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ON THE TANGENTIAL INTERPOLATION PROBLEM FOR H_2 FUNCTIONS

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ABSTRACT. The aim of this paper is to solve a matrix-valued version of the Nevanlinna-Pick interpolation problem for H_2 functions. We reduce this problem to a Nevanlinna-Pick interpolation problem for Schur functions and obtain a linear fractional transformation which describes the set of all solutions.

1. INTRODUCTION

The classical Nevanlinna-Pick interpolation problem for functions in the Hardy space of the unit disc $\mathbb{D}$ consists of the following: *given* $w_1, \dots, w_n \in \mathbb{D}$ *and* $f_1, \dots, f_n \in \mathbb{C}$, *describe the set of all functions* $f \in H_2$ *such that* (1) $\|f\|_2 \leq 1$ and (2) $f(w_i) = f_i$, $i = 1, \dots, n$. As is well known [9], [15, p. 345], a necessary and sufficient condition for a solution to exist is the nonnegativity of the matrix P with ij entry $P_{ij} = \frac{1}{1-w_i w_j^*} - f_i f_j^*$. This problem can be solved using reproducing kernels methods as in e.g. the book of Meschkowski [16]. In the matrix-valued case the situation is more involved. In this paper we solve the Nevanlinna-Pick interpolation problem in the vector Hardy classes $H_2^{p\times q}$. We denote by $\mathbb{C}^{p\times q}$ the space of all $p \times q$ matrices with complex entries, and I_p stands for the identity matrix in $\mathbb{C}^{p\times p}$ and define $H_2^{p\times q}$ to be the Hilbert space of $\mathbb{C}^{p\times q}$-valued functions with H_2 entries with inner product

$$\langle f, g\rangle_{H_2^{p\times q}} = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}(g(e^{it})^* f(e^{it}))\, dt.$$

We endow $H_2^{p\times q}$ with the matrix-valued Hermitian form

$$[f, g] = \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})^* f(e^{it})\, dt$$

and note that

$$[f, f] = \sum_{k=0}^{\infty} f_k^* f_k$$

where $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $f_k \in \mathbb{C}^{p\times q}$.

In the scalar case $\langle f, f\rangle_{H_2} = [f, f]$, but in the matrix case two constraints are possible on f, namely $\langle f, f\rangle_{H_2^{p\times q}} \leq 1$ or $[f, f] \leq I_q$. These are not equivalent in general.

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Definition 1.1. We denote by $H_2^{p\times q}(I_q)$ the set of all functions $f \in H_2^{p\times q}$ such that

$$[f, f] \leq I_q.$$

A $\mathbb{C}^{p\times q}$-valued function analytic in $\mathbb{D}$ is a Schur function if and only if the operator of multiplication by this function is a contraction from H_2^q into H_2^p. If we replace H_2^q by $\mathbb{C}^q$, we will see in §2 that the operator of multiplication by f is a contraction from $\mathbb{C}^q$ into H_2^p if and only if $[f, f] \leq I_q$. Thus, the class $H_2^{p\times q}(I_q)$ can be viewed as an analogue of Schur functions in the framework of $H_2^{p\times q}$.

We introduce the following tangential interpolation problem $\mathrm{Int}(H_2^{p\times q}(I_q))$: *Given integers $r_i \in \mathbb{N}$, given matrices $a_i \in \mathbb{C}^{r_i\times p}$, $c_i \in \mathbb{C}^{r_i\times q}$, and given points w_i in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ $(i = 1, \dots, n)$ find necessary and sufficient conditions for a function $f \in H_2^{p\times q}(I_q)$ to exist such that*

$$a_i f(w_i) = c_i \qquad (i = 1, \dots, n) \tag{1.1}$$

and describe the set of all such functions when these conditions are met.

Note that the analogous tangential Nevanlinna-Pick problem $\mathrm{Int}(\mathcal{S}^{p\times q})$ in the Schur class $\mathcal{S}^{p\times q}$ of $\mathbb{C}^{p\times q}$-valued functions analytic and contractive $(s(z)s(z)^* \leq I_P)$ in $\mathbb{D}$ has been much studied (see e.g. [11], [10],[7]). Using well-known facts about $\mathrm{Int}(\mathcal{S}^{p\times q})$ we prove in §3 the two following theorems. The first theorem generalizes to the vector case the scalar criteria mentioned above.

