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FREE IDEALS OF ONE-RELATOR GRADED LIE ALGEBRAS

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ABSTRACT. In this paper we show that a one-relator graded Lie algebra $\mathfrak{g} = L/(r)$, over a principal ideal domain K, has a homogeneous ideal $\mathfrak{h}$ with $\mathfrak{g}/\mathfrak{h}$ a free K-module of finite rank if the relator r is not a proper multiple of another element in the free Lie algebra L. As an application, we deduce that the center of a one-relator Lie algebra over K is trivial if the rank of L is greater than two. As another application, we find a new class of one-relator pro-p-groups which are of cohomological dimension 2.

STATEMENT OF RESULTS

Let K be a principal ideal domain and let L be the free Lie algebra over K on the set X. Let r be a nonzero element of L and let $\mathfrak{g} = L/(r)$, where (r) is the ideal of L generated by r. We also assume that a grading of L is given in which the elements of X are homogeneous of degree ≥ 1. The main result of this paper is the following:

Theorem 1. *If r is homogeneous of degree d and $r \notin \mathfrak{m}L$ for every maximal ideal $\mathfrak{m}$ of K, there exists a homogeneous ideal $\mathfrak{h}$ of $\mathfrak{g} = L/(r)$ and homogeneous elements $g_1, \dots, g_n \in \mathfrak{g}$ of degree $< d$ such that $\mathfrak{h}$ is a free Lie algebra and $\mathfrak{g}/\mathfrak{h}$ a free K-module with basis the images of $g_1, \dots, g_n$ in $\mathfrak{g}/\mathfrak{h}$. If* $\mathrm{Card}(X) > 2$, *or $X = \{x_1, x_2\}$ and $r \neq ax_1 + bx_2$, $c[x_1, x_2]$, the rank of $\mathfrak{h}$ is at least* 2.

Theorem 1 is proved by the elimination method, using new results on the matrix of the adjoint representation of a free Lie algebra with respect to certain *weighted* Hall bases. These results are valid over any commutative ring K and yield Theorem 1 for certain types of relators even when K is not a PID.

Theorem 1 is actually the result of an attempt to answer a question a posed to us by Hyman Bass. This question, which was prompted by joint work of Bass and Lubotsky [2], was the following: "Let $\mathfrak{g}$ be a Lie algebra (over a field K) with d generators and one (homogeneous) relator. If $d > 2$, is the center of $\mathfrak{g}$ trivial?" The analogous question for one-relator groups was answered in the affirative by Murasugi [6]. The following theorem answers this question in the affirmative when K is a field or when K is a PID and r is homogeneous.

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Whether they remain true when K is a PID and r is non-homogeneous is an open question (even for $K = \mathbb{Z}$).

Theorem 2. *Suppose that either K is a field or K is a PID and r is homogeneous. If the center of the Lie algebra $\mathfrak{g} = L/(r)$ is not trivial then $X = \{x_1, x_2\}$ and either $r = ax_1 + bx_2$ or $r = a[x_1, x_2]$.*

Corollary. *If K is a field and the center of $\mathfrak{g} = L/(r)$ is nontrivial then $\mathfrak{g}$ is isomorphic to K or $K \times K$.*

The following result was pointed out to us by Warren Dicks. We thank him for allowing us to reproduce his argument here.

Theorem 3. *If $\mathfrak{g} = L/(r)$ is a one-relator Lie algebra over a field K, there exists a free subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of finite codimension.*

To see this, let r_d be the highest degree form of r for the usual grading of L and apply Theorem 1 to get a free Lie subalgebra $\mathfrak{h}'$ of $\mathfrak{g}' = L/(r_d)$ of finite codimension. Let $\mathfrak{h}_0$ be a lifting of $\mathfrak{h}'$ to a free Lie subalgebra of L. By a result of Sirsov [8], the (graded) ideal of highest degree forms of elements of (r) is generated by r_d and so $\mathfrak{h}_0 \cap (r) = 0$. The image $\mathfrak{h}$ of $\mathfrak{h}_0$ in g is therefore a free Lie subalgebra of $\mathfrak{g}$ which is also easily seen to be of finite codimension.

In the last section of this paper we extend the elimination method to pro-p-groups. Using Theorem 1, we find a new class of one-relator pro-p-groups which are of cohomological dimension 2.

Weighted Hall Sets

Let X be a set and let $M(X)$ be the free magma on X. Every $u \in M(X)$, $u \notin X$ can be written uniquely as a product $\alpha(u)\beta(u)$ with $\alpha(u)$, $\beta(u) \in M(X)$. Let $\ell = \ell_X : M(X) \to \mathbb{N}$ be the unique mapping such that $\ell(x) = 1$ for all $x \in X$ and $\ell(uv) = \ell(u) + \ell(v)$ for all $u, v \in M(X)$; $\ell(u)$ is called the length of u. If $u_1, \dots, u_n \in M(X)$ we define $p_n = u_n \cdots u_2 u_1$ inductively by $p_1 = u_1$ and $p_{i+1} = u_{i+1} p_i$ for $i > 1$. If $u_2 = \cdots = u_n = a$ and $u_1 = b$, we denote p_n by $a^{n-1}b$. Every $u \in M(X)$ can be uniquely written in the form $u = u_1 u_2 \cdots u_n x$ with $u_i \in M(X)$, $x \in X$ and $n \geq 0$. We call this the canonical (right-normed) decompostion of u in $M(X)$.

For each $x \in X$ let $n_x \in \mathbb{N}^* = \mathbb{N} - \{0\}$ and let ω be the unique mapping of $M(X)$ into $\mathbb{N}^*$ such that $\omega(x) = n_x$ if $x \in X$ and $\omega(uv) = \omega(u) + \omega(v)$ for $u, v \in M(X)$. We will call $\omega(u)$ the weight of u with respect to the weight system $(n_x)_{x \in X}$ and ω the associated weight function. The weight system is completely determined by ω and we will ususally let ω denote it. If X is ordered we say that the weight system is compatible with this ordering if $x < y \implies \omega(x) \leq \omega(y)$.

A weighted Hall set, relative to a well-ordered set X and compatible weight system ω, is a subset H of $M(X)$ together with a well-ordering $<$ of H such that

(1) X is an ordered subset of H;
(2) For $u, v \in H$, $\omega(u) < \omega(v) \implies u < v$;
(3) For all $u \in M(X)$ with $\ell(u) > 1$, $u \in H \iff \alpha(u), \beta(u) \in H$ and $\alpha(u) < \beta(u)$ with $\alpha(u) > \alpha(\beta(u))$ if $\ell(\beta(u)) > 1$.
(4) If $u, v \in H$ with $\omega(u) = \omega(v)$ and $\ell(u), \ell(v) > 1$ then $u < v$ iff $\beta(u) < \beta(v)$ or $\beta(u) = \beta(v)$ and $\alpha(u) < \alpha(v)$.

