

ASYMPTOTIC STABILITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS BY LIAPUNOV FUNCTIONALS

BO ZHANG

ABSTRACT. We consider the asymptotic stability in a system of functional differential equations $x'(t) = F(t, x_t)$ by Liapunov functionals V . The work generalizes some well-known results in the literature in that we only require the derivative of V to be negative definite on a sequence of intervals $I_n = [s_n, t_n]$. We also show that it is not necessary to require a uniform upper bound on V for nonuniform asymptotic stability.

1. INTRODUCTION

We consider a system of functional differential equations with finite delay

$$(1.1) \quad x'(t) = F(t, x_t), \quad x \in R^n,$$

and obtain conditions on a Liapunov functional V to ensure that the zero solution of (1.1) is asymptotically stable or uniformly asymptotically stable. Our results generalize some well-known theorems in the literature in that we only require the following properties of V .

(P₁) The derivative of V along a solution of (1.1) is negative definite on a sequence of intervals $I_n = [s_n, t_n]$.

(P₂) V has a uniform upper bound on $J_n = [t_n - \sigma, t_n]$ with $\sigma > 0$.

For reference, the discussion here follows closely those of Burton [1] and Burton and Hatvani [3, 4]. Our work also has roots in the recent work of Burton and Makay [5] in which the asymptotic stability of (1.1) was obtained by a Liapunov functional V having property (P₁) and a uniform upper bound on a sequence $\{t_n\}$. A growth condition on $F(t, \varphi)$ is required in [5].

For $x \in R^n$, $|\cdot|$ denotes the Euclidean norm of x . The length of an interval $I_n = [a, b]$ is defined by $l(I_n)$. For an $n \times n$ matrix A , define the norm $|A|$ of A by $|A| = \sup\{|Ax| : |x| \leq 1\}$. For a given $h > 0$, C will be the space of continuous functions $\varphi : [-h, 0] \rightarrow R^n$ with the supremum norm $\|\varphi\| = \sup\{|\varphi(s)| : -h \leq s \leq 0\}$. C_H denotes the set of $\varphi \in C$ with $\|\varphi\| < H$. If x is a continuous function of u defined on $-h \leq u < A$, $A > 0$, and if t

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is a fixed number satisfying $0 \leq t < A$, then x_t denotes the restriction of x to the interval $[t-h, t]$ so that x_t is an element of C defined by $x_t(s) = x(t+s)$ for $-h \leq s \leq 0$. For any $\varphi \in C$ we define

$$|\varphi|_2 = \left[\int_{-h}^0 |\varphi(s)|^2 ds \right]^{1/2}.$$

In (1.1), $x'(t)$ denotes the right-hand derivative of x at t . It is assumed that $F: R^+ \times C_H \rightarrow R^n$, $R^+ = [0, +\infty)$, is continuous so that a solution will exist for each $(t_0, \varphi) \in R^+ \times C_H$. We denote by $x(t_0, \varphi)$ a solution of (1.1) with initial function $\varphi \in C_H$ where $x_{t_0}(t_0, \varphi) = \varphi$. The value of $x(t_0, \varphi)$ at t will be $x(t) = x(t, t_0, \varphi)$. For the continuation of solutions, we suppose that F takes bounded sets of $R^+ \times C_H$ into bounded sets of R^n . We also assume that $F(t, 0) = 0$ so that $x = 0$ is a solution of (1.1). Let $V: R^+ \times C_H \rightarrow R^+$ be a continuous functional and define the upper right-hand derivative of V along a solution of (1.1) by

$$V'_{(1.1)}(t, \varphi) = \limsup_{\delta \rightarrow 0^+} \{V(t + \delta, x_{t+\delta}(t, \varphi)) - V(t, \varphi)\} / \delta.$$

For reference on Liapunov's direct method and fundamental theorems of (1.1), we refer to the work of Burton [2], Hale [6], Kato [7], and Yoshizawa [8].

Definition 1.1. The zero solution of (1.1) is said to be stable if, for each $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\varphi \in C_H, \|\varphi\| < \delta, t \geq t_0]$ imply that $|x(t, t_0, \varphi)| < \varepsilon$. If δ is independent of t_0 , then the zero solution is uniformly stable.

