

THE DE BRANGES-ROVNYAK MODEL WITH FINITE-DIMENSIONAL COEFFICIENTS

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ABSTRACT. A characterization in terms of the canonical model spaces of L. de Branges and J. Rovnyak is obtained for Hilbert spaces of formal power series with vector coefficients which satisfy a difference-quotient inequality, thereby extending the closed ideal theorems of A. Beurling and P. D. Lax.

1. INTRODUCTION

This paper extends the well-known invariant subspace characterization of A. Beurling [3] and P. D. Lax [11] for the shift on the Hardy space of square summable power series with vector coefficients (cf. [10, 13–15]). The focus is instead on certain (not necessarily orthogonal) complements of contractively contained invariant manifolds of the shift. These are the spaces $\mathcal{H}(B)$ of L. de Branges and J. Rovnyak [6–8]. In the Beurling-Lax theory, the key point is a dimension inequality. The inequality is trivial when the coefficient space has infinite dimension, so the essential content is in the finite-dimensional case. Previously only special cases of the more abstract problem have been treated [6, 9], but our methods generalize an argument from [7, Theorem 6]. The main difficulty again comes down to a dimension inequality in the finite-dimensional case. The purpose here is to derive new results on the structure of $\mathcal{H}(B)$ spaces which reveal what is needed for the inequality to hold. As a consequence, we obtain a complete characterization of the spaces $\mathcal{H}(B)$.

2. $\mathcal{H}(B)$ SPACES

A basic concept in the de Branges-Rovnyak theory is complementation: A Hilbert space $\mathcal{F}$ is contained contractively in a Hilbert space $\mathcal{H}$ if $\mathcal{F}$ is a submanifold of $\mathcal{H}$ and if the inclusion map of $\mathcal{F}$ into $\mathcal{H}$ is a contraction. If $\mathcal{F}$ is contained contractively in $\mathcal{H}$, then the space complementary to $\mathcal{F}$ in $\mathcal{H}$ is the Hilbert space $\mathcal{G}$ of elements g of $\mathcal{H}$ with the property that

$$\|g\|_{\mathcal{G}}^2 = \sup\{\|g + f\|_{\mathcal{H}}^2 - \|f\|_{\mathcal{F}}^2 : f \in \mathcal{F}\}$$

is finite. The space $\mathcal{G}$ is contained contractively in $\mathcal{H}$. Moreover, $\mathcal{G}$ is the unique Hilbert space such that the inequality $\|k\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{F}}^2 + \|g\|_{\mathcal{G}}^2$ holds whenever $k = f + g$ is a decomposition of k in $\mathcal{H}$ into f in $\mathcal{F}$ and g in

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$\mathcal{E}$ and such that every element k in $\mathcal{H}$ admits a decomposition for which equality holds.

Let $\mathcal{E}$ be a finite-dimensional Hilbert space, and let $\mathcal{H}$ be a Hilbert space of formal power series $f(z)$ whose coefficients are in $\mathcal{E}$ such that

$$(1) \quad \|[f(z) - f(0)]/z\|_{\mathcal{H}}^2 \leq \|f(z)\|_{\mathcal{H}}^2 - |f(0)|_{\mathcal{E}}^2.$$

Then $\mathcal{H}$ is contained contractively in $\mathcal{E}(z)$, the Hilbert space of square summable power series $\sum a_n z^n$ with a_n in $\mathcal{E}$ and norm given by $\|\sum a_n z^n\|_{\mathcal{E}(z)}^2 = \sum |a_n|_{\mathcal{E}}^2$.

Let $B(z)$ be a power series whose coefficients are operators on $\mathcal{E}$ such that $\|B(z)f(z)\|_{\mathcal{E}(z)} \leq \|f(z)\|_{\mathcal{E}(z)}$ whenever $f(z)$ is in $\mathcal{E}(z)$. Cauchy multiplication by $B(z)$ thus defines a contraction operator on $\mathcal{E}(z)$ which will be denoted by T_B . The range $\mathcal{M}(B)$ of T_B becomes a Hilbert space in the unique norm with the property that $\|T_B f\|_{\mathcal{M}(B)} = \|f\|_{\mathcal{E}(z)}$ whenever f is orthogonal to the kernel of T_B . Furthermore, $\mathcal{M}(B)$ is contained contractively in $\mathcal{E}(z)$, and multiplication by z is a contraction on $\mathcal{M}(B)$.

