TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 1, January 1995

A NEW MEASURE OF GROWTH FOR COUNTABLE-DIMENSIONAL ALGEBRAS. I

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Dedicated to the memory of Pere Menal

ABSTRACT. A new dimension function on countable-dimensional algebras (over a field) is described. Its dimension values lie in the unit interval $[0, 1]$. Since the free algebra on two generators turns out to have dimension 0 (although conceivably some Noetherian algebras might have positive dimension!), this dimension function promises to distinguish among algebras of infinite GK-dimension.

0. INTRODUCTION

In recent times, the most prominent dimension used in the study of algebras has been the Gelfand-Kirillov dimension (GK-dimension), which measures the "growth of an algebra in terms of generators" (see [KL]). Here we present another view of "growth of an algebra", based on certain infinite matrix representations. By an *algebra* we shall always mean an associative algebra over a field, with an identity element. Some of the results in this paper were announced in [HO].

It would be unthinkable that one could have a serious study of *finite-dimensional* algebras without ever resorting to finite matrix representations. In the theory of *infinite-dimensional* algebras, however, *infinite* matrix representations have played only a very minor role. One reason for this, perhaps, is that the nice "arithmetic" functions provided by a finite matrix representation—such as trace, determinant, rank, etc.—would appear to have no cousins in the infinite-dimensional case. However a recent and surprising result by Goodearl, Menal, and Moncasi [GMM, Proposition 2.1] offers fresh hope for infinite matrix representations of *countable-dimensional* algebras A over a field F: it says that such A can be embedded in the algebra $B(F)$ of all $\omega \times \omega$ matrices over F which are simultaneously row-finite and column-finite. (Note: $\omega \times \omega$ matrices are just $\aleph_0 \times \aleph_0$ matrices with their rows and columns ordered in the standard way.) This result has been the inspiration for our work. For in any such representation of A, the elements of A now have *all* their nonzero entries relatively close to the main diagonal. This raises the question of just how closely these nonzero entries can be squeezed to the main diagonal, for a suitable embedding. To help quantify this we introduce the notion of a *growth curve* for an element

Received by the editors September 20, 1993.
1991 *Mathematics Subject Classification*. Primary 16P90, 16S50; Secondary 16S15, 16G99.

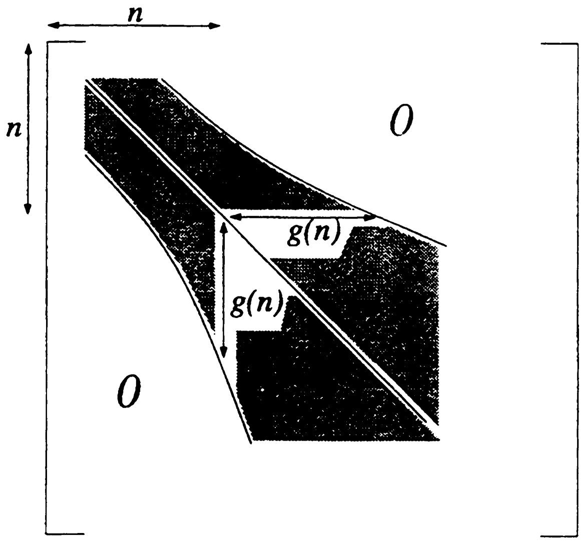

FIGURE 1

$x \in B(F)$. We say that a function $g: \mathbf{N} \to \mathbf{R}^+$ is a growth curve for $x \in B(F)$ if for each $n \in \mathbf{N}$

$$x(n, i) = 0 = x(i, n)$$

for all $i > n + g(n)$. In other words, $g(n)$ gives a bound on the "bandwidth" of x at the (n, n) position, if we interpret bandwidth as in Figure 1. (There are other interpretations of "bandwidth" of course.) Every element x of $B(F)$ has such a growth curve (simply choose $g(n)$ so that all the entries of x in the nth row and nth column are zero more than $g(n)$ places beyond the diagonal). We say that $x \in B(F)$ has *order at most* $g(n)$ *growth* (or that x has $O(g(n))$ growth), where $g: \mathbf{N} \to \mathbf{R}^+$, if there is some constant $c > 0$ such that the function $cg(n)$ is a growth curve for x. If A is a subalgebra of $B(F)$ and every $x \in A$ has $O(g(n))$ growth, then we say that the algebra A itself has $O(g(n))$ growth (but notice that the constant c in $cg(n)$ will depend on the particular $x \in A$). If A has $O(n)$ growth, then we say A has *linear growth.*

Clearly $x \in B(F)$ can be chosen so that all its growth curves $g(n)$ increase as fast as we like. However we shall show in Section 2 that any countable-dimensional algebra A can be embedded in $B(F)$ as a subalgebra of linear growth (Theorem 2.1). In general this is the furthest that we can squeeze such representations of A, at least in terms of $O(n^r)$ growth for $r \leq 1$. Indeed if A is purely infinite (that is, $A \cong A \oplus A$ as right A-modules), then any representation of A in $B(F)$ contains an element whose growth curves must all satisfy $g(n) \geq n$ for infinitely many n (Theorem 3.3).

In that case, when is "sublinear" growth possible? We begin in Section 1 by identifying a range of sublinear growths. For $0 \leq r \leq 1$ we let $G(r)$ be the set of all $x \in B(F)$ having $O(n^r)$ growth. Then, as we shall see, each $G(r)$ is a subalgebra of $B(F)$. (If $r > 1$ then this construction does not give a subalgebra.) These subalgebras have a natural filtration, which provides a powerful tool for some later proofs, both here and in [O]. In terms of these subalgebras

$G(r)$, the above results say that any countable-dimensional algebra A can be embedded in $G(1)$, while purely infinite algebras cannot be embedded in $G(r)$ for $r < 1$. This suggests the idea of using these indices r as a "dimension function" for algebras over F. If A is any countable-dimensional algebra over F, we define the *bandwidth dimension* of A to be

$$\inf\{r \in \mathbf{R},\, r \geq 0 | A \text{ embeds in } G(r)\}$$

or, equivalently,

$$\inf\{r \in \mathbf{R},\, r \geq 0 | A \text{ embeds in } B(F) \text{ with } O(n^r) \text{ growth}\}.$$

By the linear growth result, the bandwidth dimension of countable-dimensional algebras takes values in $[0, 1]$. It turns out that the bandwidth dimension values of even finitely generated algebras completely fill the unit interval $[0, 1]$. The proof of this, which is quite long, is given in [O]. In our present paper, although we don't provide examples of algebras with bandwidth dimension in $(0, 1)$, we do present a simple construction of a finitely generated subalgebra of $G(r)$, for any $0 < r \leq 1$, which cannot be embedded in $G(0)$. This is done in Section 4.

For an uncountable-dimensional algebra A, the appropriate definition for its bandwidth dimension is not yet clear. The approach we shall adopt for the time being is to use the same definition as above but with the understanding that $\inf \Phi = \infty$. An alternative, but inequivalent, approach would be to mimic GK-dimension and define the bandwidth dimension of A as the supremum of the bandwidth dimensions of its countable-dimensional subalgebras. This alternative approach is possibly the more attractive.

In Section 5 we look at the bandwidth dimension of free algebras, and we also include a discussion on the similarities and differences between bandwidth dimension and GK-dimension. That there are differences can be seen by considering the free algebra $F\{x, y\}$ on two generators. From the GK point of view, this algebra has exponential growth and therefore the *largest* possible GK-dimension, namely $+\infty$. In contrast, from our point of view, $F\{x, y\}$ embeds in the algebra $G(0)$ of finite bandwidth matrices and so has zero growth (Theorem 5.2). Accordingly, its bandwidth dimension is the *smallest* possible value, namely 0. Nevertheless there are some similarities. Very roughly, one can view the GK-dimension of a finitely generated algebra A as determining the best *lower* growth curve for the generators of A, but only relative to the *regular* representation of A and then only relative to certain bases. The bandwidth dimension of A, on the other hand, determines the best *lower* and *upper* growth curve for these generators, relative to *all* faithful matrix representations of A. (Indeed, even for the free algebra, had we restricted ourselves to just growth curves for the *regular* representation, our dimension function would also have taken the largest possible value, namely 1. See Section 5.) K. R. Goodearl has raised the interesting question of whether finite GK-dimension always implies that bandwidth dimension is 0. If that were the case, then positive bandwidth dimension might provide a natural extension to GK-dimension by distinguishing among algebras with infinite GK-dimension (that is, taking over where GK gets bad).

Bandwidth dimension behaves as one would hope with respect to subalgebras, finite subdirect products, and finite matrix algebras (see Section 5). For instance, an algebra A and its matrix algebra $M_n(A)$ have the same dimension.

However, bandwidth dimension behaves poorly on factor algebras—but this is the expected price we pay for not giving the free algebra the largest possible dimension.

There are signs that certain bandwidth dimension values may be reflected in interesting purely ring-theoretic properties. Section 3 provides one illustration of this: if a countable-dimensional algebra A has bandwidth dimension less than 1, then A is not purely infinite (Theorem 3.3). Section 6 also supports this view by showing how a growth curve restriction on a regular right self-injective ring R can result in quite strong ring properties: if R has linear growth then R must be of Type I_f, while if R has zero growth then R must have bounded index of nilpotence (Theorem 6.1).

It is not clear whether all finitely generated Noetherian algebras embed in $G(0)$. It is even conceivable that some finitely generated Noetherian algebras could have positive bandwidth dimension! The implications of this would be interesting. Similarly, the implications of a (von Neumann) regular algebra being embeddable in $G(0)$ may be interesting—does it imply direct finiteness?, unit-regularity? These, and other questions, are raised in Section 7.