If $\omega(x) = 1$ for all x, we have $\ell = \omega$ and we recover the usual definition of a Hall set. A weighted Hall set H is said to be of finite type if for any $k \in \mathbb{N}^*$ there are only finitely many $u \in H$ of weight k. This is equivalent to requiring that, for any k, $\{x \in X | \omega(x) = k\}$ be a finite set; a weight function with this property is also said to be of finite type. If $u \in M(X)$ with canonical decomposition $u_1 u_2 \cdots u_n x$, we have $u \in H$ iff $u_1, \cdots, u_n \in H$, $u_1 \geq u_2 \geq \cdots \geq u_n < x$ and $u_{i-1} < u_i \cdots u_n x$ for $2 \leq i \leq n$. If $u \in H$, we will call its canonical decomposition in $M(X)$ the canonical decomposition of u in H.

Proposition 1. *Given any well-ordered set X and a compatible weight system ω on X, there exists a weighted Hall set H relative to X, ω.*

Proof. We construct, by induction on n, well-ordered subsets W_n of $M(X)$ consisting of elements of weight n:

(a) We let W_1 consist of those elements of X which are of weight 1.

(b) Let $n \geq 2$ so that the well-ordered sets $W_1, \ldots, W_{n-1}$ are already constructed. Let $W'_{n-1} = W_1 \cup \cdots \cup W_{n-1}$ together with the well-ordering which induces the given orderings on $W_1, \ldots, W_{n-1}$ and such that $u < v$ if $\omega(u) < \omega(v)$. Let W_n consist of those elements of X of weight n together with the products ab of weight n with $a, b \in W'_{n-1}$, $a < b$ and $a > \alpha(b)$ if $\ell(b) > 1$. If ab, $a'b' \in W_n$, $x \in X \cap W_n$, let $ab < x$ and let $ab < a'b'$ if $b < b'$ or if $b = b'$ and $a < a'$. This defines a well-ordering of W_n.

The required well-ordered set H is the union of the W_i. □

Proposition 2. *Suppose that $u, v, uv \in H$. If $u_1, u_1 v \in H$ and $u < u_1$ (resp. $v_1, uv_1 \in H$ and $v < v_1$) then $uv < u_1 v$ (resp. $uv < uv_1$).*

Proof. Simply note that $u < u_1$ implies that $\omega(u) \leq \omega(u_1)$. If the inequality is strict, we have $\omega(uv) < \omega(u_1 v)$, and so $uv < u_1 v$. Otherwise, uv and $u_1 v$ have the same weight and $u < u_1$, which implies $uv < u_1 v$. □

Now let H be any weighted Hall set relative to the set X and a given weight system ω. Let x_1 be the smallest element of X; this element is also the smallest of H. Let $H_1 = H - \{x_1\}$ and let X_1 be the subset of H_1 consisting of those elements which cannot be written as a product of 2 elements of H_1. We have

$$X_1 = \{x_1^j x | x \in X, x \neq x_1, j \geq 0\}.$$

Identify $M(X_1)$ with its canonical image in $M(X)$ and let ω_1 be the restriction of ω to $M(X_1)$. Let ℓ_1 be the length function on $M(X_1)$.

Proposition 3. *The set H_1, with the induced ordering from H, is a weighted Hall set relative to X_1, ω_1. If H is of finite type then so is H_1.*

Proof. We need only prove the first assertion. Since (1), (2) and (4) follow immediately, it suffices to verify condition (3).

Let $u \in M(X_1)$ with $\ell_1(u) > 1$. Then $u = u_1 u_2$ with $u_1, u_2 \in M(X_1)$. If $u \in H_1$ then $u_1, u_2 \in H$ with $u_1 < u_2$ since $H_1 \subset H$. Since $x_1 \notin M(X_1)$ we obtain $u_1, u_2 \in H_1$. If $u_2 = u'_2 u''_2$ with $u'_2, u''_2 \in H_1$, we have $u_1 > u'_2$ since $H_1 \subset H$.

Conversely, suppose $u = u_1 u_2$ with $u_1, u_2 \in H_1$, $u_1 < u_2$. If $u_2 = u'_2 u''_2$ with $u'_2, u''_2 \in H_1$ then $u_1 > u'_2$ implies that $u \in H$ and so $u \in H_1$ since

$u \neq x_1$. If $\ell_1(u_2) = 1$ and $\ell(u_2) > 1$, we have $u_2 = x_1 v$ with $v \in H_1$. Since $u_1 \geq x_1$ we have $u \in H$ and so $u \in H_1$. □

Let u be any element of H_1 and let $u_1 u_2 \cdots u_k y$ be its canonical decomposition in H_1. Let $\gamma(u) \in H_1$ be the element $u_1 u_2 \cdots u_k y'$ with $y' = x_1 y$; this product is the canonical decomposition of $\gamma(u)$ in H_1. We will call γ the *dominance function* for H. We say that H has the *dominance property* if, for all $x \in X$ and $u \in H$ with $\omega(x) = \omega(u) > 1$ and $x < u$, we have $x < \gamma(\beta(u))$ if $\ell(u) > 1$. If $\omega = \ell$, then H has the dominance property since this condition holds vacuously. For the same reason, the weighted Hall set constructed in Proposition 1 has the dominance property.

Proposition 4. *The Hall set H has the dominance property iff, for all $u, v \in H_1$,*

$$u < v \implies \gamma(u) < \gamma(v).$$

Proof. ($\Longrightarrow$) Since the assertion is trivial if $u, v \in X_1$ or if $\omega(u) < \omega(v)$, we may assume that not both u, v lie in X_1 and that u, v have the same weight. We proceed by induction on $\omega(u)$.

If $u = u_1 u_2$, $v = v_1 v_2$ in H_1 then $\gamma(u) = u_1 \gamma(u_2)$, $\gamma(v) = v_1 \gamma(v_2)$. If $u_2 = v_2$, we are done. If $u_2 < v_2$ then by induction on the degree we have $\gamma(u_2) < \gamma(v_2)$ which gives the required result.

Now suppose $u = u_1 u_2$ in H_1 and $v \in X_1$. If $v \notin X$ then $v = x_1 w$ for some $w \in X_1$. Since $u_2 < w$ we have $\gamma(u_2) < \gamma(w) = v$ by induction on the degree. But then

$$\gamma(u) = u_1 \gamma(u_2) < x_1 v = \gamma(v).$$

We may therefore assume $v \in X$. Since

$$\omega(v) = \omega(u_1) + \omega(u_2) \geq \omega(x_1) + \omega(u_2) = \omega(\gamma(u_2)),$$

we have $v \geq \gamma(u_2)$. But $v \neq \gamma(u_2)$ as $v \in X$ and we again obtain the desired result.