The de Branges-Rovnyak space $\mathcal{H}(B)$ is defined to be the complementary space to $\mathcal{M}(B)$ in $\mathcal{E}(z)$. The space $\mathcal{H}(B)$ satisfies (1) and is an underlying space for canonical models of contractions on Hilbert space [1, 2, 12, 16, 17].

Multiplication by z is a contraction on the space $\mathcal{M}$ complementary to $\mathcal{H}$ in $\mathcal{E}(z)$. In [6] (cf. [5, Theorem 6]), de Branges extended the Beurling-Lax theorem by showing that if multiplication by z is isometric on $\mathcal{M}$, then $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(B)$. It should be further noted that when $\mathcal{E}$ is infinite dimensional, any space $\mathcal{H}$ which satisfies (1) is isometrically equal to a space $\mathcal{H}(B)$ [4, Theorem 11].

Let $\mathcal{H}(B)$ be a given space. Then $\mathcal{H}(B)$ is also contained contractively in $\mathcal{H}(zB)$. The space $\mathcal{H}(zB)$ may be obtained as those elements $h(z)$ of $\mathcal{E}(z)$ such that $[h(z) - h(0)]/z$ is in $\mathcal{H}(B)$ and $\|h(z)\|_{\mathcal{H}(zB)}^2 = \|[h(z) - h(0)]/z\|_{\mathcal{H}(B)}^2 + |h(0)|_{\mathcal{E}}^2$. The complementary space to $\mathcal{H}(B)$ in $\mathcal{H}(zB)$ is the space $B(z)\mathcal{E}$ with $\|B(z)c\|_{B(z)\mathcal{E}} = |c|_{\mathcal{E}}$ for every c orthogonal to $\mathcal{E} \cap \ker T_B$. Let us define linear transformations $J_{\pm}$ from $\mathcal{H}(B)$ into $\mathcal{E}$, with ranges denoted $\mathcal{E}_{\pm}$, as follows: $J_+ f = f(0)$ and J_- is the operator whose adjoint is given by $J_-^* c = [B(z) - B(0)]c/z$. Let $B(z) = \sum B_n z^n$, and let $\bar{B}_n$ be the adjoint of B_n on $\mathcal{E}$. Then $J_+^* c = [1 - B(z)\bar{B}(0)]c$; and since $\mathcal{E}$ is finite dimensional, $\mathcal{E}_+ = (1 - B_0\bar{B}_0)\mathcal{E}$ and $\mathcal{E}_- = (\bigvee_{n \geq 1} \bar{B}_n \mathcal{E}) \subseteq (1 - \bar{B}_0 B_0)\mathcal{E}$.

Let $R(0)$ denote the difference-quotient transformation on $\mathcal{H}(B)$, which maps $f(z)$ into $[f(z) - f(0)]/z$. Then $R(0)^* f(z) = zf(z) - B(z)J_- f$ so that $[1 - R(0)R(0)^*]f(z) = [B(z) - B(0)](J_- f)/z$ and $[1 - R(0)^* R(0)]f(z) = (J_+ f) + B(z)J_- R(0)f$. Note that if $[1 - R(0)^* R(0)]f(z) = c + B(z)c_-$ with c in $\mathcal{E}$ and c_- in $\mathcal{E}_-$, then necessarily $c = J_+ f$ and $c_- = J_- R(0)f$. Therefore, since $\dim \mathcal{E}$ is finite,

$$\text{rank}[1 - R(0)^* R(0)] = \dim\{(J_+ f, J_- R(0)f) : f \in \mathcal{H}(B)\}$$

$$(2) \quad \geq \dim \mathcal{E}_+ = \text{rank}(1 - \bar{B}_0 B_0)$$

$$(3) \quad \geq \dim \mathcal{E}_- = \text{rank}[1 - R(0)R(0)^*].$$

More precisely, the following will turn out to be a defining property of the spaces $\mathcal{H}(B)$.