Finally a word about our terminology. All rings and algebras are associative with an identity element, and all ring maps preserve the identity. The ground ring for our algebras is a field F. The ring of all $\aleph_0 \times \aleph_0$ column-finite matrices over F, with the rows and columns ordered in the standard way according to ω, is denoted by $M_\omega(F)$. For a subset X of a ring R, the left annihilator of X in R is denoted by $l_R(X)$. Similarly $r_R(X)$ denotes the right annihilator.

1. Filtered subalgebras of $B(F)$

In this section we find a family of subalgebras of $B(F)$ associated with growth curves of the form $g(n) = n^r$ where $0 \leq r \leq 1$.

Recall that a function $g\colon \mathbf{N} \to \mathbf{R}^+$ is a growth curve for $x \in B(F)$ if for each $n \in \mathbf{N}$ we have $x(n, i) = 0 = x(i, n)$ whenever $i - n > g(n)$. As we observed in the introduction, every matrix in $B(F)$ has a growth curve. We begin by calculating a growth curve for the product of two matrices in $B(F)$.

Lemma 1.1. *Suppose $x, y \in B(F)$ have g and h (respectively) as growth curves which are both increasing. Then a growth curve for the product xy is given by the function $f\colon \mathbf{N} \to \mathbf{R}^+$ where*

$$f(n) = \max\{g(n) + h(n + [g(n)]),\, h(n) + g(n + [h(n)])\}$$

and where $[\ ]$ denotes the integer part.

Proof. The (n, j) entry of xy is $\sum_{k\geq 1} x(n, k)y(k, j)$ and the largest j for which this can be nonzero cannot exceed $j = k + [h(k)]$ for $k = n + [g(n)]$. Thus for $j > n + g(n) + h(n + [g(n)])$ the (n, j) entry of xy is zero. Similarly the (i, n) entry of xy is zero whenever $i > n + h(n) + g(n + [h(n)])$. Taking the larger of these two values gives the formula for f. □

If we apply this calculation to growth curves of the form cn^r where $c > 0$ and $0 \leq r \leq 1$ we can show that the sets $G(r)$ defined in the introduction are in fact subalgebras of $B(F)$.

Proposition 1.2. *Suppose $0 \leq r \leq 1$. Let*

$$G(r) = \{x \in B(F)\colon x \text{ has } O(n^r) \text{ growth}\}$$

and for each $c \geq 0$ *let*

$$W_r(c) = \{x \in B(F)\colon\ x \text{ has } cn^r \text{ as a growth curve}\}.$$

Then:

(a) *the* $W_r(c)$ *form a chain of subspaces whose union is* $G(r)$;

(b) *for any* $c_1, c_2 \geq 0$ *we have* $W_r(c_1)W_r(c_2) \subseteq W_r(c_3)$ *where*

$$c_3 = \max\{c_1 + c_2(1+c_1)^r,\ c_2 + c_1(1+c_2)^r\};$$

(c) $G(r)$ *is a subalgebra of* $B(F)$.

Proof. (a) is trivial and (c) follows immediately from (a) and (b), so we just need to check (b). Let $x \in W_r(c_1)$ and $y \in W_r(c_2)$. We apply Lemma 1.1 with $g(n) = c_1 n^r$ and $h(n) = c_2 n^r$. Since $r \leq 1$ we have

$$g(n) + h(n + g(n)) = c_1 n^r + c_2[n + c_1 n^r]^r \leq (c_1 + c_2(1+c_1)^r)n^r$$

and similarly

$$h(n) + g(n + h(n)) \leq (c_2 + c_1(1+c_2)^r)n^r$$

as required. □

Remark 1.3. Of course c_3 can be replaced by any larger value in (b) above. In particular

$$c_3 = c_1 + c_2 + c_1 c_2$$

gives a rather simpler value that works for any $r \leq 1$. □

The multiplication law in Proposition 1.2(b) says that the subspaces $W_r(k)$, $k = 1, 2, \dots$, almost provide a filtering for the subalgebra. In fact when $r = 0$ we really do get a filtering, since in that case Proposition 1.2(b) says that

$$W_0(c_1)W_0(c_2) \subseteq W_0(c_1 + c_2).$$

This filtering will be useful later, so we record some of its properties.

Proposition 1.4. *Let* $W_0(k)$, $k = 1, 2, \dots$, *be the subspaces of* $G(0)$ *given by Proposition* 1.2.

(a) *For each* k,

$$W_0(k) = \{x \in B(F)\colon x \text{ has constant bandwidth at most } k\}.$$

(b) $W_0(0) \subseteq W_0(1) \subseteq \cdots$ *and* $G(0) = \bigcup_{k\geq 1} W_0(k)$. *Also* $W_0(c_1)W_0(c_2) \subseteq W_0(c_1 + c_2)$.

(c) $W_0(0)$ *is a subalgebra of* $G(0)$ *which is isomorphic to* $\prod_{\mathbf{N}} F$.

(d) *For each* k, $W_0(k)$ *is a finitely generated projective right and left* $W_0(0)$-*module.*

Proof. (a), (b), (c) are trivial, and (d) is easy once one splits $W_0(k)$ into a direct sum whose factors correspond to matrices having nonzero entries in exactly one sub-diagonal or super-diagonal. □

It may be worth noting here that $G(0)$ has already been studied by Tjukavkin [T], who observed that $G(0)$ is a nonregular ring in which every one-sided ideal is generated by idempotents. It is not hard to see that in fact all the $G(r)$ (where $0 \leq r \leq 1$), and indeed $B(F)$ itself, also have this property.

The other subalgebras $G(r)$ given by Proposition 1.2 also have a filtered structure, but it derives from a block matrix view of growth curves which we shall

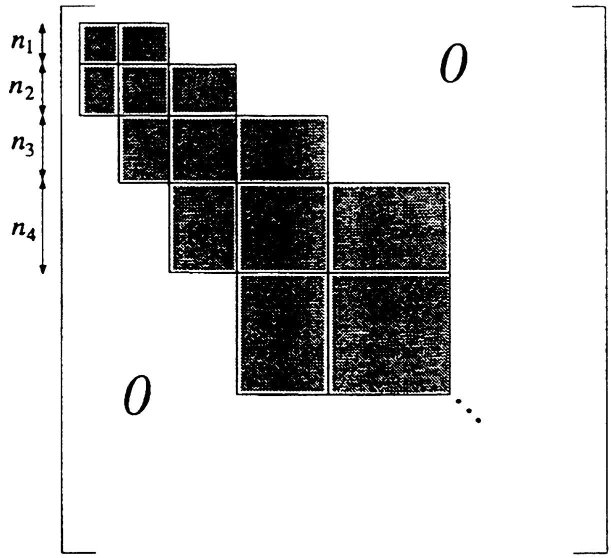

FIGURE 2

now describe. We begin with the trivial observation that, just as every $x \in B(F)$ has a growth curve, so every $x \in B(F)$ can be viewed as a block tridiagonal matrix where all the blocks down the main diagonal are square (finite) matrices (see Figure 2).

Of course the sizes of the blocks will vary for different $x \in B(F)$, or even for different $x \in G(r)$, where r is fixed, $0 \leq r \leq 1$. However, once r is fixed, it turns out (see Proposition 1.6 below) that we can choose a fixed sequence of block sizes $n_1, n_2, \ldots$ (see Figure 2), and that we can use these to represent each element of $G(r)$ as a matrix of finite block-bandwidth. (For a matrix in block form, the *block-bandwidth* is just the bandwidth measured in terms of the number of off-diagonal *blocks*, rather than the number of off-diagonal *entries*.) In such a representation an element of $G(r)$ looks like an element of $G(0)$ except that its entries are block matrices. In order to discuss our filtration of $G(r)$ we introduce the following notion.

Definition. The *spine* S determined by an increasing sequence $n_1, n_2, \ldots$ of positive integers is the natural copy of $\prod_{k=1}^{\infty} M_{n_k}(F)$ inside $B(F)$, that is, S consists of all block-diagonal matrices of the form as shown in Figure 3.

If $S \subseteq G(r)$, then we say S is a spine for $G(r)$. □

Notice that the spine determined by a sequence $\{n_k\}$ of positive integers will be a spine for $G(r)$ if and only if

$$n_{k+1} = O((n_1 + \cdots + n_k)^r).$$

Clearly $G(r)$ can have many different spines. The following result shows how to choose block sizes $n_1, n_2, \ldots$ so that the corresponding spine fits $G(r)$ as closely as possible.

Proposition 1.5. *Suppose* $0 \leq r \leq 1$. *Define the sequence* $n_1, n_2, \ldots$ *as follows:*

(a) *if* $r < 1$ *then set* $t = r/(1-r)$ *and let* $n_k = [k^t]$ *where* $[x]$ *denotes the greatest integer less than or equal to* x,

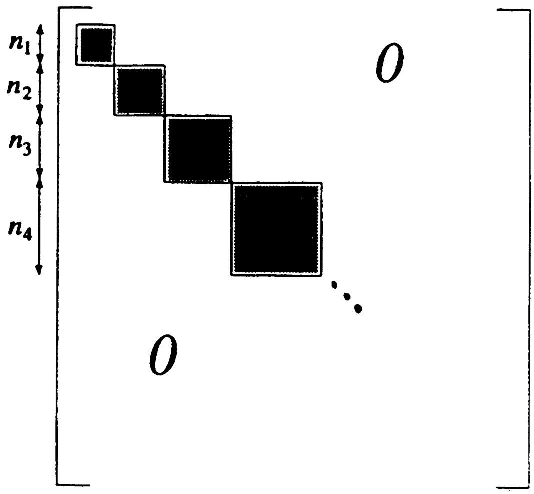

FIGURE 3

(b) *if* $r = 1$ *then let* $n_k = 2^k$.

Let S *be the spine determined by the* n_k. *Then* S *is a spine for* $G(r)$ *but is not a spine for* $G(s)$ *for any* $s < r$.

Proof. The proposition will be proved if we can find positive constants c_1, c_2 such that

$$c_1(n_1 + n_2 + \cdots + n_k)^r \leq n_{k+1} \leq c_2(n_1 + n_2 + \cdots + n_k)^r$$

is true for all large enough k.