Finally, suppose $u \in X_1$, $v = v_1 v_2$ in H_1. Since the case $u \notin X$ can be treated as above, we may assume that $u \in X$. Since H has the dominance property, we have $u < \gamma(v_2)$ and so $\gamma(u) = x_1 u < v_1 \gamma(v_2) = \gamma(v)$.

($\Longleftarrow$) Suppose $x \in X$, $u \in H$ with $x < u$, $\ell(u) > 1$ and $\omega(x) = \omega(u)$. Let $u = ab$. Then $x_1 x < a\gamma(b)$ which implies $x < \gamma(b)$ since $x \neq \gamma(b)$. □

Proposition 5. *If H has the dominance property then so has H_1.*

Proof. Let γ_1 be the dominance function for H_1, let x_2 be the smallest element of X_1, let $H_2 = H_1 - \{x_2\}$ and let X_2 be the indecomposable elements of H_1. Let $u \in X_1$, $v = v_1 v_2$ in H_1 with $u < v$ and $\omega(u) = \omega(v)$. Then $u \in H_2$ and

$$x_1 u = \gamma(u) < \gamma(v) = v_1 \gamma(v_2),$$

which yields $u \leq \gamma(v_2)$. If we could show that $\gamma(v_2) < \gamma_1(v_2)$ then we would have

$$\gamma_1(u) = x_2 u < v_1 \gamma_1(v_2) = \gamma_1(v).$$

Hence we are reduced to showing that $\gamma(v_2) < \gamma_1(v_2)$. In H_2 we have the canonical decomposition

$$v_2 = w_1 w_2 \cdots w_t z$$

with $z = x_2^i x_1^j x$, $x \in X$, $x \neq x_1$ and $j \neq 0$ if $x = x_2$. Since

$$\gamma(v_2) = w_1 w_2 \cdots w_t x_2^i x_1^{j+1} x,$$
$$\gamma_1(v_2) = w_1 w_2 \cdots w_t x_2^{i+1} x_1^j x,$$

we are reduced to showing that

$$x_2^i x_1^{j+1} x < x_2^{i+1} x_1^j x.$$

This is true if $\omega(x_1) < \omega(x_2)$ since then the right-hand side has a larger weight than the left. If x_1, x_2 have the same weight, we are reduced to showing that

$$x_1 x_1^j x < x_2 x_1^j x.$$

But this is true since $x_1 < x_2$. □

Corollary. *If H has the dominance property, is of finite type and $u \in H$, the set $\{v \in H | v \geq u\}$ is a weighted Hall set having the dominance property.*

Let $L(X)$ be the free Lie algebra on X over a commutative ring K. We have a canonical mapping ψ of $M(X)$ into $L(X)$ such that $\psi(uv) = [\psi(u), \psi(v)]$. Let H be any weighted Hall set relative to X, ω.

Proposition 6. *The mapping ψ is injective on H and the image of H is a basis of $L(X)$ as a K-module.*

Proof. Without loss of generality we may assume that X is finite. Let x_1 be the smallest element of X. By the elimination theorem (cf. [1, §2.9, Proposition 10]) we have $L(X) = Kx_1 \oplus L(X_1)$. Repeating this argument a finite number of times, we obtain the result. □

We identify H with its image in $L(X)$ and let ad be the ajoint representation of $L(X)$; we have $\mathrm{ad}(u)(v) = [u, v]$.

Theorem 4. *Suppose that H has the dominance property and $u \in H_1$. Then*

$$\mathrm{ad}(x_1)(u) = \gamma(u) + w,$$

where w a linear combination of elements of H_1 which are smaller than $\gamma(u)$.

Proof. Since the assertion is trivially true if $\ell_1(u) = 1$ we may assume that $\ell_1(u) > 1$. If $\ell_1(u) = 2$ then $u = [u_1, u_2]$ with $u_1, u_2 \in X_1$, $u_1 < u_2$, and we have $\mathrm{ad}(x_1)(u) = [\gamma(u_1), u_2] + [u_1, \gamma(u_2)]$. If $\gamma(u_1) < u_2$ then $w = [\gamma(u_1), u_2] \in H_1$ and $w < [u_1, \gamma(u_2)] = \gamma(u)$ since $u_2 < \gamma(u_2)$. If $\gamma(u_1) > u_2$ then $\mathrm{ad}(x_1)(u) = -[u_2, \gamma(u_1)] + \gamma(u)$ and $w = [u_2, \gamma(u_1)] \in H_1$, $w < \gamma(u)$ since $u_1 < u_2$ implies that $\gamma(u_1) < \gamma(u_2)$.

Assume that, if $v \in H_1$ with $2 \leq \ell_1(v) \leq k$, we have shown that $\mathrm{ad}(x_1)(v) = \gamma(v) + w$, where $w \in H_1$ is a linear combination of terms $w' \in H_1$ with $\ell_1(w') > 1$ and

(1) $w' < \gamma(v)$;
(2) $\alpha(v) \leq \alpha(w') < \beta(w') < v$;
(3) $\alpha(v) \geq \alpha(\alpha(w'))$ (resp. $\alpha(\beta(w'))$) if $\ell_1(\alpha(w')) > 1$ (resp. $\ell_1(\beta(w')) > 1$).

This holds if $k = 2$. We want to show that this then holds for $v = u$ with $\ell_1(u) = k + 1$. We divide the proof into three cases.

Case 1: $u = [x, v]$ in H_1 with $x \in X_1$, $\ell_1(v) > 1$. Then $x < v = [a, b]$, $a, b \in H_1$, $x \geq a$, and by induction $\mathrm{ad}(x_1)(v) = \gamma(v) + w$, where $w \in H_1$ is a linear combination of terms $w' = [a', b'] \in H_1$ with $\ell_1(w') > 1$ satisfying (1), (2), (3). But then

$$\begin{aligned}\mathrm{ad}(x_1)(u) &= [\gamma(x), v] + [x, \gamma(v)] + w_1 \\ &= \gamma(u) + [\gamma(x), v] + w_1,\end{aligned}$$

where w_1 is a linear combination of terms $[x, [a', b']]$ with $[a', b']$ as above. We have to show that $[\gamma(x), v]$, $[x, [a', b']]$ are linear combinations of terms $w' \in H_1$ with $\ell_1(w') > 1$ and satisfying (1), (2), (3) with v replaced by u

Consider first the term $[\gamma(x), v]$. If $\gamma(x) < v$ then $[\gamma(x), v] \in H_1$ since $\gamma(x) > x \geq a$ and $[\gamma(x), v] < \gamma(u) = [x, \gamma(v)]$ since $v < \gamma(v)$. Since $x > a = \alpha(v)$, condition (3) is satisfied, and (2) follows from $x < \gamma(x)$, $[x, v] > v$. If $\gamma(x) > v$ then $[\gamma(x), v] = -[v, \gamma(x)]$ with $[v, \gamma(x)] \in H_1$ and $[v, \gamma(x)] < \gamma(u)$ since $\gamma(x) < \gamma(v)$. Again condition (3) is satisfied since $x > a$. To prove (2) we note first that $\omega([x, v]) \geq \omega(\gamma(x))$ since $\gamma(x) = [x_1, x]$ and $x_1 < x < v$. If $\omega([x, v]) > \omega(\gamma(x))$, we have $[x, v] > \gamma(x)$, and if the weights are equal we have $[x, v] > \gamma(x) = [x_1, x]$ since $v > x$.