Theorem 1. Let $R(0)$ be the difference-quotient transformation on a given space $\mathcal{H}(B)$. Then

$$\text{rank}[1 - R(0)^*R(0)] = \dim\{c \in \mathcal{E} : B(z)c \in \mathcal{H}(B)\} + \text{rank}[1 - R(0)R(0)^*].$$

Proof. Suppose that $B(z)c$ is in $\mathcal{H}(B)$. Then $c = (J_-f) + d$ where f is in $\mathcal{H}(B)$ and $[B(z) - B(0)]d/z = 0$. Moreover,

$$(4) \quad [1 - R(0)^*R(0)]\{[R(0)^*f] + B(z)c\} = (B_0d) + B(z)J_-f.$$

Let $J_-f_1, \dots, J_-f_{s_0}$ be a basis for the subspace $\mathcal{E}' = \{c \in \mathcal{E} : B(z)c \in \mathcal{H}(B)\}$, and let $J_+g_1, \dots, J_+g_t$ be a basis for $\mathcal{E}_+$ where f_i and g_j are in $\mathcal{H}(B)$ for all i and j . Suppose that there are constants $\lambda_1, \dots, \lambda_{s_0+t}$ such that

$$\begin{aligned} 0 &= \sum_1^{s_0} \lambda_i [1 - R(0)^*R(0)]\{[R(0)^*f_i] + B(z)J_-f_i\} \\ &\quad + \sum_1^t \lambda_{s_0+j} [1 - R(0)^*R(0)]g_j. \end{aligned}$$

Equivalently by (4) we have

$$0 = \left(\sum_1^t \lambda_{s_0+j} J_+g_j \right) + B(z)J_- \left[\sum_1^t \lambda_{s_0+j} R(0)g_j + \sum_1^{s_0} \lambda_i f_i \right]$$

so that $\sum \lambda_{s_0+j} J_+g_j = 0$ and hence $\lambda_{s_0+j} = 0$ ($j = 1, \dots, t$). It follows that $\sum \lambda_i J_-f_i = 0$ and thus $\lambda_i = 0$ for all i . Therefore,

$$(5) \quad \text{rank}[1 - R(0)^*R(0)] \geq s_0 + t.$$

Let $c_i = J_-f_i$ ($i = 1, \dots, s_0$) and expand $\{c_i\}$ to a basis $c_1, \dots, c_s$ of $\{c \in \mathcal{E} : B(z)c \in \mathcal{H}(B)\}$. For every $j > s_0$ let us write $c_j = (J_-f_j) + d_j$ as above where f_j is in $\mathcal{H}(B)$ and d_j is orthogonal to $\mathcal{E}_-$. By (4), B_0d_j is in $\mathcal{E}_+$, so it is in $(B_0\mathcal{E}) \cap (1 - B_0\bar{B}_0)\mathcal{E}$. But since $\mathcal{E}$ is finite dimensional, it follows that this intersection coincides with $B_0(1 - \bar{B}_0B_0)\mathcal{E}$, and hence $B_0d_j = B_0e_j$ where e_j is in $(1 - \bar{B}_0B_0)\mathcal{E}$. Thus $d_j - e_j$ is in $\ker B_0$, which is also contained in $(1 - \bar{B}_0B_0)\mathcal{E}$, and consequently d_j is in $[(1 - \bar{B}_0B_0)\mathcal{E}] \ominus \mathcal{E}_-$.