Theorem 1.2. *The problem $\mathrm{Int}(H_2^{p\times q}(I_q))$ is solvable if and only if the block matrix*

$$K = \left(\frac{a_i a_j^*}{1 - w_i w_j^*} - c_i c_j^*\right)_{i,j=1}^{n} \tag{1.2}$$

is nonnegative.

The set of all solutions is described in the next theorem:

Theorem 1.3. *Let the matrix K given by (1.2) be strictly positive. Then, the set of all solutions f of the problem $\mathrm{Int}(H_2^{p\times q}(I_q))$ is parametrized by the linear fractional transformation*

$$f(z) = (\psi_{11}(z)\sigma(z) + \psi_{12}(z))(\psi_{21}(z)\sigma(z) + \psi_{22}(z))^{-1} \tag{1.3}$$

where the $\mathbb{C}^{(p+q)\times(p+2q)}$-valued function

$$\Psi(z) = \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix} \tag{1.4}$$

($\psi_{11}(z)$ and $\psi_{22}(z)$ are $\mathbb{C}^{p\times(p+q)}$-valued and $\mathbb{C}^{q\times q}$-valued respectively) is built from the interpolation data; it is rational and such that

$$\Psi(z)\begin{pmatrix} I_{p+q} & 0 \\ 0 & -I_q \end{pmatrix}\Psi(z)^* \leq \begin{pmatrix} I_p & 0 \\ 0 & (|z|^2 - 1)I_q \end{pmatrix} \tag{1.5}$$

for $z \in \mathbb{D}$ (with equality for z on the unit circle $\mathbb{T}$), and where the parameter σ varies in the Schur class $\mathcal{S}^{(p+q)\times q}$.

An explicit formula for $\Psi(z)$ in terms of the interpolation data is given in §3. There we also consider the degenerate case $\det K = 0$.

As was pointed out by the referee, Theorems 1.2 and 1.3 allow to solve the left interpolation problem with the constraint $\langle f, f\rangle_{H_2^{p\times q}} \leq 1$. It suffices to replace f by the $\mathbb{C}^{pq}$-valued function made from the columns of f. The referee also pointed out that the interpolation problem $\mathrm{Int}(H_2^{p\times q}(I_q))$ is a particular case of the following problem in the sense of Sarason [21]: *Given* $f_0 \in H_2^{p\times q}$ *and* θ *a* $p\times p$ *inner function, describe the elements of the set* $f_0 + \theta H_2^{p\times q}$ *for which* $[f, f] \leq I_q$. This problem can in turn be solved using the methods developed in [8].

One could think of more general interpolation problems, where interpolation conditions are also imposed form the right. The solutions of such problems have explicit description in the class of Schur functions in terms of linear fractional transformations (see [8], [7]). Explicit descriptions as (1.3) do not seem to be available in the class $H_2^{p\times q}(I_p)$. The reason is as follows: a right interpolation condition can be translated into a left interpolation condition for $\tilde{f}(z) = f(z^*)^*$. But, as will be discussed in the next section, $\tilde{f}$ need not be in $H_2^{q\times p}(I_p)$ when $f \in H_2^{p\times q}(I_q)$. This does not happen for Schur functions: if S is a Schur function so is $\widetilde{S}$.

In §4, we set the problem $\mathrm{Int}(H_2^{p\times q}(I_q))$ in the framework of a general interpolation problem in reproducing kernel Hilbert spaces. In §5, we explain how the interpolation problem $\mathrm{Int}(H_2^{p\times q})(I_p)$ can be considered in a nonpositive framework.

2. Characterization of the class $H_2^{p\times q}(I_q)$

In this section we give two characterizations of the class $H_2^{p\times q}(I_q)$: one in terms of a nonnegative kernel, the second one in terms of Schur functions. We first recall that a $\mathbb{C}^{p\times p}$-valued function $K(z, w)$ defined on $\Omega\times\Omega$ ($\Omega\subset\mathbb{C}$) is called nonnegative if $K(z, w) = K(w, z)^*$ and if, furthermore, for every choice of integer m and of $w_1, \dots, w_m \in \Omega$, the Hermitian block-matrix with ij block $K(w_i, w_j)$ is nonnegative. A $\mathbb{C}^{p\times q}$-valued function analytic in $\mathbb{D}$ is a Schur function if and only if, as already mentioned, the operator M_S of multiplication by S is a contraction from H_2^q into H_2^p, or equivalently, if and only if the function $H_S(z, w) = \frac{I_p - S(z)S(w)^*}{1 - zw^*}$ is nonnegative in $\mathbb{D}$; see, e.g. [10] for a proof of these facts. The analogue here is:

Theorem 2.1. *Let* f *be a* $\mathbb{C}^{p\times q}$*-valued function analytic in* $\mathbb{D}$. *Then* f *belongs to* $H_2^{p\times q}(I_q)$ *if and only if the operator of multiplication by* f *is a contraction operator from* $\mathbb{C}^q$ *into* H_2^p, *or equivalently, if and only if the kernel*

$$K_f(z, w) = \frac{I_p}{1 - zw^*} - f(z)f(w)^* \tag{2.1}$$

is nonnegative.

Proof. Let f be in $H_2^{p\times q}(I_q)$. We define a linear transformation $M_f : \mathbb{C}^q \to H_2^{p\times 1}$ by

$$M_f v = f(z)v \qquad (v \in \mathbb{C}^q). \tag{2.2}$$

Then, with $x(z)=\sum_{i=1}^m \frac{\alpha_i}{1-z\nu_i^*}$ $(\alpha_i\in\mathbb{C}^{p\times 1}\,,\ \nu_i\in\mathbb{D})$,

$$M_f^*x=\sum_{i=1}^m f(\nu_i)^*\alpha_i$$

and

$$\|f(z)v\|^2_{H_2^{p\times 1}}=v^*\left(\sum_{i=0}^\infty f_i^*f_i\right)v.$$

Since f belongs to $H_2^{p\times q}(I_q)$,

$$\|f(z)v\|^2_{H_2^{p\times 1}}\le v^*v=\|v\|^2_{\mathbb{C}^q}$$

and so $\|M_f\|=\|M_f^*\|\le 1$. Therefore, $I-M_fM_f^*\ge 0$ and a straightforward computation shows that

$$\langle (I-M_fM_f^*)x\,,\,x\rangle_{H_2^{p\times 1}}=\sum_{i\,,j=1}^m \alpha_i^*\left(\frac{I_p}{1-\nu_i\nu_j^*}-f(\nu_i)f(\nu_j)^*\right)\alpha_j$$

for x as above and implies the nonnegativity of $K_f(z\,,w)$.

To prove the converse, let us introduce a linear densely defined transformation $T:H_2^{p\times 1}\to\mathbb{C}^q$ by

$$T\frac{\alpha}{1-zw^*}=f(w)^*\alpha\qquad (\alpha\in\mathbb{C}^q\,,\ w\in\mathbb{D}).$$

It is readily checked that T is a well-defined contraction. Hence, it has a unique extension which is a contraction (and is still denoted by T) to all of $H_2^{p\times 1}$. Furthermore, $T^*v=f(z)v$ for all $v\in\mathbb{C}^q$ and in view of (2.2), $T^*=M_f$. Since $\|M_f\|=\|T^*\|\le 1$, the same computation as above leads to

$$\sum_{i=0}^\infty f_i^*f_i\le I_q$$

and therefore, $f\in H_2^{p\times q}(I_q)$. □

We note a number of differences between functions in $\mathcal{S}^{p\times q}$ and in $H_2^{p\times q}(I_q)$. For S a Schur function, the kernel $H_S(z\,,w)$ is nonnegative if and only if for every $z\in\mathbb{D}$, the matrix $H_S(z\,,z)$ is nonnegative. Furthermore, H_S is nonnegative in $\mathbb{D}$ if and only if $H_{\tilde{S}}$ is nonnegative in $\mathbb{D}$; see [4]. The kernels K_f do not share these properties: the example $f(z)=(1-z^2)^{-1/2}$ shows that $K_f(z\,,z)$ can be nonnegative for every $z\in\mathbb{D}$ while the kernel $K_f(z\,,w)$ is not nonnegative in $\mathbb{D}$. One can also give examples of functions f for which K_f is nonnegative while $K_{\tilde{f}}$ is not. Take for instance $p=1$, $q=2$, and $f(z)=(1\,,z)$.

Before turning to the next characterization we recall that $\mathbb{C}^{p\times q}$-valued function g is of bounded type if it can be expressed in the form $g(z)=\frac{\hat{g}(z)}{d(z)}$ where $\hat{g}(z)\in H_\infty^{p\times q}$ and $d(z)\in H_\infty$. Note also that every $H_2^{p\times q}(I_q)$ function is of bounded type.