Definition 1.2. The zero solution of (1.1) is asymptotically stable (AS) if it is stable and if, for each $t_0 \in R^+$, there exists $\delta = \delta(t_0) > 0$ such that $\|\varphi\| < \delta$ implies that $x(t, t_0, \varphi) \rightarrow 0$ as $t \rightarrow +\infty$. The zero solution of (1.1) is uniformly asymptotically stable (UAS) if it is uniformly stable and if there is a $\sigma > 0$ and if for each $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that $[t_0 \in R^+, \|\varphi\| < \sigma, t \geq t_0 + T]$ imply that $|x(t, t_0, \varphi)| < \varepsilon$.

Definition 1.3. $W: R^+ \rightarrow R^+$ is called a wedge if W is continuous and strictly increasing with $W(0) = 0$. Throughout this paper W, W_j ($j = 0, 1, 2, \dots$) will denote the wedges.

Definition 1.4. A continuous function $G: R^+ \rightarrow R^+$ is convex downward if $G([t+s]/2) \leq [G(t) + G(s)]/2$ for all $t, s \in R^+$.

Jensen's inequality. Let W be convex downward and let $f, p: [a, b] \rightarrow R^+$ be continuous with $\int_a^b p(s) ds > 0$. Then

$$\int_a^b p(s) ds W \left[\frac{\int_a^b p(s) f(s) ds}{\int_a^b p(s) ds} \right] \leq \int_a^b p(s) W(f(s)) ds.$$

2. MAIN THEOREMS

The following lemmas will be used in the proof of our results.

Lemma 2.1. *If W_1 is a wedge, then for any $L > 0$ there is a convex downward wedge W_0 such that $W_0(r) \leq W_1(r)$ for all $r \in [0, L]$. In fact, $W_0(r) = \int_0^r W_1(s) ds/L$ will suffice.*

Lemma 2.2 [4, p. 286]. *Let $x : [t_0 - h, +\infty) \rightarrow R^n$ be a bounded continuous function and $\{t_n\}$ be an increasing sequence of real numbers with $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ (short notation $\{t_n\} \uparrow \infty$) such that $|x_{t_n}|_2 \rightarrow 0$ as $n \rightarrow +\infty$. Then there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a sequence $\{s_k\}$ with $s_k \in [t_{n_k} - h, t_{n_k}]$ and $|t_{n_k} - s_k| < \frac{1}{k}$ such that $|x(s_k)| + |x_{s_k}|_2 \rightarrow 0$ as $k \rightarrow +\infty$.*

Theorem 2.1. *Suppose that there exist a continuous functional $V : R^+ \times C_H \rightarrow R^+$, wedges W, W_i ($i = 1, 2, 3$), a constant $\sigma > 0$ and a sequence $\{t_n\} \uparrow \infty$ such that*

- (i) $W_1(|\varphi(0)|) \leq V(t, \varphi)$, $V(t, 0) = 0$ for all $t \in R^+$ and $V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ for $t \in [t_n - \sigma, t_n]$,
- (ii) $V'_{(1.1)}(t, x_t) \leq 0$ for all $t \geq t_0$ and $V'_{(1.1)}(t, x_t) \leq -W(|x(t)|)$ for $t \in [t_n - h, t_n]$, where $x(t) = x(t, t_0, \varphi)$ is any solution of (1.1) with $x_t \in C_H$ and $t_n - h \geq t_0$.

Then the zero solution of (1.1) is AS.

Proof. Let $t_0 \in R^+$ and $\varepsilon > 0$. Then there exists $\delta > 0$ ($\delta < H$) such that $V(t_0, \varphi) < W_1(\varepsilon)$ whenever $\|\varphi\| < \delta$. Let $x(t) = x(t, t_0, \varphi)$ be a solution of (1.1) with $\|\varphi\| < \delta$. It then follows that $W_1(|x(t)|) \leq V(t, x_t) \leq V(t_0, \varphi) < W_1(\varepsilon)$ and, therefore, $|x(t)| < \varepsilon$ for $t \geq t_0$. Thus the zero solution of (1.1) is stable.