Now $\{d_j : j > s_0\}$ is linearly independent: For suppose $\sum \alpha_j d_j = 0$. Then $\sum_{j>s_0} \alpha_j c_j = \sum_{j>s_0} \alpha_j J_-f_j$ is in $\mathcal{E}'$, so there exist β_i such that $\sum_{j>s_0} \alpha_j c_j = \sum_{i \leq s_0} \beta_i c_i$. Since $\{c_i\}$ is linearly independent, $\alpha_j = 0$ for all j , and hence

$$\begin{aligned} t &= \dim \mathcal{E}_+ = \text{rank}(1 - B_0\bar{B}_0) = \text{rank}(1 - \bar{B}_0B_0) \\ &= \dim\{[(1 - \bar{B}_0B_0)\mathcal{E}] \ominus \mathcal{E}_-\} + \dim \mathcal{E}_- \\ &\geq (s - s_0) + \text{rank}[1 - R(0)R(0)^*]. \end{aligned}$$

In conjunction with (5) we have

$$\text{rank}[1 - R(0)^*R(0)] \geq s + \text{rank}[1 - R(0)R(0)^*].$$

To verify the reverse inequality, it suffices to show that there exist $r = \text{rank}[1 - R(0)^*R(0)] - \text{rank}[1 - R(0)R(0)^*]$ linearly independent vectors a_i in $\mathcal{E}$ such that $B(z)a_i$ is in $\mathcal{H}(B)$. By inequalities (2) and (3), it follows that $r = r_0 + r_1$ where $r_0 = \text{rank}[1 - R(0)^*R(0)] - \dim \mathcal{E}_+$ and $r_1 = \dim\{[\text{ran}(1 - \bar{B}_0B_0)] \ominus \mathcal{E}_-\}$.

Suppose that $r_0 > 0$ and recall the basis $\{J_+g_j\}$ of $\mathcal{E}_+$. As above, $\{[1 - R(0)^*R(0)]g_j\}$ is linearly independent, so if $\mathcal{G}$ is its span, then there are r_0 vectors $[1 - R(0)^*R(0)]\hat{g}_i$ ($i = 1, \dots, r_0$), with $\hat{g}_i$ in $\mathcal{H}(B)$, which form a basis of $\text{ran}[1 - R(0)^*R(0)] \ominus \mathcal{G}$. Now there exist constants λ_{ij} such that $J_+\hat{g}_i = \sum_{j=1}^r \lambda_{ij}J_+g_j$ for each i . Let us define $a_i = J_-R(0)(\hat{g}_i - \sum_j \lambda_{ij}g_j)$ for $i = 1, \dots, r_0$. Then $B(z)a_i = [1 - R(0)^*R(0)](\hat{g}_i - \sum_j \lambda_{ij}g_j)$ is in $\mathcal{H}(B)$, and $\{a_1, \dots, a_{r_0}\}$ is linearly independent: Suppose that $\sum \mu_i a_i = 0$. Then

$$\sum \mu_i [1 - R(0)^*R(0)]\hat{g}_i = \sum_i \mu_i [1 - R(0)^*R(0)] \left(\sum_j \lambda_{ij}g_j \right)$$

which must be zero since it is in both $\mathcal{G}$ and $\mathcal{G}^\perp$. Therefore $\mu_i = 0$ for every i .

Next, suppose that $r_1 > 0$ and let $\hat{d}_1, \dots, \hat{d}_{r_1}$ be a basis of $[\text{ran}(1 - \bar{B}_0 B_0)] \ominus \mathcal{E}_-$. Then $B(z)\hat{d}_j = B_0\hat{d}_j$ and $\hat{d}_j = (1 - \bar{B}_0 B_0)b_j$ for some b_j in $\mathcal{E}$. Let $\hat{f}_j(z) = [1 - B(z)\bar{B}(0)]B_0b_j$ and define $a_{r_0+j} = \hat{d}_j + J_-R(0)\hat{f}_j$ for $j = 1, \dots, r_1$. Then $B(z)a_{r_0+j} = [1 - R(0)^*R(0)]\hat{f}_j$ is in $\mathcal{H}(B)$.

