

NORM ESTIMATES FOR RADially SYMMETRIC SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. The semilinear elliptic equation $\Delta u + f(u) = 0$ in R^n with the condition $\lim_{|x| \rightarrow \infty} u(x) = 0$ is studied, where $n \geq 2$ and $f(u)$ has a superlinear and subcritical growth at $u = \pm\infty$. For example, the functions $f(u) = |u|^{p-1}u - u$ ($1 < p < \infty$ if $n = 2$, $1 < p < (n+2)/(n-2)$ if $n \geq 3$) and $f(u) = u \log |u|$ are treated. The L^2 and H^1 norm estimates $C_1(k+1)^{n/2} \leq \|u\|_{L^2} \leq \|u\|_{H^1} \leq C_2(k+1)^{n/2}$ are established for any radially symmetric solution u which has exactly $k \geq 0$ zeros in the interval $0 \leq |x| < \infty$. Here $C_1, C_2 > 0$ are independent of u and k .

1. INTRODUCTION

In this paper we give the $L^2(\mathbf{R}^n)$ and $H^1(\mathbf{R}^n)$ norm estimates for radially symmetric solutions of the semilinear elliptic problem

$$(1.1) \quad \Delta u + f(u) = 0, \quad x \in \mathbf{R}^n,$$

$$(1.2) \quad \lim_{|x| \rightarrow \infty} u(x) = 0,$$

where $n \geq 2$ and $f(u)$ is a continuous function. We are interested in the power nonlinearity $f(u) = |u|^{p-1}u - u$ and the logarithmic nonlinearity $f(u) = u \log |u|$. The solution of the problem (1.1)–(1.2) is considered as a bound state of the nonlinear Schrödinger equation

$$(1.3) \quad -i \frac{\partial v}{\partial t} = \Delta v + g(v),$$

or a standing wave solution of the nonlinear Klein–Gordon equation

$$(1.4) \quad \frac{\partial^2 v}{\partial t^2} = \Delta v - m^2 v + g(v).$$

Here $g(v)$ is a continuous function from $\mathbf{C}$ to $\mathbf{C}$ satisfying

$$(1.5) \quad g(re^{i\theta}) = e^{i\theta} g(r) \quad (r, \theta \in \mathbf{R}), \quad g(\mathbf{R}) \subset \mathbf{R}.$$

In equation (1.3) or (1.4), the solution v of the form $v(x, t) = e^{i\lambda t} w(x)$ is called a bound state or a standing wave solution if $w \not\equiv 0$ is a square integrable

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real-valued function. Then equation (1.3) is reduced to the semilinear elliptic equation

$$(1.6) \quad \Delta w + g(w) - \lambda w = 0,$$

which becomes equation (1.1) by setting $f(w) = g(w) - \lambda w$. Equation (1.4) is also rewritten as equation (1.6) in which λ is replaced by $\lambda^2 - m^2$. Condition (1.2) is caused by $w \in L^2(\mathbf{R}^n)$. In equation (1.6) with a special type of the nonlinear term $g(w) = |w|^{p-1}w$ or $w \log |w|$, we employ the change of variable $w(x) = \lambda^{1/(p-1)}u(\sqrt{\lambda}x)$ or $w(x) = e^\lambda u(x)$, respectively. Then equation (1.6) is transformed into the equation

$$(1.7) \quad \Delta u + |u|^{p-1}u - u = 0,$$

or

$$(1.8) \quad \Delta u + u \log |u| = 0.$$

It has been proved by many authors [1, 2, 4, 7, 8, 10] that under suitable assumptions on $f(u)$, for example, $f(u) = |u|^{p-1}u - u$ with $1 < p < (n+2)/(n-2)$ or $f(u) = u \log |u|$, equation (1.1)–(1.2) has infinitely many radially symmetric solutions. More precisely, (see Grillakis [2] and Kajikiya [7]) equation (1.1)–(1.2) possesses a sequence $\{u_k\}_{k=0}^\infty \subset C^2(\mathbf{R}^n) \cap H^1(\mathbf{R}^n)$ of radially symmetric solutions such that u_k has exactly k zeros in the interval $0 \leq |x| < \infty$ and

$$\lim_{k \rightarrow \infty} \|u_k\|_{H^1} = \lim_{k \rightarrow \infty} L(u_k) = +\infty.$$

Here $H^1(\mathbf{R}^n)$ and $\|\cdot\|_{H^1}$ denote the usual L^2 Sobolev space of the first order and the corresponding norm, respectively. The functional $L(u)$ stands for the Lagrangian of (1.1), i.e.,

$$(1.9) \quad L(u) = \int_{\mathbf{R}^n} \left(\frac{1}{2} |\nabla u|^2 - F(u) \right) dx, \quad \text{where } F(u) = \int_0^u f(s) ds.$$

The above result raises a question: What is the relation between the Sobolev norm or the Lagrangian of radially symmetric solutions and the number of their zeros? The purpose of this paper is to solve the question. Our answer is as follows. Any radially symmetric solution u of (1.7) or (1.8) with (1.2) is estimated as

$$(1.10) \quad C_1(k+1)^{n/2} \leq \|u\|_{L^2} \leq \|u\|_{H^1} \leq C_2(k+1)^{n/2},$$

$$(1.11) \quad C_3(k+1)^n \leq L(u) \leq C_4(k+1)^n,$$

provided that u has exactly k (≥ 0) zeros in the interval $0 \leq |x| < \infty$. Here $\|u\|_{L^2}$ denotes the L^2 -norm of u . Each C_i is a positive constant independent of u and k . It will be proved later that any radially symmetric solution of (1.7) or (1.8) with (1.2) has at most a finite number of zeros.

To obtain the estimates (1.10) and (1.11), this paper is organized into six sections. In Section 2, we impose several assumptions on the nonlinear term $f(u)$ and state our theorem strictly. In Section 3, we prepare some technical lemmas which will play an important role in the subsequent sections. In Section 4, we investigate the various properties of radially symmetric solutions. In Section 5, we give the lower estimate for the Sobolev norms of solutions via

the number of zeros. In Section 6, we prove the upper estimate for the Sobolev norms of solutions.

2. MAIN RESULT

In this section, we state the main result. In order to discuss radially symmetric solutions $u = u(r)$, $r = |x|$, we convert (1.1)–(1.2) to the singular boundary value problem for the ordinary differential equation

$$(2.1) \quad u'' + \frac{n-1}{r}u' + f(u) = 0, \quad r > 0,$$

$$(2.2) \quad u'(0) = 0, \quad \lim_{r \rightarrow \infty} u(r) = 0.$$

In the following, we impose several assumptions on the function $f(s)$.

(f1) Suppose that $f(s)$ is continuous and that there exists a unique local solution to any initial value problem for equation (2.1).

It is well-known that assumption (f1) is fulfilled if $f(s)$ is locally Lipschitz continuous. The function $f(u) = u \log |u|$ in (1.8) is not locally Lipschitz continuous, however, condition (f1) is satisfied. This has been proved in [7, Appendix]. Next, we impose conditions on the sign of $f(s)$ and on the behavior of $f(s)$ as $s \rightarrow 0$.

(f2) There exist positive constants a and b such that

$$f(s) > 0, \quad s \in (-b, 0) \cup (a, \infty), \quad f(s) < 0, \quad s \in (-\infty, -b) \cup (0, a).$$

(f3) There exists a constant $q \geq 1$ such that

$$-\infty \leq \limsup_{s \rightarrow 0} \frac{f(s)}{|s|^{q-1}s} < 0 \quad \text{and} \quad \limsup_{s \rightarrow 0} \left| \frac{sf(s)}{F(s)} \right| < \infty,$$

where $F(s) = \int_0^s f(r)dr$.

