CURVATURE CONDITIONS ON RIEMANNIAN MANIFOLDS
WITH BROWNIAN HARMONICITY PROPERTIES

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ABSTRACT. The time and place that Brownian motion on a Riemannian manifold first exits a normal ball of radius ϵ is considered and a general procedure is given for computing asymptotic expansions, as ϵ decreases to zero, for joint moments of the first exit time and place random variables. It is proven that asymptotic versions of exit time and place distribution properties that hold on harmonic spaces are equivalent to certain curvature conditions for harmonic spaces. In particular, it is proven that an asymptotic mean value condition involving first exit place is equivalent to certain levels of curvature conditions for harmonic spaces depending on the order of the asymptotics. Also, it is proven that an asymptotic uncorrelated condition for first exit time and place is equivalent to weaker curvature conditions at corresponding orders of asymptotics.

1. INTRODUCTION

It is a problem of interest to determine what properties of a Riemannian manifold can be deduced from properties of Brownian motion on the manifold. We will establish that certain curvature conditions for the manifold are equivalent to properties of the distributions of the times and places that Brownian motion first exits from normal balls in the manifold.

Let M be an n-dimensional Riemannian manifold. Let $B_m(\epsilon)$ denote the geodesic ball of radius ϵ about m in M. Let $S_m(\epsilon)$ denote the boundary of $B_m(\epsilon)$. Let (X_t) be Brownian motion on $B_m(\epsilon)$, started from m. The infinitesimal generator of (X_t) is $\frac{1}{2}\Delta$, where Δ is the Laplace-Beltrami operator on functions. Let T_ϵ be the first exit time of X_t from $B_m(\epsilon)$:

$$(1.1) \qquad T_\epsilon = \inf\{t > 0 : d(m, X_t) \geq \epsilon\}.$$

For simplicity, when there is no ambiguity, we will use T for T_ϵ.

We examine the marginal and joint distributions of the first exit time and place, T_ϵ and X_{T_ϵ}. In particular, we consider properties involving these distributions that hold on harmonic spaces and establish curvature conditions that are equivalent to asymptotic versions of these properties as $\epsilon \downarrow 0$.

We do this by extending the perturbation method of [GP] and [L1]. We give a general technique by which any joint moment of T_ϵ and X_{T_ϵ} can be expanded in terms of curvature expressions and powers of ϵ.

In the remainder of this section we give curvature conditions for harmonic spaces and state our results. In Section 2, we introduce the family of homogeneous harmonic polynomials which are useful working on the unit ball and unit sphere in $\mathbb{R}^n$. In Section 3, we describe how a perturbation method can be used to expand joint moments of T and X_T in powers of ϵ using the solutions of systems of Poisson equations evaluated at the origin. In Section 4, we give a procedure for evaluating the solutions of the systems of Poisson equations at the origin. Using the techniques developed in the previous sections, in Section 5 we compute expansions of joint moments of T and X_T in powers of ϵ in terms of curvature tensors. These expansions are then used in Section 6 to give precise equivalences between asymptotic Brownian properties and curvature conditions.

Harmonic spaces. Let $r = d(\cdot, m)$ denote the distance function from $m \in M$ and let S denote the unit sphere in the tangent space $T_m M$. The manifold M is said to be *locally harmonic* if, for all $m \in M$, there exists a ball $B_m(\epsilon)$ such that Δr^2 is a function depending only on r inside $B_m(\epsilon)$. Equivalently, M is locally harmonic if, for all $m \in M$, for all smooth functions f defined on S, and for sufficiently small $\epsilon > 0$,

$$(1.2) \qquad E_m\left[f(\exp_m^{-1}(X_T)/\epsilon)\right] = \int_S f(z)\, d\sigma(z),$$

where $d\sigma$ denotes the uniform probability measure on S. In particular, if u is harmonic in $B_m(\epsilon)$ and $u(x) = f(\exp_m^{-1}(x)/\epsilon)$ on $S_m(\epsilon)$ then the expectation on the left in (1.2) equals $u(m)$. Then (1.2) is a mean value property and gives an equivalent definition of locally harmonic spaces. See [RWW], [K].

The class of locally harmonic spaces includes Riemannian manifolds locally isometric to Euclidean space or rank one symmetric spaces [B]. Nonsymmetric examples of harmonic spaces have also been found [DR]. One approach to classifying harmonic spaces is to consider curvature conditions for these spaces.

We fix a point $m \in M$ and represent all tensors with respect to normal coordinates at m. No distinction is made between upper and lower indices and we use the summation convention for any repeated indices, unless otherwise noted. Let R_{ijkl} denote the Riemann curvature tensor, $\rho_{ij} = R_{iaja}$ denote the Ricci curvature tensor, and $\tau = \rho_{aa}$ denote scalar curvature. The metric tensor is denoted g_{ij} and equals the Kronecker delta δ_{ij} at m.

For a locally harmonic manifold M, the second order curvature condition,

$$(1.3) \qquad (\rho_{ij})_m = (1/n)(\tau g_{ij})_m,$$

holds for each $m \in M$ and τ_m is constant on M; i.e.,

$$(1.4) \qquad (\nabla_i \tau)_m = 0$$

holds for each $m \in M$. A manifold with these properties is said to be an *Einstein manifold*. If (1.3) holds for each $m \in M$ and $n > 2$, then the scalar curvature is constant by Schur's theorem.

Let $\sum_{P(ijkl)}$ denote summation over all permutations of the indices i, j, k, and l. At the next level of curvature conditions for locally harmonic manifold M, the fourth order condition,

$$(1.5) \qquad \sum_{P(ijkl)} \left(R_{ipjq}R_{kplq}\right)_m = H_m \sum_{P(ijkl)} (g_{ij}g_{kl})_m,$$

holds for each $m \in M$ and

$$(1.6) \qquad H_m = \frac{(2\|\rho\|^2 + 3\|R\|^2)_m}{2n(n+2)}$$

is constant on M. A consequence of the curvature conditions (1.3) and (1.5) is the *super-Einstein* condition,

$$(1.7) \qquad (R_{ipqr}R_{jpqr})_m = (1/n)(\|R\|^2 g_{ij})_m.$$

If M is an Einstein manifold, (1.7) holds for each $m \in M$, and $n > 4$, then $\|R\|^2$ and H_m are constant by a Schur type theorem [B; p. 165]. Information about curvature conditions for harmonic spaces can be found in [RWW] or [B].

Statement of results. Consider the asymptotic mean value condition (AMVC):

$$(1.8) \qquad E_m\left[f(\exp_m^{-1}(X_T)/\epsilon)\right] = \int_S f(z)\, d\sigma(z) + o(\epsilon^k), \quad \epsilon \downarrow 0,$$

for every f, a smooth function on S, and for all $m \in M$. It is shown in [L1] that if AMVC holds with $k = 2$, then the second order curvature condition (1.3) holds on M. We shall also show that the converse is true. If $n = 2$ and AMVC holds with $k = 3$ then M has constant scalar curvature, and hence constant curvature [P2]. This problem was also considered in [KO1] using a different approach. We also show, conversely, that constant scalar curvature and (1.3) imply AMVC with $k = 3$.

We extend the results of [GP], [L1], and [P2] to give an asymptotic expansion of the left side of (1.8) up to order ϵ^5, $\epsilon \downarrow 0$. We then use this to show that curvature conditions (1.3), (1.4), and (1.5) hold on M if and only if AMVC holds with $k = 4$. Furthermore, AMVC holds with $k = 5$ if and only if, in addition, H_m is constant on M. This also is an improvement of the results of [KO1].

For M a locally harmonic space, Brownian motion also has the property that X_T and T are independent random variables if $\epsilon > 0$ is sufficiently small [KO2], [L2]. This property is weaker than harmonicity. For example the metric product of S^3 and H^3, the three-dimensional spaces with constant curvature 1 and -1, respectively, has the independence property but is not an Einstein manifold and hence not harmonic [Hu].

Weaker curvature conditions have been found for manifolds with this independence property. Consider the asymptotic uncorrelated condition (AUC):
(1.9)

$$E_m\left[Tf(\exp_m^{-1}(X_T)/\epsilon)\right] - E_m[T]\, E_m\left[f(\exp_m^{-1}(X_T)/\epsilon)\right] = o(\epsilon^{k+2}), \quad \epsilon \downarrow 0,$$

for every f, a smooth function on S, and for all $m \in M$. It has been shown that if AUC holds with $k = 3$ then M has constant scalar curvature [L1]. In [KO2] it is shown that if T and X_T are independent for all normal balls at each $m \in M$ then M has constant scalar curvature,
(1.10)

$$(3\nabla_p\nabla_p\rho_{ij} - 4\rho_{ip}\rho_{jp} + 2R_{ipjq}\rho_{pq} + 2R_{ipqr}R_{jpqr})_m = (1/n)[(2\|R\|^2 - 2\|\rho\|^2)g_{ij}]_m$$

holds for every $m \in M$ and $\|R\|^2 - \|\rho\|^2$ is constant on M. If M is an Einstein manifold, then the super-Einstein condition follows from (1.10).

We extend the method of [GP] and [L1] and give an asymptotic expansion of the left side of (1.9) up to order ϵ^7, $\epsilon \downarrow 0$, and show that we actually have

equivalences between the curvature conditions above and AUC for appropriate values of k. We show that AUC holds with $k = 3$ if and only if M has constant scalar curvature, AUC holds with $k = 4$ if and only if M has constant scalar curvature and (1.10) holds on M, and AUC holds with $k = 5$ if and only if scalar curvature and $\|R\|^2 - \|\rho\|^2$ are constant and (1.10) holds on M.