Finally, $\{a_i : i = 1, \dots, r = r_0 + r_1\}$ is linearly independent: Suppose that there are constants $\nu_1, \dots, \nu_r$ such that

$$0 = \sum \nu_i a_i = \sum_1^{r_0} \nu_i a_i + \sum_1^{r_1} \nu_{r_0+j} [\hat{d}_j + J_-R(0)\hat{f}_j].$$

It follows that $\sum_1^{r_1} \nu_{r_0+j} \hat{d}_j = 0$ since a_i ($1 \leq i \leq r_0$) and $J_-R(0)\hat{f}_j$ ($1 \leq j \leq r_1$) are in $\mathcal{E}_-$, and $\hat{d}_j$ is orthogonal to $\mathcal{E}_-$ for every j . Therefore $\nu_{r_0+j} = 0$ ($j = 1, \dots, r_1$), and consequently $\sum_1^{r_0} \nu_i a_i = 0$ so that $\nu_i = 0$ for all i . $\square$

3. THE CHARACTERIZATION

Let $\mathcal{H}$ be a space which satisfies (1), and let $\mathcal{H}'$ be the Hilbert space of all power series $h(z)$ such that $[h(z) - h(0)]/z$ is in $\mathcal{H}$ with $\|h(z)\|_{\mathcal{H}'}^2 = \|[h(z) - h(0)]/z\|_{\mathcal{H}}^2 + |h(0)|_{\mathcal{E}}^2$. Then $\mathcal{H}'$ satisfies (1), and $\mathcal{H}$ is contained contractively in $\mathcal{H}'$. Let $\mathcal{R}$ be the complementary space to $\mathcal{H}$ in $\mathcal{H}'$, and let $i_{\mathcal{H}}$ and $i_{\mathcal{R}}$ denote the respective inclusion maps of $\mathcal{H}$ and $\mathcal{R}$ into $\mathcal{H}'$. Then every h in $\mathcal{H}'$ admits the unique decomposition $h = (i_{\mathcal{H}}^* h) + (i_{\mathcal{R}}^* h)$ where $\|h\|_{\mathcal{H}'}^2 = \|i_{\mathcal{H}}^* h\|_{\mathcal{H}}^2 + \|i_{\mathcal{R}}^* h\|_{\mathcal{R}}^2$.

A fundamental result from the theory of $\mathcal{H}(B)$ spaces is: $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(B)$ if and only if the dimension of $\mathcal{R}$ does not exceed the dimension of $\mathcal{E}$ [6]. More generally, if $\mathcal{E} \subseteq \tilde{\mathcal{E}}$ and $\dim \mathcal{R} \leq \dim \tilde{\mathcal{E}}$, then $\mathcal{H}$ is a space $\mathcal{H}(\tilde{B})$ where the coefficients of $\tilde{B}(z)$ act on $\tilde{\mathcal{E}}$.

Lemma. Let $\mathcal{F}$ be the subspace of elements of $\mathcal{H}$ for which equality holds in (1). Then $\mathcal{R}$ and $\mathcal{H} \cap \mathcal{R}$ are contained in $\mathcal{H}' \ominus \mathcal{F}$ and $\mathcal{H} \ominus \mathcal{F}$ respectively. Moreover, $\dim \mathcal{R} = \dim \mathcal{H}' \ominus \mathcal{F}$ and $\dim \mathcal{H} \cap \mathcal{R} = \dim \mathcal{H} \ominus \mathcal{F}$.

Proof. As in [9], $\mathcal{F}$ is a (closed) subspace of $\mathcal{H}$ and is contained isometrically in $\mathcal{H}'$. Therefore for any f in $\mathcal{F}$ and g in $\mathcal{R}$, we have

$$\langle f, g \rangle_{\mathcal{H}'} = \langle f, i_{\mathcal{R}} g \rangle_{\mathcal{H}'} = \langle i_{\mathcal{R}}^* f, g \rangle_{\mathcal{R}} = \langle 0, g \rangle_{\mathcal{R}} = 0.$$

Hence $\mathcal{R}$ is a subset of $\mathcal{H}' \ominus \mathcal{F}$.

The restriction of $i_{\mathcal{R}}^*$ to $\mathcal{H}' \ominus \mathcal{F}$ is linear and continuous and has trivial kernel: if $i_{\mathcal{R}}^* h = 0$ for some h in $\mathcal{H}' \ominus \mathcal{F}$, then $i_{\mathcal{R}}^* h = h$, so h is also in $\mathcal{F}$, and thus $h = 0$. It follows that $\dim \mathcal{H}' \ominus \mathcal{F} = \dim i_{\mathcal{R}}^*(\mathcal{H}' \ominus \mathcal{F}) \leq \dim \mathcal{R}$, and hence $\dim \mathcal{R} = \dim \mathcal{H}' \ominus \mathcal{F}$.