Now consider the terms $[x, [a', b']]$. If $x \geq a'$, we have $[x, [a', b']] \in H_1$ and $[x, [a', b']] < [x, \gamma(v)]\gamma(u) =$ since $[a', b'] < \gamma(v)$. Since $\omega([x, v]) \geq \omega([a', b'])$, we may assume in proving (2) that $[x, v]$ and $[a', b']$ have equal weight. But then $[x, v] > [a', b']$ since $v > b'$; this gives (2) and condition (3) follows from $x \geq a'$. Now suppose that $x < a'$. Then $[x, a']$, $[x, b'] \in H_1$ by (3) since $x \geq a$ and we have

$$[x, [a', b']] = [[x, a'], b'] + [a', [x, b']].$$

If $[x, a'] < b'$ then $[[x, a'], b'] \in H_1$ since $[x, a'] > x \geq a$ and $a \geq \alpha(b')$ if $\ell_1(b') > 1$. Now (1) holds since $b' < \gamma(v)$ implies that $[[x, a'], b'] < [x, \gamma(v)] = \gamma(u)$ and (3) holds since $x \geq a$ and $a \geq \alpha(b')$ if $\ell_1(b') > 1$. To prove (2), we have to show $[x, v] > b'$; but this is trivially true since $v > b'$. If $[x, a'] > b'$ then $[b', [x, a']] \in H_1$ and (3) follows as above. To prove (1), we have to show that $[b', [x, a']] < [x, [a, \gamma(b)]]$. Since $b' > x$, we have $\omega(b') \geq \omega(x)$ and so $\omega([x, a']) \leq \omega([a, \gamma(b)])$. If we have strict inequality, we are done; so we may assume equality. But $b' < \gamma(b)$ since $[a', b'] < \gamma(v) = [a, \gamma(b)]$ and $a \leq a'$. So $a' < b' < \gamma(b)$ which implies $[x, a'] < [a', \gamma(b)]$ and we are done. Finally, $[a', [x, b']] \in H_1$ and the same arguments as above yield (1),(2),(3).

Case 2: $u = [v, x]$ in H_1 with $x \in X_1$, $\ell_1(v) > 1$. Then $v = [a, b]$ in H_1 and $\mathrm{ad}(x_1)(u) = \gamma(u) + [\gamma(v), x] + w_1$, where w_1 is a linear combination of terms $[w', x]$ with $w' = [a', b']$ satisfying the same conditions as in Case 1. We have to show that $[[a', b'], x]$, $[\gamma(v), x]$ are linear combinations of terms $w' \in H_1$ with $\ell_1(w') > 1$ and satisfying (1), (2), (3) with $v = u$.

Consider first the term $[\gamma(v), x]$. If $\gamma(v) < x$ then $[\gamma(v), x] \in H_1$ and is smaller than $\gamma(u) = [v, \gamma(x)]$ since $x < \gamma(x)$. Condition (2) holds since $[v, x] > x$, and (3) holds since $v > a$. If $\gamma(v) > x$ then $[\gamma(v), x] = -[x, \gamma(v)] = -[x, [a, \gamma(b)]$ and $[x, [a, \gamma(b)] \in H_1$ since $x > a$. Moreover, $[x, \gamma(v)] < [v, \gamma(x)] = \gamma(u)$ since $v < x$ implies $\gamma(v) < \gamma(x)$, and $v > a$ gives

(3). To prove (2) we have to show $[v, x] > \gamma(v)$. But $\omega([v, x]) \geq \omega(\gamma(v))$ and equality would imply $\omega(x) = \omega(x_1)$ which is impossible since $x > v$ and $\omega(v) > \omega(a) \geq \omega(x_1)$.

Now consider the terms $[[a', b'], x]$. If $[a', b'] < x$, we have $[[a', b'], x] \in H_1$ and $[[a', b'], x] < [v, \gamma(x)] = \gamma(u)$, which gives (1), and (2) follows from $[v, x] > x$. Since $v > b' > a'$ we have (3). If $[a', b'] > x$, we have $[[a', b'], x] = -[x, [a', b']]$. Since $x > v > b' > a'$, we have $[x, [a', b']] \in H_1$, and $[x, v'] < [v, \gamma(x)] = \gamma(u)$ since $[a', b'] < \gamma(v) < \gamma(x)$. This gives (1) and (3). Condition (2) follows from $\omega([v, x]) > \omega([a', b'])$.

Case 3: $u = [u_1, u_2]$ in H_2 with $\ell_1(u_1), \ell_1(u_2) > 1$. Then $u_1 = [a, b]$, $u_2 = [c, d]$ in H_1 and

$$\operatorname{ad}(x_1)(u) = \gamma(u) + [\gamma(u_1), u_2] + w_1 + w_2,$$

where w_1 (resp. w_2) is a linear combination of terms $[[a', b'], u_2] \in H_1$ (resp. $[u_1, [c', d']] \in H_1$) with $w' = [a', b'] \in H_1$, (resp. $w' = [c', d'] \in H_1$) satisfying $\ell_1(w') > 1$ and (1), (2), (3) with $v = u_1$ (resp. $v = u_2$). We have to show that the terms $[\gamma(u_1), u_2]$, $[[a', b'], u_2]$, $[u_1, [a', b']]$ are linear cominations of terms $w' \in H_1$ with $\ell_1(w') > 1$ and satisfying (1), (2), (3) with $v = u$.

Consider first the term $w = [\gamma(u_1), u_2]$. If $\gamma(u_1) < u_2$ then $w \in H_1$ as $\gamma(u_1) > u_1 > c$. Since $w = [[a, \gamma(b)], [c, d]]$ and $u_1 > a, c$ condition (3) holds. Also $w < [u_1, \gamma(u_2)] = \gamma(u)$, which gives (1), and $[u_1, u_2] > u_2$ gives (2). If $\gamma(u_1) > u_2$ then $w = -[u_2, [a, \gamma(b)]]$. Now $u_2 > a$ since $\omega(u_2) > \omega(a)$ and so $z = [u_2, [a, \gamma(b)]] \in H_1$. Now $z < \gamma(u)$ since $z = [u_2, \gamma(u_1)]$ and $\gamma(u_1) < \gamma(u_2)$. Finally, $u_1 \geq a, c$ yields (3), and $\omega([u_1, u_2]) > \omega([a, \gamma(b)])$, $u_1 < u_2$, gives (2).