Next let $t_0 \geq 0$ and find $\delta > 0$ of stability for $\varepsilon_0 = \min\{H, 1\}$. If $\|\varphi\| < \delta$, then $|x(t, t_0, \varphi)| < \varepsilon_0$ for $t \geq t_0$. We will show that $x(t, t_0, \varphi) \rightarrow 0$ as $t \rightarrow +\infty$. Without loss of generality, we may assume that $t_{n-1} + h \leq t_n$ for $n = 1, 2, \dots$. By Lemma 2.1 there exists a convex downward wedge W_4 such that $W_4(r^2) \leq W(r)$ for $0 \leq r \leq 1$. For $t \geq t_n$, we have

$$\begin{aligned} V(t, x_t) &\leq V(t_0, \varphi) - \sum_{j=1}^n \int_{t_j-h}^{t_j} W(|x(s)|) ds \\ &\leq V(t_0, \varphi) - \sum_{j=1}^n \int_{t_j-h}^{t_j} W_4(|x(s)|^2) ds. \end{aligned}$$

Apply Jensen's inequality to obtain

$$(2.1) \quad V(t, x_t) \leq V(t_0, \varphi) - \sum_{j=1}^n h W_4\left(\frac{1}{h} |x_{t_j}|_2^2\right).$$

This implies that

$$\sum_{j=1}^{+\infty} W_4\left(\frac{1}{h} |x_{t_j}|_2^2\right) < +\infty \quad \text{and} \quad |x_{t_n}|_2 \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

By Lemma 2.2, it follows that there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a sequence $\{s_k\}$ with $s_k \in [t_{n_k} - h, t_{n_k}]$ and $t_{n_k} - s_k < \frac{1}{k}$ such that

$$(2.2) \quad |x(s_k)| + |x_{s_k}|_2 \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

Without loss of generality, we may assume that $s_k \in [t_{n_k} - \sigma, t_{n_k}]$ for $k = 1, 2, \dots$. For the given $\varepsilon > 0$ there exists $K > 0$ such that $W_2(|x(s_k)|) + W_3(|x_{s_k}|_2) < W_1(\varepsilon)$. We then have

$$W_1(|x(t)|) \leq V(t, x_t) \leq V(s_k, x_{s_k}) < W_1(\varepsilon) \text{ and } |x(t)| < \varepsilon$$

for $t \geq t_{n_k} \geq s_k$. Thus the proof is complete.

Theorem 2.2. Suppose that there exist a continuous functional $V : R^+ \times C_H \rightarrow R^+$, wedges W, W_i ($i = 1, 2, 3$), a constant $\sigma > 0$ and a sequence $\{t_n\} \uparrow \infty$ such that

- (i) $W_1(|\varphi(0)|) \leq V(t, \varphi)$, $V(t, 0) = 0$ for all $t \in R^+$ and $V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ for $t \in [t_n - \sigma, t_n]$.
- (ii) $V'_{(1.1)}(t, x_t) \leq 0$ for all $t \geq t_0$ and $V'_{(1.1)}(t, x_t) \leq -W(|x_t|_2)$ for $t \in I_n$ and $x_t \in C_H$, where $x(t) = x(t, t_0, \varphi)$ is any solution of (1.1) with $t_n - h \geq t_0$ and I_n is a sequence of intervals $I_n = [s_n, t_n]$ with $\sum_{n=1}^{\infty} l(I_n) = +\infty$.

Then the zero solution of (1.1) is AS.

Proof. The fact that the zero solution of (1.1) is stable follows from the proof of Theorem 2.1. We now show that the zero solution of (1.1) is AS. Let $t_0 \geq 0$ and find $\delta > 0$ of stability for $\varepsilon_0 = \min\{H, 1\}$. Thus if $\|\varphi\| < \delta$, then $|x(t, t_0, \varphi)| < \varepsilon_0$ for $t \geq t_0$. We claim that

