\mu \circ \alpha^{-1})$ *for* $\mu \in M(S)$, (ii) $\alpha^n = \mathrm{id}$ *on* S, *and* (iii) $\chi(\alpha(x)\alpha^2(x) \cdots \alpha^n(x)) = 1$ *for* $x \in S$.

Proof. Let γ and α be as in the conclusion of Theorem 7. Then α is a homeomorphism. (To see this, simply consider T^{-1}.) Since T is linear, we must have $T\mu = (\gamma\mu) \circ \alpha^{-1}$. Set $\chi = \gamma \circ \alpha^{-1}$, so χ is a continuous character of S and satisfies (i).

Now define $D : C_0(S) \to C_0(S)$ by setting $Df = (\chi f) \circ \alpha$. Then (i) shows that

$$\int f d(T\mu) = \int \chi f d(\mu \circ \alpha^{-1}) = \int Df\, d\mu \ \forall f \in C_0(S) \quad \text{and} \quad \mu \in M(S).$$

In other words, T is the adjoint mapping of D. Moreover,

$$(D^n f)(x) = \chi(\alpha(x)\alpha^2(x) \cdots \alpha^n(x)) f(\alpha^n x)$$

by induction. Therefore $T^n = I$ if and only if α and χ satisfy (ii) and (iii), which completes the proof.

We say that a measurable subset E of G is *Haar-perfect* if each relatively open nonempty subset of E has positive Haar measure. An example of a Haar-perfect closed subsemigroup with nondense interior is obtained by taking $G = \mathbb{R}$ and $S = E \cup [2, \infty)$, where E is any nonempty Lebesgue-perfect compact subset of $[1, 2]$ with empty interior.

Theorem 9. *Suppose* (i) *$S \subset G$ is a Haar-perfect closed subsemigroup containing e and each of S' and S'' is a Haar-perfect closed subsemigroup of G',* (ii) *each of $A : M_a(S) \to M_a(S')$ and $B : M_a(S) \to M_a(S'')$ is an isometric additive surjection, and* (iii) *A and B satisfy*

(J) $$B(\mu * \nu + \nu * \mu) = B\mu * A\nu + A\nu * B\mu \ \forall \mu, \nu \in M_a(S).$$

Then there exist a unique continuous homomorphism $\gamma : S \to \mathbb{T}$ and a unique homeomorphic quasi-isomorphism $\alpha : S \to S'$ which satisfy either (a) *or* (b) *of Theorem* 4 *with $M(S)$ replaced by $M_a(S)$. Moreover, there exist $c \in \mathbb{T}$ and $z \in S''$ such that $B\mu = (c\delta_z) * A\mu = A\mu * (c\delta_z)$ for each $\mu \in M_a(S)$.*

Proof. (I) Arguing as in the proof of Lemma 4.3, one checks that A extends uniquely to a tightly continuous mapping $A' : M(S) \to M(S')$. (This requires Lemma 4.2 for G', but *not* for G.) Moreover, A' is a norm-decreasing additive mapping which satisfies (J) with A replaced by A' for $\mu \in M_a(S)$ and $\nu \in M(S)$.

Similarly, B extends uniquely to a tightly continuous mapping $B' : M(S) \to M(S')$, which is a norm-decreasing additive mapping satisfying

(1) $$B'(\mu * \nu + \nu * \mu) = B'\mu * A'\nu + A'\nu * B'\mu \ \forall \mu, \nu \in M(S).$$

Similarly, B^{-1} extends to a unique tightly continuous mapping $B^{\sim} : M(S'') \to M(S)$, which is norm-decreasing and additive.

(II) We shall show, without using the assumption "$e \in S$", that each of A' and B' is an isometric surjection.

To this end, choose and fix any $\mu_0 \in M(S)$ such that $a\mu_0 \geq 0$ for some nonzero $a \in \mathbb{C}$. Let $\nu \in M_a(S)$ be any probability measure; such a ν exists by (i). Since $M_a(S)$ is a two-sided ideal in $M(S)$ and each of A and B is an isometry, we infer from (1) that

$$\begin{aligned} 2\|\mu_0\| = \|\mu_0 * \nu + \nu * \mu_0\| &= \|B(\mu_0 * \nu + \nu * \mu_0)\| \\ &\leq 2\|B'\mu_0\| \cdot \|A\nu\| = 2\|B'\mu_0\|, \end{aligned}$$

and so $\|B'\mu_0\| = \|\mu_0\|$.