Now consider the term $w_1' = [[a', b'], u_2]$. If $[a', b'] < u_2$ then $w_1' \in H_1$ since $[a', b'] > u_1 \geq c$. Since $u_2 < \gamma(u_2)$, $w_1' < \gamma(u) = [u_1, \gamma(u_2)]$, which gives (1). We have (2) since $u_1 < [a', b']$, $[u_1, u_2] > u_2$, and (3) holds since $u_1 > a', c$. If $[a', b'] > u_2$ then $w_2' = -[u_2, [a', b']]$ and $[u_2, [a', b']] \in H_1$ since $u_2 > u_1 > a'$. Since $[a', b'] < \gamma(u_1) < \gamma(u_2)$, we have (1). Condition (2) holds since $\omega([u_1, u_2]) > \omega([a', b']$, $u_1 < u_2$, and (3) holds as above.

Finally, consider the term $w_2' = [u_1, [c', d']]$. If $u_1 \geq c'$, we have $w_2' \in H_1$, and (1) holds since $[c', d'] < \gamma(u_2)$. Condition (2) holds since $\omega([u_1, u_2]) > \omega([c', d'])$, and (3) holds since $u_1 \geq a, c'$. If $u_1 < c'$ we have

$$w_2' = [[u_1, c'], d'] + [c', [u_1, d']$$

with $[u_1, c']$, $[u_1, d'] \in H_1$ since $u_1 \geq c$ and $c \geq \alpha(c')$ (resp. $\alpha(d')$) if $\ell_1(c') > 1$ (resp. $\ell_2(d') > 1$). We have $[c', [u_1, d'] \in H_1$, and (1) holds since $u_1 < c'$ implies that $[u_1, d'] < [c', d'] < \gamma(u_2)$. Condition (3) holds since $u_1 \geq c$, and (2) holds since $u_2 > d'$ implies that $[u_1, u_2] > [u_1, d']$. Now consider the term $w = [[u_1, c'], d']$. If $[u_1, c'] < d'$ then $w \in H_1$, and (1) holds since $d' < \gamma(u_2)$. Condition (2) holds since $u > u_2 > d'$ and (3) follows from $u_1 \geq c$. If $[u_1, c'] > d'$ then $z = -w = [d', [u_1, c']] \in H_1$, and (1) follows from the above as $[u_1, c'] < [u_1, d']$. We have (2) since $u_1 < c' < d'$, $[u_1, c'] < [u_1, d'] < [u_1, u_2] = u$; we use that $d' < u_2$. Condition (3) follows from $u_1 \geq c$. □

Proof of Theorem 1

Choose an ordering of X which is compatible with the given weight system ω on X and let H be a weighted Hall set with respect to X, ω which has the dominance property. We can write $r = cs + t$, where $c \in K - \{0\}$, $s \in H$ and t is a linear combination of elements of H which are strictly less than s. Assume that c is a unit of K; this is true in the case that K is a field. At this point we should like to point out that, under this assumption our proof is valid over any commutative ring K. Without loss of generality, we can assume $c = 1$. If r is a linear combination of elements of X then $\mathfrak{g}$ is free; so we may assume that r is not linear form. If $X = \{x_1, x_2\}$ and $r = [x_1, x_2]$ then again the theorem holds trivially. We may therefore assume that r is not linear and that either Card(X) > 2 or $X = \{x_1, x_2\}$ and $r \neq c[x_1, x_2]$. We may also assume, without loss of generality, that X is a finite set. Let $h_1 < h_2 < \dots$ be the elements of H and, for $i \geq 1$, let $H_i = \{h \in H_{i-1} | h > h_i\}$, where $H_0 = H$. Let X_i be the set of indecomposable elements of H_i; for $i \geq 1$ we have (setting $X_0 = X$)

$$X_i = \{\mathrm{ad}(h_i)^j(x) | j \geq 0, \ x \in X_{i-1}, x \neq h_i\}.$$

If ω_i is the restriction of ω to $M(X_i)$ then H_i is, relative to X_i and ω_i, a weighted Hall set of finite type which has the dominance property. Let γ_i be the dominance function for H_i and let $\delta_i = \mathrm{ad}(h_i)$. Then, for all $u \in H_i$, $\delta_i(u) = \gamma_i(u) + w$ where w is a linear comination of elements of H_i which are strictly less than $\gamma_i(u)$.

Since $r \in L(X_1)$, we have $s \in H_1$ and we have the canonical decompositions in H_1

$$s = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_m)(x),$$

$$s_i = \gamma_1^i(s) = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_m)(\delta_1^i(x)),$$

and $\delta_1^i(r) = s_i + w_i$ where w_i is a linear combination of elements of H_1 which are strictly less than s_i. If $\mathfrak{r}$ is the ideal of L generated by r then $\mathfrak{r} \subset L(X_1)$ and $\mathfrak{r}$ is the ideal of $L(X_1)$ generated by the elements $r_i = \delta_1^i(r)$.

If $m = 0$, i.e. if $\ell_1(s) = 1$, then the images of the elements r_i in the free K-module $L(X_2)/[L(X_2), L(X_2)]$ are part of a basis and so the r_i are part of a basis of $L(X_1)$ (cf. [4, Proposition 2]). It follows that $\mathfrak{h} = L(X_1)/\mathfrak{r}$ is a free Lie algebra over K and $\mathfrak{g}/\mathfrak{h} \cong Kh_1$ with $\omega(h_1) < d$ since r is not linear. Under our hypotheses on r, $x > h_3$, so the rank of $\mathfrak{h}$ is ≥ 2.

Suppose that $m > 0$ and let $h \in H_1$ be the smallest element of X_1 that appears in any of the elements $\delta_1^i(r)$. Then $h_2 \leq h \leq u_m$. If $h > h_2$ then all the elements r_i are in $L(X_2)$, the elements u_i are in H_2, $x \in X_2$, and $r_{ij} = \delta_2^j(r_i) = s_{ij} + w_{ij}$ where $s_{ij} = \gamma_2^j(s_i) \in H_2$ with canonical decomposition in H_2

$$s_{ij} = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_m)(\delta_2^j\delta_1^i(x))$$

and w_{ij} is a linear combination of elements of H_2 which are strictly smaller than s_{ij}. The same thing happens if $h = h_2 < u_m$. Indeed, in this case, $u_1, \dots, u_m \in L_2$ and $w_{ij} \in H_2$ since a nonzero scalar multiple of h_2 cannot appear as a term in the decomposition of w_{ij} as a linear combination of elements of H_1 because $\omega(h_2) < \omega(w_{ij}) = \omega(s_{ij})$ since $\ell_2(s_{ij}) > 1$. If

$h = h_2 = u_m = \cdots = u_{m-k+1} < u_{m-k}$ then s, $s_{ij} \in H_2$ with canonical decompostions in H_2

$$s = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_{m-k})(\delta_2^k(x)),$$

$$s_{ij} = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_{m-k})(\delta_2^{j+k}\delta_1^i(x)),$$

and, as above, w_{ij} is a linear combination of elements of H_2 if $m \neq k$. If $m = k$, we have

$$\omega(w_{ij}) = \omega(s_{ij}) > \omega(h_2)$$

and so $w_{ij} \in H_2$.