Next, let g be in $\mathcal{H} \cap \mathcal{R}$. Then g is in $\mathcal{H}' \ominus \mathcal{F}$ but also in $\mathcal{H} \ominus \mathcal{F}$ since for any f in $\mathcal{F}$

$$\langle f, g \rangle_{\mathcal{H}} = \langle i_{\mathcal{H}}^* f, g \rangle_{\mathcal{H}} = \langle f, i_{\mathcal{H}} g \rangle_{\mathcal{H}'} = \langle f, g \rangle_{\mathcal{H}'} = 0.$$

Therefore $(\mathcal{H} \cap \mathcal{R}) \subseteq (\mathcal{H} \ominus \mathcal{F})$. Finally $\dim \mathcal{H} \cap \mathcal{R} = \dim \mathcal{H} \ominus \mathcal{F}$ as above since $i_{\mathcal{R}}^*(\mathcal{H} \ominus \mathcal{F})$ is contained in $\mathcal{H} \cap \mathcal{R}$. $\square$

The following will distinguish the spaces $\mathcal{H}(B)$.

Corollary 1. *Let $\mathcal{F}(B)$ be the subspace of elements of a given space $\mathcal{H}(B)$ for which equality holds in (1). Then*

$$\dim J_+ \mathcal{F}(B) = \dim(\mathcal{C} \cap \ker T_B) + \text{rank}[1 - R(0)R(0)^*].$$

Proof. Since $B(z)\mathcal{C}$ is finite dimensional, the lemma implies that $\mathcal{H}(B) \ominus \mathcal{F}(B)$ coincides with $\mathcal{H}(B) \cap B(z)\mathcal{C}$. By (1), the kernel of $1 - R(0)^*R(0)$ is contained in $\mathcal{F}(B)$ and is exactly the kernel of the restriction of J_+ to $\mathcal{F}(B)$. Thus since $1 - R(0)^*R(0)$ has finite rank and

$$J_+ \mathcal{F}(B) = J_+ \{\text{ran}[1 - R(0)^*R(0)] \cap \mathcal{F}(B)\},$$

it follows that

$$\begin{aligned} \text{rank}[1 - R(0)^*R(0)] &= \dim\{\text{ran}[1 - R(0)^*R(0)] \cap \mathcal{F}(B)\} \\ &\quad + \dim[\mathcal{H}(B) \ominus \mathcal{F}(B)] \\ &= \dim J_+ \mathcal{F}(B) + \dim[\mathcal{H}(B) \cap B(z)\mathcal{C}]. \end{aligned}$$

The corollary now follows from Theorem 1 since we also have

$$\begin{aligned} \text{rank}[1 - R(0)^*R(0)] &= \dim(\mathcal{C} \cap \ker T_B) + \dim[\mathcal{H}(B) \cap B(z)\mathcal{C}] \\ &\quad + \text{rank}[1 - R(0)R(0)^*]. \quad \square \end{aligned}$$

By [7, Lemma 4], equality holds in (1) for a given space $\mathcal{H}(B)$ if and only if $\mathcal{H}(B)$ contains no nonzero element of the form $B(z)c$ with c in $\mathcal{C}$. An immediate consequence of the above results is

Corollary 2. *Let $\mathcal{H}(B)$ be a given space. Then $\text{rank}[1 - R(0)^*R(0)] = \text{rank}[1 - R(0)R(0)^*]$ if and only if equality holds in (1) for every $f(z)$ and there is no nonzero vector c such that $B(z)c = 0$.*

We now have the proposed characterization.