The next theorem was proved in the scalar case and for $\|f\|_2=1$ by D. Sarason in [25, p. 500] (see also [22], [23], [24]). Sarason's proof is based on the Herglotz's representation formula for functions with real positive part in $\mathbb{D}$ and does not seem to extend to the general case $[f, f] \leq I_q$. The method presented here is based on the notion of positivity of kernels and allows extensions to the nonpositive case, as illustrated in the last section.

Theorem 2.2. *Let f be a $\mathbb{C}^{p\times q}$-valued function analytic in $\mathbb{D}$. Then f belongs to $H_2^{p\times q}(I_q)$ if and only if it can be written as*

$$f(z) = s_1(z)(I_q - zs_2(z))^{-1} \tag{2.3}$$

for some Schur function

$$S(z) = \begin{pmatrix} s_1(z) \\ s_2(z) \end{pmatrix} \in \mathcal{S}^{(p+q)\times q}. \tag{2.4}$$

Proof. Let f admit a representation (2.3) with a Schur function S defined by (2.4). Writing

$$A(z) = (I_p \ \ zf(z)) \tag{2.5}$$

and taking into account (2.1), (2.3) we have

$$K_f(z, w) = A(z)\frac{I_{p+q} - S(z)S(w)^*}{1 - zw^*}A(w)^*.$$

Since S is of the Schur class, the kernel $K_f(z, w)$ is nonnegative (see e.g. [10]) and by Theorem 2.1, $f \in H_2^{p\times q}(I_q)$. Conversely, let f be in $H_2^{p\times q}(I_q)$. By Theorem 2.1 the kernel $K_f(z, w)$ given by (2.1) is nonnegative in $\mathbb{D}$. Substituting (2.5) into (2.1) we obtain

$$K_f(z, w) = \frac{A(z)A(w)^* - f(z)f(w)^*}{1 - zw^*} \geq 0. \tag{2.6}$$

Since A and f are of bounded type, by a result of R. Leech and M. Rosenblum, it follows from (2.6) that

$$f(z) = A(z)S(z) \tag{2.7}$$

for some $S \in \mathcal{S}^{(p+q)\times q}$ (see [2], [19, p. 107]; the existence of S is a consequence of a version of the commutant lifting theorem due to Rosenblum [18]). Substituting (2.5) into (2.7) we obtain

$$(I_p \ \ zf(z))S(z) = f(z) \tag{2.8}$$

or, equivalently, $s_1(z)+zf(z)s_2(z) = f(z)$ which immediately implies (2.3). □

We note that a given f may have a number of different representations of the form (2.3), as is illustrated by the example $f(z) = \frac{1}{3-z}$, which corresponds to the two choices $s_1(z) = \frac{1}{3}$, $s_2(z) = -\frac{1}{3}$ and $s_1(z) = f(z)$, $s_2(z) = 0$.

3. Interpolation problem in $H_2^{p\times q}(I_q)$

Using Theorem 2.2 we reduce the initial problem $\mathrm{Int}(H_2^{p\times q}(I_q))$ (see Introduction) to an interpolation problem $\mathrm{Int}(\mathcal{S}^{(p+q)\times q})$ in the Schur class $\mathcal{S}^{(p+q)\times q}$.

Lemma 3.1. *Let h and S be the functions defined by* (2.3), (2.4) *which belong to $H_2^{p\times q}(I_q)$ and $\mathcal{S}^{(p+q)\times q}$ respectively. Then, h satisfies* (1.1) *if and only if S satisfies the following interpolation conditions*

$$(a_i\ w_ic_i)S(w_i) = c_i \qquad (i = 1, \dots, n). \tag{3.1}$$

Proof. Let h satisfy (1.1). Multiplying (2.8) by a_i on the left and setting into the obtained equality $z = w_i$ we get (3.1), thanks to (1.1).