Proceeding inductively, assume that $r \in L(X_n)$ and that for $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $n \geq 1$,

$$r_i = \delta_n^{i_n}\cdots\delta_2^{i_2}\delta_1^{i_1}(r) = s_i + w_i,$$

where $s_i \in H_n$ with canonical decomposition in H_n

$$s_i = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_{m-k})(y_i)$$

with

$$y_i = \delta_n^{i_n+k_n}\cdots\delta_2^{i_2+k_2}\delta_1^{i_1}(x) \quad (k_i \geq 0,\ \text{uniquely determined by } s),$$

and w_i a linear combination of elements of H_n which are strictly less than s_i.

If $k = m$ the elements $s_i \in X_n$ are distinct and so the images of the elements r_i in the free K-module $L(X_n)/[L(X_n), L(X_n)]$ are part of a basis. Hence the r_i are part of a basis of the free Lie algebra $L(X_n)$. Since they also generate $\mathfrak{r}$ as an ideal of $L(X_n)$ we obtain that $\mathfrak{h} = L(X_n)/\mathfrak{r}$ is a free Lie algebra with

$$\mathfrak{g}/\mathfrak{h} \cong Kh_1 \oplus \cdots \oplus Kh_n.$$

Since $h_n = u_1$, we have $\omega(h_n) < d$ and so the components of $\mathfrak{g}/\mathfrak{h}$ are zero in degrees $\geq d$. The rank of $\mathfrak{h}$ is infinite since, in this case, $s_i \notin X_1$ and so the the r_i together with the elements $\delta_1^j(h_2) > h_n$ are part of a basis of $L(X_n)$.

Remark. In the case $m = k$ it is possible to continue the elimination if $r \in L(X_{n+1})$. This happens iff either r is not a linear form in $L(X_n)$ or r is a linear form in $L(X_n)$ which does not have a scalar multiple of h_{n+1} as a term. In this case, we get

$$r_{i,i_{n+1}} = \delta_{n+1}^{i_{n+1}}(r_i) = s_{i,i_{n+1}} + w_{i,i_{n+1}}$$

with $s_{i,i_{n+1}} \in X_{n+1}$,

$$s_{i,i_{n+1}} = \delta_{n+1}^{i_{n+1}}\delta_n^{i_n+k_n}\cdots\delta_2^{i_2+k_2}\delta^{i_1}(x),$$

and $w_{i,i_{n+1}}$ a linear combination of elements of H_{n+1} which are stricly less than $s_{i,i_{n+1}}$. We make use of the fact that $L(X_{n+1})$ is an ideal of $L(X)$.

If $k < m$ let h be the smallest element of X_n that appears in any of the r_i. Then $h_{n+1} \leq h \leq u_{m-k}$. If $h > h_{n+1}$ then u_i, $w_i \in H_{n+1}$, $y_i \in X_{n+1}$ and

$$\delta_{n+1}^{i_{n+1}}(r_i) = s_{i,i_{n+1}} + w_{i,i_{n+1}}$$

with $s_{i,i_{n+1}} \in H_n$ with canonical decomposition in H_{n+1}

$$s_{i,i_{n+1}} = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_{m-k})\delta_{n+1}^{i_{n+1}}(y_i)$$

and $w_{i,i_{n+1}}$ a linear combination of elements of H_{n+1} which are strictly less than $s_{i,i_{n+1}}$. The same is true if $h = h_{n+1} < u_{m-k}$. Indeed, a nonzero scalar multiple of h_{n+1} cannot appear as a term in the decomposition of w_i as a linear combination of elements of H_n; otherwise, $\omega(h_{n+1}) = \omega(s_i)$ contradicting $\ell_n(s_i) \geq 2$. If $h = h_{n+1} = u_{m-k} = \cdots u_{m-k-\ell+1} < u_{m-k-\ell}$ then again $s_{i,i_{n+1}} \in H_{n+1}$ with canonical decompostion in H_{n+1}

$$s_{i,i_{n+1}} = \mathrm{ad}(u_1)\mathrm{ad}(u_2)\cdots\mathrm{ad}(u_{m-k-\ell})\delta_{n+1}^{i_{n+1}+\ell}(y_i)$$

and $w_{i,i_{n+1}} \in L(X_{n+1})$ since

$$\omega(w_{i,i_{n+1}}) = \omega(s_{i,i_{n+1}}) > \omega(h_{n+1}).$$

If n is smallest with $s \in X_n$, it follows that the above procedure gives a free subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in n steps. Since $L(X_n)$ is an ideal of L, we obtain that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. The free subalgebra $\mathfrak{h}$ is of infinite rank if $n > 1$ or $n = 1$ and $\mathrm{Card}(X) > 2$. If $n = 1$ we have $\mathfrak{h}$ of finite rank ≥ 2 if $\mathrm{Card}(X) = 2$. Finally, we may take g_i to be the image of h_i in $\mathfrak{g}$ since

$$\mathfrak{g}/\mathfrak{h} \cong Kh_1 \oplus Kh_2 \oplus \cdots \oplus Kh_n$$

and $\omega(h_i) < d$ for $1 \leq i \leq n$.

Remarks. The natural number n and the Lie element s could have been chosen more optimally as follows: Let n be smallest with $r \in L(X_n)$, $r \notin [L(X_n), L(X_n)]$ and such that the largest linear term of r in $L(X_n)$ is a unit multiple of $s \in X_n$. The terms of r which are in $[L(X_n), L(X_n)]$ cause no problems since $L(X_n)$ is an ideal of $L(X)$ and so they can be ignored. Also note that, if n is largest with $r \in L(X_n)$, then r is a linear form in $L(X_n)$.