In the case $r = 1$, $n_1 + n_2 + \cdots + n_k = 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 2 = n_{k+1} - 2$ and so $c_1 = 1$ and $c_2 = 2$ will do the trick.

Now suppose $r < 1$ and consider the sequence $n_k = [k^t]$. For the moment we shall just assume that $t > 0$: the reason for the correct value of t will appear during the course of the proof. Let $f\colon \mathbf{R}^+ \to \mathbf{R}^+$ be the function $f(x) = x^t$ and notice that f is strictly increasing. Hence for any integer $n \geq 0$ we have

$$[n^t] \leq n^t \leq \int_n^{n+1} f(x)dx \leq (n+1)^t \leq [(n+1)^t] + 1.$$

Adding up all such inequalities for $n = 0, 1, \ldots, k$ gives

$$(*) \qquad \sum_{n=1}^{k}[n^t] \leq \int_0^{k+1} f(x)\,dx = \frac{(k+1)^{t+1}}{t+1} \leq \left(\sum_{n=1}^{k+1}[n^t]\right) + (k+1).$$

From the left-most $\leq$ in $(*)$, we have

$$(k+1)^{t+1} \geq (t+1)\sum_{n=1}^{k}[n^t]$$

and therefore

$$(k+1)^t \geq (t+1)^{t/(t+1)} \left\{\sum_{n=1}^{k}[n^t]\right\}^{t/(t+1)}.$$

Hence for all large enough k we have

$$[(k+1)^t] \geq c_1 \left\{\sum_{n=1}^{k}[n^t]\right\}^{t/t+1}$$

where c_1 is a suitable positive constant. Hence the correct value of t must satisfy $\frac{t}{t+1} = r$ or equivalently $t = \frac{r}{1-r}$ as claimed in the statement of the proposition.

Now consider the right-most $\leq$ in $(*)$. From this we get

$$\begin{aligned}\sum_{n=1}^{k}[n^t] &\geq \frac{(k+1)^{t+1}}{t+1} - [(k+1)^t] - (k+1) \\ &= (k+1)^{t+1}\left\{\frac{1}{t+1} - \frac{[(k+1)^t]}{(k+1)^t}\cdot\frac{1}{k+1} - \frac{1}{(k+1)^t}\right\} \\ &\geq \frac{1}{2(t+1)}(k+1)^{t+1}\end{aligned}$$

for all large enough k, since the second and third terms in $\{\dots\}$ both tend to zero as $k \to \infty$. Hence, using the same value of t as before, we get

$$[(k+1)^t] \leq (k+1)^t \leq c_2 \left\{\sum_{n=1}^{k}[n^t]\right\}^r$$

for all large enough k (where $c_2 = (2(t+1))^r$ is a positive constant). This completes the proof. □

For a given $r \in [0, 1]$ let $n_1, n_2, \dots$ be the positive integers as in 1.5 (that is, $n_k = [k^t]$ if $r < 1$, otherwise $n_k = 2^k$), and for $d = 0, 1, 2, \dots$ set

$$X_r(d) = \{x \in B(F)\colon\ x \text{ has block-bandwidth at most } d\},$$

where block-bandwidth is measured relative to diagonal block sizes $n_1, n_2, \dots$. The next proposition shows that the $X_r(d)$ provide a filtration of $G(r)$, thereby generalizing Proposition 1.4. (Notice that $X_0(d) = W_0(d)$, but $X_r(d) \neq W_r(d)$ for $r > 0$.)

Proposition 1.6. *For each* $r \in [0, 1]$, *the subspaces* $X_r(d)$ *defined above satisfy:*

(a) $X_r(0) \subseteq X_r(1) \subseteq X_r(2) \subseteq \cdots$ *and* $G(r) = \bigcup_{d\geq 1} X_r(d)$. *Also* $X_r(d_1)X_r(d_2) \subseteq X_r(d_1 + d_2)$, *so we have a filtration of* $G(r)$.

(b) $X_r(0)$ *is the spine for* $G(r)$ *determined by* $n_1, n_2, \dots$ *and so* $X_r(0) \cong \prod_{k=1}^{\infty} M_{n_k}(F)$.

(c) *For each* d, $X_r(d)$ *is a finitely generated projective right and left* $X_r(0)$-*module.*

Proof. First suppose $0 \leq r < 1$. Let $t = \frac{r}{1-r}$ and $n_k = [k^t]$ for each $k \in \mathbf{N}$. We can find positive constants c_1, c_2 such that

$$(1) \qquad c_1 k^t \leq n^r \leq c_2(k+1)^t$$

for all positive integers k, and for all positive integers n for which the (n, n) diagonal position of the spine determined by $n_1, n_2, \dots$ lies in the kth diagonal block. The latter condition on n is simply

$$\sum_{m=1}^{k-1}[m^t] < n \leq \sum_{m=1}^{k}[m^t].$$

For from the proof of 1.5 we have

$$\frac{1}{2(t+1)}k^{t+1} \le \sum_{m=1}^{k-1}[m^t] \le n \le \sum_{m=1}^{k}[m^t] \le \frac{1}{t+1}(k+1)^{t+1}$$

for all large k, whence taking $t/(t+1)$ powers leads to (1) (notice that $r = t/(t+1)$).

Now let c be a given positive constant. Let $d = 2[cc_2 + 1]$. Then for (n, n) in the kth block we have from (1) that

$$cn^r \le cc_2(k+1)^t \le dn_{k+1} \le n_{k+1} + n_{k+2} + \cdots + n_{k+d}.$$

It follows that $W_r(c) \subseteq X_r(d)$. This shows $G(r) \subseteq \bigcup_{d\ge 1} X_r(d)$.

To establish the reverse containment, we show

$$X_r(d) \subseteq W_r\left(\frac{1}{c_1}(d+1)^{t+1}\right) \tag{2}$$

for all positive integers d. Let $x \in X_r(d)$. The bandwidth of x at any (n, n) position within the kth block is at most

$$\begin{aligned}[][k^t] + [(k+1)^t] + \cdots + [(k+d)^t] &\le (d+1)(k+d)^t \\ &\le (d+1)^{t+1}k^t \le \frac{1}{c_1}(d+1)^{t+1}n^r \quad \text{by (1).}\end{aligned}$$

This shows $x \in W_r(\frac{1}{c_1}(d+1)^{t+1})$, which proves (2).

We have now shown that $G(r) = \bigcup_{d\ge 1} X_r(d)$ when $0 \le r < 1$. By a similar (but easier) proof, this also holds for $r = 1$; in fact $W_1(c) \subseteq X_1(c)$ and $X_1(d) \subseteq W_1((d+1)2^{d+1})$. The remaining statements in 1.6 are proved in an analogous fashion to their 1.4 counterparts. □

As a corollary we have the following useful formula for the powers $(W_r(c))^m$ when $r < 1$:

Corollary 1.7. *Let $0 \le r < 1$ and let $c \in \mathbf{R}^+$. Then there is a positive constant d such that*

$$(W_r(c))^m \subseteq W_r(dm^{1/(1-r)})$$

for all positive integers m.

Proof. Let c_1 be the positive constant used in the proof of 1.6. Given $c \in \mathbf{R}^+$, by the proof of 1.6 we have $W_r(c) \subseteq X_r(d_1)$ for some $d_1 \in \mathbf{N}$. By 1.6(a)

$$(W_r(c))^m \subseteq (X_r(d_1))^m \subseteq X_r(d_1 m).$$

Employing (2) of the proof of 1.6 and noting that there $t + 1 = 1/(1-r)$, we deduce that

$$X_r(d_1 m) \subseteq W_r\left(\frac{1}{c_1}(d_1 m + 1)^{1/(1-r)}\right)$$

and so

$$(W_r(c))^m \subseteq W_r(dm^{1/(1-r)})$$

for some positive d which is independent of m. □

Thus when $0 \le r < 1$ the powers of the subspaces $W_r(c)$ grow "at most polynomially of degree $1/(1-r)$". This throws up an important distinction between sublinear growth and linear growth, because for $r = 1$ it is easily seen that the powers of $W_r(c)$ have genuine exponential growth.

2. Linear growth

Here our aim is to establish the following, somewhat surprising, linear growth result which improves the Goodearl, Menal, and Moncasi embedding [GMM, Proposition 2.1] mentioned in the Introduction.

Theorem 2.1. *Every countable-dimensional algebra A over a field F has linear growth, that is, A can be embedded in $G(1)$. Thus every countable-dimensional algebra has its bandwidth dimension in* $[0, 1]$.

There are two key results which will lead us to the proof of this theorem. The first (Theorem 2.3) calculates the block sizes of a simultaneous block tridiagonal form for a finite number of given linear transformations of a countable-dimensional vector space. The second (Theorem 2.4) is that every countable-dimensional algebra can be embedded in a finitely generated algebra. This was established in 1989 by O'Meara, Vinsonhaler, and Wickless [OVW]. The following elementary lemma is required for Theorem 2.3.

Lemma 2.2. *Let U be a subspace of a vector space V over F, and let $x_1, x_2, \dots, x_k \in \mathrm{End}_F(V)$. Then*

$$[U: x_1^{-1}(U) \cap x_2^{-1}(U) \cap \cdots \cap x_k^{-1}(U) \cap U] \leq k[V: U].$$

Proof. We have the linear transformation

$$\theta: U \to (V/U) \oplus (V/U) \oplus \cdots \oplus (V/U)$$

from U into k copies of V/U given by

$$u \to (x_1(u) + U, x_2(u) + U, \dots, x_k(u) + U),$$

whose kernel is $x_1^{-1}(U) \cap \cdots \cap x_k^{-1}(U) \cap U$. Now

$$[U: \ker\theta] \leq \dim((V/U) \oplus \cdots \oplus (V/U)) = k[V: U]$$

and the result follows. □

Theorem 2.3. *Let V be any countably-infinite-dimensional vector space over a field F, and let $x_1, \dots, x_k \in \mathrm{End}_F(V)$. Then there exist finite-dimensional subspaces $U_0 = \{0\}$, $U_1, U_2, \dots, U_n, \dots$ of V such that:*

(1) $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n \oplus \cdots$;
(2) $\dim U_n = (2k+1)^{n-1}$ *for all* $n \geq 1$;
(3) $x_i(U_n) \subseteq U_{n-1} \oplus U_n \oplus U_{n+1}$ *for* $i = 1, \dots, k$ *and* $n \geq 1$.