The following assumptions imply that $f(s)$ has a superlinear and subcritical growth order in the neighborhood of $s = \pm\infty$.

$$(f4) \quad \lim_{s \rightarrow \pm\infty} \frac{f(s)}{s} = \infty.$$

$$(f5) \quad \begin{aligned} 0 < \liminf_{s \rightarrow \pm\infty} \frac{sf(s)}{F(s)} &\leq \limsup_{s \rightarrow \pm\infty} \frac{sf(s)}{F(s)} < \infty && \text{if } n = 2, \\ 0 < \liminf_{s \rightarrow \pm\infty} \frac{sf(s)}{F(s)} &\leq \limsup_{s \rightarrow \pm\infty} \frac{sf(s)}{F(s)} < \frac{2n}{n-2} && \text{if } n \geq 3. \end{aligned}$$

It is easy to check that all of the conditions (f1)–(f5) are satisfied by the functions, for example, $f(s) = |s|^{p-1}s - s$, $f(s) = |s|^{p-1}s - |s|^{q-1}s$, where $1 < q < p < \infty$ if $n = 2$ and $1 < q < p < (n+2)/(n-2)$ if $n \geq 3$, and $f(s) = s \log |s|$. Before stating the main theorem, we present our earlier result which guarantees the existence of solutions.

Theorem 0 ([7, Theorem 4]). *Suppose that assumptions (f1), (f2), (f4) and (f5) hold. Then there exist sequences $\{u_k\}_{k=0}^\infty$ and $\{v_k\}_{k=0}^\infty$ of solutions to (2.1)–(2.2) such that u_k and v_k have exactly $k(\geq 0)$ zeros in $(0, \infty)$ and $u_k(0) > 0 > v_k(0)$.*

For any solution u of (2.1)–(2.2), it will be proved later on that $\nabla u \in L^2(\mathbf{R}^n)$ and $F(u) \in L^1(\mathbf{R}^n)$ and moreover that u has at most a finite number of zeros in $(0, \infty)$. Therefore the main result below makes sense.

Theorem 1. *Under assumptions (f1)–(f5), there exist positive constants C_i ($1 \leq i \leq 4$) such that*

$$C_1(k+1)^{n/2} \leq \|\nabla u\|_{L^2} \leq C_2(k+1)^{n/2}, \quad C_3(k+1)^n \leq L(u) \leq C_4(k+1)^n$$

for any solution u of (2.1)–(2.2) which has exactly $k(\geq 0)$ zeros. Here $L(u)$ is defined by (1.9). Moreover, if $f(u) = |u|^{p-1}u - u$, where $1 < p < \infty$ when $n = 2$ and $1 < p < (n+2)/(n-2)$ when $n \geq 3$, or if $f(u) = u \log |u|$, then we obtain the estimate

$$C_5(k+1)^{n/2} \leq \|u\|_{L^2} \leq \|u\|_{H^1} \leq C_6(k+1)^{n/2}.$$

Here C_5 and C_6 are also positive constants independent of u and k .

From the above result we see that the H^1 -norm of solution grows like $k^{n/2}$ as $k \rightarrow \infty$. The growth rate $n/2$ depends only on the dimension n , does not on the growth order of $f(u)$ as $u \rightarrow \pm\infty$.

On the other hand, in [5, 6] we have studied equation (2.1) in the unit ball:

$$(2.3) \quad u'' + \frac{n-1}{r}u' + f(u) = 0, \quad 0 < r < 1,$$

$$(2.4) \quad u'(0) = u(1) = 0,$$

where $f(u)$ behaves like $|u|^{p-1}u$ as $u \rightarrow \pm\infty$ and $1 < p < \infty$ for $n = 2$ and $1 < p < (n+2)/(n-2)$ for $n \geq 3$. Then we have obtained the estimate

$$C_1(k+1)^{(p+1)/(p-1)} \leq \|u\|_{H^1} \leq C_2(k+1)^{(p+1)/(p-1)}$$

for any solution u of (2.3)–(2.4) with k zeros in $(0, 1)$. Here $\|u\|_{H^1}$ denotes the H^1 -norm of u on the unit ball and C_1 and C_2 are independent of u and k . In this case, the growth rate $(p+1)/(p-1)$ depends only on the growth order of $f(u)$, does not on the dimension n .

3. PRELIMINARY LEMMAS

From here to the end of the paper, we always suppose that all of assumptions (f1) through (f5) hold. In this section, we prepare some technical but crucial lemmas. Let $F(s)$ be defined in assumption (f3), that is,

$$F(s) = \int_0^s f(r)dr.$$

Definition 3.1. By assumptions (f2) and (f4) the function $F(s)$ has exactly two zeros except for $s = 0$. We denote these zeros by $-B$ and A . That is,

$$-B < -b < 0 < a < A \quad \text{and} \quad F(A) = F(-B) = 0.$$

From the above definition it follows that

$$(3.1) \quad F(A) = F(-B) = F(0) = 0,$$

$$(3.2) \quad \begin{aligned} F(s) &< 0 & \text{for } s \in (-B, A) \setminus \{0\}, \\ F(s) &> 0 & \text{for } s \in (-\infty, -B) \cup (A, \infty). \end{aligned}$$

In what follows, we present three lemmas which will be used in Sections 5 and 6.

Lemma 3.2. *For any A_1 in (A, ∞) and any $-B_1$ in $(-\infty, -B)$ there exists a constant $\theta_0 \in (0, 1)$ such that*

$$F(s) \leq 2F(\theta s) \leq 2F(s) \quad \text{when } \theta s \in (-\infty, -B_1] \cup [A_1, \infty) \text{ and } \theta \in (\theta_0, 1).$$

Proof. We show that there exists a positive constant C depending only on A_1 and B_1 such that

$$(3.3) \quad (1 + C \log \theta)F(s) \leq F(\theta s)$$

provided that $\theta \in (0, 1)$ and $\theta s \in (-\infty, -B_1] \cup [A_1, \infty)$.

By assumption (f5) together with (3.2) there exists a constant $C > 0$ such that $sf(s) \leq CF(s)$ for $s \in (-\infty, -B_1] \cup [A_1, \infty)$. Let $\theta \in (0, 1)$ and $\theta s \geq A_1 > 0$. Then we have

$$F(s) = F(\theta s) + \int_{\theta s}^s f(r)dr \leq F(\theta s) + C \int_{\theta s}^s \frac{F(r)}{r} dr.$$

Since $F(r)$ is strictly increasing in (a, ∞) by (f2), the last integral is estimated as

$$\int_{\theta s}^s \frac{F(r)}{r} dr \leq -F(s) \log \theta.$$

Therefore we obtain

$$F(s) \leq F(\theta s) - C(\log \theta)F(s).$$

The same argument as above is also valid for $\theta s \in (-\infty, -B_1]$. Thus we have (3.3). In the inequality (3.3) we choose θ_0 to be sufficiently closed to 1 such that $-1/2 < C \log \theta_0 < 0$. Then we obtain the first inequality of the lemma. The second one follows from the monotonicity of $F(s)$. Hence the proof is complete.