2. WORKING ON THE UNIT BALL

We will translate the problem from the geodesic ball about $m \in M$ to the unit ball in $T_m M$, the tangent space at m. We consider values of ϵ less than ϵ_1 where ϵ_1 is chosen such that the exponential map is non-singular in $B_0(\epsilon_1)$, the ball of radius ϵ_1 in $T_m M$. Let B and S denote the unit ball and unit sphere, respectively, in $T_m M$. We define $\Phi_\epsilon : C^\infty(\overline{B}) \to C^\infty(\overline{B_m(\epsilon)})$ by

$$(2.1) \qquad\qquad (\Phi_\epsilon f)(\exp_m x) \doteq f(x/\epsilon).$$

Working in the unit ball, we find it convenient to use homogeneous harmonic polynomials defined as follows. For $x, z \in \mathbb{R}^n$, let

$$(2.2) \qquad\qquad K(x, z) \doteq \frac{|z|^2 - |x|^2}{|x - z|^n}.$$

Any kth derivative of K,

$$(2.3) \qquad\qquad F(z) = \frac{\partial^k}{\partial x_{i_k} \dots \partial x_{i_1}} K(x, z)\bigg|_{x=0}$$

(where repetitions among the indices $i_1, i_2, \dots, i_k$ are possible) is harmonic and homogeneous of degree $2 - k - n$ and therefore, the Kelvin transform, $|z|^{n+2k-2} F(z)$, is harmonic and homogeneous of degree k. We define the homogeneous harmonic polynomial $Q_{i_1 \dots i_k}$ with coefficients $Q_{i_1 \dots i_k}^{a_1 \dots a_k}$ as follows:

$$(2.4) \quad \begin{aligned} Q_{i_1 \dots i_k}(z) &= \sum_{a_1, \dots, a_k = 1}^{n} Q_{i_1 \dots i_k}^{a_1 \dots a_k} z_{a_1} \cdots z_{a_k} \\ &\doteq \frac{|z|^{n+2k-2}}{n(n+2) \cdots (n+2k-2)} \cdot \frac{\partial^k}{\partial x_{i_k} \dots \partial x_{i_1}} K(x, z)\bigg|_{x=0} \end{aligned}$$

In particular, we take the version of the coefficients $Q_{i_1 \dots i_k}^{a_1 \dots a_k}$ symmetrized with respect to $a_1, \dots, a_k$.

By induction, one can show that partial derivatives of K with respect to x have the form

$$(2.5) \quad \begin{aligned} &\frac{\partial^k}{\partial x_{i_k} \dots \partial x_{i_1}} K(x, z) \\ &= \frac{n(n+2) \cdots (n+2k-2)(z_{i_1} - x_{i_1}) \cdots (z_{i_k} - x_{i_k})(|z|^2 - |x|^2)}{|x - z|^{n+2k}} \\ &\quad + \frac{|z|^2 p(x, z)}{|x - z|^{n+2k-2}} + \frac{|x|^2 q(x, z)}{|x - z|^{n+2k-2}} + \sum_{j=1}^{k} \frac{x_{i_j} r_j(x, z)}{|x - z|^{n+2k-2}} \end{aligned}$$

where p, q, and r_j, $j = 1, \dots, k$, are polynomials in x and z and depend on $i_1, \dots, i_k$. It follows that

$$(2.6) \qquad\qquad Q_{i_1 \dots i_k}(z) = z_{i_1} \cdots z_{i_k} + |z|^2 \hat{Q}_{i_1 \dots i_k}(z)$$

where $\hat{Q}_{i_1 \ldots i_k}$ is a homogeneous polynomial of degree $k - 2$. Since $Q_{i_1 \ldots i_k}$ is harmonic and homogeneous of degree k, it is the harmonic projection of the monomial $z_{i_1} \cdots z_{i_k}$ in the space of homogeneous polynomials of degree k. Furthermore, the polynomials $Q_{i_1 \ldots i_k}(z)$ span the space of harmonic homogeneous polynomials of degree k. Since the restrictions of harmonic homogeneous polynomials to S form a dense set in the space of continuous functions on S, we need only consider functions f from this class in conditions (1.8) and (1.10). See [VK, Section 9.2.3].

Let $\sum_{P(i_1 \ldots i_k)}$ denote summation over the permutations of the indices $i_1, \ldots, i_k$. Then we have from [VK, *ibid.*] that $Q_{i_1 \ldots i_k}(z)$ equals

(2.7)
$$z_{i_1} \cdots z_{i_k}$$
$$+ \sum_{j=1}^{[k/2]} \sum_{P(i_1 \ldots i_k)} \frac{(-1)^j |z|^{2j} \delta_{i_1 i_2} \cdots \delta_{i_{2j-1} i_{2j}} z_{i_{2j+1}} \cdots z_{i_k}}{2^j j! (k - 2j)! (n + 2k - 4)(n + 2k - 6) \cdots (n + 2k - 2j - 2)}.$$

The coefficient $Q_{i_1 \ldots i_k}^{a_1 \ldots a_k}$ equals

(2.8)
$$\frac{1}{k!} \sum_{P(a_1 \ldots a_k)} \left\{ \delta_{i_1 a_1} \cdots \delta_{i_k a_k} \right.$$
$$\left. + \sum_{j=1}^{[k/2]} \sum_{P(i_1 \ldots i_k)} \frac{(-1)^j \delta_{i_1 i_2} \cdots \delta_{i_{2j-1} i_{2j}} \delta_{a_1 a_2} \cdots \delta_{a_{2j-1} a_{2j}} \delta_{i_{2j+1} a_{2j+1}} \cdots \delta_{i_k a_k}}{2^j j! (k - 2j)! (n + 2k - 4)(n + 2k - 6) \cdots (n + 2k - 2j - 2)} \right\}.$$

In particular, we have

$$Q_i(z) = z_i,$$
$$Q_{ij}(z) = z_i z_j - \frac{1}{n} \delta_{ij} |z|^2,$$
$$Q_{ijk}(z) = z_i z_j z_k - \frac{1}{2(n + 2)} \sum_{P(ijk)} \delta_{ij} z_k |z|^2,$$

(2.9)
$$Q_{ijkl}(z) = z_i z_j z_k z_l - \frac{1}{4(n + 4)} \sum_{P(ijkl)} \delta_{ij} z_k z_l |z|^2$$
$$+ \frac{1}{8(n + 2)(n + 4)} \sum_{P(ijkl)} \delta_{ij} \delta_{kl} |z|^4,$$

$$Q_{ijklh}(z) = z_i z_j z_k z_l z_h - \frac{1}{12(n + 6)} \sum_{P(ijklh)} \delta_{ij} z_k z_l z_h |z|^2$$
$$+ \frac{1}{8(n + 4)(n + 6)} \sum_{P(ijklh)} \delta_{ij} \delta_{kl} z_h |z|^4.$$

3. ASYMPTOTIC EXPANSIONS FROM PDE'S

We use approximate solutions of partial differential equations to compute asymptotic expansions of $E_m[T^l(\Phi_\epsilon f)(X_T)]$ in ϵ. Then we will find the approximate solutions as solutions of systems of Poisson equations with boundary

conditions. We only consider the cases $l = 0, 1$ but this method can be extended for any positive integer l.

Proposition 3.1. *Let* $A = \frac{1}{2}\Delta$. *Let* f *be a smooth function on* S. *Suppose* $\hat{u}_\epsilon$ *and* $\hat{v}_\epsilon$ *satisfy the following as* $\epsilon \downarrow 0$:

$$(3.1) \qquad \begin{aligned} \Phi_\epsilon^{-1} A \Phi_\epsilon \hat{u}_\epsilon &= O(\epsilon^{k-1}) \text{ in } B, \quad \hat{u}_\epsilon = f \text{ on } S, \\ \Phi_\epsilon^{-1} A \Phi_\epsilon \hat{v}_\epsilon + \hat{u}_\epsilon &= O(\epsilon^{k+1}) \text{ in } B, \quad \hat{v}_\epsilon = 0 \text{ on } S. \end{aligned}$$

Then

$$(3.2) \qquad \begin{aligned} E_m[(\Phi_\epsilon f)(X_T)] &= \hat{u}_\epsilon(0) + O(\epsilon^{k+1}), \\ E_m[T(\Phi_\epsilon f)(X_T)] &= \hat{v}_\epsilon(0) + O(\epsilon^{k+3}). \end{aligned}$$

Set $f \equiv 1$ and $\hat{u}_\epsilon \equiv 1$, then Proposition 3.1 implies:

Corollary 3.2. *Suppose* $\hat{v}_\epsilon$ *satisfies the following as* $\epsilon \downarrow 0$:

$$(3.3) \qquad \Phi_\epsilon^{-1} A \Phi_\epsilon \hat{v}_\epsilon + 1 = O(\epsilon^{k+1}) \text{ in } B, \quad \hat{v}_\epsilon = 0 \text{ on } S.$$

Then

$$(3.4) \qquad E_m[T] = \hat{v}_\epsilon(0) + O(\epsilon^{k+3}).$$

Perturbation method. Before proceeding with the proof of Proposition 3.1, we describe how solutions of (3.1) can be found using systems of Poisson equations. The operator $A = \frac{1}{2}\Delta$ has the following homogeneous decomposition with respect to normal coordinates (x^j) [GP]:

$$(3.5) \qquad A = \frac{1}{2}\Delta_{-2} + \frac{1}{2}\sum_{i=0}^{\infty}\Delta_i \qquad (\Delta_{-1} = 0).$$

Here Δ_{-2} is the ordinary Euclidean Laplacian and Δ_i is a second order linear differential operator with polynomial coefficients of the form $\Delta_i = q_i^j \partial_j + q_i^{jk} \partial_j \partial_k$ where q_i^j and q_i^{jk} are respectively $(i + 1)$-degree and $(i + 2)$-degree homogeneous polynomials in (x^j). Δ_i maps k-degree homogeneous polynomials to $(k + i)$-degree homogeneous polynomials.