Proof. Let $\{w_1, \dots, w_n, \dots\}$ be a fixed basis for V. Set $U_0 = \{0\}$. We shall establish, by induction, the existence of subspaces U_n, V_n of V for $n = 1, 2, \dots,$ with the U_n finite-dimensional, such that the following properties hold for all $n \geq 1$:

(i) $V = U_1 \oplus \cdots \oplus U_n \oplus V_n$;
(ii) $V_n = U_{n+1} \oplus V_{n+1}$;
(iii) $w_n \in U_1 + \cdots + U_n$;
(iv) $x_i(V_{n+1}) \subseteq V_n$ for $i = 1, \dots, k$;
(v) $x_i(U_n) \subseteq U_{n-1} \oplus U_n \oplus U_{n+1}$ for $i = 1, \dots, k$;
(vi) $\dim U_n = 2k(\dim U_1 + \cdots + \dim U_{n-1}) + 1$.

For $n = 1$ we simply take $U_1 = \langle w_1 \rangle$ and let V_1 be any complement of U_1 in V. Now suppose $n \geq 1$ and that we have constructed $U_1, \ldots, U_n, V_1, \ldots, V_n$ satisfying the above properties. The construction of U_{n+1}, V_{n+1} involves several steps.

Firstly, let

$$X = x_1^{-1}(V_n) \cap x_2^{-1}(V_n) \cap \cdots \cap x_k^{-1}(V_n) \cap V_n.$$

By Lemma 2.2 and (i), $[V_n : X] \leq k[V : V_n] = k(\dim U_1 + \cdots + \dim U_n)$. Let $y \in V_n$ be the projection of w_{n+1} on V_n relative to the decomposition $V = (U_1 + \cdots + U_n) \oplus V_n$. Choose a subspace V'_{n+1} of X such that $[X : V'_{n+1}] \leq 1$ and $y \notin V'_{n+1}$. Write

$$V_n = U'_{n+1} \oplus V'_{n+1}$$

for some subspace U'_{n+1} containing y. Note that $w_{n+1} \in U_1 + \cdots + U_n + U'_{n+1}$ and $\dim U'_{n+1} = [V_n : V'_{n+1}] \leq [V_n : X] + 1$. Hence

$$\dim U'_{n+1} \leq k(\dim U_1 + \cdots + \dim U_n) + 1.$$

For $i = 1, \ldots, k$ we have by induction, using (ii) and (iv), that

$$x_i(U_n) \subseteq x_i(V_{n-1}) \subseteq V_{n-2} = U_{n-1} \oplus V_{n-1} = U_{n-1} \oplus U_n \oplus V_n.$$

Therefore $x_i(U_n)$ is a subspace of $(U_{n-1} + U_n + U'_{n+1}) \oplus V'_{n+1}$. Let Y_i be the projection of $x_i(U_n)$ on V'_{n+1} relative to this decomposition, and let

$$U''_{n+1} = Y_1 + \cdots + Y_k.$$

Notice that $\dim Y_i \leq \dim x_i(U_n) \leq \dim(U_n)$, whence $\dim U''_{n+1} \leq k(\dim U_n)$. Set

$$U_{n+1} = U'_{n+1} \oplus U''_{n+1},$$

and write

$$V'_{n+1} = U''_{n+1} \oplus V_{n+1}$$

for some subspace V_{n+1}.

Since $V_n = U'_{n+1} \oplus V'_{n+1} = U'_{n+1} \oplus U''_{n+1} \oplus V_{n+1} = U_{n+1} \oplus V_{n+1}$, we have (ii), and hence also (i) for $n+1$. From $w_{n+1} \in U_1 + \cdots + U_n + U'_{n+1} \subseteq U_1 + \cdots + U_n + U_{n+1}$, we get (iii). Property (iv) follows from $x_i(V_{n+1}) \subseteq x_i(X) \subseteq V_n$. Also $x_i(U_n) \subseteq (U_{n-1} \oplus U_n \oplus U'_{n+1}) \oplus U''_{n+1} = U_{n-1} \oplus U_n \oplus U_{n+1}$ gives (v). For (vi), observe that

$$\begin{aligned} \dim U_{n+1} &= \dim U'_{n+1} + \dim U''_{n+1} \\ &\leq (k(\dim U_1 + \cdots + \dim U_n) + 1) + k(\dim U_n) \\ &\leq 2k(\dim U_1 + \cdots + \dim U_n) + 1. \end{aligned}$$

By expanding U_{n+1} to include more of V_{n+1}, we can arrange our choice of U_{n+1} and V_{n+1} such that in addition to properties (i), (ii), ..., (v), we have

$$\dim U_{n+1} = 2k(\dim U_1 + \cdots + \dim U_n) + 1.$$

This completes the induction.

Property (1) is an immediate consequence of (i) and (iii). The recursive relation (vi), together with $\dim U_1 = 1$, yields (2). Finally (3) is just (v). □

Theorem 2.4. *Every countable-dimensional algebra A over a field can be embedded in some finitely generated algebra (in a 2-generator algebra in fact).*

Proof. [OVW] established the corresponding result for countable rings but, as pointed out in the Introduction to [OVW], the same techniques also work

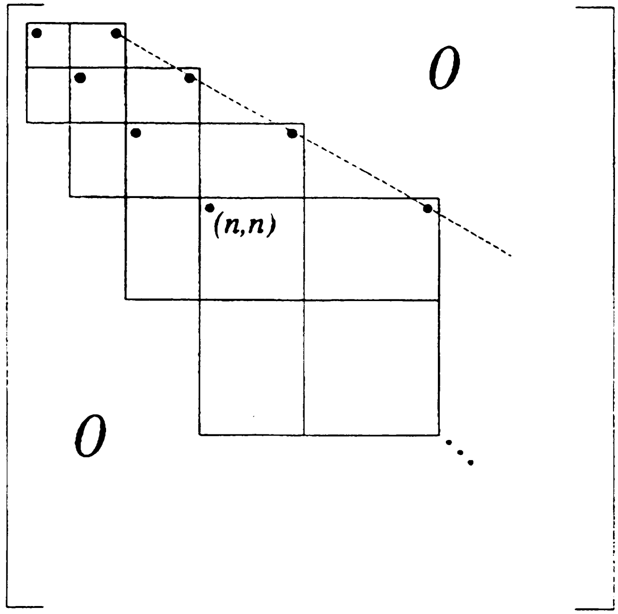

FIGURE 4

for algebras. Specifically, using the notation of the proof of the Theorem in [OVW], the modifications required are as follows: Suppose R is a countable-dimensional algebra. Let $\{r_1, r_2, \dots, r_n, \dots\}$ be a basis for R. Define the maps

$$R \xrightarrow{\phi_1} eSe \xrightarrow{\phi_2} fTf \xrightarrow{\phi_3} T$$

exactly as before, and note that these are now algebra maps. Hence $\theta = \phi_3\phi_2\phi_1 : R \to T$ is an algebra embedding whose image is contained in the subalgebra generated by $\phi(a)$, $\phi(b)$, f, v, w. □

Proof of Theorem 2.1. By Theorem 2.4 we can reduce to the case where A is a subalgebra of $Q = M_\omega(F)$ generated by a finite number of elements, say $x_1, x_2, \dots, x_k$ (we could even take $k = 2$). By Theorem 2.3, there is a similarity transformation of Q under which all the x_i are simultaneously in block tridiagonal form (see Figure 4) and where the sizes of the diagonal blocks are

$$1, 2k+1, (2k+1)^2, \dots, (2k+1)^n, \dots .$$

Consider the dotted piecewise linear curve obtained by joining the outside corners of the upper blocks. This is clearly an upper growth curve for all the x_i. Viewed as a growth curve, it has the equation

$$g(n) = 4k(k+1)n - (4k^2 + 2k - 1)$$

because at the (n, n) position for any n of the form

$$\begin{aligned} n &= \text{row index of the first entry of the } m\text{th diagonal block} \\ &= \text{sum of the first } m-1 \text{ block sizes } + 1 \\ &= 1 + (2k+1) + (2k+1)^2 + \cdots + (2k+1)^{m-2} + 1 \\ &= \frac{(2k+1)^{m-1} - 1}{2k} + 1, \end{aligned}$$

the bandwidth of the tridiagonal block structure is

$$(2k+1)^{m-1} + (2k+1)^m - 1 = 4k(k+1)n - (4k^2 + 2k - 1) = g(n).$$

Hence the growth curve is a straight line! Clearly $g(n)$ is also a lower growth curve. This shows that the similarity transformation puts all the x_i in $G(1)$. Therefore since $x_1, \ldots, x_k$ generate A, and $G(1)$ is a subalgebra of $M_\omega(F)$ by Proposition 1.2, the image of A lies inside $G(1)$ as well. □

It is worth noting that the proof of Theorem 2.1 shows that linear growth for any finitely generated subalgebra A of $M_\omega(F)$ can be demonstrated using just a similarity transformation. We shall see examples later (Theorem 5.2 and Example 5.4) of finitely generated subalgebras of $M_\omega(F)$ which turn out to have sublinear growth (even embed in $G(0)$) but for which no similarity transformation can embed them in a $G(r)$ for any $r < 1$. (See also Remark 4.2.)