Lemma 3.3. *Let $A_1 \in (A, \infty)$ and $-B_1 \in (-\infty, -B)$. Then there exist positive constants C_1, C_2 and p such that*

$$1 < p < \infty \quad \text{when } n = 2, \quad 1 < p < \frac{n+2}{n-2} \quad \text{when } n \geq 3,$$

$$(3.4) \quad sf(s) \leq C_1 F(s) \leq C_2 |s|^{p+1} \quad \text{for } s \in (-\infty, -B_1] \cup [A_1, \infty).$$

Proof. We choose a suitable constant $p > 1$ by (f5) such that

$$\limsup_{s \rightarrow \pm\infty} \frac{sf(s)}{F(s)} < p + 1 < \infty \quad \text{if } n = 2,$$

$$\limsup_{s \rightarrow \pm\infty} \frac{sf(s)}{F(s)} < p + 1 < \frac{2n}{n-2} \quad \text{if } n \geq 3.$$

In other words, there exists an $R > 0$ such that

$$\frac{sF'(s)}{F(s)} = \frac{sf(s)}{F(s)} < p + 1 \quad \text{for } |s| \geq R.$$

Solving the differential inequality, we obtain (3.4) for $|s| \geq R$. For s in $[-R, -B_1] \cup [A_1, R]$, we use $F(s) > 0$ to get (3.4). This completes the proof.

In Section 5, we use a nonlinear variant of Prüfer transformation to estimate the Sobolev norms of solutions. To do so, we need the next lemma.

Lemma 3.4. *Let $-b_1$ and a_1 be fixed in $(-B, -b)$ and in (a, A) , respectively. Then there exist two continuous functions $g(s)$ and $h(s)$ satisfying the following conditions.*

(i) $f(s) = g(s) - h(s)$ for $s \in \mathbf{R}$, $f(s) = g(s)$ for $s \in (-\infty, -b_1] \cup [a_1, \infty)$.

(ii) It holds that $sg(s), G(s), H(s) > 0$ for any $s \neq 0$, where

$$G(s) = \int_0^s g(r)dr \quad \text{and} \quad H(s) = \int_0^s h(r)dr.$$

(iii) The function $f(s)/g(s)$ is bounded above.

(iv) $\frac{1}{2}H(s) \geq h(s)\frac{G(s)}{g(s)}$ for $s \in \mathbf{R}$.

Proof. By assumption (f3) there exist positive constants C, δ and r such that

$$C|F(s)| \geq |sf(s)| \quad \text{and} \quad F(s) \leq -\delta|s|^{q+1} \quad \text{for } s \in (-r, r).$$

Since $|F(s)| = -F(s) > 0$ for $s \in [-b_1, a_1] \setminus \{0\}$ by (3.2), the above inequality gives a constant C_0 such that

$$(3.5) \quad C_0|F(s)| \geq |sf(s)| \quad \text{for } s \in [-b_1, a_1]$$

and

$$(3.6) \quad C_0|F(s)| \geq |s|^{q+1} \quad \text{for } s \in [-b_1, a_1].$$

We now choose μ so large that

$$(3.7) \quad \mu > 2q - 1 \quad \text{and} \quad \frac{\mu + 1}{2C_0} > \max \{f(a_1)a_1^{-q}, |f(-b_1)|b_1^{-q}, 2\}.$$

We moreover define $\bar{a}, \bar{b} > 0$ by

$$(3.8) \quad \bar{a} = f(a_1)a_1^{-\mu} \quad \text{and} \quad \bar{b} = -f(-b_1)b_1^{-\mu}.$$

Then we define the function $g(s)$ by

$$(3.9) \quad g(s) = \begin{cases} f(s), & s \in (-\infty, -b_1] \cup [a_1, \infty), \\ \bar{b}|s|^{\mu-1}s, & s \in [-b_1, 0], \\ \bar{a}|s|^{\mu-1}s, & s \in [0, a_1]. \end{cases}$$

We set $h(s) = g(s) - f(s)$. Then it is clear that g and h are continuous and satisfy condition (i). From the definition of $g(s)$ it follows that $sg(s), G(s) > 0$ for $s \neq 0$. Noting that $H(s) = G(s) - F(s)$ and $F(s) < 0$ for $s \in (0, a_1]$, we see that $H(s) > 0$ for $s \in (0, a_1]$. Since $H'(s) = h(s) = 0$ ($s \geq a_1$), it holds that $H(s) = H(a_1) > 0$ for $s \geq a_1$. Thus we have that $H(s) > 0$ for $s > 0$. The same argument implies that $H(s) > 0$ for $s < 0$ as well.

We prove assertion (iii). From the definition of $g(s)$ it follows that $f(s)/g(s) < 0$ for $s \in (0, a)$ and $f(s)/g(s) = 1$ for $s \in [a_1, \infty)$. Since $g(s) > 0$ on $[a, a_1]$, the function $f(s)/g(s)$ is bounded above on $(0, \infty)$. The boundedness above on $(-\infty, 0)$ follows from the same argument as above.

We show assertion (iv). Since $H(s) > 0, h(s) = 0$ for $s \in (-\infty, -b_1] \cup [a_1, \infty)$, the inequality (iv) holds for s in these intervals. Let $s \in (0, a_1)$. Then we use (3.9) to rewrite inequality (iv) as

$$\frac{1}{2}(\bar{a}s^{\mu+1} - (\mu + 1)F(s)) \geq \bar{a}s^{\mu+1} - sf(s).$$

We note that $|F(s)| = -F(s) > 0$. Therefore

$$(3.10) \quad (\mu + 1)|F(s)| \geq \bar{a}s^{\mu+1} - 2sf(s) \quad \text{for } s \in (0, a_1).$$

To show this, it is sufficient to prove the inequalities

$$(3.11) \quad \frac{\mu + 1}{2}|F(s)| \geq \bar{a}s^{\mu+1}$$

and

$$(3.12) \quad \frac{\mu + 1}{2}|F(s)| \geq 2|sf(s)|$$

for $s \in (0, a_1)$. The above inequality (3.12) follows readily from (3.5) and (3.7). We show (3.11). It follows from (3.7) and (3.8) that

$$\bar{a}s^{\mu+1} \leq \bar{a}a_1^{\mu-q}s^{q+1} = f(a_1)a_1^{-q}s^{q+1} \leq \frac{\mu + 1}{2C_0}s^{q+1} \quad \text{for } s \in (0, a_1).$$

This together with (3.6) yields (3.11). Therefore we obtain (3.10) and inequality (iv) for $s \geq 0$. The same method as above remains valid for $s \leq 0$. The proof is thereby complete.

4. PROPERTY OF SOLUTIONS

In this section, we investigate the property of solutions in detail. We first consider the initial value problem for (2.1) at $r = 0$,

$$(4.1) \quad u'' + \frac{n-1}{r}u' + f(u) = 0, \quad r > 0,$$

$$(4.2) \quad u'(0) = 0, \quad u(0) = \lambda.$$

Definition 4.1. We denote the solution of (4.1)–(4.2) by $u(r, \lambda)$.

We prove the global existence of the solution $u(r, \lambda)$ and the continuous dependence on the initial data λ .

Lemma 4.2. (i) For any $\lambda \in \mathbf{R}$, the solution $u(r, \lambda)$ can be extended to $r = \infty$. Moreover, we have for any $\Lambda > 0$,

$$\sup\{|u(r, \lambda)| + |u'(r, \lambda)| + |u''(r, \lambda)| : 0 \leq r < \infty, \quad |\lambda| \leq \Lambda\} < \infty.$$

(ii) The solution depends continuously on λ . More precisely, for any $\lambda_0 \in \mathbf{R}$ and $R > 0$, the solution $u(\cdot, \lambda)$ converges to $u(\cdot, \lambda_0)$ in $C^2[0, R]$ as $\lambda \rightarrow \lambda_0$.