We now consider the general case. Let n be largest with $r \in L(X_n)$. Then r is a homogeneous linear form in $L(X_n)$ whose image in $L(X_n) \otimes (K/\mathfrak{m})$ is nonzero for any maximal ideal of K. It follows that the homogeneous components of $\mathfrak{h}$ and $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ are locally free of constant finite rank and hence that $\mathfrak{h}$ and $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ are free K-modules. Since $\mathfrak{h} \otimes (K/\mathfrak{m})$ is a free Lie algebra over $K/\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of K by [9, Satz 5] or [4, Proposition 2], it follows that $\mathfrak{h}$ is free over K by [4, Proposition 2] and the following Lemma:

Lemma. *Let $\mathfrak{g}$ be a graded Lie algebra over a principal ideal domain K such that each homogeneous component is a finitely generated free K-module and such that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is also K-free. If the homology group $H_2(\mathfrak{g}, K/\mathfrak{m}) = 0$ for each maximal ideal $\mathfrak{m}$ of K then $H_2(\mathfrak{g}, K) = 0$.*

Proof. We choose a presentation $\mathfrak{g} = L/\mathfrak{r}$ together with a grading of the free Lie algebra L such that $L/[L, L]$ is isomorphic to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ as graded modules. Using the Hochschild-Serre spectral sequence for the extension $\mathfrak{g} = L/\mathfrak{r}$:

$$H_2(L, K) \to H_2(\mathfrak{g}, K) \to H_0(\mathfrak{g}, H_1(\mathfrak{r}, K)) \to H_1(L, K) \to H_1(\mathfrak{g}, K) \to 0,$$

cf. [3, p. 351], and the fact that $H_2(L, K) = 0$, $H_1(\mathfrak{g}, K) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ and

$$H_0(\mathfrak{g}, H_1(\mathfrak{r}, K)) = \mathfrak{r}/[\mathfrak{r}, L],$$

we obtain that $H_2(\mathfrak{g}, K)$ is isomorphic to $\mathfrak{r}/[\mathfrak{r}, L]$. Since $H_2(\mathfrak{g}, K/\mathfrak{m})$ is isomorphic to $(L/[\mathfrak{r}, L]) \otimes (K/\mathfrak{m})$ we obtain that $(\mathfrak{r}/[\mathfrak{r}, L]) \otimes (K/\mathfrak{m}) = 0$ for all maximal ideals $\mathfrak{m}$ of K and hence that $\mathfrak{r}/[\mathfrak{r}, L] = 0$ since each homogeneous component of $\mathfrak{r}/[\mathfrak{r}, L]$ is a finitely generated K-module. □

The following is a corollary of the proof.

Corollary. *Let K be an arbitrary commutative ring and let r be a homogeneous element of $L(X)$. Suppose that for some n we have $r \in L(X_n)$, $r \notin [L(X_n), L(X_n)]$ and that the largest linear term of r in $L(X_n)$ has a coefficient which is a unit of K. Then the elements*

$$\mathrm{ad}(h_n)^{i_n} \cdots \mathrm{ad}(h_2)^{i_2}\mathrm{ad}(h_1)^{i_1}(r) \quad (i_1, \dots i_n \geq 0)$$

are part of a basis of $L(X_n)$. In particular, Theorem 1 holds in this case.

Proof of Theorem 2

We may assume that $X = \{x_1, \dots, x_n\}$. Let

$$r = r_1 + r_2 + \cdots + r_d$$

be the decomposition of r into its homogeneous components r_i of degree i for the natural grading of L.

Lemma. *If K is a field and the center of $\mathfrak{g} = L/(r)$ is nonzero then so is the center of $L/(r_d)$.*

Proof. Let π be an indeterminate over K and let L' (resp. L'') be $L \otimes_K K[\pi]$ (resp. $L \otimes_K K(\pi)$). Similarly define $\mathfrak{g}'$, $\mathfrak{g}''$. If ψ is the automorphism of L'' defined by $\psi(x_i) = x_i/\pi$, let $s = \pi^d \psi(r)$. Then

$$s = r_d + \pi r_{d-1} + \cdots + \pi^{d-1} r_1$$

with $s, t \in L'$ and

$$\mathfrak{g}'' \cong (L'/(s)) \otimes_{K[\pi]} K(\pi).$$

By [4, Theorem 3], $L'/(s)$ is a free $k[\pi]$ module. Now suppose that the center of $\mathfrak{g}$ is nonzero. Then the center of $\mathfrak{g}_\pi = L'/(s)$ is nonzero and hence so is the center of $\mathfrak{g}_\pi/\pi\mathfrak{g}_\pi \cong L/(r_d)$. □

Suppose K is a field and that the center of $\mathfrak{g}$ is not trivial. Then, by the above Lemma, the center of $L/(r_d)$ is not zero. If $d = 1$ we must have Card(X) $= 2$. Suppose that either Card(X) > 2 or Card(X) $= 2$, $d > 2$ and let z be a nonzero homogeneous element of the center of $L/(r_d)$. We first show that the degree of this element must be $\geq d$. Let x_1, x_2 be the first two elements of X. Since $[x_1, z] \in \mathfrak{r}$ the degree of z is $\geq d - 1$. If the degree of z is $d - 1$ then $[x_i, z] = a_i r$ with $a_i \in K$, $a_i \neq 0$. Hence $[a_2 x_1, z] = [a_1 x_2, z]$ which gives $[z, a_2 x_1 - a_1 x_2] = 0$. This implies that $z = a x_1 + b x_2$ and $r = c[x_1, x_2]$. But then, by hypothesis, we must have Card(X) > 2. If x_3 is the third element of X_3 the above argument yields $r = d[x_1, x_3]$ which is a contradiction. Hence the degree of z is at least d which implies, by Theorem 1, that $z \in \mathfrak{h}$ and hence that $z = 0$ since the rank of $\mathfrak{h}$ is at least 2. This contradiction means that we must have Card(X) $= 2$ and $d = 1$ or $d = 2$. If $d = 1$ we are done so we may assume $d = 2$. If $r_1 = 0$ we are again done, so we may assume $r = a x_1 + b x_2 + c[x_1, x_2]$ with $c \neq 0$ and one of a, b nonzero. After a linear change of variables we have $r = x_1 + [x_1, x_2]$. But, in this case, the center of $\mathfrak{g}$ is zero; so this case does not arise.

We now consider the case where K is a principal ideal domain and r is homogeneous. Without loss of generality, we may assume that $d > 2$ or Card(X) > 2. Suppose first that r is not a proper multiple of another element of L.

Then, by [4, Corollary to Theorem 2], $\mathfrak{g} = L/(r)$ is a free K-module. If K' is the quotient field of K, the center of $\mathfrak{g}$ is zero iff the center of $\mathfrak{g} \otimes_K K'$ is zero which gives Theorem 2.