3. Bandwidth dimension of purely infinite algebras

In this section we show that, in general, we cannot achieve sublinear growth (that is, $O(n^r)$ growth for some $r < 1$) for countable-dimensional algebras (Theorem 3.3). Thus the embedding found in Theorem 2.1 is the best possible.

The idea is to view elements of $B(F)$ as block tridiagonal matrices acting on a vector space decomposition

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n \oplus \cdots$$

as in the proof of Theorem 2.3. The following lemma gives us a condition which ensures that the subspaces grow too quickly for sublinear growth.

Lemma 3.1. *Let $x, z \in \operatorname{End}_F(V)$ be such that x, z are both one-to-one, but $\operatorname{Im} x \cap \operatorname{Im} z = 0$. Suppose there are finite-dimensional subspaces $U_1, U_2, \ldots$ of V such that $V = \bigoplus_{i \geq 1} U_i$ and such that for both the maps $f = x$ and z we have*

$$(*) \qquad f\left(\bigoplus_{i=1}^{k} U_i\right) \subseteq \bigoplus_{i=1}^{k+1} U_i$$

for each $k \geq 1$. Then for each $k \geq 2$ we have

$$\dim U_k \geq \sum_{i=1}^{k-1} \dim U_i.$$

Proof. For each k let $W_k = \bigoplus_{i=1}^{k} U_i$ so that by $(*)$ we have $xW_{k-1}, zW_{k-1} \subseteq W_k$. Also $xW_{k-1} \cap zW_{k-1} \subseteq \operatorname{Im} x \cap \operatorname{Im} z = 0$. Since x, z are both one-to-one this implies

$$\begin{aligned} \dim W_k &\geq \dim(xW_{k-1} + zW_{k-1}) \\ &= \dim(xW_{k-1}) + \dim(zW_{k-1}) \\ &= \dim W_{k-1} + \dim W_{k-1} \end{aligned}$$

and so

$$\dim U_k = \dim W_k - \dim W_{k-1} \geq \dim W_{k-1}$$

as desired. □

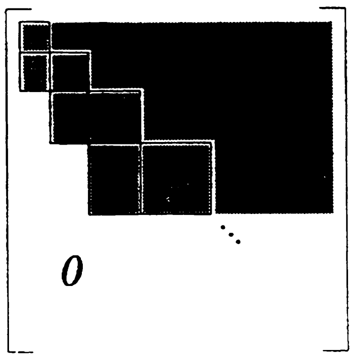

FIGURE 5

The hypothesis (∗) says that x, z are simultaneously in "block upper Hessenberg" form (see Figure 5). This form is possible for any finite collection of column-finite matrices, as long as the block sizes are chosen properly. It determines a lower growth curve for x and z.

Lemma 3.1 gives an estimate for the bandwidth of x and z (at least below the main diagonal). In the notation of the proof let $n = \sum_{i=1}^{k-1} \dim U_i$. Then we can estimate how far down the nth column we need to go before all the entries for x and z become zero. Indeed, in the nth column the entries are zero beyond the $(n + \dim U_k)$th row. Lemma 3.1 says that $\dim U_k \geq n$. (See Figure 6.)

If the subdiagonal blocks are chosen to be as short as possible, then there must be a nonzero entry in the final row of each of these blocks. If we are looking for the smallest possible bandwidth, then the "best" place for such a nonzero entry is the bottom right-hand corner of each block (this gives the smallest deviation from the main diagonal). For the (n, n) entry in the diagram we would thus have a bandwidth of $\dim U_k \geq n$. Hence if $g(n)$ is a growth curve for x and z, there are infinitely many n for which $g(n) \geq n$. Therefore any common growth curve for the maps x, z in Lemma 3.1 must be at least linear. All

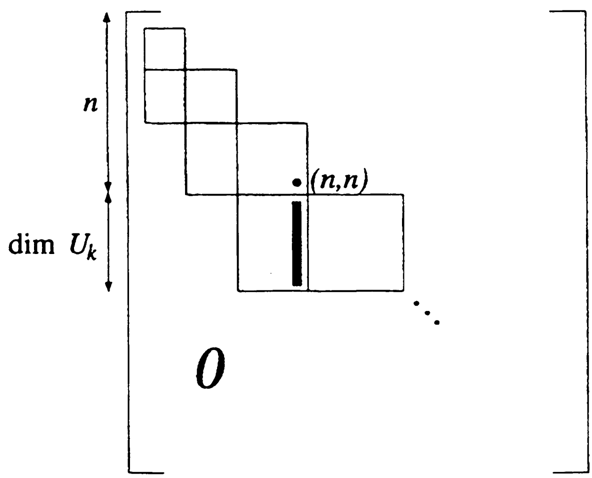

FIGURE 6

that remains is to find a way of ensuring that any (faithful) representation in $\mathrm{End}_F(V)$ of a suitable algebra A always contains such a pair of maps.

Proposition 3.2. *Let A be a countable-dimensional algebra over F containing elements w, x, y, z satisfying $yx = 1 = wz$ and $yz = 0$. Then any copy of A inside $B(F)$ has at least linear growth, and so A has bandwidth dimension one.*

Proof. The same equations will be satisfied by the images $\overline{w}, \overline{x}, \overline{y}, \overline{z}$ of w, x, y, z in $B(F)$. The first two equations force $\overline{x}$ and $\overline{z}$ to be one-to-one, and if $\overline{x}(v_1) = \overline{z}(v_2) \in \mathrm{Im}\,\overline{x} \cap \mathrm{Im}\,\overline{z}$ then $v_1 = \overline{y}\,\overline{x}(v_1) = \overline{y}\,\overline{z}(v_2) = 0$ and so $\mathrm{Im}\,\overline{x} \cap \mathrm{Im}\,\overline{z} = 0$. So the result follows from Lemma 3.1 and the above discussion. □

It is easy to construct matrices w, x, y, z in $B(F)$ which satisfy the hypotheses of Proposition 3.2. More generally, these hypotheses are satisfied by two fairly large classes of algebras. One of these classes needs a name:

Definition. We say that a ring R is *purely infinite* if $R \cong R \oplus R$ as right R-modules. □

If R is also regular and right self-injective, then this usage agrees with that in [G, pp. 116–117].

Theorem 3.3. *Let A be a countable-dimensional algebra over F such that either*

(i) *A is purely infinite, or*
(ii) *A is regular and $A \oplus A \lesssim A$ as right A-modules.*

Then every copy of A in $B(F)$ has at least linear growth, and so A has bandwidth dimension one.

Proof. In either case A contains a pair of orthogonal idempotents e, f such that $eA \cong A_A \cong fA$. By [Jac2, Proposition 4, p. 51] there are elements $w, x, y, z \in A$ such that $yx = 1$, $xy = e$ and $wz = 1$, $zw = f$. But then $0 = ef = xyzw$ and so $yz = 0$. The theorem thus follows from Proposition 3.2. □

If A is not regular then we cannot weaken condition (i) to say $A \oplus A \lesssim A$. For example, consider the free algebra A on two generators. Then $A \oplus A \lesssim A$, because A is not a right Ore domain, but the conditions of Proposition 3.2 are not satisfied (since A is a domain, $yz = 0$ forces $y = 0$ or $z = 0$). And in any event we shall see in Section 5 that A can in fact be embedded in $G(0)$! A construction in [O] provides examples of countable-dimensional algebras A which have bandwidth dimension 1 (or any $r \in [0, 1]$ for that matter) and which are also "directly finite", that is, $xy = 1$ implies $yx = 1$ for all $x, y \in A$.

4. Some algebras not embeddable in $G(0)$

To date, all our examples of countable-dimensional algebras A have had bandwidth dimension either 0 or 1. In this section we construct for each $0 < r \leq 1$ a finitely generated subalgebra A of $G(r)$ which cannot be embedded in $G(0)$. Although a much stronger result than this is given in [O], namely the construction of a finitely generated subalgebra of $G(r)$ with bandwidth dimension precisely r, the present construction is much simpler. It provides a quick demonstration that there are countable-dimensional algebras A which "fit in the middle of $G(0)$ and $G(1)$". However all we can say (at present) about the bandwidth dimension of our A is that it lies in $[0, r]$.

The following proposition will help us construct some transformations in $\mathrm{End}_F(V)$ which cannot be represented by a matrix in $G(0)$.

Proposition 4.1. *Let V be a countably-infinite-dimensional vector space over F and suppose $x \in \mathrm{End}_F(V)$ has a matrix with finite bandwidth n relative to some basis of V.*

(a) *If $\ker x = 0$, then $\mathrm{Im}\, x$ has codimension at most n in V.*
(b) *If $\mathrm{Im}\, x = V$, then $\ker x$ has dimension at most n.*

Proof. We can represent x by a block tridiagonal matrix where all the blocks are $n \times n$ matrices. This means we have a sequence $U_1, U_2, \ldots$ of subspaces of V such that $V = \bigoplus_k U_k$, $\dim U_k = n$ for each k, and

$$xU_1 \subseteq U_1 \oplus U_2, \quad xU_k \subseteq U_{k-1} \oplus U_k \oplus U_{k+1} \qquad (k > 1).$$

Suppose firstly that $\ker x = 0$. To prove (a) consider any finite-dimensional subspace W of V such that $W \cap \mathrm{Im}\, x = 0$. It is enough to show that $\dim W \leq n$. By choosing k large enough we can assume that $W \subseteq U_1 \oplus \cdots \oplus U_k$. Now $x(U_1 \oplus \cdots \oplus U_{k-1}) \subseteq U_1 \oplus \cdots \oplus U_k$ and $W \cap x(U_1 \oplus \cdots \oplus U_{k-1}) = 0$. So comparing dimensions gives

$$\begin{aligned} kn &= \dim(U_1 \oplus \cdots \oplus U_k) \\ &\geq \dim W + \dim(x(U_1 \oplus \cdots \oplus U_{k-1})) \\ &= \dim W + \dim(U_1 \oplus \cdots \oplus U_{k-1}) \quad (\text{since } \ker x = 0) \\ &= \dim W + (k-1)n \end{aligned}$$

and the result follows.