Proof. This lemma has been proved in our earlier paper [7, Lemma 3.1]; however, for the sake of completeness we give the proof in the following. Let $\Lambda > 0$ be given and λ satisfy $|\lambda| \leq \Lambda$. The local solution $u(r) = u(r, \lambda)$ of (4.1)–(4.2) is uniquely defined because of assumption (f1). We define the energy function $E(r)$ by

$$(4.3) \quad E(r) = E(r, \lambda) \equiv \frac{1}{2}u'(r, \lambda)^2 + F(u(r, \lambda)),$$

$$F(s) = \int_0^s f(\tau)d\tau.$$

From an easy computation it follows that

$$(4.4) \quad E'(r) = -\frac{n-1}{r}u'(r)^2 \leq 0,$$

which implies that $E(r)$ is decreasing. Moreover, if $u(r)$ is not a stationary solution, then the zeros of $u'(r)$ are isolated and therefore $E(r)$ is strictly decreasing. Hence we have

$$\frac{1}{2}u'(r)^2 + F(u(r)) = E(r) \leq E(0) \leq \max_{|\lambda| \leq \Lambda} F(\lambda).$$

Noting that $\lim_{s \rightarrow \pm\infty} F(s) = \infty$ by (f4), we have a constant $C_\Lambda > 0$ such that

$$(4.5) \quad |u(r, \lambda)|, \quad |u'(r, \lambda)| \leq C_\Lambda \quad \text{for } r \geq 0 \text{ and } |\lambda| \leq \Lambda,$$

which implies that u can be extended to $r = \infty$.

We show u'' to be bounded. Equation (4.1) is rewritten as

$$(4.6) \quad (r^{n-1}u')' = -r^{n-1}f(u).$$

Integrating this relation over $[0, r]$, we get

$$(4.7) \quad r^{n-1}u'(r) = - \int_0^r f(u(s))s^{n-1}ds,$$

which together with (4.5) implies

$$(4.8) \quad |u'(r)| \leq \frac{C_1}{n}r, \quad \text{where } C_1 = \max_{|s| \leq C_\Lambda} |f(s)|.$$

Using this inequality in (4.1), we obtain

$$|u''(r)| \leq \frac{n-1}{r}|u'| + |f(u)| \leq \frac{n-1}{n}C_1 + C_1$$

for $r \geq 0$ and $|\lambda| \leq \Lambda$. This inequality and (4.5) prove assertion (i).

We prove assertion (ii). Let $R > 0$ and $\lambda_0 \in \mathbf{R}$ be given. Let $\{\lambda_i\}$ be a sequence converging to λ_0 . We write $u_i(r) = u(r, \lambda_i)$ for $i \geq 0$. Instead of showing the convergence of $\{u_i\}$ to u_0 in $C^2[0, R]$, we will prove the equivalent statement: *For any subsequence $\{v_j\}$ of $\{u_i\}$, there exists a subsequence $\{w_k\}$ of $\{v_j\}$ such that w_k converges to u_0 in $C^2[0, R]$.*

Let $\{v_j\}$ be any subsequence of $\{u_i\}$. Applying Ascoli-Arzelà's theorem together with assertion (i), we choose a subsequence $\{w_k\}$ of $\{v_j\}$ such that w_k converges to a certain limit w_0 in $C^1[0, R]$. We now recall that w_k satisfies (4.7), that is,

$$(4.9) \quad r^{n-1}w'_k(r) = - \int_0^r f(w_k(s))s^{n-1}ds.$$

Letting $k \rightarrow \infty$, we have

$$(4.10) \quad r^{n-1}w'_0(r) = - \int_0^r f(w_0(s))s^{n-1}ds,$$

which implies that $w_0 \in C^2(0, R]$ and w_0 satisfies (4.1). Since $w'_0(0) = \lim_{k \rightarrow \infty} w'_k(0) = 0$ and $w_0(0) = \lim_{k \rightarrow \infty} w_k(0) = \lambda_0$, it follows from (f1) that $w_0(r) \equiv u_0(r)$. We prove that w''_k converges uniformly to $u''_0 = w''_0$. The identities (4.9) and (4.10) yield

$$\frac{1}{r}|w'_k(r) - w'_0(r)| \leq n^{-1} \max_{0 \leq s \leq R} |f(w_k(s)) - f(w_0(s))|.$$

Therefore $(1/r)w'_k(r)$ converges uniformly on $[0, R]$, so does the right-hand side of

$$w''_k = -\frac{n-1}{r}w'_k - f(w_k(r)).$$

Thus w_k converges to $w_0 = u_0$ in $C^2[0, R]$ and this completes the proof.

The next lemma gives us information about zeros, zero derivatives and local extrema of solutions.

Lemma 4.3. *Let u satisfy (4.1) on $[0, \infty)$ and be not a stationary solution. Then we have:*

- (i) *If $u'(r_0) = 0$ for some r_0 , then $u''(r_0) \neq 0$ and $u(r)$ has a strict local maximum or minimum at r_0 .*
- (ii) *If $u(r)$ has a local maximum (or minimum) at r_0 , then $u(r) < u(r_0)$ (or $u(r) > u(r_0)$) for all $r \in (r_0, \infty)$.*
- (iii) *If $u(r_1) = u(r_2) = 0$ and $u(r) > 0$ (or < 0) for $r \in (r_1, r_2)$ with some r_1, r_2 , then there exists a unique $\xi \in (r_1, r_2)$ such that $u'(\xi) = 0$. Furthermore, it holds that $u(\xi) > A$ (or $u(\xi) < -B$) and that $u'(r) > 0$ (or < 0) for $r \in [r_1, \xi)$ and $u'(r) < 0$ (or > 0) for $r \in (\xi, r_2]$.*

Proof. If $u'(r_0) = u''(r_0) = 0$ for some $r_0 \geq 0$, then $f(u(r_0)) = 0$ by (4.1), hence $u(r) \equiv u(r_0)$ is a stationary solution. This contradicts the assumption of the lemma. Therefore we have assertion (i).

To prove (ii), it is sufficient to show that

$$u(r) \neq u(r_0) \quad \text{for } r \in (r_0, \infty) \quad \text{if } u'(r_0) = 0.$$

Since $E(r)$ is strictly decreasing and $u'(r_0) = 0$, we have $F(u(r_0)) = E(r_0) > E(r) \geq F(u(r))$ for $r > r_0$. Hence $u(r_0) \neq u(r)$.

We prove (iii). Let $u(r_1) = u(r_2) = 0$ and $u(r) > 0$ for $r \in (r_1, r_2)$. By Rolle's theorem there exists a point $\xi \in (r_1, r_2)$ such that $u'(\xi) = 0$. On the other hand, since $u(r)$ has only simple zeros and $E(r)$ is strictly decreasing, we have

$$E(r) > E(r_2) = \frac{1}{2}u'(r_2)^2 > 0 \quad \text{for } r \in (r_1, r_2).$$

Then it follows that $0 < E(\xi) = F(u(\xi))$. Therefore our assumption $u(r) > 0$ implies that $u(\xi) > A$. We substitute $r = \xi$ into (4.1) to get $u''(\xi) = -f(u(\xi)) < 0$ since $u(\xi) > A$. Hence we obtain the fact that $u''(\xi) < 0$ if $u'(\xi) = 0$ at some $\xi \in (r_1, r_2)$. This fact proves the uniqueness of zeros of u' and moreover that $u'(r) > 0$ for $r \in [r_1, \xi)$ and $u'(r) < 0$ for $r \in (\xi, r_2]$. The proof is thereby complete.