Proposition 3.3. *Let* f *be a smooth function on* S. *Suppose that* u_j *and* v_j, $0 \le j \le k$, *satisfy*

$$\Delta_{-2} u_0 = 0 \text{ in } B, \quad u_0 = f \text{ on } S,$$

$$(3.6) \qquad \Delta_{-2} u_j + \sum_{i=2}^{j} \Delta_{i-2} u_{j-i} = 0 \text{ in } B, \quad u_j = 0 \text{ on } S \quad (1 \le j \le k),$$

$$\Delta_{-2} v_j + \sum_{i=2}^{j} \Delta_{i-2} v_{j-i} + 2u_j = 0 \text{ in } B, \quad v_j = 0 \text{ on } S \quad (0 \le j \le k).$$

Let $\hat{u}_\epsilon$ *and* $\hat{v}_\epsilon$ *be defined as follows:*

$$(3.7) \qquad \hat{u}_\epsilon = \sum_{j=0}^{k} \epsilon^j u_j, \quad \hat{v}_\epsilon = \sum_{j=0}^{k} \epsilon^{j+2} v_j.$$

Then $\hat{u}_\epsilon$ *and* $\hat{v}_\epsilon$ *are solutions of* (3.1).

Proof. On S, $\hat{u}_\epsilon = u_0 + 0 = f$ and $\hat{v}_\epsilon = 0$. In B, the homogeneous decomposition is

$$(3.8) \qquad \Phi_\epsilon^{-1} A \Phi_\epsilon = \frac{1}{2} \epsilon^{-2} \Delta_{-2} + \frac{1}{2} \sum_{i=0}^{\infty} \epsilon^i \Delta_i,$$

where normal coordinates are identified with coordinates in the tangent space. Therefore we have, using (3.6) and (3.7),

$$
\begin{aligned}
(3.9) \qquad \Phi_\epsilon^{-1} A \Phi_\epsilon \hat{u}_\epsilon &= \frac{1}{2} \sum_{j=0}^{k} \epsilon^{j-2} \left[\Delta_{-2} u_j + \sum_{i=2}^{j} \Delta_{i-2} u_{j-i} \right] \\
&\quad + \frac{1}{2} \sum_{j=0}^{k} \epsilon^{k-1} \left[\sum_{i=k-j+1}^{\infty} \epsilon^{i+j-k-1} \Delta_{i-2} u_j \right] \\
&= \frac{1}{2} \sum_{j=0}^{k} \epsilon^{k-1} (\tilde{\Delta}_{k-j-1} u_j),
\end{aligned}
$$

where $\tilde{\Delta}_{k-j-1}$ is a second order linear differential operator with bounded coefficients.

By repeated application of Schauder's estimate, $\|u_j\|_2 \le K_j \|f\|_2$, where the constant K_j does not depend on f. Therefore, $|\Phi_\epsilon^{-1} A \Phi_\epsilon \hat{u}_\epsilon| \le K \epsilon^{k-1} \|f\|_2 = O(\epsilon^{k-1})$.

Similarly, using repeated applications of Schauder's estimate, we have

$$(3.10) \qquad \Phi_\epsilon^{-1} A \Phi_\epsilon \hat{v}_\epsilon + \hat{u}_\epsilon = \frac{1}{2} \sum_{j=0}^{k} \epsilon^{k+1} (\tilde{\Delta}_{k-j-1} v_j) = O(\epsilon^{k+1}). \qquad \square$$

Proof of Proposition 3.1. We first prove the following lemma using the perturbation method.

Lemma 3.4. $E_m[T] = O(\epsilon^2)$ *and* $E_m[T^2] = O(\epsilon^4)$.

Proof. Let v and w be solutions of

$$
\begin{aligned}
(3.11) \qquad \Delta_{-2} v + 2 &= 0 \text{ in } B, \quad v = 0 \text{ on } S, \\
\Delta_{-2} w + 2v &= 0 \text{ in } B, \quad w = 0 \text{ on } S,
\end{aligned}
$$

Let $\tilde{v}_\epsilon = \Phi_\epsilon(\epsilon^2 v)$ and $\tilde{w}_\epsilon = \Phi_\epsilon(\epsilon^4 w)$. Then by Proposition 3.3,

$$
\begin{aligned}
(3.12) \qquad A\tilde{v}_\epsilon + 1 &= O(\epsilon) \text{ in } B_m(\epsilon), \quad \tilde{v}_\epsilon = 0 \text{ on } S_m(\epsilon), \\
A\tilde{w}_\epsilon + \tilde{v}_\epsilon &= O(\epsilon^3) \text{ in } B_m(\epsilon), \quad \tilde{w}_\epsilon = 0 \text{ on } S_m(\epsilon).
\end{aligned}
$$

Therefore by Dynkin's formula we have

$$
\begin{aligned}
(3.13) \qquad 0 = E_m[\tilde{v}_\epsilon(X_T)] &= \tilde{v}_\epsilon(m) + E_m\left[\int_0^T (A\tilde{v}_\epsilon)(X_s)\,ds \right] \\
&= \epsilon^2 v(0) - E_m\left[\int_0^T (1 + O(\epsilon))\,ds \right] \\
&= \epsilon^2 v(0) - (1 + O(\epsilon)) E_m[T].
\end{aligned}
$$

Thus $E_m[T] = O(\epsilon^2)$. Also by Dynkin's formula,

$$0 = E_m[\tilde{w}_\epsilon(X_T)] = \tilde{w}_\epsilon(m) + E_m\left[\int_0^T (A\tilde{w}_\epsilon)(X_s)\,ds\right]$$

$$(3.14) \qquad = \tilde{w}_\epsilon(m) - E_m\left[\int_0^T \{\tilde{v}_\epsilon(X_s) + O(\epsilon^3)\}\,ds\right]$$

$$= \tilde{w}_\epsilon(m) - E_m\left[\int_0^T \tilde{v}_\epsilon(X_s)\,ds\right] + O(\epsilon^5).$$

By Itô's formula and (3.14) we have

$$0 = E_m[T\tilde{v}_\epsilon(X_T)] = 0 + E_m\left[\int_0^T \{s(A\tilde{v}_\epsilon)(X_s) + \tilde{v}_\epsilon(X_s)\}\,ds\right]$$

$$(3.15) \qquad = E_m\left[\int_0^T \{-s(1 + O(\epsilon)) + \tilde{v}_\epsilon(X_s)\}\,ds\right]$$

$$= -(1 + O(\epsilon))E_m[T^2]/2 + \tilde{w}_\epsilon(m) + O(\epsilon^5)$$

$$= -(1 + O(\epsilon))E_m[T^2]/2 + \epsilon^4 w(0) + O(\epsilon^5).$$

Thus $E_m[T^2] = O(\epsilon^4)$. □

Proof of Proposition 3.1. Let $\tilde{u}_\epsilon$ and $\tilde{v}_\epsilon$ be defined as follows:

$$(3.16) \qquad \tilde{u}_\epsilon = \Phi_\epsilon \hat{u}_\epsilon, \qquad \tilde{v}_\epsilon = \Phi_\epsilon \hat{v}_\epsilon,$$

where $\hat{u}_\epsilon$ and $\hat{v}_\epsilon$ are defined as in (3.7). Then we have

$$(3.17) \qquad \begin{aligned} A\tilde{u}_\epsilon = O(\epsilon^{k-1}) \text{ in } B_m(\epsilon), &\quad \tilde{u}_\epsilon = \Phi_\epsilon f \text{ on } S_m(\epsilon), \\ A\tilde{v}_\epsilon + \tilde{u}_\epsilon = O(\epsilon^{k+1}) \text{ in } B_m(\epsilon), &\quad \tilde{v}_\epsilon = 0 \text{ on } S_m(\epsilon). \end{aligned}$$

By Dynkin's formula,

$$E_m[(\Phi_\epsilon f)(X_T)] = E_m[\tilde{u}_\epsilon(X_T)]$$

$$= \tilde{u}_\epsilon(m) + E_m\left[\int_0^T (A\tilde{u}_\epsilon)(X_s)\,ds\right]$$

$$(3.18) \qquad = \hat{u}_\epsilon(0) + E_m\left[\int_0^T O(\epsilon^{k-1})\,ds\right]$$

$$= \hat{u}_\epsilon(0) + O(\epsilon^{k+1}).$$

By Itô's formula and Dynkin's formula,

$$E_m[T(\Phi_\epsilon f)(X_T)] = E_m[T\tilde{u}_\epsilon(X_T)]$$

$$= 0 + E_m\left[\int_0^T \{s(A\tilde{u}_\epsilon)(X_s) + \tilde{u}_\epsilon(X_s)\}\,ds\right]$$

$$(3.19) \qquad = E_m\left[\int_0^T \{\tilde{u}_\epsilon(X_s) + sO(\epsilon^{k-1})\}\,ds\right]$$

$$= E_m\left[\int_0^T \tilde{u}_\epsilon(X_s)\,ds\right] + O(\epsilon^{k+3})$$

$$= \hat{v}_\epsilon(0) + O(\epsilon^{k+3}). \quad \square$$

4. Evaluating solutions at 0

Pizetti formulas. In the previous section, we reduced the problem to a system of Poisson equations with boundary conditions. We derive tools to evaluate solutions of these equations at zero. We give Pizetti's formula in $\mathbb{R}^n$ [CH, pp. 287–289]:

Lemma 4.1. *Let σ_r denote the uniform probability measure on $S_0(r)$. Suppose $\Delta_{-2}^{m+1} u = 0$, then*

$$(4.1) \qquad \int_{S_0(r)} u \, d\sigma_r = u(0) + \sum_{h=1}^{m} \frac{1}{2^h h! \, n(n+2) \cdots (n+2h-2)} (\Delta_{-2}^h u)(0) r^{2h}.$$

We define the following dimension constants for $k \geq 0$:

$$(4.2) \qquad \phi_h^k \doteq \begin{cases} 1, & h = 0, \\ \dfrac{1}{2^h h! \, (n+2k)(n+2k+2) \cdots (n+2k+2h-2)}, & h > 0. \end{cases}$$

As shown in [L1], Lemma 4.1 implies

Corollary 4.2. *Suppose $\Delta_{-2}^{m+1} u = 0$ in B and $u = 0$ on S, then*

$$(4.3) \qquad u(0) = - \sum_{h=1}^{m} \phi_h^0 (\Delta_{-2}^h u)(0).$$

We extend this formula to derivatives of u. Let

$$(4.4) \qquad Q_{i_1 \ldots i_k}(\nabla) \doteq \sum_{j_1, \ldots, j_k=1}^{n} Q_{i_1 \ldots i_k}^{j_1 \ldots j_k} \partial_{j_1} \ldots \partial_{j_k}.$$

We have then:

Lemma 4.3. *Suppose $\Delta_{-2}^{m+1} u = 0$ in B and $u = 0$ on S, then*

$$(4.5) \qquad [Q_{i_1 \ldots i_k}(\nabla) u](0) = - \sum_{h=1}^{m} \phi_h^k [Q_{i_1 \ldots i_k}(\nabla) \Delta_{-2}^h u](0).$$

Proof. For simplicity we write Q for $Q_{i_1 \ldots i_k}$. Let $U(x) = Q(x) u(x)$. Since Q and its derivatives are harmonic, by repeated application of the product rule for the Laplacian, we have the following formula for $\Delta_{-2}^h U$:

$$(4.6) \qquad \Delta_{-2}^h U = \sum_{j=0}^{h} \binom{h}{j} 2^j \sum_{a_1, \ldots, a_j=1}^{n} (\partial_{a_1} \ldots \partial_{a_j} Q)(\partial_{a_1} \ldots \partial_{a_j} \Delta_{-2}^{h-j} u).$$

Since Q is a homogeneous polynomial of degree k,

$$(4.7) \qquad \begin{aligned} (\partial_{a_1} \ldots \partial_{a_j} Q)(0) &= 0, & j &< k, \\ \partial_{a_1} \ldots \partial_{a_j} Q &= k! \, Q_{i_1 \ldots i_k}^{a_1 \ldots a_k}, & j &= k, \\ \partial_{a_1} \ldots \partial_{a_j} Q &= 0, & j &> k. \end{aligned}$$

Therefore,

$$(4.8) \qquad (\Delta_{-2}^h U)(0) = \begin{cases} \dfrac{h! \, 2^k}{(h-k)!} [Q(\nabla) \Delta_{-2}^{h-k} u](0), & h \geq k, \\ 0, & h < k. \end{cases}$$

By (4.6) and (4.7), $\Delta_{-2}^{m+k+1} U = 0$ in B. Also $U = 0$ on S. $U(0) = 0$ since $Q(0) = 0$. Therefore by Corollary 4.2,

$$(4.9) \qquad 0 = -\sum_{h=k}^{m+k} \frac{h!\, 2^k}{(h-k)!} \phi_h^0 [Q(\nabla) \Delta_{-2}^{h-k} u](0).$$

Equation (4.5) follows by replacing the h with $h + k$ and solving for $[Q(\nabla)u](0)$. $\square$

In particular, we can now compute derivatives of u at 0 in terms of derivatives of $\Delta_{-2}u$ at 0.

Corollary 4.4. *Suppose $\Delta_{-2}^{m+1} u = 0$ in B and $u = 0$ on S. Then*

(4.10)

$$(\partial_{i_1} \cdots \partial_{i_k} u)(0) = -\sum_{h=1}^m \phi_h^k (\partial_{i_1} \cdots \partial_{i_k} \Delta_{-2}^h u)(0)$$

$$-\sum_{j=1}^{[k/2]} \sum_{h=0}^{m-j} \sum_{P(i_1 \ldots i_k)} \frac{\phi_h^k \,(-1)^j\, \delta_{i_1 i_2} \cdots \delta_{i_{2j-1} i_{2j}} \,(\partial_{i_{2j+1}} \cdots \partial_{i_k} \Delta_{-2}^{h+j} u)(0)}{2^j\, j!\, (k-2j)!\, (n+2k-4)(n+2k-6) \cdots (n+2k-2j-2)}.$$

Reduction process. The functions u_j ($1 \le j \le k$) and v_j ($0 \le j \le k$) defined by (3.6) above are solutions of Poisson equations of the form

$$(4.11) \qquad \Delta_{-2}V_j + \sum_{i=2}^j \Delta_{i-2}V_{j-i} + 2U_j = 0 \text{ in } B,$$

with boundary condition $V_j = 0$ on S. Here $\Delta_{-2}V_j$ can be expressed in terms of U_j and V_i, $i < j$. If there exists m such that $\Delta_{-2}^{m+1}V = 0$ then Corollaries 4.2 and 4.4 show how to express V_j and its derivatives evaluated at 0 in terms of U_j and V_i, $i < j$, and their derivatives evaluated at 0. We develop formulas to assist in the reduction process and to verify that the hypothesis of Corollaries 4.2 and 4.4 is satisfied by the solutions of (3.6).

Define the differential operator A_k^j as follows:

$$(4.12) \qquad A_k^j \doteq \begin{cases} \Delta_k, & j = 0, \\ \Delta_{-2}A_k^{j-1} - A_k^{j-1}\Delta_{-2}, & j \ge 1. \end{cases}$$

It follows easily by induction that each A_k^j has the following properties:

Lemma 4.5. *A_k^j is a linear differential operator, of order $2 + j$, with polynomial coefficients of degree $\le 2 + k - j$ and $\ge 1 + k - 2j$ (where a polynomial of negative degree is understood to be the 0 polynomial). Furthermore,*

$$(4.13) \qquad \Delta_{-2}^m \Delta_k = \sum_{j=0}^m \binom{m}{j} A_k^j \Delta_{-2}^{m-j}.$$

We now check the hypothesis of Corollaries 4.2 and 4.4.

Lemma 4.6. *Suppose for $0 \leq j \leq k$, u_j and v_j satisfy (3.6). Then $\Delta_{-2}^{2k+1} u_k = 0$ and $\Delta_{-2}^{2k+2} v_k = 0$ in B.*

Proof. We proceed by induction. First consider u_j, $0 \leq j \leq k$. By (3.6), $\Delta_{-2} u_0 = 0$ in B. Suppose $\Delta_{-2}^{2j+1} u_j = 0$ in B, $j < k$. Then by (3.6) and Lemma 4.5,

$$(4.14) \qquad \Delta_{-2}^{2k+1} u_k = -\sum_{i=2}^{k} \sum_{h=0}^{2k} \binom{2k}{h} A_{i-2}^h \Delta_{-2}^{2k-h} u_{k-i}.$$

Now $\Delta_{-2}^{2k-h} u_{k-i} = 0$ for $i \geq 1$ and $h \leq i$. By Lemma 4.5, $A_{i-2}^h = 0$ for $h > i$. Thus $\Delta_{-2}^{2k+1} u_k = 0$ in B.

By (3.6), $\Delta_{-2}^2 v_0 = 0$ in B. Suppose $\Delta_{-2}^{2k+1} u_k = 0$ and $\Delta_{-2}^{2j+2} v_j = 0$, $j < k$. Then,

$$(4.15) \qquad \Delta_{-2}^{2k+2} v_k = -\sum_{i=2}^{k} \sum_{h=0}^{2k+1} \binom{2k+1}{h} A_{i-2}^h \Delta_{-2}^{2k+1-h} v_{k-i} - 2\Delta_{-2}^{2k+1} u_k = 0$$

in B. $\square$

Lemma 4.6 implies that we may apply Corollaries 4.2 and 4.4 to the solutions of (3.6) and thus we may ultimately express each of $u_i(0)$ and $v_i(0)$ in terms of u_0 and its derivatives evaluated at 0. In Section 5, we give the results of this reduction process for $0 \leq i \leq 4$. Before we do this, however, we give formulas for computing u_0 and its derivatives at 0.

Derivatives of u_0 at 0. Let u_0 denote the solution of

$$(4.16) \qquad \Delta_{-2} u_0 = 0 \text{ in } B, \quad u_0 = f \text{ on } S$$

as before. Then by the Poisson formula,

$$(4.17) \quad (\partial_{i_1} \partial_{i_2} \ldots \partial_{i_k} u_0)(0) = \int_S n(n+2) \cdots (n+2k-2) Q_{i_1 i_2 \ldots i_k}(z) f(z) \, d\sigma(z)$$

where σ is the uniform probability measure on S.

When a harmonic homogeneous polynomial $Q_{p_1 \ldots p_m}(z)$ is substituted for $f(z)$, we have an explicit computation for the integral.

Lemma 4.7.

$$(4.18) \qquad \begin{aligned} \int_S n(n+2) \cdots (n+2k-2) & Q_{i_1 \ldots i_k}(z) Q_{p_1 \ldots p_m}(z) \, d\sigma(z) \\ &= \begin{cases} k! \, Q_{i_1 \ldots i_k}^{p_1 \ldots p_k}, & k = m, \\ 0, & k \neq m. \end{cases} \end{aligned}$$

where $Q_{i_1 \ldots i_k}^{p_1 \ldots p_k}$ are the coefficients given in (2.8).