Conversely, let S satisfy (3.1). Substituting the decomposition (2.4) into (3.1) we obtain

$$a_is_1(w_i) + w_ic_is_2(w_i) = c_i$$

or, equivalently,

$$a_is_1(w_i)(I_q - w_is_2(w_i))^{-1} = c_i \qquad (i = 1, \dots, n)$$

which in view of (2.3), coincides with (1.1). □

As is well known [10], there exists $S \in \mathcal{S}^{(p+q)\times q}$ satisfying the interpolation conditions (3.1) if and only if the matrix

$$K = \left(\frac{(a_i\ w_ic_i)(a_j\ w_jc_j)^* - c_ic_j^*}{1 - w_iw_j^*}\right)_{i,j=1}^{n}$$

is nonnegative. This matrix can be rewritten in the form (1.2). Now Theorem 1.1 is an immediate consequence of Lemma 3.1.

When $K > 0$, the set of all functions S satisfying (3.1) is parametrized by the linear fractional transformation

$$S(z) = (\theta_{11}(z)\sigma(z) + \theta_{12}(z))(\theta_{21}(z)\sigma(z) + \theta_{22}(z))^{-1} \tag{3.2}$$

where the resolvent matrix $\Theta = (\theta)_{ij}$ is given by

$$\Theta(z) = I_{p+2q} + (z-1)M^*(I - zW^*)^{-1}K^{-1}(I - W)^{-1}MJ, \tag{3.3}$$

with

$$W = \operatorname{diag}(w_iI_{r_i})_{i=1}^{n}, \qquad J = \begin{pmatrix} I_{p+q} & 0 \\ 0 & -I_q \end{pmatrix}, \tag{3.4}$$

$$M = \begin{pmatrix} a_1 & w_1c_1 & c_1 \\ \vdots & \vdots & \vdots \\ a_n & w_nc_n & c_n \end{pmatrix} \tag{3.5}$$

and the parameter $\sigma(z)$ varying in $\mathcal{S}^{(p+q)\times q}$ (see [12], [10]).

Using the identity

$$K - WKW^* = MJM^* \tag{3.6}$$

we easily obtain that

$$\Theta(z)J\Theta(z)^* - J = (|z|^2 - 1)M^*(I - zW^*)^{-1}K^{-1}(I - z^*W)^{-1}M \tag{3.7}$$

and hence, $\Theta(z)$ is J-inner in $\mathbb{D}$, i.e.,

$$\Theta(z)J\Theta(z)^* \le J \quad (z \in \mathbb{D}), \qquad \Theta(z)J\Theta(z)^* = J \quad (z \in \mathbb{T}).$$

It follows from (2.4) that

$$s_1(z) = (I_p\ O_{p\times q})S(z), \qquad s_2(z) = (O_{q\times p}\ I_q)S(z). \tag{3.8}$$

Substituting (3.2) into (3.8) and (3.8) into (2.3) we obtain

(3.9)
$$\begin{aligned} h(z) &= (I_p\ O_{p\times q})(\theta_{11}(z)\sigma(z)+\theta_{12}(z)) \\ &\quad \times ((\theta_{21}(z) - z(O_{q\times p}\ I_q)\theta_{11}(z))\sigma(z) + \theta_{22}(z) - z(O_{q\times p}\ I_q)\theta_{12}(z))^{-1} \\ &= (\psi_{11}(z)\sigma(z)+\psi_{12}(z))(\psi_{21}(z)\sigma(z)+\psi_{22}(z))^{-1}, \end{aligned}$$

where

$$\Psi(z) = \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -zI_q & I_q \end{pmatrix} \Theta(z). \tag{3.10}$$

Substituting (3.3) into (3.10) we obtain

$$\begin{aligned} \Psi(z) &= \begin{pmatrix} I_p & 0 & 0 \\ 0 & -zI_q & I_q \end{pmatrix} \\ &\quad + (z-1)\begin{pmatrix} \frac{a_1^*}{1-zw_1^*} & \cdots & \frac{a_n^*}{1-zw_n^*} \\ c_1^* & \cdots & c_n^* \end{pmatrix} K^{-1}(I-W)^{-1}MJ \end{aligned} \tag{3.11}$$

which in view of (3.9) is an explicit formula for the resolvent matrix appearing in Theorem 1.3. It follows from (3.10), (3.11) that ψ_{12} and ψ_{12} are rational functions and ψ_{21}, ψ_{22} are linear ones. Substituting (3.10) into (3.7) we get (1.5) which ends the proof of Theorem 1.3.