In the following lemma, we investigate the asymptotic behavior of solutions to (2.1)–(2.2) as $r \rightarrow \infty$ and proves the decay of the energy. Moreover we show that $\nabla u \in L^2(\mathbf{R}^n)$ and $F(u) \in L^1(\mathbf{R}^n)$, therefore the Lagrangian $L(u)$ is well-defined and Theorem 1 makes sense.

Lemma 4.4. *Let $u \not\equiv 0$ be a solution of (4.1) on $[0, \infty)$ which converges to 0 as $r \rightarrow \infty$. Denote by $E(r)$ the energy associated with u . Then the following assertions hold :*

- (i) *u has at most a finite number of zeros in $[0, \infty)$.*
- (ii) *$\lim_{r \rightarrow \infty} u'(r) = \lim_{r \rightarrow \infty} u''(r) = \lim_{r \rightarrow \infty} E(r) = 0$ and $E(r) > 0$ for $r \geq 0$.*
- (iii) *$\nabla u \in L^2(\mathbf{R}^n)$ and $uf(u), F(u) \in L^1(\mathbf{R}^n)$.*

(iv) $\sup_{r \geq 0} |u'(r)|r^{n-1} < \infty$ and $\lim_{r \rightarrow \infty} E(r)r^n = 0$.

Proof. Assertion (i) readily follows from Lemma 4.3 (iii) since $u(r)$ tends to 0 as $r \rightarrow \infty$. We prove assertion (ii). Since $F(s)$ is bounded below, so is $E(r)$. Recall that $E(r)$ is strictly decreasing, hence it has a certain limit E_∞ as $r \rightarrow \infty$. Then we have

$$\frac{1}{2}u'(r)^2 = E(r) - F(u(r)) \longrightarrow E_\infty \quad \text{as } r \rightarrow \infty,$$

which implies that $E_\infty \geq 0$ and $\lim_{r \rightarrow \infty} |u'(r)| = \sqrt{2E_\infty}$. If $E_\infty > 0$, then $\lim_{r \rightarrow \infty} u'(r) = \sqrt{2E_\infty}$ or $-\sqrt{2E_\infty}$. But this contradicts the boundedness of $u(r)$. Thus it follows that $E_\infty = 0$ and $\lim_{r \rightarrow \infty} u'(r) = 0$. Moreover it holds that $E(r) > 0$ for $r \geq 0$ since $E(r)$ is strictly decreasing. Letting $r \rightarrow \infty$ in (4.1), we see that $\lim_{r \rightarrow \infty} u''(r) = 0$.

We prove assertion (iii). By assertion (i) there exists a constant $R > 0$ such that

$$(A) \quad 0 < u(r) < a \quad (r \geq R),$$

or

$$(B) \quad -b < u(r) < 0 \quad (r \geq R).$$

We deal with case (A) only. However case (B) can be treated in the same way. Assertion (ii) means

$$(4.11) \quad E(r) = \frac{1}{2}u'(r)^2 + F(u(r)) > 0.$$

Since $F(u(r)) < 0$ in case (A), it holds that $u'(r)^2 > 0$, and so $u'(r) < 0$ for $r \geq R$.

On the other hand, we rewrite (4.1) as

$$(r^{n-1}u')' = -r^{n-1}f(u).$$

The right-hand side is positive because of (A). Therefore $r^{n-1}u'(r)$ (< 0) is increasing and $|u'(r)|r^{n-1}$ is decreasing in $[R, \infty)$. Hence we have

$$(4.12) \quad |u'(r)|r^{n-1} \leq C_0 \equiv |u'(R)|R^{n-1} \quad \text{for } r \geq R.$$

This inequality gives

$$\int_R^s u'(r)^2 r^{n-1} dr \leq C_0 \int_R^s |u'(r)| dr = -C_0 \int_R^s u'(r) dr \leq C_0 u(R).$$

Letting $s \rightarrow \infty$, we deduce that

$$\int_R^\infty u'(r)^2 r^{n-1} dr \leq C_0 u(R) < \infty.$$

Thus we obtain $\nabla u \in L^2(\mathbf{R}^n)$.

We show that $uf(u) \in L^1(\mathbf{R}^n)$. Multiplying (4.1) by $u(r)r^{n-1}$ and integrating it over $[R, r]$, we obtain

$$\begin{aligned} - \int_R^r uf(u)s^{n-1} ds &= - \int_R^r |u'|^2 s^{n-1} ds + u'(r)u(r)r^{n-1} - u'(R)u(R)R^{n-1} \\ &\leq -u'(R)u(R)R^{n-1}. \end{aligned}$$

Here we have used that $u'(r) < 0$ and $u(r) > 0$ for $r \geq R$ in case (A). Since $|uf(u)| = -uf(u)$ for $0 < u < a$, we have

$$\int_R^\infty |uf(u)|s^{n-1}ds \leq |u'(R)u(R)|R^{n-1}.$$

We show that $F(u) \in L^1(\mathbb{R}^n)$. Note that $F(u) < 0$ for $r \geq R$ in case (A). Then (4.11) implies

$$|F(u(r))| = -F(u(r)) \leq \frac{1}{2}u'(r)^2 \quad \text{for } r \geq R.$$

Hence the assertion $F(u) \in L^1$ follows from $\nabla u \in L^2$.

Finally, we prove assertion (iv). From (4.12) it follows that $\sup_{r \geq 0} |u'(r)|r^{n-1} < \infty$. We show that $\lim_{r \rightarrow \infty} E(r)r^n = 0$. Since $F(u(r)) < 0$ for $r \geq R$, we have that $0 < E(r) \leq u'(r)^2/2$. Therefore it is sufficient to prove that $\lim_{r \rightarrow \infty} u'(r)^2r^n = 0$. If $n \geq 3$, it follows from (4.12) that

$$u'(r)^2r^n \leq C_0^2 r^{-(n-2)} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Let $n = 2$. We want to show that $u'(r)^2r^n = (u'(r)r)^2$ tends to 0 as $r \rightarrow \infty$. Recall that $|u'(r)|r$ is decreasing in $[R, \infty)$, which has been proved in the previous statement to (4.12) with $n = 2$. Therefore we set $\delta = \lim_{r \rightarrow \infty} |u'(r)|r$. If $\delta > 0$, then $-u'(r) = |u'(r)| \geq \delta/r$ for $r \geq R$. Integrating both sides, we obtain

$$-u(r) + u(R) \geq \delta \log(r/R) \rightarrow \infty \quad \text{as } r \rightarrow \infty,$$

which contradicts that $\lim_{r \rightarrow \infty} u(r) = 0$. Hence $\lim_{r \rightarrow \infty} u'(r)r = 0$. This completes the proof.

Definition 4.5. We denote by S the set of all solutions $u \not\equiv 0$ of (2.1)–(2.2), that is,

$$S = \{u \in C^2[0, \infty) : u(r) \not\equiv 0 \text{ satisfies (2.1) and (2.2)}\}.$$

Moreover we define

$$S^+ = \{u \in S : u(0) > 0\}, \quad S^- = \{u \in S : u(0) < 0\},$$

$$S_k = \{u \in S : u \text{ has exactly } k \text{ zeros in } [0, \infty)\},$$

$$S_k^+ = S_k \cap S^+ \quad \text{and} \quad S_k^- = S_k \cap S^- \quad \text{for } k \geq 0.$$

We note that S_k^+ and S_k^- are nonempty by Theorem 0. Lemma 4.4 (i) implies that $S = \bigcup_{k=0}^\infty S_k$ and $S^\pm = \bigcup_{k=0}^\infty S_k^\pm$.