If r is a proper multiple of another element of L, we can write $r = cs$ with c a nonunit of K and s an element of L which is not a proper multiple. Let $\mathfrak{s} = (s)$. Then $(r) = c\mathfrak{s}$ and $\mathfrak{g} = L/c\mathfrak{s}$. Since $L/\mathfrak{s}$ is a free K-module by [4, Corollary to Theorem 2], we obtain that $\mathfrak{s}$ is a free Lie-algebra over K, cf. [9, Satz 5]. Hence $\mathfrak{s}/c\mathfrak{s}$ is a free Lie algebra over K/Kc. Since this Lie algebra is of infinite rank, its center is zero, and since the center of $L/\mathfrak{s}$ is zero by the first part we see that the center of $\mathfrak{g}$ must also be zero. □

Application to pro-p-groups

Let X be a well-ordered set together with a locally finite weight function ω which is compatible with the ordering of X. We may therefore assume $X = \{x_1, x_2, ...\}$ with $x_i < x_j$ if $i < j$. Let $F = F(X)$ be the free pro-p-group on the set X, cf. [7, §1.5]. Let H be a weighted Hall set with respect to X, ω having the dominance property and let the sets $X_n \subset M(X)$ $(n \geq 0)$ be defined as in the proof of Theorem 1. Using the embedding of $M(X)$ into F obtained by means of the operation $[x, y] = x^{-1}y^{-1}xy = x^{-1}y^x$ on F, we identify X_n with its image in F. We also define $\operatorname{ad}(x)(y) = [x, y]$. The following result is the elimination theorem for pro-p-groups:

Proposition 7. *Let $f : F(X) \to F(T)$ be the homomorhism of pro-p-groups defined by $f(x) = x$ for $x \in T$ and $f(x) = 1$ for $x \in X - T$. Then* $\ker(f)$ *is a free pro-p-group with basis the set X_T consisting of elements*

$$x_{i,j} = \operatorname{ad}(x_{j_1})\operatorname{ad}(x_{j_2}) \cdots \operatorname{ad}(x_{j_k})(x_i)$$

with $x_{j_1}, x_{j_2}, \ldots, x_{j_k} \in T$ $(k \geq 0)$ and $x_i \in X - T$.

Proof. For the natural right action of F on $N = \ker(f)$ we have

$$x_{ij}^{x_\ell} = x_{ij} x_{i,j'}^{-1}$$

with $j' = (\ell, x_{j_1}, \ldots, x_{j_k})$ if $j = (x_{j_1}, \ldots, x_{j_k})$. This also defines a right action of $F(T)$ on $F(X_T))$. Let h_0 be the homomorphism of $F(X_T)$ into N induced by the identity map on X_T and let S be the semidirect product of $F(X_T)$ by $F(T)$. Since h_0 is compatible with the actions of $F(T)$ on $F(X_T)$ and N it extends to a map h of S into $F(X)$. If g is the homomorphism of $F(X)$ into S with $g(x) = (1, x)$ for $x \in X - T$ and $g(x) = (x, 1)$ for $x \in T$ then $hg = 1$ and $gh = 1$. □

Corollary. *If f is the homomorphism of $F(X)$ into $F(x_1)$ defined by $f(x_1) = x_1$ and $f(x) = 1$ for $x \in X$, $x \neq x_1$, then* $\ker(f) = F(X_1)$.

If F_n is the closed subgroup of F generated by X_n then F_n is a normal subgroup of F and is a free pro-p-group with basis X_n. The weight function ω defines a filtration of F and induces one on F_n (cf. [5]) so that, if $gr(F_n)$ is the associated Lie algebra, we have $gr(F_n) = L(X_n)$ over $\mathbb{Z}_p$.

Theorem 5. *Let $r \in F$ and suppose that $r \in F_n$, $r \notin F_n^p[F_n, F_n]$. Let G_i be the image of F_i in G. Then G_n is a normal free pro-p-subgroup of G*

and $\Gamma_i = G_{i-1}/G_i \cong \mathbb{Z}_p$ *for* $0 < i \leq n$. *Moreover,* $R/[R, R]$ *is a free* $\mathbb{Z}_p[[G]]$-*module of rank 1, where* $\mathbb{Z}_p[[G]]$ *is the completed* $\mathbb{Z}_p$-*algebra of* G. *In particular,* G *is of cohomological dimension* ≤ 2.

Proof. For $i > 0$ let g_i be the image of h_i in Γ_{i-1}, where h_i is the smallest element of X_{i-1}. Then g_i is a generator of Γ_i. Let $\gamma_i = 1 - g_i$ in $\mathbb{Z}_p[[G]]$. For $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$ let

$$r_i = \mathrm{ad}(h_n)^{i_n} \cdots \mathrm{ad}(h_2)^{i_2}\mathrm{ad}(h_1)^{i_1}(r)$$

and let ρ (resp. ρ_i) be the image of r (resp. r_i) in $R/[R, R]$. The elements r_i generate R as a closed normal subgroup of F_n and

$$\rho_i = \rho \cdot (\gamma_1^{i_1}\gamma_2^{i_2} \cdots \gamma_n^{i_n})$$

for the natural right action of $\mathbb{Z}_p[[G]]$ on $R/[R, R]$. We now show that the images of the ρ_i in the free commutative pro-p-group $F_n/[F_n, F_n]$ are part of a basis. To see this let $\overline{\rho_i}$ (resp. $\overline{\rho}$) be the initial forms of the images of ρ_i (resp. ρ) in $F_n/F_n^p[F_n, F_n]$, with respect to the filtration defined by ω. It suffices to show that the $\overline{\rho_i}$ can be completed to a homogeneous basis of $gr(F_n/F_n^p[F_n, F_n]$. But this follows from the Corollary to Theorem 1 with $K = \mathbb{F}_p$ and $r = \overline{\rho}$; note that in $L(X) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z}$

$$\overline{\rho_i} = \mathrm{ad}(h_n)^{i_n} \cdots \mathrm{ad}(h_1)^{i_1}(\overline{\rho}).$$

If S is the set of the elements r_i, we obtain that S is part of a basis Y of the free pro-p-group F_n. This yields $G_n = F(T)$ with $T = Y - S$. If $I = \mathbb{N}^n$ then, by the elimination theorem, the mapping which sends the element $u = (u_i)_{i\in I} \in Z_p[[G_n]]^I$ to the element $\sum \rho_i \cdot u_i \in R/[R, R]$ is an isomorphism of $\mathbb{Z}_p[[G_n]]$-modules. But every element of $\mathbb{Z}_p[[G]]$ can be uniquely written in the form

$$\sum \gamma_1^{i_1}\gamma_2^{i_2} \cdots \gamma_n^{i_n} u_{i_1 i_2 \ldots i_n}$$

with $u_{i_1 i_2 \ldots i_n} \in \mathbb{Z}_p[[G_n]]$. This implies that $R/[R, R]$ is a free $\mathbb{Z}_p[[G]]$-module. By [4, Proposition 1], we obtain that G is of cohomological dimension ≤ 2. □

As an example, cd(G) $= 2$ if r is the relator

$$[x_1, [x_1, x_2]]^p \mathrm{ad}([x_1, x_2])^m([x_1, [x_1, x_2]]).$$

This follows from the fact the hypothesis of Theorem 5 is satisfied with $n = 3$. The best one could do for this relator, using the results of [4], was to prove that cd(G) $= 2$ for $p > \frac{2}{3}m + 1$.

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