We now suppose that K is degenerate ($\operatorname{rank} K = r < \sum_{i=1}^{n} r_i = l$). Let $e_{i_1}, \ldots, e_{i_r}$ be vectors from the canonical basis of $\mathbb{C}^l$ such that

$$\operatorname{Lin}\{e_{i_j},\ j=1,\ldots,r\} \cap \ker K = \{0\} \tag{3.12}$$

where Lin stands for linear span and $\ker K = \{c \in \mathbb{C}^{1\times l} : cK = 0\}$. Let Q be the element of $\mathbb{C}^{r\times l}$ defined by

$$Q = \begin{pmatrix} e_{i_1} \\ \vdots \\ e_{i_r} \end{pmatrix}. \tag{3.13}$$

In view of (3.12), (3.13), $\operatorname{rank} QKQ^* = \operatorname{rank} K = r$, $QKQ^* > 0$, and therefore, the pseudoinverse matrix

$$K^{[-1]} = Q^*(QKQ^*)^{-1}Q \in \mathbb{C}^{l\times l} \tag{3.14}$$

is well defined. Moreover (see [1]), the set of all functions $S \in \mathcal{S}^{(p+q)\times q}$ satisfying (3.1) is parametrized by the linear fractional transformation (3.2) with the resolvent matrix

$$\Theta(z) = I_{p+2q} + (z-1)M^*(I - zW^*)^{-1}K^{[-1]}(I-W)^{-1}MJ \tag{3.15}$$

with W, J, M, and $K^{[-1]}$ defined by (3.4), (3.5), (3.14), and parameter $\sigma(z)$ of the form

$$\sigma(z) = U\begin{pmatrix} I_\mu & 0 \\ 0 & \hat{\sigma}(z) \end{pmatrix} V \tag{3.16}$$

with fixed unitary matrices $U \in \mathbb{C}^{(p+q)\times(p+q)}$, $V \in \mathbb{C}^{q\times q}$ depending only on the interpolation data and

$$\mu = \operatorname{rank} P_{\ker K}(I-W)^{-1}M \tag{3.17}$$

($P_{\ker K}$ stands for the orthogonal projection onto the subspace $\ker K$), and $\hat{\sigma}(z)$ varying in the Schur class $\mathcal{S}^{(p+q-\mu)\times(q-\mu)}$.

Remark 3.2. It follows from the identity (3.6) that

$$\begin{aligned}(I-W)^{[-1]}MJM^*(I-W^*)^{-1} \\ &= (I-W)^{-1}K(I-W^*)^{-1} - ((I-W)^{-1}-I)K((I-W^*)^{-1}-I) \\ &= (I-W)^{-1}K + K(I-W^*)^{-1} - K\end{aligned}$$

which implies

$$P_{\ker K}(I-W)^{-1}MJM^*(I-W)^*)^{-1}P_{\ker K} = 0.$$

Therefore the subspace $\mathcal{G} = \operatorname{Ran}(P_{\ker K}(I-W)^{-1}M)$ is J-neutral in $\mathbb{C}^{1\times(p+2q)}$ and so (see [6]) (3.17) implies

$$\dim \mathcal{G} = \operatorname{rank} P_{\ker K}(I-W)^{-1}M = \mu \leq q.$$

Substituting (3.15) into (3.10) we obtain the following analogue of Theorem 1.3:

Theorem 3.3. *Let the matrix K defined by* (1.2) *be nonnegative,* $\operatorname{rank} K = r$; *and let Q be the matrix defined by* (3.12), (3.13). *Then the set of all the solutions of the problem* $\operatorname{Int}(H_2^{p\times q}(I_q))$ *is parametrized by the linear fractional transformation* (1.3) *with the resolvent matrix*

$$\begin{aligned}\Psi(z) = &\begin{pmatrix} I_p & 0 & 0 \\ 0 & -zI_q & I_q \end{pmatrix} \\ &+ (z-1)\begin{pmatrix} \dfrac{a_1^*}{1-zw_1^*} & \cdots & \dfrac{a_n^*}{1-zw_n^*} \\ c_1^* & \cdots & c_n^* \end{pmatrix} K^{[-1]}(I-W)^{-1}MJ\end{aligned}$$

(*with M, J, W, and $K^{[-1]}$ given by* (3.4), (3.5), (3.14)) *and the parameter $\sigma(z)$ of the form* (3.16).