Combining Lemmas 4.3 and 4.4, we see the outline of the graph of solutions to (2.1)–(2.2) in the following lemma.

Lemma 4.6. Let $u \in S_k^+$. If $k = 0$, hence u is a positive solution, then $u'(r) < 0$ for $r \in (0, \infty)$. If $k \geq 1$ and $\{r_i\}_{i=1}^k$ ($0 < r_1 < r_2 < \dots < r_k$) denote the zeros of u , then u' has exactly $k+1$ zeros $\{t_i\}_{i=0}^k$ in $[0, \infty)$ such that $0 = t_0 < r_1 < t_1 < \dots < r_k < t_k$ and

$$u'(r) < 0 \quad \text{for } r \in (t_{2i}, t_{2i+1}), \quad u'(r) > 0 \quad \text{for } r \in (t_{2i+1}, t_{2i+2}),$$

$$(-1)^k u'(r) < 0 \quad \text{for } r \in (t_k, \infty),$$

$$u(t_{2i}) > A \quad \text{and} \quad u(t_{2i+1}) < -B \quad \text{for } i \geq 0.$$

The same result as above holds also for $u \in S_k^-$ after exchanging $u'(r) < 0$ for $u'(r) > 0$ and replacing the last two inequalities by $u(t_{2i}) < -B$ and $u(t_{2i+1}) > A$.

Proof. Let $u \in S_0^+$. Hence $u(r)$ is a positive solution of (4.1) which tends to 0 as $r \rightarrow \infty$. If $u''(0) > 0$, then Lemma 4.3 (ii) implies that $u(r) > u(0) > 0$ for all $r > 0$. This contradicts that $\lim_{r \rightarrow \infty} u(r) = 0$. Hence $u''(0) < 0$, and so $u'(r) < 0$ for $r > 0$ sufficiently small. If $u'(T) = 0$ at some first $T \in (0, \infty)$, then $u(r)$ has a local minimum at $r = T$. In fact, since $u'(r) < 0$ for $r < T$ and $u'(T) = 0$, it holds that $u''(T) \geq 0$, and therefore $u''(T) > 0$ by Lemma 4.3 (i). Then it follows from Lemma 4.3 (ii) that $u(r) > u(T) > 0$ for $r > T$, which contradicts that $\lim_{r \rightarrow \infty} u(r) = 0$. Consequently, we see that $u'(r) < 0$ for all $r > 0$. For $u \in S_k^+$ also, the above discussion together with Lemma 4.3 proves Lemma 4.6.

We have already proved in Lemma 4.2 that for each λ , the energy $E(r, \lambda)$ is decreasing in r and bounded below since $u(r, \lambda)$ and $u'(r, \lambda)$ are also bounded. Therefore $E(r, \lambda)$ has a limit as $r \rightarrow \infty$.

Lemma 4.7. *There exist constants A_* and B_* such that $A_* > A$, $B_* > B$ and*

$$\lim_{r \rightarrow \infty} E(r, \lambda) < 0 \quad \text{for any } \lambda \in (-B_*, A_*) \setminus \{0\}.$$

Proof. For $\lambda \in (-B, A) \setminus \{0\}$, we have

$$E(r, \lambda) \leq E(0, \lambda) = F(\lambda) < 0 \quad \text{for } r \geq 0.$$

Hence, for such λ the assertion holds. Let $\lambda = A$ and consider the energy $E(r, A)$. Since $u(r, A)$ is not a stationary solution, the corresponding energy is strictly decreasing. Hence we get

$$E(1, A) < E(0, A) = F(A) = 0.$$

Using Lemma 4.2 (ii), we find a constant A_* ($> A$) such that $E(1, \lambda) < 0$ for $\lambda \in [A, A_*]$, therefore $\lim_{r \rightarrow \infty} E(r, \lambda) < 0$. The existence of B_* is proved in the same way as above.

The next result is a direct consequence of Lemma 4.4 (ii) and Lemma 4.7.

Lemma 4.8. *It holds that*

$$u(0) \geq A_* \quad \text{for } u \in S^+ \quad \text{and} \quad u(0) \leq -B_* \quad \text{for } u \in S^-.$$

We need the lemma below in Definition 4.11 later on.

Lemma 4.9. *For any $\lambda > A$ (or $\lambda < -B$) there exists an $r > 0$ such that $u(r, \lambda) = A$ (or $u(r, \lambda) = -B$).*

Proof. We first set $\varepsilon_0 = \inf\{f(s) : s \geq A\}$, which is positive because of (f2) and (f4). Let $\lambda > A$ and $u(r)$ denote $u(r, \lambda)$. We next write $C = \sup\{|u'(r)| : r \geq 0\}$ by Lemma 4.2. In contradiction to the assertion of the lemma, suppose that $u(r) > A$ for all $r \geq 0$. Then (4.1) implies

$$u'' = -\frac{n-1}{r}u' - f(u) \leq \frac{n-1}{r}C - \varepsilon_0 \leq -\frac{\varepsilon_0}{2} \quad \text{for } r \text{ large enough,}$$

which implies that $\lim_{r \rightarrow \infty} u'(r) = -\infty$. This contradicts the boundedness of $u'(r)$ and the proof is complete.

Definition 4.10. Let A_* and B_* be determined by Lemma 4.7. We define positive constants A_1 , B_1 , θ_0 and θ in the following way. First, we fix A_1 and B_1 such that $-B_* < -B_1 < -B$ and $A < A_1 < A_*$. Secondly, the constant $\theta_0 \in (0, 1)$ is determined by Lemma 3.2. Lastly, we define the constant $\theta \in (0, 1)$ such that $\max(A_1/A_*, B_1/B_*, \theta_0) < \theta < 1$. Therefore the conclusions of Lemmas 3.2 and 3.3 hold.

We recall that equation (4.1) has a singularity at $r = 0$. To estimate the integral of $u'(r)^2 r^{n-1}$ over $[0, \infty)$, we divide the interval $[0, \infty)$ into a small neighborhood of $r = 0$ and the other interval. To this end, we introduce the notation $t(\lambda)$. Then the interval $[0, t(\lambda))$ means the small neighborhood of $r = 0$.

Definition 4.11. For any $\lambda \in (-\infty, -B_*] \cup [A_*, \infty)$ we define $t(\lambda)$ by

$$t(\lambda) = \min\{r > 0 : u(r, \lambda) = \theta\lambda\}.$$

The set on the right-hand side is nonempty. Indeed, if $\lambda \geq A_*$, then the definition of θ implies that $\theta\lambda > A_1 > A$, hence $u(0, \lambda) = \lambda > \theta\lambda > A$. Therefore Lemma 4.9 implies that the set is nonempty, and so $t(\lambda)$ is well-defined. From the definition of $t(\lambda)$ it follows that

$$(4.13) \quad \lambda \geq u(r, \lambda) \geq \theta\lambda > A_1 \quad \text{when } \lambda \geq A_* \text{ and } r \in [0, t(\lambda)],$$

$$(4.14) \quad \lambda \leq u(r, \lambda) \leq \theta\lambda < -B_1 \quad \text{when } \lambda \leq -B_* \text{ and } r \in [0, t(\lambda)].$$

In the two lemmas below, we present a few estimates for $t(\lambda)$, which will be useful for proving Lemma 4.14 later on and computing the integral of $u'(r)^2 r^{n-1}$ over $[0, t(\lambda)]$.

Lemma 4.12. We replace θ by a constant slightly less than 1, if necessary. Then it holds that $0 < t(\lambda) < 1$ for $\lambda \in (-\infty, -B_*] \cup [A_*, \infty)$.