$$(2.3) \quad \liminf_{n \rightarrow +\infty} |x_{t_n}|_2 = 0.$$

Suppose that there exists $K > 0$ and a constant $\alpha > 0$ such that $|x_{t_n}|_2 \geq \alpha$ for all $n \geq K$. Define $P(t) = \int_{t-h}^t |x(s)|^2 ds$. Then $P'(t) = |x(t)|^2 - |x(t-h)|^2$ and there exists a constant $L > 0$ such that $|P'(t)| \leq L$ for all $t \geq t_0$. For $|t_n - t| \leq \alpha^2/2L$ we have $|P(t_n) - P(t)| \leq L|t_n - t| \leq \alpha^2/2$ and $P(t) \geq P(t_n) - \alpha^2/2 \geq \alpha^2/2$. This implies that $|x_t|_2 \geq \alpha/2$ for $t \in J_n = [t_n - \alpha^2/2L, t_n]$. We consider the following cases.

Case 1. There exist a constant $\gamma > 0$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $l(I_{n_k}) \geq \gamma$ with $n_1 \geq K$. We then choose $L > 0$ sufficiently large such that $J_{n_k} \subset I_{n_k}$. Let $t \geq t_{n_m}$ and integrate (ii) from t_0 to t to obtain

$$(2.4) \quad V(t, x_t) \leq V(t_0, \varphi) - \sum_{k=1}^m \int_{J_{n_k}} W(|x_s|_2) ds \\ \leq V(t_0, \varphi) - mW(\alpha/2)\alpha^2/2L \rightarrow -\infty \text{ as } m \rightarrow +\infty,$$

a contradiction.

Case 2. Suppose that $l(I_n) \rightarrow 0$ as $n \rightarrow +\infty$. We may assume that $I_n \subset J_n$ for all $n \geq K$. Thus for $t \geq t_n$ we have

$$V(t, x_t) \leq V(t_0, \varphi) - \sum_{j=K}^n \int_{I_j} W(|x_s|_2) ds \\ \leq V(t_0, \varphi) - W(\alpha/2) \sum_{j=K}^n l(I_j) \rightarrow -\infty \text{ as } n \rightarrow +\infty,$$

a contradiction. This proves (2.3). By Lemma 2.2, we again conclude that (2.2) holds for a sequence $\{s_k\}$ with $s_k \in [t_{n_k} - \sigma, t_{n_k}]$, where $\{t_{n_k}\}$ is a subsequence

of $\{t_n\}$. Therefore, the zero solution of (1.1) is asymptotically stable by the proof following (2.2) in Theorem 2.1.

We now discuss the uniform asymptotic stability. Our result generalizes a well-known theorem of Burton [1].

Theorem 2.3. Suppose that there exist a continuous functional $V : R^+ \times C_H \rightarrow R^+$, wedges W, W_i ($i = 1, 2, 3$), a positive constant J and a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \leq J$ such that

- (i) $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ for all $(t, \varphi) \in R^+ \times C_H$,
- (ii) $V'_{(1.1)}(t, x_t) \leq 0$ for all $t \geq t_0$ and $V'_{(1.1)}(t, x_t) \leq -W(|x(t)|)$ for $t \in [t_n - h, t_n]$, where $x(t) = x(t, t_0, \varphi)$ is any solution of (1.1) with $x_t \in C_H$ and $t_n - h \geq t_0$.

Then the solution of (1.1) is UAS.

Proof. There exists a wedge W^* such that $W_2(r) + W_3(r\sqrt{h}) \leq W^*(r)$ for all $0 \leq r \leq 1$. The uniform stability is clear from (i). Next choose $\delta > 0$ of uniform stability for $\varepsilon_0 = \min\{H, 1\}$. Thus, if $x(t) = x(t, t_0, \varphi)$ is a solution of (1.1), then $|x(t, t_0, \varphi)| < \varepsilon_0$ whenever $\|\varphi\| < \delta$ and $t \geq t_0$. Moreover, $V(t, x_t) \leq W^*(\delta)$ for all $t \geq t_0$. We will follow the proof of Theorem 2.1. For any $\varepsilon > 0$ find $\eta > 0$ such that $W_2(\eta) + W_3(\eta) < W_1(\varepsilon)$. From (2.1) we have

$$(2.5) \quad V(t, x_t) \leq W^*(\delta) - \sum_{j=1}^n h W_4\left(\frac{1}{h} |x_{t_j}|_2^2\right).$$