To prove (b) we essentially use a dual argument. Fix k and consider $W = (U_1 \oplus \cdots \oplus U_k) \cap \ker x$. It is enough to show that $\dim W \leq n$ (independent of k). Let $\pi \colon V \to U_1 \oplus \cdots \oplus U_{k-1}$ be the projection map determined by $V = \bigoplus_i U_i$. Then

$$\pi x \left(\bigoplus_{k+1}^{\infty} U_i \right) \subseteq \pi \left(\bigoplus_{k}^{\infty} U_i \right) = 0$$

and so

$$\bigoplus_{i=1}^{k-1} U_i = \pi V = \pi x V = \pi x \left(\bigoplus_{i=1}^{k} U_i \right).$$

Now looking at the dimensions gives

$$\begin{aligned} \dim \left(\bigoplus_{i=1}^{k} U_i \right) &= \dim \left(\bigoplus_{i=1}^{k-1} U_i \right) + \dim \left(\ker(\pi x) \cap \bigoplus_{i=1}^{k} U_i \right) \\ &\geq \dim \left(\bigoplus_{i=1}^{k-1} U_i \right) + \dim W \end{aligned}$$

and so $\dim W \leq n$ as required. □

Remark 4.2. We saw at the end of Section 2 that not only do finitely generated subalgebras of $M_\omega(F)$ embed in $G(1)$, they are in fact similar to subalgebras of $G(1)$. Every one-generator algebra A can be embedded as a subalgebra of $G(0)$ (and so has bandwidth dimension zero). To see this, notice that A is either finite dimensional (in which case the result is trivial) or else isomorphic to the polynomial algebra $F[x]$, and in that case we can map the generator to the standard shift matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ & \ddots & \ddots & \end{bmatrix}.$$

However Proposition 4.1 implies that there are one-generator subalgebras A of $M_\omega(F)$ which are not similar to any subalgebra of $G(0)$. Indeed we can let A be the subalgebra generated by any matrix x corresponding to a one-to-one linear transformation whose range has infinite codimension. □

Example 4.3. Suppose $0 < r \leq 1$. Then there is a four-generator subalgebra A of $G(r)$ such that A cannot be embedded in $G(0)$.

Proof. Let V be a countable-dimensional vector space over F with basis $\{v_n : n \in \mathbf{N}\}$ and let $Q = \mathrm{End}_F(V)$. Let $f : \mathbf{N} \to \mathbf{N}$ be the function given by

$$f(n) = n + [n^r]$$

where $[a]$ is the greatest integer less than or equal to a. Let A be the subalgebra of Q generated by w, x, y, z where

$$x(v_n) = v_{n+1} \quad \text{for all } n \geq 1,$$
$$y(v_n) = \begin{cases} 0 & \text{if } n = 1, \\ v_{n-1} & \text{if } n > 1, \end{cases}$$
$$z(v_n) = v_{f(n)} \quad \text{for all } n \geq 1,$$
$$w(v_n) = \begin{cases} 0 & \text{if } n \notin \mathrm{Im}\, f, \\ v_k & \text{if } n = f(k). \end{cases}$$

Notice that w is well defined since f is a strictly increasing function (and so one-to-one). If we represent elements of Q in terms of the given basis then w, x, y, z all belong to $G(r)$, and so A is a subalgebra of $G(r)$.

For each integer $n \geq 1$ let $e_n = x^{n-1}y^{n-1} - x^n y^n$. We now assemble a list of relations which are satisfied by w, x, y, z and the e_n, the idea being that these same relations have to be satisfied in any other copy of A in Q. First it is easy to check that

(i) $yx = 1 = wz$, and

(ii) $e_1 = 1 - xy \neq 0$.

Next, by [Jac1] we see that

(iii) the e_n are nonzero, orthogonal, pairwise equivalent idempotents.

Notice also that for each n, the transformation e_n is just the natural projection of V onto the subspace spanned by v_n. Hence

(iv) $we_n = 0 = e_n z$ for all $n \notin \mathrm{Im}\, f$. Notice here that, since the function f is of the form $f(n) = n + g(n)$ where g is an unbounded, increasing function, there are infinitely many $n \notin \operatorname{Im} f$. Indeed each time g increases in value, say $g(n) < g(n+1)$, the function f skips a value in $\mathbf{N}$ since

$$f(n) = n + g(n) < n + g(n+1) < (n+1) + g(n+1) = f(n+1).$$

Now any embedding of A in $G(0)$ will produce elements w, x, y, z and $\{e_n\}$ in $G(0)$ which satisfy the relations (i)–(iv) above. By (i) the map w would then be onto, and by (ii), (iii), and (iv) w would have an infinite-dimensional kernel. But by Proposition 4.1 such a map cannot be in $G(0)$, and so A cannot be embedded in $G(0)$. □

5. Bandwidth dimension of free algebras

The main result of this section is that the free algebra on any finite or countably infinite number of generators can be embedded in $G(0)$ and so has bandwidth dimension 0 (Theorem 5.2). This result is rather surprising since the free algebra is often thought of as being "large": its GK-dimension is ∞, for example. At the end of this section we shall look in more detail at the differences between GK-dimension and bandwidth dimension.

The following lemma contains the key idea for representing the free algebra in terms of matrices with small bandwidth. We recall the notation e_{ij} for the matrix unit with all entries zero except for a one in the (i, j) position.

Lemma 5.1. *Let F be any field (or indeed any ring with identity) and let $F\{x, y\}$ be the free algebra in two indeterminates x, y. Suppose $w = a_1 a_2 \cdots a_n$ is any word in x, y of degree n. Then there is an algebra homomorphism $\varphi: F\{x, y\} \to M_{n+1}(F)$ such that:*

(i) *the only nonzero entries of $\varphi(x)$ and $\varphi(y)$ are on the superdiagonal, and these are all 1's;*
(ii) $\varphi(w) = e_{1,n+1}$;
(iii) *if v is any other word in x, y having degree n then $\varphi(v) = 0$;*
(iv) *if v is any word in x, y of degree greater than n, then $\varphi(v) = 0$;*
(v) *if $\rho \in F\{x, y\}$ then the $(1, n+1)$ entry of $\varphi(\rho)$ is the coefficient of w in ρ.*

Proof. To construct φ it is enough to specify the images of x, y. To do this, partition the set $\{1, 2, \ldots, n\}$ as $X \cup Y$ where $i \in X \Leftrightarrow a_i = x$ and $i \in Y \Leftrightarrow a_i = y$. Set

$$\varphi(x) = \sum_{i \in X} e_{i,i+1} \quad \text{and} \quad \varphi(y) = \sum_{i \in Y} e_{i,i+1}.$$

Then (i) is certainly true.

Let N be the set of strictly upper triangular matrices in $M_{n+1}(F)$. Then $\varphi(x), \varphi(y) \in N$ and $N^n \subseteq F e_{1,n+1}$. Hence if $v = b_1 b_2 \cdots b_n$ is any word of degree n in x, y then the matrix $\varphi(v)$ has zero entries except in the $(1, n+1)$ position. This $(1, n+1)$ entry is given by

$$\sum_{i_k} \overline{b}_1(1, i_1)\overline{b}_2(i_1, i_2) \cdots \overline{b}_n(i_{n-1}, n+1)$$

where each $\overline{b}_k = \varphi(b_k)$. But as each $\overline{b}_i$ is $\varphi(x)$ or $\varphi(y)$, the terms in the above sum can be nonzero only if $i_1 = 2$, $i_2 = 3, \ldots, i_{n-1} = n$ (because of (i)). So the $(1, n+1)$ entry of $\varphi(v)$ is simply

$$\bar{b}_1(1, 2)\bar{b}_2(2, 3)\cdots\bar{b}_n(n, n+1)$$

and this term can be nonzero only if $\bar{b}_i(i, i+1) \neq 0$ for each i. By the definition of $\varphi(x)$ and $\varphi(y)$ we thus have $\bar{b}_i = \varphi(x)$ when $i \in X$ and $\bar{b}_i = \varphi(y)$ when $i \in Y$. Hence the $(1, n+1)$ entry is nonzero only when $v = w$ and then the entry is simply 1. Thus (ii) and (iii) are proven.

Since $N^{n+1} = 0$ any word of higher degree is mapped to zero by φ. This proves (iv). Finally since $\varphi(x)$, $\varphi(y)$ are zero except on the superdiagonal, no word of smaller degree than n can yield a nonzero $(1, n+1)$ entry. Hence (v) is true too. □

Theorem 5.2. *Let F be a field and let $F\{x, y\}$ be the free algebra in two (noncommuting) indeterminates. Then $F\{x, y\}$ can be embedded as a subalgebra of $G(0)$. In particular $F\{x, y\}$ has bandwidth dimension 0.*

Proof. It is enough to embed $F\{x, y\}$ as a subalgebra of a direct product $\prod_{k=1}^{\infty} M_{n_k}(F)$ in such a way that each component of the images of x, y is a matrix whose only nonzero entries lie on the superdiagonal.