Proof. Since the polynomials Q are harmonic,

$$(4.19) \quad \Delta_{-2}^h (Q_{i_1 \ldots i_k} Q_{p_1 \ldots p_m}) = 2^h \sum_{a_1, \ldots, a_h = 1}^{n} (\partial_{a_1} \ldots \partial_{a_h} Q_{i_1 \ldots i_k})(\partial_{a_1} \ldots \partial_{a_h} Q_{p_1 \ldots p_m}).$$

Since the polynomials Q are homogeneous

$$(4.20) \qquad (\partial_{a_1} \ldots \partial_{a_h} Q_{i_1 \ldots i_k})(0) = \begin{cases} k! \, Q_{i_1 \ldots i_k}^{a_1 \ldots a_k}, & h = k, \\ 0, & h \neq k. \end{cases}$$

Since for harmonic function Q,

$$(4.21) \qquad Q_{p_1 \ldots p_k}(\nabla) Q = \partial_{p_1} \ldots \partial_{p_k} Q,$$

we have

$$(4.22) \qquad \begin{aligned} k! \sum_{a_1, \ldots, a_k = 1}^{n} Q_{i_1 \ldots i_k}^{a_1 \ldots a_k} Q_{p_1 \ldots p_k}^{a_1 \ldots a_k} &= Q_{p_1 \ldots p_k}(\nabla) Q_{i_1 \ldots i_k} \\ &= \partial_{p_1} \ldots \partial_{p_k} Q_{i_1 \ldots i_k} \\ &= k! \, Q_{i_1 \ldots i_k}^{p_1 \ldots p_k}. \end{aligned}$$

Therefore (4.18) follows by Pizetti's formula, (4.19), (4.20), and (4.22). $\square$

5. RESULTS OF REDUCTION PROCEDURE

The operators A_i^j in terms of curvature. We use the summation convention for repeated indices in the following. For convenience we define the following tensors involving curvature with respect to a normal coordinate system:

$$(5.1) \qquad \begin{aligned} (R \# R)_{ijkl} &\doteq R_{iajb} R_{kalb}, & (R \# \nabla \rho)_{ijk} &\doteq R_{iajb} \nabla_b \rho_{ka}, \\ (\rho * \nabla \rho)_{ijk} &\doteq \rho_{ia} \nabla_a \rho_{jk}, & (R * R)_{ij} &\doteq R_{iabc} R_{jabc}, \\ (R \# \rho)_{ij} &\doteq R_{iajb} \rho_{ab}, & (\rho * \rho)_{ij} &\doteq \rho_{ia} \rho_{ja}, \\ (\rho \# \nabla \rho)_i &\doteq \rho_{ab} \nabla_b \rho_{ia}, & (\rho * \nabla \tau)_i &\doteq \rho_{ia} \nabla_a \tau, \\ \|R\|^2 &\doteq R_{abcd} R_{abcd}, & \|\rho\|^2 &\doteq \rho_{ab} \rho_{ab}. \end{aligned}$$

We note the following symmetries:

$$(5.2) \qquad \begin{aligned} (R \# R)_{ijkl} &= (R \# R)_{klij} = (R \# R)_{jilk}, \\ (R * R)_{ij} &= (R * R)_{ji}, \\ (R \# \rho)_{ij} &= (R \# \rho)_{ji}, \\ (\rho * \rho)_{ij} &= (\rho * \rho)_{ji}, \end{aligned}$$

and the following relations:

$$(5.3) \qquad \begin{aligned} (R \# R)_{iaja} &= 2(R \# R)_{iaaj} = (R * R)_{ij}, \\ (R \# R)_{ijaa} &= (R \# \rho)_{ij}, \\ (R * R)_{aa} &= \|R\|^2, \\ (R \# \rho)_{aa} &= (\rho * \rho)_{aa} = \|\rho\|^2. \end{aligned}$$

The first few Δ_i have been computed [GP], [KO2]:

$$\Delta_0 = (1/3)R_{ipjq}(m)\,x^p x^q\,\partial_i\partial_j - (2/3)\rho_{ip}(m)\,x^p\,\partial_i\,,$$

$$\Delta_1 = (1/6)(\nabla_r R_{ipjq})(m)\,x^p x^q x^r\,\partial_i\partial_j$$
$$+ (1/12)[\nabla_i\rho_{pq} - 6\nabla_q\rho_{ip}](m)\,x^p x^q\,\partial_i\,,$$

$$\Delta_2 = (1/60)[3\nabla_s\nabla_r R_{ipjq} + 4R_{rish}R_{pjqh}](m)\,x^p x^q x^r x^s\,\partial_i\partial_j$$

(5.4)
$$+ (1/180)[9\nabla_i\nabla_r\rho_{pq} - 36\nabla_r\nabla_q\rho_{pi} + 4\rho_{rh}R_{hpiq}$$
$$- 16(R\,\#\,R)_{ipqr}](m)\,x^p x^q x^r\,\partial_i\,,$$

$$\Delta_3 = (1/90)[\nabla_t\nabla_s\nabla_r R_{ipjq} + 3\nabla_t(R_{rish}R_{pjqh})](m)\,x^p x^q x^r x^s x^t\,\partial_i\partial_j$$
$$+ (1/360)[6\nabla_i\nabla_s\nabla_r\rho_{pq} - 20\nabla_s\nabla_r\nabla_q\rho_{pi} + 10R_{risk}\nabla_k\rho_{pq}$$
$$+ 3\nabla_i(R\,\#\,R)_{pqrs} - 20\nabla_s(R\,\#\,R)_{ipqr}](m)\,x^p x^q x^r x^s\,\partial_i\,.$$

Using the definitions in (4.12) we compute the following operators:
(5.5)
$$A_0^1 = -(2/3)\rho_{ij}\,\partial_i\partial_j\,,$$

$$A_0^j = 0\,,\quad j \geq 2\,,$$

$$A_1^1\big|_0 = -(1/3)\nabla_i\tau\,\partial_i\,,$$

$$A_1^2 = -2\nabla_k\rho_{ij}\,\partial_i\partial_j\partial_k\,,$$

$$A_1^j = 0\,,\quad j \geq 3\,,$$

$$A_2^1\big|_0 = 0\,,$$

$$A_2^2\big|_0 = -(2/45)[27\nabla_i\nabla_j\tau + 9\nabla_p\nabla_p\rho_{ij} + 38(\rho * \rho)_{ij}$$
$$- 34(R\,\#\,\rho)_{ij} + 6(R * R)_{ij}]\partial_i\partial_j\,,$$

$$A_2^3 = -(8/15)[9\nabla_l\nabla_k\rho_{ij} + 2(R\,\#\,R)_{ijkl}]\partial_i\partial_j\partial_k\partial_l\,,$$

$$A_2^j = 0\,,\quad j \geq 4\,,$$

$$A_3^1\big|_0 = 0\,,$$

$$A_3^2\big|_0 = -(1/45)[18\nabla_i\Delta\tau + 40(\rho * \nabla\tau)_i - 13\nabla_i(\|\rho\|^2)$$
$$+ 60(\rho\,\#\,\nabla\rho)_i + 3\nabla_i(\|R\|^2)]\partial_i\,,$$

$$A_3^3\big|_0 = -(4/15)[12\nabla_k\nabla_j\nabla_i\tau + 9\nabla_k\nabla_p\nabla_p\rho_{ij}$$
$$+ 33\nabla_k(\rho * \rho)_{ij} - 5(\rho * \nabla\rho)_{ijk}$$
$$- 19\nabla_k(R\,\#\,\rho)_{ij} - 50(R\,\#\,\nabla\rho)_{ijk} + 6\nabla_k(R * R)_{ij}]\partial_i\partial_j\partial_k\,,$$

$$A_3^4 = -(16/3)[2\nabla_h\nabla_l\nabla_k\rho_{ij} + \nabla_h(R\,\#\,R)_{ijkl}]\partial_i\partial_j\partial_k\partial_l\partial_h\,,$$

$$A_3^j = 0\,,\quad j \geq 5\,.$$

We note that the operators A_0^2, A_1^3, A_2^4, and A_3^5 are zero because of curvature identities. We also compute the following compositions of operators:

(5.6)

$$A_0^1 A_0 \big|_0 = (4/9)[2(\rho * \rho)_{ij} - (R \# \rho)_{ij}] \partial_i \partial_j \,,$$

$$A_0^1 A_0^1 = (4/9)\rho_{ij}\rho_{kl}\, \partial_i \partial_j \partial_k \partial_l \,,$$

$$A_0^1 A_1 \big|_0 = (1/18)[-\nabla_i(\|\rho\|^2) + 12(\rho \# \nabla \rho)_i] \partial_i \,,$$

$$A_0^1 A_1^1 \big|_0 = (2/9)[\rho_{ij}\nabla_k \tau + 4\nabla_k(\rho * \rho)_{ij} - 2\rho_{pq}\nabla_k R_{piqj}] \partial_i \partial_j \partial_k \,,$$

$$A_0^1 A_1^2 = (4/3)\rho_{ij}\nabla_h \rho_{kl}\, \partial_i \partial_j \partial_k \partial_l \partial_h \,,$$

$$A_1^1 A_0 \big|_0 = (2/9)(\rho * \nabla \tau)_i\, \partial_i \,,$$

$$A_1^1 A_0^1 \big|_0 = (2/9)\rho_{ij}\nabla_k \tau\, \partial_i \partial_j \partial_k \,,$$

$$A_1^2 A_0 \big|_0 = (4/3)[\nabla_k(\rho * \rho)_{ij} + (\rho * \nabla \rho)_{ijk} - R_{piqj}\nabla_k \rho_{pq} - 2(R \# \nabla \rho)_{ijk}] \partial_i \partial_j \partial_k.$$