4. A general interpolation problem

The referee suggested an alternative way to solve the interpolation problem $\operatorname{Int}(H_2^{p\times q}(I_q))$, using extensions of operators. This leads us to a problem described here. We first recall the following result: if $K(z, w)$ is a $\mathbb{C}^{p\times p}$-valued nonnegative function on a set Ω, there exists a (uniquely defined) Hilbert space $H(K)$ with reproducing kernel K, i.e., such that:

1. For every $w \in \Omega$ and $c \in \mathbb{C}^p$, the function $z \to K(z, w)c$ belongs to $H(K)$.
2. For every w and c as above and $x \in H(K)$,

$$\langle x, K(\cdot, w)c\rangle_{H(K)} = c^*x(w).$$

We refer to [5], [20] and [26] for further information on these spaces.

The left-sided interpolation problems (with conditions (1.1)) in Schur classes and in the classes $H_2^{p\times q}(I_q)$ are particular cases of the following problem: *Given two functions $K_1(z, w)$ and $K_2(z, w)$, nonnegative for z, w in a set Ω and respectively $\mathbb{C}^{p\times p}$- and $\mathbb{C}^{q\times q}$-valued, find all functions $f\colon \Omega \mapsto \mathbb{C}^{p\times q}$ such that* (1) *the interpolation conditions* (1.1) *hold, and* (2) *the operator of multiplication* *by f is a contraction from $H(K_2)$ into $H(K_1)$.* We will call this problem $I(K_1, K_2)$.

The Schur case corresponds to $\Omega = \mathbb{D}$ and $K_1(z, w) = \frac{I_p}{1-zw^*}$ and $K_2(z, w) = \frac{I_q}{1-zw^*}$ while the case considered in this paper corresponds to $\Omega = \mathbb{D}$ and $K_1(z, w) = \frac{I_p}{1-zw^*}$ and $K_2(z, w) = I_q$. Following the proof of Theorem 2.1 it is not difficult to prove that the multiplication by f is a contraction from $H(K_2)$ into $H(K_1)$ if and only if the function

$$K_1(z, w) - f(z)K_2(z, w)f(w)^*$$

is nonnegative in Ω. Hence a necessary condition for the interpolation problem $I(K_1, K_2)$ to have a solution is that the block matrix

$$K = (a_i K_1(w_i, w_j)a_j^* - c_i K_2(w_i, w_j)c_j^*)_{i,j=1}^n$$

is nonnegative.

This condition is in general not sufficient. Indeed there may be no nonzero functions f for which multiplication by f sends $H(K_2)$ into $H(K_1)$. For instance, take $p = q = 1$ and $K_1(z, w) = 1$ and $K_2(z, w) = \frac{1}{1-zw^*}$. On the other hand, in this case, the choice $a = 1$, $c = \frac{1}{2}$, and $w = 0$ leads to a nonnegative (1×1) matrix K.

Let us suppose that the matrix K is nonnegative, and let us consider the span S of the columns of the functions $z \to K_1(z, w_i)a_i^*$. Define an operator T by

$$T(K_1(\cdot, w_i)a_i^* e) = K(\cdot, w_i)c_i^* e$$

where e spans $\mathbb{C}^{r_i}$. From the positivity of the matrix K follows that T is a well-defined contraction.

Theorem 4.1. *Let us assume that for every point of interpolation w_i, the matrix $K_2(w_i, w_i)$ is strictly positive. Let f be a solution of the interpolation problem $I(K_1, K_2)$. Then, M_f^* is a contractive extension of T. Conversely, let T_e be an extension of T and suppose that T_e is of the form $T_e = M_f^*$ for some function f. Then, f is a solution to $I(K_1, K_2)$.*

Proof. It is readily checked that, for $c \in \mathbb{C}^p$,

$$M_f^* K_1(\cdot, w)c = K_2(\cdot, w)f(w)^* c.$$

Hence M_f^* is an extension of T, contractive since M_f is assumed contractive.

Conversely, let T_e be an extension of T of the assumed form. Then, since $T_e = M_f^*$,

$$\begin{aligned}\langle T_e K_1(\cdot, w_i)a_i^* c, K_2(\cdot, w)d\rangle_{H(K_2)} &= \langle K_1(\cdot, w_i)a_i^* c, f(\cdot)K_2(\cdot, w)d\rangle_{H(K_1)} \\ &= d^* K_2(w, w_i)f(w_i)^* a_i^* c,\end{aligned}$$

where $w \in \Omega$, $c \in \mathbb{C}^{r_i}$, $d \in \mathbb{C}^q$. On the other hand, since T_e extends T,

$$\begin{aligned}\langle T_e K_1(\cdot, w_i)a_i^* c, K_2(\cdot, w)d\rangle_{H(K_2)} &= \langle K_2(\cdot, w_i)c_i^* c, K_2(\cdot, w)d\rangle_{H(K_2)} \\ &= d^* K_2(w, w_i)c_i^* c.\end{aligned}$$