Let $w_1, w_2, \ldots, w_k, \ldots$ be a list of all the words in x, y (i.e. monomials in $F\{x, y\}$). This is possible because there are only countably many such words. For each k, let d_k be the degree of w_k and let $n_k = d_k + 1$. Let $\varphi_k\colon F\{x, y\} \to M_{n_k}(F)$ be the algebra homomorphism given by Lemma 5.1 for the word $w = w_k$. Let $\varphi\colon F\{x, y\} \to \prod_k M_{n_k}(F)$ be the algebra homomorphism determined by the φ_k. Then Lemma 5.1(v) ensures that φ is an embedding, and Lemma 5.1(i) ensures that the images of x, y have the required form. □

Remarks. (1) In terms of a concrete realization of $F\{x, y\}$ inside $G(0)$, the proof of Theorem 5.2 shows that we can choose x and y as block diagonal $\omega \times \omega$ matrices where the blocks are finite and matching, and constitute all pairs of finite matrices of the following form: the first is an $n \times n$ matrix whose only nonzero entries lie on the superdiagonal and are all 1, while the second is an $n \times n$ matrix of the same form but with the 1's in complementary positions on the superdiagonal. For example, one such matching pair of 5×5 blocks is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(2) Note that Theorem 5.2 also holds for a free algebra on any finite or countably infinite number of indeterminates because such algebras can be embedded in $F\{x, y\}$. □

In terms of bandwidth dimension then, the free algebra $F\{x, y\}$ is "small". On the other hand we saw in Theorem 3.3 that any purely infinite algebra A has bandwidth dimension one and so, presumably, is "large". In this context it may be worth pointing out the following, probably very well-known, result.

Proposition 5.3. *If A is a purely infinite algebra over F, then A contains a subalgebra isomorphic to the free algebra on two generators.*

Proof. Since $A \cong A \oplus A$ as right A-modules, we have seen that A contains elements w, x, y, z such that $yx = 1 = wz$ and $yz = 0$. But then $r_A(x) = 0 = r_A(z)$ and $xA \cap zA = 0$. Now by the standard argument used in [Jat, p. 45] for non-Ore domains, it can be seen that x, z generate a copy of the free algebra (with identity) inside A. □

Thus the free algebra on two generators is certainly "smaller" than any purely infinite algebra, and Theorem 5.2 quantifies the difference. On the other hand the free algebra has infinite GK-dimension, so this may be a good place to examine more closely the similarities and differences between bandwidth dimension and GK-dimension.

Recall firstly how the GK-dimension of a finitely generated F-algebra A is calculated (see [KL]). We begin with a finite-dimensional subspace U of A which contains 1 and generates A as an F-algebra. Then the GK-dimension of A is given by

$$GK\text{-}\dim A = \limsup_k \left(\frac{\log(\dim U^k)}{\log k} \right)$$

where, as usual, U^k is the subspace of A generated by all products of k elements of U.

We can interpret this calculation in terms of the "block upper Hessenberg" form which we observed after Lemma 3.1, and so use it to find a "lower growth curve" for the generators of A. Consider the regular representation of A where A acts via left multiplication on itself. Then $U \subseteq U^2 \subseteq U^3 \subseteq \cdots$ is an increasing chain of subspaces of A whose union is A, and for any $u \in U$ we clearly have $uU^k \subseteq U^{k+1}$. Hence these subspaces let us put all the elements of U simultaneously into block upper Hessenberg form (see Figure 7) where the block sizes shown are determined by the condition

$$\dim U^k = n_1 + n_2 + \cdots + n_k.$$

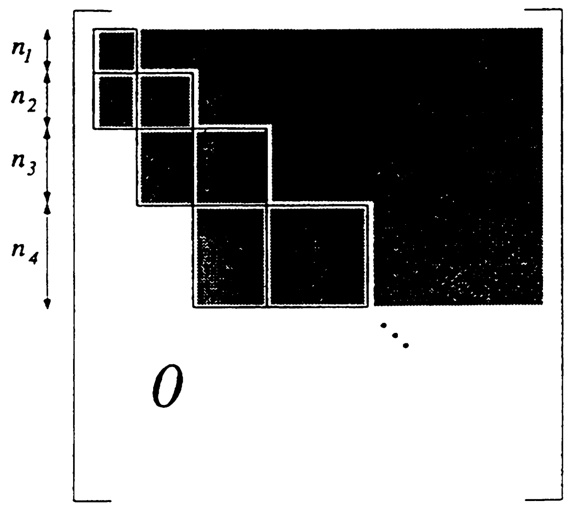

FIGURE 7

In terms of this representation, the above calculation of $GK\text{-}\dim A$ essentially seeks to express $\dim U^k$ (= the total size of the first k diagonal blocks) as a polynomial in k (= the number of blocks). This is because $\dim U^k$ "looks increasingly like" k^t if $GK\text{-}\dim A = t$. In this sense GK-dimension measures how far we need to go down each column before all the entries are zero in the regular representation of the generators of A. Thus loosely speaking, we can view $GK\text{-}\dim A$ as determining a "lower growth curve" for the generators of A. This view of GK-dimension allows us to draw some comparisons with bandwidth dimension.

1. The GK-dimension of A tells us about a *lower* growth curve for the generators of A, whereas the bandwidth dimension of A gives us an *upper and lower* growth curve for these generators. (This might lead us to expect the bandwidth dimension to be larger than the GK-dimension.)
2. The lower growth curves associated with GK-dimension are relative to the regular representation of A and, indeed, are relative to bases of A which correspond to ascending subspaces
$$U \subseteq U^2 \subseteq U^3 \subseteq \cdots$$
where U generates A as an F-algebra. On the other hand the growth curves determined by the bandwidth dimension are the "best possible" among all possible (faithful) matrix representations. (This factor would tend to make the bandwidth dimension smaller than the GK-dimension.)
3. GK-dimension measures the growth by comparing the total size of the first k blocks with k, but the bandwidth dimension measures the growth by comparing the size of the next block with the total size of all the preceding blocks. (This probably just results in a different scale being used for the two dimensions.)

Because of (1) and (3) it is difficult to make more precise comparisons between the two dimensions: the regular representation of A need not give rise to row-finite matrices, and so there may not be any upper growth curves at all. For some algebras, however, we do get an upper growth curve "free of charge" and we can compare the actual calculation of the two dimensions. In the following two examples there is a close connection between the number k of diagonal blocks (at a given stage) and their total size. With this type of example, we can use the lower growth curve on its own to estimate the GK-dimension.

Example 5.4. Let A be the free algebra $F\{x, y\}$ on two generators. For each integer $k \geq 0$ let U_k be the subspace of A spanned by all monomials of degree exactly k, and notice that $\dim U_k = 2^k$.

To calculate $GK\text{-}\dim A$ we use $U = U_0 \oplus U_1$ and find that

$$\begin{aligned}\dim U^k &= \dim(U_0 \oplus U_1 \oplus \cdots \oplus U_k)\\ &= 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1\end{aligned}$$

and this gives exponential growth when compared with the number of blocks k. Thus $GK\text{-}\dim A = \infty$.

If we represent the generators x, y relative to a basis which corresponds to the decomposition $A = U_0 \oplus U_1 \oplus \cdots$ then we get lower triangular matrices (since $xU_k \subseteq U_{k+1}$ and $yU_k \subseteq U_{k+1}$) so the lower growth curve will also be an upper growth curve. If we compare the size of the $(k+1)$st block (namely, 2^{k+1}) with the total size of the preceding k blocks (namely, $1+2+\cdots+2^k = 2^{k+1}-1$) we see that the growth curve is linear. Indeed, it wouldn't make any difference what basis we chose for A here: the actions of x, y on A make x, y one-to-one maps such that $\operatorname{Im} x \cap \operatorname{Im} y = 0$ and so Lemma 3.1 shows that we must get at least linear growth. That is, if we restricted ourselves to the regular representation, the bandwidth dimension of the free algebra A would take the largest possible value, 1. But by allowing ourselves to choose from all possible matrix representations we can find a much slimmer representation, as Theorem 5.2 shows. □

Example 5.5. Let A be the polynomial algebra $F[x, y]$ in two (commuting) indeterminants. Once again, for each $k \geq 0$, let U_k be the subspace generated by all monomials of degree exactly k. This time, $\dim U_k = k+1$ but otherwise the calculations are similar to those in Example 5.4. Thus we calculate $GK\text{-}\dim A$ using $U = U_0 \oplus U_1$ and find that

$$\dim U^k = \tfrac{1}{2}(k+1)(k+2).$$

This time we have quadratic growth and $GK\text{-}\dim A = 2$. Once again the lower growth curve is also an upper growth curve, and the size of the $(k+1)$st block is $k+2$ whereas the total size of the preceding blocks is $\frac{1}{2}(k+1)(k+2)$. This gives a growth curve of the form $ck^{1/2}$ and so we would get a bandwidth dimension of $\frac{1}{2}$ if we restricted ourselves to the regular representation and to this type of basis.

A similar calculation with $A = F[x_1, \ldots, x_n]$ would give $GK\text{-}\dim A = n$ and the growth curve for A would suggest a bandwidth dimension of $1 - \frac{1}{n}$. But in fact, since $F[x_1, \ldots, x_n]$ embeds in $\prod_{k=1}^{\infty} M_k(F)$, Corollary 6.5 of the next section shows that $F[x_1, \ldots, x_n]$ has bandwidth dimension 0. (At least when F is infinite, $F[x_1, \ldots, x_n]$ actually embeds in $G(0)$. We can even choose $x_1, \ldots, x_n$ to be diagonal matrices.) □

As with GK-dimension, bandwidth dimension behaves nicely for subalgebras, finite subdirect products, and finite matrix rings. Namely

1. if the algebra A embeds in the algebra B, then the bandwidth dimension of A is at most the bandwidth dimension of B;
2. the bandwidth dimension of a finite subdirect product cannot exceed that of its factors;
3. for any algebra A and positive integer n, the algebras A and $M_n(A)$ have the same bandwidth dimension.

On the other hand, GK-dimension has the very useful property that if I is an ideal of the algebra A, then $GK\text{-}\dim A/I \leq GK\text{-}\dim A$. But this property fails very badly for bandwidth dimension. Of course, this is the expected price we pay for not giving the free algebra the largest possible dimension! For example, take A to be the free algebra on a countably infinite set and choose I so that A/I is purely infinite. Then A has bandwidth dimension 0 but A/I has bandwidth dimension 1 (Theorems 3.3 and 5.2).