This implies that there exists a positive integer $K = K(\delta)$ such that $|x_{t_k}|_2 < \eta/2$ for some k with $1 \leq k \leq K$. Consequently, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $|x_{t_{n_k}}|_2 < \eta/2$ and $t_{n_k} - t_{n_{k-1}} \leq KJ$ with $t_{n_0} = t_0$ for $k = 1, 2, \dots$. For brevity, we rename $\{t_{n_k}\}$ by $\tau_k = t_{n_k}$. Since $P(t) = \int_{t-h}^t |x(s)|^2 ds$ is uniformly continuous on R^+ , there exists $\gamma > 0$ ($\gamma < h$) such that $|x_t|_2 < \eta$ on $[\tau_k - \gamma, \tau_k]$ for $k = 1, 2, \dots$. Let M be a positive integer such that $W^*(\delta) - MW(\eta)\gamma < 0$. We claim that there exists an integer m with $1 \leq m \leq M$ and a $s_m \in [\tau_m - \gamma, \tau_m]$ with $|x(s_m)| < \eta$. In fact, if $|x(s)| \geq \eta$ on $[\tau_k - \gamma, \tau_k]$ for $k = 1, 2, \dots, M$, then

$$\begin{aligned} V(\tau_M, x_{\tau_M}) &\leq W^*(\delta) - \sum_{k=1}^M \int_{\tau_k - \gamma}^{\tau_k} W(|x(s)|) ds \\ &\leq W^*(\delta) - W(\eta)M\gamma < 0, \end{aligned}$$

a contradiction. Thus, such s_m exists and

$$\begin{aligned} V(t, x_t) &\leq V(s_m, x_{s_m}) \leq W_2(|x(s_m)|) + W_3(|x_{s_m}|_2) \\ &\leq W_2(\eta) + W_3(\eta) < W_1(\varepsilon) \end{aligned}$$

for $t \geq \tau_M \geq s_m$. This implies that $|x(t)| < \varepsilon$ for $t \geq t_0 + T \geq \tau_M$, $T = KJM$, and the zero solution of (1.1) is UAS.

Our next theorem is a refinement of a simple version of Theorem 3 in [3] on uniform asymptotic stability.

Theorem 2.4. Suppose that there exist a continuous functional $V : R^+ \times C_H \rightarrow R^+$, wedges W, W_i ($i = 1, 2, 3$), positive constants σ, J and a sequence $\{t_n\} \uparrow \infty$ with $t_n - t_{n-1} \leq J$ such that

- (i) $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|\varphi|_2)$ for all $(t, \varphi) \in R^+ \times C_H$,

- (ii) $V'_{(1.1)}(t, x_t) \leq 0$ for all $t \geq t_0$ and $V'_{(1.1)}(t, x_t) \leq -W(|x_t|_2)$ for $t \in [t_n - \sigma, t_n]$, where $x(t) = x(t, t_0, \varphi)$ is any solution of (1.1) with $x_t \in C_H$ and $t_n - \sigma \geq t_0$.

Then the zero solution of (1.1) is UAS.

Proof. There exists a wedge W^* such that $W_2(r) + W_3(r\sqrt{h}) \leq W^*(r)$ for all $0 \leq r \leq 1$. The zero solution of (1.1) is uniformly stable by (i). Next choose $\delta > 0$ of uniform stability for $\varepsilon_0 = \min\{H, 1\}$. Thus if $x(t) = x(t, t_0, \varphi)$ is a solution of (1.1), then $|x(t, t_0, \varphi)| < \varepsilon_0$ whenever $\|\varphi\| < \delta$ and $t \geq t_0$. Moreover, $V(t, x_t) \leq W^*(\delta)$ for all $t \geq t_0$. We will follow the proof of Theorem 2.2. For any $\varepsilon > 0$, choose $\eta > 0$ such that $W_2(\eta) + W_3(\eta) < W_1(\varepsilon)$. We first claim that there exist a constant $K = K(\delta)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ with $t_{n_k} - t_{n_{k-1}} \leq KJ$ and