Let B be a differential operator with polynomial coefficients. Then $B|_0$ consists of the terms of B with constant coefficients. Suppose that all terms in $B|_0$ are of order k such that $B|_0 = b^{i_1 \cdots i_k}\partial_{i_1} \cdots \partial_{i_k}$. Then define the linear operators L by

$$(5.7) \qquad L_{k-2j}^k(B) = b^{i_1 \dots i_k} \sum_{P(i_1 \dots i_k)} \frac{\delta_{i_1 i_2} \cdots \delta_{i_{2j-1} i_{2j}}\, \partial_{i_{2j+1}} \cdots \partial_{i_k}}{2^j\, j!\, (k-2j)!} \,,$$

if $j \le k/2$. We use the following constants and operators derived from the operators in (5.5) and (5.6) using L:

(5.8)

$$L_0^2 A_0^1 = -(2/3)\tau \,,$$

$$L_0^2(A_0^1 A_0) = (4/9)\|\rho\|^2 \,,$$

$$L_2^4(A_0^1 A_0^1) = (8/9)[\tau \rho_{ij} + 2(\rho * \rho)_{ij}] \partial_i \partial_j \,,$$

$$L_1^3(A_1^2) = -4\nabla_i \tau \partial_i \,,$$

$$L_1^3(A_1^2)A_0 \big|_0 = (8/3)(\rho * \nabla \tau)_i\, \partial_i \,,$$

$$L_1^3(A_0^1 A_1^1) = (2/9)[\tau \nabla_i \tau + 6(\rho * \nabla \tau)_i + \nabla_i(\|\rho\|^2) + 12(\rho \# \nabla \rho)_i] \partial_i \,,$$

$$L_3^5(A_0^1 A_1^2) = (4/3)[\tau \nabla_k \rho_{ij} + 2\rho_{ij}\nabla_k \tau + 2\nabla_k(\rho * \rho)_{ij} + 2(\rho * \nabla \rho)_{ijk}] \partial_i \partial_j \partial_k \,,$$

$$L_1^3(A_1^1 A_0) = (2/9)[\tau \nabla_i \tau + 2(\rho * \nabla \tau)_i] \partial_i \,,$$

$$L_1^3(A_1^2 A_0) = (2/3)[4(\rho * \nabla \tau)_i + \nabla_i(\|\rho\|^2) + 4(\rho \# \nabla \rho)_i] \partial_i \,,$$

$$L_1^3(A_1^2)A_0^1 = (8/3)\rho_{ij}\nabla_k \tau\, \partial_i \partial_j \partial_k \,,$$

$$L_1^3(L_1^3(A_1^2)A_0^1) = (8/3)[\tau \nabla_i \tau + 2(\rho * \nabla \tau)_i] \partial_i \,,$$

$$L_0^2(A_2^2) = -(4/45)[18\Delta \tau + 2\|\rho\|^2 + 3\|R\|^2] \,,$$

$$L_2^4(A_2^3) = -(8/15)[27\nabla_i \nabla_j \tau + 9\nabla_p \nabla_p \rho_{ij} - 14(R \# \rho)_{ij} \\ + 18(\rho * \rho)_{ij} + 6(R * R)_{ij}] \partial_i \partial_j \,,$$

$$L_1^3(A_3^3) = (4/15)[-54\nabla_i \Delta \tau - 70(\rho * \nabla \tau)_i + 14\nabla_i(\|\rho\|^2) \\ - 80(\rho \# \nabla \rho)_i - 9\nabla_i(\|R\|^2)] \partial_i \,,$$

$$L_3^5(A_3^4) = (32/3)[-4\nabla_k \nabla_j \nabla_i \tau - 3\nabla_k \nabla_p \nabla_p \rho_{ij} - 6\nabla_k(\rho * \rho)_{ij} \\ + 3\nabla_k(R \# \rho)_{ij} + 10(R \# \nabla \rho)_{ijk} - 2\nabla_k(R * R)_{ij}] \partial_i \partial_j \partial_k.$$

Solutions of Poisson equations at 0. The procedure described in Section 4 may now be used to express the values of the solutions of (3.6) at 0. The details of these calculations are too long to include here. We have the following for $u_i(0)$:

(5.9)

$$u_1(0) = 0,$$

$$u_2(0) = \frac{(A_0^1 u_0)(0)}{8n(n+2)},$$

$$u_3(0) = \frac{(A_1^1 u_0)(0)}{8n(n+2)} + \frac{(A_1^2 u_0)(0)}{48n(n+2)(n+4)},$$

$$u_4(0) = -\frac{L_0^2(A_0^1)(A_0^1 u_0)(0)}{16n^2(n+2)(n+4)} + \frac{(A_0^1 A_0 u_0 + A_2^2 u_0)(0)}{48n(n+2)(n+4)}$$

$$+ \frac{(3A_0^1 A_0^1 u_0 + A_2^3 u_0)(0)}{384n(n+2)(n+4)(n+6)},$$

$$u_5(0) = -\frac{L_0^2(A_0^1)(A_1^1 u_0)(0)}{16n^2(n+2)(n+4)} - \frac{L_0^2(A_0^1)(A_1^2 u_0)(0)}{64n^2(n+2)(n+4)(n+6)}$$

$$+ \frac{[(n+8)A_1^1 A_0 u_0 - L_1^3(A_1^2)A_0 u_0](0)}{48n(n+2)^2(n+4)}$$

$$+ \frac{[3(n+10)A_1^1 A_0^1 u_0 - 4L_1^3(A_1^2)A_0^1 u_0](0)}{384n(n+2)^2(n+4)(n+6)}$$

$$+ \frac{(A_0^1 A_1 u_0 + A_3^2 u_0)(0)}{48n(n+2)(n+4)} + \frac{(3A_0^1 A_1^1 u_0 + A_1^2 A_0 u_0 + A_3^3 u_0)(0)}{384n(n+2)(n+4)(n+6)}$$

$$+ \frac{(10A_0^1 A_1^2 u_0 + A_3^4 u_0)(0)}{3840n(n+2)(n+4)(n+6)(n+8)}.$$

For $v_i(0)$:

(5.10)

$$v_0(0) = \frac{1}{n} u_0(0),$$

$$v_1(0) = 0,$$

$$v_2(0) = \frac{1}{n} u_2(0) - \frac{L_0^2(A_0^1)}{4n^2(n+2)} u_0(0),$$

$$v_3(0) = \frac{1}{n}u_3(0) - \frac{L_0^2(A_0^1)}{4n^2(n+2)}u_1(0) - \frac{[L_1^3(A_1^2)u_0 - 6A_1^1u_0](0)}{24n(n+2)^2(n+4)},$$

$$v_4(0) = \frac{1}{n}u_4(0) - \frac{L_0^2(A_0^1)}{4n^2(n+2)}u_2(0)$$

$$+ \left[\frac{L_0^2(A_0^1)L_0^2(A_0^1)}{8n^3(n+2)(n+4)} - \frac{L_0^2(A_0^1A_0 + A_2^2)}{24n^2(n+2)(n+4)} \right] u_0(0)$$

$$+ \frac{[6L_0^2(A_0^1)(A_0^1u_0) - L_2^4(3A_0^1A_0^1 + A_2^3)u_0](0)}{192n(n+2)(n+4)^2(n+6)}$$

$$+ \frac{(A_0^1A_0u_0 + A_2^2u_0)(0)}{24n(n+2)(n+4)^2(n+6)},$$

$$v_5(0) = \frac{1}{n}u_5(0) - \frac{L_0^2(A_0^1)}{4n^2(n+2)}u_3(0)$$

$$+ \left[\frac{L_0^2(A_0^1)L_0^2(A_0^1)}{8n^3(n+2)(n+4)} - \frac{L_0^2(A_0^1A_0 + A_2^2)}{24n^2(n+2)(n+4)} \right] u_1(0)$$

$$+ \frac{L_0^2(A_0^1)[(n-6)A_1^1u_0 + L_1^3(A_1^2)u_0](0)}{32n^2(n+2)^2(n+4)(n+6)}$$

$$+ \frac{[4L_1^3(L_1^3(A_1^2A_0^1)u_0 - 3(n+10)L_1^3(A_1^1A_0^1)u_0](0)}{192n(n+2)^3(n+4)(n+6)}$$

$$- \frac{(3n^2 + 28n + 68)[L_1^3(A_1^2)A_0^1u_0](0)}{48n(n+2)^3(n+4)^2(n+6)^2(n+8)}$$

$$+ \frac{n^3 + 32n^2 + 244n + 560)(A_1^1A_0^1u_0)(0)}{64n(n+2)^3(n+4)^2(n+6)^2(n+8)}$$

$$+ \frac{L_0^2(A_0^1)(A_1^2u_0)(0)}{192n(n+2)(n+4)(n+6)^2(n+8)}$$

$$- \frac{[L_3^5(10A_0^1A_1^2 + A_3^4)u_0](0)}{1920n(n+2)(n+4)(n+6)^2(n+8)}$$

$$- \frac{(3n^2 + 28n + 68)[L_1^3(A_1^2)A_0^1u_0](0)}{48n(n+2)^3(n+4)^2(n+6)^2(n+8)}$$

$$+ \frac{(n^3 + 32n^2 + 244n + 560)(A_1^1A_0^1u_0)(0)}{64n(n+2)^3(n+4)^2(n+6)^2(n+8)}$$

$$+ \frac{(3A_0^1A_1^1u_0 + A_1^2A_0u_0 + A_3^3u_0)(0)}{192n(n+2)(n+4)(n+6)^2(n+8)}.$$

Formulas concerning the joint distribution of exit time and place. The first expansion is given in [GP].