Proof. We rewrite (4.1) as

$$(r^{n-1}u')' = -r^{n-1}f(u).$$

Integrating both sides twice, we have

$$u(0) - u(r) = \int_0^r \frac{1}{s^{n-1}} \int_0^s f(u(\tau)) \tau^{n-1} d\tau ds.$$

Let $u = u(r, \lambda)$ and substitute $r = t(\lambda)$. Then we get

$$(4.15) \quad (1 - \theta)\lambda = \int_0^{t(\lambda)} \frac{1}{s^{n-1}} \int_0^s f(u) \tau^{n-1} d\tau ds.$$

On the other hand, by assumption (f4) we have a constant $R > 0$ such that $f(s) \geq 2ns$ for $s \geq R$. Let $\lambda \geq A_*$ and $\theta \in (1/2, 1)$. If $\lambda \geq 2R$, then (4.13) implies that $R \leq \theta\lambda \leq u(r) \leq \lambda$ for $r \in (0, t(\lambda))$, and so (4.15) yields

$$(1 - \theta)\lambda \geq 2n \int_0^{t(\lambda)} \frac{1}{s^{n-1}} \int_0^s u(\tau) \tau^{n-1} d\tau ds \geq \theta\lambda t(\lambda)^2,$$

hence,

$$t(\lambda)^2 \leq (1 - \theta)/\theta < 1.$$

If $A_* \leq \lambda \leq 2R$, then $A_1 \leq \theta\lambda \leq u(r) \leq 2R$ for $r \in (0, t(\lambda))$. We now set $f_0 = \min\{f(s) : A_1 \leq s \leq 2R\} > 0$, and hence (4.15) implies

$$2(1 - \theta)R \geq f_0 \int_0^{t(\lambda)} \frac{1}{s^{n-1}} \int_0^s \tau^{n-1} d\tau ds = \frac{f_0}{2n} t(\lambda)^2.$$

Consequently, choosing θ to be sufficiently closed to 1, we have $t(\lambda) < 1$. For $\lambda \leq -B_*$ also, our method remains valid.

From now on, we always fix θ for which the conclusion of Lemma 4.12 is valid. In the next lemma, we estimate $t(\lambda)$ from below.

Lemma 4.13. *There exists a positive constant C such that*

$$Ct(\lambda) \geq |\lambda|F(\lambda)^{-1/2} \quad \text{for } \lambda \in (-\infty, -B_*] \cup [A_*, \infty).$$

Proof. Let $\lambda \geq A_*$ and $u(r) = u(r, \lambda)$. By (3.4) there exists a constant $C_1 > 0$ such that $sf(s) \leq C_1 F(s)$ for $s \in [A_1, \infty)$. We note that $A_1 \leq \theta\lambda \leq u(r) \leq \lambda$ for $r \in (0, t(\lambda))$ by (4.13). Then the right-hand side of (4.15) is estimated as

$$\int_0^{t(\lambda)} \frac{1}{s^{n-1}} \int_0^s f(u) \tau^{n-1} d\tau ds \leq C_1 \int_0^{t(\lambda)} \frac{1}{s^{n-1}} \int_0^s \frac{F(u)}{u} \tau^{n-1} d\tau ds \leq \frac{C_1 F(\lambda)}{2n\theta\lambda} t(\lambda)^2.$$

This inequality together with (4.15) implies the desired conclusion. In case of $\lambda \leq -B_*$ the above argument is also valid and the proof is complete.

We prove that the energy diverges to $+\infty$ as $\lambda \rightarrow \pm\infty$, which will be used in Section 6.

Lemma 4.14. *We have*

$$\lim_{\lambda \rightarrow \pm\infty} E(R, \lambda) = \infty \quad \text{for any } R \geq 0.$$

Proof. Since $E(r)$ is decreasing, it is sufficient to prove the lemma for $R \geq 1$. Let $R \geq 1$. We fix $\lambda \in \mathbf{R}$ arbitrarily and write $u(r) = u(r, \lambda)$ and $E(r) = E(r, \lambda)$. The lemma is proved by using Pohozaev's identity [9]:

$$(4.16) \quad 2E(R)R^n + (n-2)u'(R)u(R)R^{n-1} = \int_0^R (2nF(u) - (n-2)uf(u))r^{n-1} dr.$$

This relation is obtained in the following way. First, multiplying (4.1) by $u(r)r^{n-1}$ and integrating the resultant identity over $[0, R]$, we obtain

$$(4.17) \quad \int_0^R |u'|^2 r^{n-1} dr = \int_0^R uf(u)r^{n-1} dr + u'(R)u(R)R^{n-1}.$$

Next, multiplying (4.1) by $u'(r)r^n$ and integrating it over $[0, R]$, we have

$$(4.18) \quad \frac{n-2}{2} \int_0^R |u'|^2 r^{n-1} dr + E(R)R^n = n \int_0^R F(u)r^{n-1} dr.$$

Lastly, combining (4.17) and (4.18), we obtain (4.16).

On the other hand, there exist constants $\nu > 0$ and $C > 0$ such that

$$(4.19) \quad 2nF(s) - (n-2)sf(s) \geq \nu F(s) - C \quad \text{for } s \in \mathbf{R}.$$

Indeed, if $n = 2$, we choose $\nu = 2n = 4$ to get (4.19). If $n \geq 3$, we have a small constant $\nu > 0$ by (f5) such that

$$\frac{sf(s)}{F(s)} \leq \frac{2n - \nu}{n - 2} \quad \text{for } |s| \text{ large enough,}$$

which implies (4.19).

Since $\lim_{s \rightarrow \pm\infty} f(s)/s = \infty$, one finds a constant $C_0 > 0$ such that

$$(4.20) \quad s^2/2 \leq F(s) + C_0 \quad \text{for } s \in \mathbf{R}.$$

Hence we have

$$(4.21) \quad u'(R)u(R) \leq \frac{1}{2}u'(R)^2 + \frac{1}{2}u(R)^2 \leq E(R) + C_0.$$

Substituting (4.19) and (4.21) into (4.16), we obtain

$$(4.22) \quad C_1 E(R) + C_2 \geq \int_0^R F(u)r^{n-1} dr,$$

where C_1 and C_2 are positive constants depending only on R . We will estimate the right-hand side. Let $t = t(\lambda)$ be defined by Definition 4.11. From (4.20) it follows that $F(s) + C_0 \geq 0$ for $s \in \mathbf{R}$. Adding $(C_0/n)R^n$ to both sides of (4.22), we have

$$(4.23) \quad C_1 E(R) + C_3 \geq \int_0^R (F(u) + C_0)r^{n-1} dr,$$

where $C_3 = C_2 + (C_0/n)R^n$. Note that the integrand on the right-hand side is nonnegative and that $t(\lambda) \leq 1 \leq R$ by Lemma 4.12. Therefore we have

$$(4.24) \quad \int_0^R (F(u) + C_0)r^{n-1} dr \geq \int_0^{t(\lambda)} F(u)r^{n-1} dr.$$

Let $\lambda \geq A_*$. By (4.13) we have that $A_1 \leq \theta\lambda \leq u(r) \leq \lambda$ and $F(\theta\lambda) \leq F(u(r))$ for $r \in (0, t(\lambda))$. Hence Lemma 3.2 with $s = \lambda$ implies that

$$(4.25) \quad \int_0^{t(\lambda)} F(u)r^{n-1} dr \geq \int_0^{t(\lambda)} F(\theta\lambda)r^{n-1} dr \geq \frac{1}{2n}F(\lambda)t(\lambda)^n.$$

We use Lemmas 4.13 and 3.3 to estimate the right-hand side as

$$(4.26) \quad (2n)^{-1}F(\lambda)t(\lambda)^n \geq C_4 F(\lambda)^{-(n-2)/2} \lambda^n \geq C_5 \lambda^{n-(p+1)(n-2)/2}.$$

Combining (4.23), (4.24), (4.25) and (4.26), we obtain

$$C_1 E(R) + C_3 \geq C_5 \lambda^{n-(p+1)(n-2)/2} \longrightarrow \infty \quad \text{as } \lambda \rightarrow \infty.$$

Here we have used that $n - (p+1)(n-2)/2 > 0$ by Lemma 3.3. The argument above remains valid for the case where $\lambda \leq -B_*$. Thus the proof is complete.