$$(2.6) \quad |x_{t_{n_k}}|_2 < \eta/2.$$

Since $P(t) = \int_{t-h}^t |x(s)|^2 ds$ is uniformly continuous on R^+ , there exists $\gamma > 0$ ($\gamma < \min\{\sigma, h\}$) such that

$$(2.7) \quad |P(t) - P(s)| < \eta^2/8$$

whenever $|t - s| < \gamma$. Let K be the first positive integer such that $W^*(\delta) - \gamma KW(\eta/4) < 0$. We will show that there exists an integer n_1 , $1 \leq n_1 \leq K$, with $|x_{t_{n_1}}|_2 < \eta/2$. Suppose that $|x_{t_n}| \geq \eta/2$ for $n = 1, 2, \dots, K$. By (2.7) with $t = t_n$, we have $|x_s|_2 \geq \eta/4$ for all $s \in [t_n - \gamma, t_n]$. Integrate (ii) from t_0 to t_K to obtain

$$\begin{aligned} V(t_K, x_{t_K}) &\leq W^*(\delta) - \sum_{j=1}^K \int_{t_j-\gamma}^{t_j} W(|x_s|_2) ds \\ &\leq W^*(\delta) - \gamma KW(\eta/4) < 0, \end{aligned}$$

a contradiction. Using the same argument, we obtain a subsequence $\{x_{t_{n_k}}\}$ satisfying (2.6). Moreover, $|x_s|_2 < \eta$ for $s \in [t_{n_k} - \gamma, t_{n_k}]$ by (2.7). We again rename $\{t_{n_k}\}$ by $\tau_k = t_{n_k}$. Next find a positive integer M such that $W^*(\delta) - MW(\eta\sqrt{\gamma/2})\gamma/2 < 0$. We show that there exists m , $1 \leq m \leq M$, with $s_m \in [\tau_m - \gamma, \tau_m]$ and $|x(s_m)| < \eta$. In fact, if $|x(s)| \geq \eta$ on $[\tau_k - \gamma, \tau_k]$ for $k = 1, 2, \dots, M$, then for $t \in [\tau_k - \gamma/2, \tau_k]$ we have

$$\begin{aligned} \int_{t-h}^t |x(s)|^2 ds &\geq \int_{\tau_k-\gamma}^t |x(s)|^2 ds \\ &\geq \int_{\tau_k-\gamma}^{\tau_k-\gamma/2} |x(s)|^2 ds \geq \gamma\eta^2/2 \end{aligned}$$

and $|x_t|_2 \geq \eta\sqrt{\gamma/2}$. This then yields

$$\begin{aligned} V(\tau_M, x_{\tau_M}) &\leq W^*(\delta) - \sum_{k=1}^M \int_{\tau_k-\gamma/2}^{\tau_k} W(|x_s|_2) ds \\ &\leq W^*(\delta) - MW(\eta\sqrt{\gamma/2})\gamma/2 < 0, \end{aligned}$$

a contradiction. Thus, such s_m exists and

$$\begin{aligned} V(t, x_t) &\leq V(s_m, x_{s_m}) \leq W_2(|x(s_m)|) + W_3(|x_{s_m}|_2) \\ &\leq W_2(\eta) + W_3(\eta) < W_1(\varepsilon) \end{aligned}$$

for $t \geq \tau_M \geq s_m$. We then have $|x(t)| < \varepsilon$ for $t \geq t_0 + T \geq \tau_M$, $T = KJM$, and the zero solution of (1.1) is UAS.

3. EXAMPLE

Consider the scalar equation

$$(3.1) \quad x'(t) = -a(t)x(t) + b(t)x(t-1)$$

where $a(t) = 3(|\sin(\pi t/2)| - \sin(\pi t/2))^2 g^2(t)$,

$$b(t) = 2(|\sin(\pi t/2)| - \sin(\pi t/2))(|\cos(\pi t/2)| + \cos(\pi t/2))g(t)g(t-1)$$

and $g: R \rightarrow R^+$ is any continuous function with $1 \leq g(t)$ for all $t \in [4k - \frac{3}{2}, 4k - \frac{1}{2}]$ and $g(t) \leq B$ on $[4k - \frac{3}{2} - \sigma, 4k - \frac{1}{2}]$ for some positive numbers σ and B , where $k = 1, 2, \dots$. Then the zero solution of (3.1) is AS.