Proposition 5.1. *For small* $\epsilon > 0$,

$$(5.11) \quad E_m[T] = a_0\epsilon^2 + a_1\epsilon^4\tau_m + a_2\epsilon^6\{6\Delta\tau + (5/n)\tau^2 - \|\rho\|^2 + \|R\|^2\}_m + O(\epsilon^8),$$

where a_0, a_1, *and* a_2 *are positive constants that depend only on the dimension,* n.

We note that by Corollary 3.2 and Proposition 3.3, when $u_0 \equiv 1$,

$$(5.12) \quad E_m[T] = \epsilon^2 v_0(0) + \epsilon^3 v_1(0) + \epsilon^4 v_2(0) + \epsilon^5 v_3(0) + \epsilon^6 v_4(0) + O(\epsilon^8).$$

Substituting the formulas (5.9) and (5.10), we get (5.11) with the following values of a_0, a_1, and a_2:

$$(5.13) \quad a_0 = \frac{1}{n}, \quad a_1 = \frac{1}{6n^2(n+2)}, \quad a_2 = \frac{1}{90n^2(n+2)(n+4)}.$$

We give an expansion for $E_m[(\Phi_\epsilon f)(X_T)]$. This extends the results of [L1] and [P2].

Proposition 5.2. *Let* f *be a smooth function on* S. *For small* $\epsilon > 0$,

(5.14)
$$E_m[(\Phi_\epsilon f)(X_T)]$$
$$= \int_S f(z)\, d\sigma(z) + b_1\epsilon^2 \int_S (\rho_{ij})_m\, Q_{ij}(z)\, f(z)\, d\sigma(z)$$
$$+ \epsilon^3 \int_S [b_2(\nabla_i\tau)_m\, Q_i(z) + b_3(\nabla_k\rho_{ij})_m\, Q_{ijk}(z)]f(z)\, d\sigma(z)$$
$$+ \epsilon^4 \int_S \{b_4[9\nabla_i\nabla_j\tau + 3\nabla_q\nabla_q\rho_{ij} + (10/n)\tau\rho_{ij}$$
$$+ 6(\rho * \rho)_{ij} - 8(R \# \rho)_{ij} + 2(R * R)_{ij}]_m\, Q_{ij}(z)$$
$$+ b_5[18\nabla_l\nabla_k\rho_{ij} - 5\rho_{ij}\rho_{kl} + 4(R \# R)_{ijkl}]_m\, Q_{ijkl}(z)\}f(z)\, d\sigma(z)$$
$$+ \epsilon^5 \int_S \Big\{ \{b_6\tau\nabla_i\tau + b_7(\rho * \nabla\tau)_i$$
$$+ b_8[12\nabla_i(\Delta\tau) - 7\nabla_i(\|\rho\|^2) + 20(\rho \# \nabla\rho)_i + 2\nabla_i(\|R\|^2)]\}_m\, Q_i(z)$$
$$+ \{b_9\tau\nabla_k\rho_{ij} + b_{10}(n-2)\rho_{jk}\nabla_i\tau + b_{11}[12\nabla_k\nabla_j\nabla_i\tau$$
$$+ 9\nabla_k\nabla_h\nabla_h\rho_{ij} + 18\nabla_k(\rho * \rho)_{ij} - 10(\rho * \nabla\rho)_{ijk}$$
$$- 14\nabla_k(R \# \rho)_{ij} - 40(R \# \nabla\rho)_{ijk} + 6\nabla_k(R * R)_{ij}]\}_m\, Q_{ijk}(z)$$
$$+ b_{12}[4\nabla_h\nabla_l\nabla_k\rho_{ij} - 5\rho_{kl}\nabla_h\rho_{ij}$$
$$+ 2\nabla_h(R \# R)_{ijkl}]_m\, Q_{ijklh} \Big\}f(z)\, d\sigma(z)$$
$$+ O(\epsilon^6),$$

where b_i *are nonzero constants that depend only on the dimension,* n.

Proof. By Propositions 3.1 and 3.3,
(5.15)
$$E_m[(\Phi_\epsilon f)(X_T)] = u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + \epsilon^3 u_3(0) + \epsilon^4 u_4(0) + \epsilon^5 u_5(0) + O(\epsilon^6)$$

Substituting formulas (5.9) and then using (4.17) we get (5.14) with the following values of $b_1, b_2, \ldots, b_{12}$:

(5.16)

$$b_1 = -\frac{1}{12}, \quad b_2 = -\frac{1}{24(n+2)}, \quad b_3 = -\frac{1}{24},$$

$$b_4 = -\frac{1}{360(n+4)}, \quad b_5 = -\frac{1}{1440}, \quad b_6 = -\frac{1}{72n(n+2)(n+4)},$$

$$b_7 = -\frac{1}{72(n+2)^2}, \quad b_8 = -\frac{1}{1440(n+2)(n+4)}, \quad b_9 = -\frac{1}{48n(n+6)},$$

$$b_{10} = \frac{1}{288(n+2)(n+6)}, \quad b_{11} = -\frac{1}{1440(n+6)}, \quad b_{12} = -\frac{1}{1440}.$$

We give an expansion involving the joint distribution of T and X_T. This extends the results of [L1].

Proposition 5.3. *Let f be a smooth function on S. For small $\epsilon > 0$,*

$$E_m[T(\Phi_\epsilon f)(X_T)] - E_m[T]E_m[(\Phi_\epsilon f)(X_T)]$$

$$= c_1\epsilon^5 \int_S (\nabla_i \tau)_m \, Q_i(z) f(z) \, d\sigma(z)$$

$$+ c_2\epsilon^6 \int_S [9\nabla_i\nabla_j\tau + 3\nabla_h\nabla_h\rho_{ij} - 4(\rho * \rho)_{ij}$$

$$+ 2(R \# \rho)_{ij} + 2(R * R)_{ij}]_m \, Q_{ij}(z) f(z) \, d\sigma(z)$$

(5.17)
$$+ \epsilon^7 \int_S \Big\{ \{c_3\tau\nabla_i\tau + c_4(\rho * \nabla\tau)_i$$

$$+ c_5[6\nabla_i(\Delta\tau) - \nabla_i(\|\rho\|^2) + \nabla_i(\|R\|^2)]\}_m \, Q_i(z)$$

$$+ \{c_6\rho_{jk}\nabla_i\tau + c_7[4\nabla_k\nabla_j\nabla_i\tau + 3\nabla_k\nabla_h\nabla_h\rho_{ij} - 4\nabla_k(\rho * \rho)_{ij}$$

$$+ 2\nabla_k(R \# \rho)_{ij} + 2\nabla_k(R * R)_{ij}]\}_m \, Q_{ijk}(z) \Big\} f(z) \, d\sigma(z)$$

$$+ O(\epsilon^8),$$

where $c_1, c_2, \dots, c_7$ are nonzero constants that depend only on the dimension, n.

Proof. As before, we use Propositions 3.1 and 3.3. We then substitute formula (5.10) and, using (4.17), we get (5.17) with constants

(5.18)
$$c_1 = \frac{1}{12(n+2)^2(n+4)}, \qquad c_2 = \frac{1}{360(n+4)^2(n+6)},$$

$$c_3 = \frac{5n+6}{72n(n+2)^3(n+4)(n+6)}, \qquad c_4 = -\frac{1}{6(n+2)^3(n+4)(n+6)},$$

$$c_5 = \frac{1}{144(n+2)^2(n+4)(n+6)}, \qquad c_6 = -\frac{1}{144(n+2)^2(n+4)},$$

$$c_7 = \frac{1}{720(n+6)^2(n+8)}. \qquad \square$$

6. Curvature conditions

Asymptotic mean value condition. We now use Proposition 5.2 to derive curvature conditions for manifolds satisfying the asymptotic mean value condition

(AMVC). Recall that for positive integer k AMVC is defined as follows:

$$(1.8) \qquad E_m\left[f(\exp_m^{-1}(X_T)/\epsilon)\right] = \int_S f(z)\,d\sigma(z) + o(\epsilon^k), \quad \epsilon \downarrow 0,$$

for every f, a smooth function on S, and for all $m \in M$.

Theorem 6.1, below, gives relationships between AMVC and the curvature conditions (expressed in normal coordinates at m):

$$(6.1a) \qquad \left(\rho_{pq} - \frac{1}{n}\tau\, g_{pq}\right)_m = 0, \quad 1 \le p, q \le n,$$

$$(6.1b) \qquad (\nabla_p \tau)_m = 0, \quad 1 \le p \le n,$$

(6.1c)

$$\sum_{P(pqrs)}\left[(R \# R)_{pqrs} - \frac{2\|\rho\|^2 + 3\|R\|^2}{2n(n+2)}\,g_{pq}\,g_{rs}\right]_m = 0, \quad 1 \le p, q, r, s \le n,$$

$$(6.1d) \qquad \left[\nabla_p(\|R\|^2)\right]_m = 0, \quad 1 \le p \le n.$$

Theorem 6.1. *A Riemannian manifold M and Brownian motion X on M satisfy AMVC*

(1) *for $k = 2$ if and only if (6.1a) holds for every $m \in M$;*
(2) *for $k = 3$ if and only if (6.1a–b) hold for every $m \in M$;*
(3) *for $k = 4$ if and only if (6.1a–c) hold for every $m \in M$;*
(4) *for $k = 5$ if and only if (6.1a–d) hold for every $m \in M$.*

Proof. As noted in Section 2, it is sufficient to consider only functions of the form $f(z) = Q_{p_1 \ldots p_k}(z)$. Letting $f(z) = Q_{pq}(z)$ in the ϵ^2 term of (5.14) and applying Lemma 4.7, part (1) follows.