Theorem 2. *Let $\mathcal{H}$ be a Hilbert space of formal power series which satisfies (1), and let $\mathcal{F}$ be the subspace of those series for which equality holds in (1). Then $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(B)$ if and only if the dimension of the space of constant coefficients of elements of $\mathcal{F}$ is at least the rank of $1 - TT^*$ where T is the difference-quotient transformation on $\mathcal{H}$.*

Proof. Any space $\mathcal{H}(B)$ has the stated property by Corollary 1.

Conversely, suppose that $\mathcal{H}$ is a space which satisfies (1) and the dimension hypothesis. Let $\mathcal{H}'$, $\mathcal{R}$, $i_{\mathcal{H}}$ and $i_{\mathcal{R}}$ be defined as above, and let $f(z)$ and $g(z)$ be in $\mathcal{H}$. Since

$$\langle i_{\mathcal{H}}^* z f(z), g(z) \rangle_{\mathcal{H}} = \langle z f(z), i_{\mathcal{H}'} g(z) \rangle_{\mathcal{H}'} = \langle f(z), T g(z) \rangle_{\mathcal{H}},$$

it follows that $T^* f(z) = i_{\mathcal{H}}^* z f(z)$.

Let S denote the difference-quotient transformation on $\mathcal{H}'$. Then

$$\begin{aligned} (1 - TT^*)f(z) &= f(z) - T i_{\mathcal{H}}^* z f(z) \\ &= f(z) - S[z f(z) - i_{\mathcal{H}}^* z f(z)] = S i_{\mathcal{H}}^* z f(z). \end{aligned}$$

More generally, $S\mathcal{R}$ is contained in the range of $1 - TT^*$: Let $g(z)$ be in $\mathcal{R}$ such that $g(z)$ is orthogonal to $i_{\mathcal{H}}^* z f(z)$ for every $f(z)$ in $\mathcal{H}$. Then

$$0 = \langle g(z), i_{\mathcal{H}}^* z f(z) \rangle_{\mathcal{H}} = \langle g(z), z f(z) \rangle_{\mathcal{H}'} = \langle S g(z), f(z) \rangle_{\mathcal{H}}$$

for every $f(z)$ in $\mathcal{H}$. Letting $f(z) = S g(z)$, we conclude that $g(z)$ is constant. Hence $S\mathcal{R} = S \vee \{i_{\mathcal{H}}^* z f(z) : f(z) \in \mathcal{H}\}$, which is contained in $(1 - TT^*)\mathcal{H}$ since the rank of $1 - TT^*$ is finite by the hypothesis.

It follows that $\mathcal{R}$ is finite dimensional since

$$\dim \mathcal{R} \leq \dim S\mathcal{R} + \dim \ker S \leq \text{rank}(1 - TT^*) + \dim \mathcal{E}.$$

Thus by the lemma $\mathcal{R} = \mathcal{H}' \ominus \mathcal{F}$.

Furthermore, since $\mathcal{H}'$ contains $\mathcal{E}$, the kernel of the restriction of S to $\mathcal{H}' \ominus \mathcal{F}$ is $\mathcal{E} \ominus \{f(0) : f(z) \in \mathcal{F}\}$. Hence, we have that

$$\begin{aligned} \dim \mathcal{R} &= \dim[\mathcal{E} \ominus \{f(0) : f(z) \in \mathcal{F}\}] + \dim S\mathcal{R} \\ &\leq \dim \mathcal{E} - \dim\{f(0) : f(z) \in \mathcal{F}\} + \text{rank}(1 - TT^*) \\ &\leq \dim \mathcal{E} \end{aligned}$$

by the hypothesis. Therefore, $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(B)$. $\square$

Finally, any space which satisfies (1) is at least a reducing subspace of $R(0)$ on some space $\mathcal{H}(B)$.

Corollary 3. Let $\mathcal{H}$, $\mathcal{F}$ and T be defined as in Theorem 2, but assume on the other hand that

$$\delta = \text{rank}(1 - TT^*) - \dim\{f(0) : f(z) \in \mathcal{F}\}$$

is finite and positive. If $\tilde{\mathcal{E}}$ is any Hilbert space with dimension at least δ , then $\mathcal{H} \oplus \tilde{\mathcal{E}}(z)$ is isometrically equal to a space $\mathcal{H}(B)$.

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