Proof. Define

$$V(t, \varphi) = \frac{1}{2}|\varphi(0)|^2 + \frac{1}{3} \int_{-1}^0 a(t+s)\varphi^2(s) ds$$

for $(t, \varphi) \in R^+ \times C$ and $W_1(r) = r^2/2$. It follows that

$$(3.2) \quad W_1(|\varphi(0)|) \leq V(t, \varphi).$$

If $x(t) = x(t, t_0, \varphi)$ is a solution of (3.1), then

$$V(t, x_t) = \frac{1}{2}|x(t)|^2 + \frac{1}{3} \int_{t-1}^t a(s)|x(s)|^2 ds$$

and

$$\begin{aligned} V'_{(3.1)}(t, x_t) &= x(t)[-a(t)x(t) + b(t)x(t-1)] \\ &\quad + \frac{1}{3}a(t)x^2(t) - \frac{1}{3}a(t-1)x^2(t-1) \\ &= -\frac{2}{3}a(t)x^2(t) + b(t)x(t)x(t-1) \\ &\quad - \frac{1}{3}a(t-1)x^2(t-1). \end{aligned}$$

Notice that $a(t-1) = 3(|\cos(\pi t/2)| + \cos(\pi t/2))^2 g^2(t-1)$ and $b(t)x(t)x(t-1) \leq \frac{1}{3}a(t)x^2(t) + \frac{1}{3}a(t-1)x^2(t-1)$. Thus,

$$(3.3) \quad V'_{(3.1)}(t, x_t) \leq -\frac{1}{3}a(t)x^2(t)$$

for $t \geq t_0$ and

$$(3.4) \quad V'_{(3.1)}(t, x_t) \leq -2x^2(t) \quad \text{on } [t_k - 1, t_k]$$

where $t_k = 4k - \frac{1}{2}$. Define $W_2(r) = r^2/2$ and $W_3(r) = 4B^2r^2$. Since $g(t) \leq B$ on $[t_k - 1 - \sigma, t_k]$, it follows that

$$(3.5) \quad V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|\varphi|_2) \quad \text{for } t \in [t_k - \sigma, t_k].$$

By (3.2) through (3.5), all conditions of Theorem 2.1 are satisfied. Therefore, the zero solution of (3.1) is AS.

Remark 3.1. For $V(t, \varphi)$ defined above, it is clear that $V'_{(3.1)}(t, x_t) \equiv 0$ on $[4(k-1), 4k-2]$. If we choose $g(4k - \frac{1}{4}) = k$, then $V(t, \varphi)$ is not bounded for fixed φ .

Remark 3.2. If g is bounded in the above example, then the zero solution of (3.1) is UAS.

Remark 3.3. The asymptotic stability for (3.1) discussed above will not follow from Hale [6, p. 108] because he requires that $a(t) \geq \delta > 0$ and $(2a(t) - \delta)\delta - b^2(t) \geq \gamma$. It will not follow from the work of Burton and Hatvani [4, Corollaries 3.1–3.3] because they require that $\int_{t-1}^t a^2(s) ds$ be bounded in Corollary 3.1, $\int_{t-1}^t b^4(s) ds$ be bounded in Corollary 3.2, and $\liminf_{t \rightarrow +\infty} a(t) > 0$ in Corollary 3.3. It will not follow from the result of Burton and Makay [5] because they require a growth condition on $a(t)$ and $b(t)$.

Remark 3.4. For brevity, we omit examples for Theorems 2.2 and 2.4. We refer to [3, p. 84] and [4, p. 289] for equations with integral terms such as

$$x'(t) = -a(t)x(t) + b(t) \int_{t-h}^t \lambda(s)x(s) ds.$$

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FAYETTEVILLE STATE UNIVERSITY,
FAYETTEVILLE, NORTH CAROLINA 28301

E-mail address: bozhang@hazel.fsufay.edu