To prove part (2), letting $f(z) = Q_p(z)$ in the ϵ^3 term of (5.14) and applying Lemma 4.7, we have that AMVC for $k = 3$ also implies (6.1b) for $m \in M$. We note that (6.1a–b) imply

$$(6.2) \qquad (\nabla_r \rho_{pq})_m = 0, \quad 1 \le p, q, r \le n.$$

Thus the converse also follows.

To prove part (3), we first note that if (6.1a–b) hold for every $m \in M$,

$$(6.3) \qquad \nabla\tau = 0 = \nabla\rho.$$

Letting $f(z) = Q_{pq}(z)$ in the ϵ^4 term of (5.14), applying Lemma 4.7, and using (6.3), we have that AMVC for $k = 4$ implies

$$(6.4) \qquad \begin{aligned} &\left[(5/n)\tau\rho_{pq} + 3(\rho * \rho)_{pq} - 4(R \# \rho)_{pq} + (R * R)_{pq}\right]_m \\ &\qquad = (1/n)\left[((5/n)\tau^2 - \|\rho\|^2 + \|R\|^2)\,g_{pq}\right]_m. \end{aligned}$$

Using (6.1a), this reduces to

$$(6.5) \qquad \left[(R * R)_{pq}\right]_m = (1/n)\left[\|R\|^2\, g_{pq}\right]_m, \quad 1 \le p, q \le n.$$

Now letting $f(z) = Q_{pqrs}(z)$ in the ϵ^4 term of (5.14), applying Lemma 4.7, and using (6.3), we have that AMVC for $k = 4$ implies

(6.6)

$$\sum_{P(pqrs)} \left\{ 4(R \# R)_{pqrs} - 5\rho_{pq}\rho_{rs} \right.$$

$$- \frac{1}{n+4} \left[12(R * R)_{rs} + 8(R \# \rho)_{rs} - 20(\rho * \rho)_{rs} - 10\tau\rho_{rs} \right] g_{pq}$$

$$\left. + \frac{1}{(n+2)(n+4)} \left[6\|R\|^2 - 6\|\rho\|^2 - 5\tau^2 \right] g_{pq} \, g_{rs} \right\}_m = 0.$$

Using (6.1a) and (6.5), this reduces to (6.1c).

To prove the converse, we note that that (6.1a) and (6.1c) imply (6.5) [B, pp. 164–165] and thus also (6.6). If (6.1a–b) hold for every $m \in M$ then (6.3) also holds. Therefore if (6.1a–c) hold for every $m \in M$, AMVC holds for $k = 4$.

Finally, to prove part (4), letting $f(z) = Q_p(z)$ in the ϵ^5 term of (5.14), applying Lemma 4.7, and using (6.3), we have that AMVC for $k = 5$ implies (6.1d) for every $m \in M$. Conversely, from (6.1a–d), (6.3), and (6.5) it follows that all parts of the ϵ^5 term of (5.14) vanish. Therefore AMVC holds for $k = 5$ if (6.1a–d) hold for every $m \in M$. □

Asymptotic uncorrelated condition.   Proceeding as above, we use Proposition 5.3 to derive curvature conditions for manifolds satisfying the asymptotic uncorrelated condition (AUC). Recall that for positive integer k AUC is defined as follows:

(1.9)

$$E_m \left[Tf(\exp_m^{-1}(X_T)/\epsilon) \right] - E_m[T] E_m \left[f(\exp_m^{-1}(X_T)/\epsilon) \right] = o(\epsilon^{k+2}), \quad \epsilon \downarrow 0,$$

for every f, a smooth function on S, and for all $m \in M$.

Theorem 6.2 gives relationships between AUC and the curvature conditions (expressed in normal coordinates at m):

(6.7a) $(\nabla_p \tau)_m = 0, \quad 1 \le p \le n,$

(6.7b)

$$\left[3\nabla_h \nabla_h \rho_{pq} - 4(\rho * \rho)_{pq} + 2(R \# \rho)_{pq} + 2(R * R)_{pq} \right.$$

$$\left. - \frac{-2\|\rho\|^2 + 2\|R\|^2}{n} g_{pq} \right]_m = 0,$$

$$1 \le p, q \le n,$$

(6.7c) $\left[\nabla_p \left(\|R\|^2 - \|\rho\|^2 \right) \right]_m = 0, \quad 1 \le p \le n.$

These conditions are overall weaker than the conditions (6.1a–d).

Theorem 6.2. *A Riemannian manifold M and Brownian motion X on M satisfy AUC*

   (1) *for $k = 3$ if and only if (6.7a) holds for every $m \in M$;*
   (2) *for $k = 4$ if and only if (6.7a–b) hold for every $m \in M$;*
   (3) *for $k = 5$ if and only if (6.7a–c) hold for every $m \in M$.*

Proof. As before, we only consider functions of the form $f(z) = Q_{p_1 \ldots p_k}(z)$. Letting $f(z) = Q_p(z)$ in the ϵ^5 term of (5.17) and applying Lemma 4.7, part (1) follows. We note that if (6.7a) holds for every $m \in M$ or, equivalently, if AUC holds for $k \geq 3$, then τ is constant. Using this fact, letting $f(z) = Q_{pq}(z)$ in the ϵ^6 term of (5.17), and applying Lemma 4.7, part (2) follows.

Finally, to prove part (3), letting $f(z) = Q_p(z)$ in the ϵ^7 term of (5.17), applying Lemma 4.7, and using the fact that τ is constant, we have that AUC for $k = 5$ implies (6.7c) for every $m \in M$. Conversely, if (6.7a,c) hold for every $m \in M$, τ and $\|R\|^2 - \|\rho\|^2$ are constant. It follows from (6.7b) that

$$
(6.8) \qquad \{\nabla_r [3\nabla_h \nabla_h \rho_{pq} - 4(\rho * \rho)_{pq} + 2(R \# \rho)_{pq} + 2(R * R)_{pq}]\}_m = 0,
$$
$$
1 \leq p, q, r \leq n.
$$

Therefore, all parts of the ϵ^7 term of (5.17) vanish and thus AUC holds for $k = 5$ if (6.7a–c) hold for every $m \in M$. $\square$

The first statement below follows immediately since the independence of T and X_T implies AUC for all positive integers k. The second statement follows applying the condition (6.1a) to (6.7b–c).

Corollary 6.3. *If T and X_T are independent random variables for small positive values of ϵ, then conditions (6.7a–c) hold for every $m \in M$. If in addition, M is an Einstein manifold, condition (6.5) holds for every $m \in M$ and $\|R\|^2$ is constant on M.*

7. Remarks

These results can be extended in the following ways. First, the reduction procedure given in Section 4 can be applied to the solutions of larger systems of Poisson equations of the same form as (3.6):
$$(7.1)$$
$$
\Delta_{-2} u_0^{(0)} = 0 \text{ in } B, \quad u_0^{(0)} = f \text{ on } S,
$$
$$
\Delta_{-2} u_j^{(0)} + \sum_{i=2}^{j} \Delta_{i-2} u_{j-i}^{(0)} = 0 \text{ in } B, \quad u_j^{(0)} = 0 \text{ on } S \quad (1 \leq j \leq k),
$$
$$
\Delta_{-2} u_j^{(1)} + \sum_{i=2}^{j} \Delta_{i-2} u_{j-i}^{(1)} + 2u_j^{(0)} = 0 \text{ in } B, \quad u_j^{(1)} = 0 \text{ on } S \quad (0 \leq j \leq k),
$$
$$
\vdots
$$
$$
\Delta_{-2} u_j^{(l)} + \sum_{i=2}^{j} \Delta_{i-2} u_{j-i}^{(l)} + 2u_j^{(l-1)} = 0 \text{ in } B, \quad u_j^{(l)} = 0 \text{ on } S \quad (0 \leq j \leq k).
$$

Then asymptotic expansions of $E_m[T^l(\Phi_\epsilon f)(X_T)]$ in ϵ can be computed as

$$
(7.2) \qquad E_m[T^l(\Phi_\epsilon f)(X_T)] = l! \sum_{j=0}^{k} \epsilon^{j+2l} u_j^{(l)}(0) + O(\epsilon^{k+2l+1}).
$$

Additional terms in the asymptotic expansions (5.11), (5.14), and (5.17) can be computed provided one computes enough terms in the homogeneous decomposition of the Laplacian. It is possible, in principle, to compute any term of the

expansion in ϵ of $E_m[T^l(\Phi_\epsilon f)(X_T)]$. In practice, the size of this task increases extremely rapidly as either l or the level of asymptotics increases.

Finally, these methods can be extended to any diffusion with infinitesimal generator A which admits a homogeneous decompostion with respect to normal coordinates (x^j):

$$(7.3) \qquad A = \frac{1}{2}\Delta_{-2} + \frac{1}{2}\sum_{i=-1}^{\infty} A_i,$$

where Δ_{-2} is the ordinary Euclidean Laplacian and A_i is a second order linear differential operator with polynomial coefficients of the form $A_i = q_i^j \partial_j + q_i^{jk}\partial_j\partial_k$ where q_i^j and q_i^{jk} are respectively $(i+1)$-degree and $(i+2)$-degree homogeneous polynomials in (x^j).

We also note that due to the large number of calculations involved, the calculations leading to all terms in (5.11), all terms except the ϵ^5 term in (5.14), and all terms except the ϵ^7 term in (5.17) were first computed by hand and then checked using computer symbolic algebra programs. The last term in each of (5.14) and (5.17) is based in part on computations made by computer. The programs used in these computations were *Mathematica*(TM), Version 2.0, a mathematical software system developed and distributed by Wolfram Research, Inc., and *MathTensor*(TM), a *Mathematica*-based tensor analysis program developed and distributed by Leonard Parker and Steven Christensen, founders of MathSolutions, Inc.

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