Next choose any net (μ_n) in $M_a(S)$ such that $\|\mu_n\| \leq \|\mu_0\|$ for each n and $\mu_n \to \mu_0$ weak-$*$. Such a net is necessarily a tight net [6]. Hence $B\mu_n \to B'\mu_0$ weak-$*$ by the tight continuity of B' (or by the definition of B') and $\|B\mu_n\| = \|\mu_n\| \leq \|\mu_0\| = \|B'\mu_0\|$ for each n. Therefore $(B\mu_n)$ is a tight net for $B'\mu_0$ in $M_a(S'')$. It follows from the tight continuity of $B^\sim$ that

$$B^\sim(B'\mu_0) = *\text{-}\lim_n B^\sim(B\mu_n) = *\text{-}\lim_n \mu_n = \mu_0. \tag{2}$$

Since $\mu_0 \in M(S)$ was an arbitrary measure such that $a\mu_0 \geq 0$ for some nonzero $a \in \mathbb{C}$ and since each of $B^\sim$ and B' is real-linear, we infer from (2) that $B^\sim(B'\mu) = \mu$ for all $\mu \in M(S)$. Similarly, $B'(B^\sim\mu') = \mu'$ for all $\mu' \in M(S'')$. Since each of B' and $B^\sim$ is norm-decreasing, we conclude that $B' : M(S) \to M(S'')$ is an isometric surjection. Similarly $A' : M(S) \to M(S')$ is an isometric surjection.

(III) Accordingly, S, A', and B' fulfill all the hypotheses of Theorem 4. Hence there exist a unique continuous homomorphism $\gamma : S \to \mathbb{T}$ and a unique continuous quasi-isomorphism $\alpha : S \to S'$ that satisfy either (a) or (b) of Theorem 4 with A replaced by A'. Moreover, α is a homeomorphism by Corollary 6. Since A' is an extension of A, this completes the proof.

Remark 10. (i) In light of Theorem 9 and its proof, both Theorem 7 and Corollary 8 hold with $M(S)$ replaced by $M_a(S)$ whenever S is a nonempty Haar-perfect closed subsemigroup. Corollary 8 and its "conjugately-linear" version are generalizations of the characterization of isometric involutions on $M(G)$ and on $M_a(G)$ by Patterson [7].

(ii) The author has been unable to remove the assumption "$e \in \overline{S}$" in Theorem 4 and in Theorem 9. However, he conjectures that the isometry condition on A (and on B) can be replaced by the weaker condition that A is an injection with $\|A\| < (1+\sqrt{2})/2$. This conjecture is based upon the author's result [9] on idempotent measures and W. Rudin's observation in [8, 4.6.3(c)].

(iii) Let G be an arbitrary locally compact group, and let K be a compact infinite group. Define $A\mu = B\mu = \mu \times \lambda$ for $\mu \in M(G)$, where λ is the norm-one Haar measure on K. Then A is a linear isometric isomorphism of $M(G)$ onto a weak-$*$ closed subalgebra of $M(G \times K)$. However, there exist no mappings γ, α as in the conclusion of Theorem 4. Hence the "L-subspace" assumption in Theorem 4 is *not* superfluous.

(iv) Suppose G is connected, $\mu \in M(G)$, and $D := \{x \in G \mid \mu*\delta_x+\delta_x*\mu = 0\}$ has nonempty interior. Then $\mu = 0$. (This is obvious from the proof of Lemma 4.2.)

(v) Suppose the subsemigroup S contains $e \in G$. Then the "square-preserving" assumption on T in Theorem 7 may be replaced by the "inverse-preserving" assumption: $T(\mu^{-1}) = (T\mu)^{-1}$ whenever $\mu \in M(S)$ is invertible in $M(S)$.

(vi) *The Referee's Proof of Lemma* 4.2: Let $T_y(\mu) = \delta_{y-1} * \mu * \delta_y$. Then $T_y(\mu) = -\mu$ for $y = x$ and x^2 by $(*)$. Hence

$$\mu = T_x(-\mu) = T_x T_x(\mu) = -\mu,$$

and so $\mu = 0$, as desired.

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