Comparing these two expressions and taking into account that $K_2(w_i, w_i) > 0$, we obtain that f satisfies the interpolation conditions (1.1). □

The set of all contractive extensions of T can be parametrized (see [17]). In general, it seems difficult to decide whether there are extensions of the form M_f^* to the operator T. For the special case considered in the previous sections, as remarked by the referee, all extensions of T are of the form M_f^*. For a related problem we refer to [13].

5. The nonpositive case

The approach presented in this paper suggests extensions to the nonpositive framework. To be more precise, let us first recall that a $\mathbb{C}^{p\times p}$-valued function $K(z, w)$ defined on $\Omega \times \Omega$ has κ negative squares if and only if $K(z, w)^* = K(w, z)^*$ and if furthermore, for every choice of integer m, of points $w_1, \dots, w_m \in \Omega$ and vectors $\xi_1, \dots, \xi_m \in \mathbb{C}^p$, the $m \times m$ Hermitian matrix with ij entry $\xi_i^* K(w_i, w_j)\xi_j$ has at most κ strictly negative eigenvalues and exactly κ such eigenvalues for some choice of $m, w_1, \dots, w_m, \xi_1, \dots, \xi_m$; see [14]. The Schur classes $\mathcal{S}^{p\times q}$ have been extended by M. G. Krein and H. Langer to classes $\mathcal{S}_\kappa^{p\times q}$ by the requirements that S is meromorphic in $\mathbb{D}$ and that the kernel $\frac{I_p - S(z)S(w)^*}{1-zw^*}$ has κ negative squares for z, w in the domain of holomorphy of S. An element in $\mathcal{S}_\kappa^{p\times q}$ can be written as $S = S_0 B^{-1}$, where $S_0 \in \mathcal{S}^{p\times q}$ and where B is a $q \times q$ Blaschke product of degree κ. In particular, elements of these extended Schur classes are of bounded type in $\mathbb{D}$.

Definition 5.1. The $(p \times q)$-valued function f meromorphic in $\mathbb{D}$ is said to be in $H_{2,\kappa}^{p\times q}$ if it is of bounded type and the kernel (2.1) has κ negative squares in the domain of analyticity of f in $\mathbb{D}$.

The hypothesis that f is of bounded type cannot be dispensed with; the fact that the kernel (2.1) has a finite number of negative squares does not imply that f is of bounded type, as is seen by taking $p = q = 1$ and any function f analytic in $\mathbb{D}$ but not of bounded type. The kernel (2.1) has then one negative square (if it was positive, f would be in H_2). The representation theorem, Theorem 2.2, extends to the classes $H_{2,\kappa}^{p\times q}$. In the representation (2.3), S has now κ negative squares, and in the proof, one needs the analogue of the result of Leech and Rosenblum when the kernel (2.6) has now a finite number of negative squares: such an analogue was proved in [3, Theorem 4.6]. Thus:

Theorem 5.2. *Let f be a $\mathbb{C}^{p\times q}$-valued function of bounded type in $\mathbb{D}$. Then the kernel K_f has κ negative squares if and only if $f(z) = s_1(z)(I_q - zs_2(z))^{-1}$ where*

$$S(z) = \begin{pmatrix} s_1(z) \\ s_2(z) \end{pmatrix} \in \mathcal{S}_\kappa^{(p+q)\times q}.$$

Thus,

$$f(z) = s_{10}(z)(B(z) - zs_{20}(z))^{-1}$$

where $S = S_0 B^{-1}$, with $S_0 = \begin{pmatrix} s_{10} \\ s_{20} \end{pmatrix} \in \mathcal{S}^{(p+q)\times q}$ and B is a $\mathbb{C}^{q\times q}$-valued Blaschke product of degree κ.

One can then study the tangential interpolation problem considered in §3 in the classes $H_{2,\kappa}^{p\times q}$, by reducing the problem to an interpolation problem in the class $\mathcal{S}_\kappa^{(p+q)\times q}$ and resorting to the results of Ball and Helton [8].

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