6. Growth curve restrictions on regular self-injective rings

There are good signs that certain bandwidth dimension values may be reflected in interesting purely ring-theoretic properties. The main theorem (6.1) of this section is an illustration of how a growth curve restriction can result in quite strong ring properties. For background on regular self-injective rings, the reader is referred to [G], particularly Chapters 9 and 10.

Theorem 6.1. *Let R be a (von Neumann) regular, right self-injective ring.*

(1) *If R has **zero growth** over some field F (that is, R embeds in $G(0)$), then R must have bounded index of nilpotence.*
(2) *If R has **linear growth** over some field F (that is, R embeds in $G(1)$), then R must be of Type I_f.*

We shall give parallel proofs of the two parts of the theorem, via three lemmas. The key to the proof is the filtration of $G(r)$ supplied by the $X_r(d)$, in the cases $r = 0$ and $r = 1$ (Proposition 1.6). The difference between the two cases is that for $r = 0$ the spine $X_r(0)$ is a regular self-injective ring of bounded index of nilpotence, whereas for $r = 1$ the spine is only of Type I_f.

Lemma 6.2. *Let $G(r) = \bigcup_{d\geq 1} X_r(d)$ be the filtration given in 1.6. Let $S = X_r(0)$ be the corresponding spine.*

(1) *If $r = 0$, then each $X_0(d)$ $(= W_0(d))$ has a bound ($2d + 1$ in fact) on the number of independent nonzero pairwise isomorphic S-submodules.*
(2) *If $0 \leq r \leq 1$, then for each nonzero S-submodule Y of $X_r(d)$, there is a bound on the $n \in \mathbf{N}$ for which $Y \cong nZ$ for some S-module Z.*

Proof. Since $S \cong \prod_{k=1}^{\infty} M_{n_k}(F)$ for some n_k, S is a regular self-injective ring of Type I_f. By 1.6(c) each $X_r(d)$ is a finitely generated projective S-module. Hence $X_r(d)$ is a nonsingular injective Type I_f right S-module. In the case $r = 0$, S has bounded index of nilpotence, whence (1) follows from [G, Corollaries 7.3, 7.13]. The general case in (2) follows from [G, Proposition 2.4] and the fact that the endomorphism ring of a nonsingular injective Type I_f module, being a Type I_f ring, is a direct product of regular rings of bounded index of nilpotence [G, Theorem 10.24]. □

Lemma 6.3. *Suppose R is a regular right self-injective subring of $B(F)$. Let $g \in R$ be an idempotent.*

(1) *If $Rg \subseteq W_0(d)$ for some $d \in \mathbf{N}$, then gRg has bounded index of nilpotence.*
(2) *If $Rg \subseteq X_1(d)$ for some $d \in \mathbf{N}$, then gRg is of Type I_f.*

Proof. (1) Suppose $Rg \subseteq W_0(d)$. Let $W = W_0(d)$. By (1) of Lemma 6.2 there is a bound, t say, on the number of independent nonzero isomorphic left S-submodules of W. Then t is also a bound on the number of independent nonzero isomorphic principal left ideals of gRg, because such left ideals will generate independent isomorphic left S-submodules of W. By [G, Corollary 7.3], gRg has bounded index of nilpotence.

(2) Suppose $Rg \subseteq X_1(d)$. Let $X = X_1(d)$ and let $B = gRg$. Observe that for any $h \in B$ and any decomposition

$$Bh = Bh_1 \oplus \cdots \oplus Bh_n$$

where the h_i are orthogonal idempotents of B with $Bh_1 \cong Bh_2 \cong \cdots \cong Bh_n$, we have

$$SBh = SBh_1 \oplus \cdots \oplus SBh_n$$

with $SBh_1 \cong SBh_2 \cong \cdots \cong SBh_n$ as left S-modules. Let $h \in B$ be the central idempotent such that $B(1-h)$ is the Type I_f part of B. Suppose $h \neq 0$. By (2) of Lemma 6.2 there exists $t \in \mathbf{N}$ such that $SBh \not\cong tZ$ for any S-module Z. But since Bh is the direct sum of a Type II_f ring and a purely infinite ring, $Bh \cong t(Bh_1)$ for some $h_1 \in B$ [G, Theorem 10.16 and Proposition 10.28]. Now our earlier observation implies $SBh \cong t(SBh_1)$, which is a contradiction. Hence $h = 0$ and so B must be of Type I_f. □

Lemma 6.4. *Let R be a regular right self-injective ring. If R has unbounded index of nilpotence (respectively R is not of Type I_f), then R contains an infinite set $\{g_n\}_1^\infty$ of orthogonal idempotents such that each g_nRg_n has unbounded index of nilpotence (respectively no g_nRg_n is of Type I_f).*

Proof. The two statements of the lemma follow, for example, from the Type decomposition of a regular right self-injective ring [G, Theorem 10.22]. For the statement on unbounded index, use [G, Theorem 10.19, Corollary 7.3] on the purely infinite part, [G, Proposition 10.28] on the Type II_f part, and [G, Theorem 10.24] on the Type I_f part. For the other statement, if $R(1-h)$ is the Type I_f part where $h \neq 0$, then any infinite set $\{g_n\}_1^\infty$ of nonzero orthogonal idempotents of Rh will do the trick. □

Proof of Theorem 6.1. (1) Suppose $R \subseteq G(0)$. By way of a contradiction, suppose R has unbounded index of nilpotence. Lemma 6.4 then gives an infinite set $\{g_n\}_1^\infty$ of orthogonal idempotents of R such that each g_nRg_n has unbounded index. Each $g_n \in W_0(d_n)$ for some positive integer d_n because $R \subseteq G(0) = \bigcup W_0(d)$. Furthermore we can arrange for the d_n to form a strictly increasing sequence. From (1) of Lemma 6.3, $Rg_n \not\subseteq W_0(2d_n)$, whence for each n we can choose $x_n \in Rg_n$ such that $x_n \notin W_0(2d_n)$. By right self-injectivity of R, there is an $x \in R$ with $xg_n = x_n$ for all n (this is the critical use of injectivity). But $x \in W_0(m)$ for some m, and so from Proposition 1.4(b) we have that $x_n = xg_n$ implies $x_n \in W_0(m)W_0(d_n) \subseteq W_0(m+d_n)$, which for large n contradicts $x_n \notin W_0(2d_n)$. Therefore R must have bounded index.

(2) This parallels (1) by using the second halves of Lemmas 6.3 and 6.4, together with Proposition 1.6(a). Thus assuming the result is false, we choose the g_n so that g_nRg_n is not of Type I_f, and we choose $x_n \in Rg_n$ such that $x_n \notin X_1(2d_n)$. An element $x \in X_1(m)$ satisfying $xg_n = x_n$ for all n will again provide a contradiction $x_n \in X_1(m)X_1(d_n) \subseteq X_1(m+d_n) \subseteq X_1(2d_n)$ for large n. □

Corollary 6.5. *Let $\{n_k\}$ be any unbounded sequence of positive integers and let*

$$A = \prod_{k=1}^{\infty} M_{n_k}(F).$$

Then A does not have zero ($O(1)$) growth. However the bandwidth dimension of A is 0.

Proof. A is a regular right self-injective ring of unbounded index, hence A does not embed in $G(0)$ by Theorem 6.1(1). Let r be any positive real number. We can embed A in $B(F)$ so that n^r is a growth curve for A, simply by padding out the usual block diagonal representation of A, and repeating each block often enough until the increasing curve n^r has allowed a bandwidth large enough to accommodate the next block. Hence

$$\inf\{r|A \text{ embeds in } G(r)\} = 0$$

which says that the bandwidth dimension of A is 0. □

It is certainly not the case that all regular subrings of $G(0)$ have bounded index (for example, consider the subalgebra of $G(0)$ consisting of all $\omega \times \omega$ matrices with an arbitrary finite block in the top left corner and scalars down the diagonal). In fact regular subrings of $G(0)$ need not even have all their primitive factors artinian. For example we can obtain a copy of $\varinjlim M_{2^n}(F)$ in $G(0)$ by considering all matrices of the form

$$\begin{bmatrix} B & & & O \\ & B & & \\ & & B & \\ O & & & \ddots \end{bmatrix}$$

where $B \in M_{2^n}(F)$ for some n (this is a simple regular algebra which is not artinian [G, Example 8.1]).

7. Questions

There are many basic questions concerning bandwidth dimension which we have not yet answered, and, in some cases, have not even considered. We now list some of the more prominent ones.

(1) If a finitely generated algebra A has finite GK-dimension, must A embed in $G(0)$? If not, does A at least have zero bandwidth dimension?

(2) Do all finitely generated Noetherian algebras embed in $G(0)$? If not, can some have positive bandwidth dimension? If the answer to the first part is "yes", then the algebra $G(0)$ of finite bandwidth matrices might provide a good "universe" for constructing Noetherian algebras with interesting pathology.

(3) Let G be a finitely generated group and let $A = F[G]$ be the group algebra over F. What does the bandwidth dimension of A say about G? What are the possible bandwidth dimensions of such A? The case where G is polycyclic-by-finite (hence A Noetherian) but not nilpotent-by-finite (hence A of infinite GK-dimension) might provide a good test case for question (2).

(4) How does bandwidth dimension behave under a skew polynomial extension $A[y; \tau, \sigma]$? For instance, if A has zero growth, does $A[y; \tau, \sigma]$ have zero growth? If this were true then, of course, we could answer (2) for many Noetherian algebras.

(5) Let A be a (von Neumann) regular subalgebra of $G(0)$. Must A be directly finite ($xy = 1$ implies $yx = 1$)? Must A be unit-regular? The latter is not true if we require only that A has zero bandwidth dimension because it is known [GMM, Corollary 2.3] that the free regular algebra on a countable set over a countable field F embeds in $\prod_{n=1}^{\infty} M_n(F)$. Incidentally, it was for the proof of this result that Goodearl, Menal, and Moncasi needed the fact that countable-dimensional algebras embed in $B(F)$. This in turn was our starting point!

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