Recall the fact that $E(r) > 0$ for $r \geq 0$ and $u \in S$. Then we have an a priori estimate connected with $|u'(r)|$ and $|u(r)|$.

Lemma 4.15. *There exist positive constants a_0 and C such that*

$$(4.27) \quad C|u'(r)| \geq |u(r)|^{(q+1)/2}$$

provided that $|u(r)| \leq a_0$ and $u \in S$. Here q is the constant in (f3).

Proof. By (f3) there exist positive constants m and a_0 such that $0 < a_0 < \min(a, b)$ and $F(s) \leq -m|s|^{q+1}$ for $|s| \leq a_0$. Let $u \in S$ and $r \geq 0$ be a point satisfying $|u(r)| \leq a_0$. Since $E(r) = |u'|^2/2 + F(u) > 0$ by Lemma 4.4 (ii), we have that $|u'|^2/2 - m|u|^{q+1} > 0$, which implies (4.27).

5. PROOF OF THE LOWER ESTIMATE

In this section we prove the proposition below.

Proposition 5.1. *There exists a constant $C > 0$ such that*

$$C(k+1)^n \leq \int_0^\infty |u'|^2 r^{n-1} dr \quad \text{for } u \in S_k.$$

The proof of Proposition 5.1 is based on a nonlinear variant of Prüfer transformation. See [3, p. 332] for the usual linear Prüfer transformation. Our goal is to find the relation between the L^2 -norm of ∇u for a solution u and the number of zeros of u . The former is a continuous quantity, but the latter is a discrete one. To connect these quantities, we rewrite equation (2.1) to the first order system

$$\begin{cases} u' = v, \\ v' = -\frac{n-1}{r}v - f(u). \end{cases}$$

Then the solution $(u(r), v(r))$ draws a smooth curve which does not contain the origin in the (u, v) -phase plain. Here we choose v -axis and u -axis as axes of abscissa and ordinate, respectively. Then we will prove later on that if $u(\tau) = 0$ at some τ , then the orbit $(u(r), v(r))$ intersects the v -axis at τ and moves counterclockwise around the origin as r increases in a small neighborhood of τ . Therefore, the number of zeros can be represented by the winding number of the orbit as r varies over the interval $[0, \infty)$. To calculate the winding number, we introduce polar coordinates and define the argument φ as below.

Definition 5.2. For $u \in S$ we define the functions $\rho(r)$ and $\varphi(r)$ by

$$(5.1) \quad u' = \rho \cos \varphi,$$

$$(5.2) \quad Y(u) = \rho \sin \varphi,$$

$$(5.3) \quad \varphi(0) = \frac{\pi}{2},$$

where $Y(s) = \operatorname{sgn}(s)\sqrt{G(s)}$ and $G(s)$ is defined by Lemma 3.4.

Observing Lemma 3.4 and (3.9), we see that $sY(s) > 0$ for $s \neq 0$ and moreover that $Y \in C^1(\mathbf{R})$, $Y'(s) = (1/2)|g(s)|G(s)^{-1/2}$ for $s \neq 0$ and $Y'(0) = 0$. The relations (5.1), (5.2) and (5.3) determine the functions $\rho(r)$ and $\varphi(r)$ of class $C^1[0, \infty)$ uniquely. In fact, since any nontrivial solution $u \neq 0$ of (2.1)–(2.2) has only simple zeros, we get

$$(5.4) \quad \rho(r) = \left(u'(r)^2 + Y(u)^2\right)^{1/2} = (u'(r)^2 + G(u))^{1/2} > 0 \quad \text{for } r \in [0, \infty).$$

Therefore $\rho(r)$ and $\varphi(r)$ are well-defined. Since the left-hand sides of (5.1) and (5.2) are continuously differentiable, the functions $\rho(r)$ and $\varphi(r)$ are of class $C^1[0, \infty)$ as well.

The definitions (5.1) and (5.2) transform (2.1) into a system of differential equations for (ρ, φ) . In particular, φ' satisfies the following equation.

Lemma 5.3. *For $u \in S$ we have*

$$(5.5) \quad \rho^2 \varphi' = Y'(u) \left\{ |u'|^2 + 2f(u) \frac{G(u)}{g(u)} \right\} + \frac{n-1}{r} u' Y(u).$$

Proof. Differentiating (5.1) with respect to r , we have

$$u'' = \rho' \cos \varphi - \rho \varphi' \sin \varphi.$$

Substituting this relation into (2.1) yields

$$(5.6) \quad \frac{n-1}{r} u' + f(u) = -\rho' \cos \varphi + \rho \varphi' \sin \varphi.$$

Next, we differentiate (5.2) to obtain

$$(5.7) \quad Y'(u)u' = \rho' \sin \varphi + \rho \varphi' \cos \varphi.$$

Multiplying (5.6) by (5.2) and (5.7) by (5.1), and then summing up these identities, we obtain

$$\rho^2 \varphi' = Y'(u)|u'|^2 + \frac{n-1}{r} u' Y(u) + f(u)Y(u).$$

Substituting $Y(u) = 2Y'(u)G(u)/g(u)$ into the last term on the above identity, we get the desired relation (5.5).

The next lemma implies that the orbit $(u', Y(u))$, hence (u', u) also, winds counterclockwise around the origin as r increases in a small neighborhood of each zero of u .

Lemma 5.4. *Let $u \in S$. Then for any zero τ of u there is an $\varepsilon > 0$ such that*

$$\varphi'(r) > 0 \quad \text{if } 0 < |r - \tau| < \varepsilon.$$

Therefore $\varphi(r)$ is strictly increasing in some neighborhoods of zeros of $u(r)$.

Proof. Since u has only simple zeros, $u'(\tau) \neq 0$ and $\tau \neq 0$. Substituting $Y(u) = 2Y'(u)G(u)/g(u)$ into (5.5), we get

$$\frac{\rho^2}{Y'(u)} \varphi' = |u'|^2 + \{2f(u) + \frac{2(n-1)}{r} u'\} \frac{G(u)}{g(u)}.$$

From the definition (3.9) of $g(s)$ it follows that

$$\frac{G(u(r))}{g(u(r))} = \frac{1}{\mu+1} u(r) \longrightarrow 0 \quad \text{as } r \rightarrow \tau,$$

and so,

$$\lim_{r \rightarrow \tau} \frac{\rho^2}{Y'(u)} \varphi' = u'(\tau)^2 > 0.$$

We note that $Y'(u) = (1/2)|g(u)|G(u)^{-1/2} > 0$ for $u \neq 0$. Therefore, if $|r - \tau| > 0$ is sufficiently small, then we obtain $\rho(r)^2 \varphi'(r) > 0$. This completes the proof.

In the following lemma, we show that for any solution u , the value of $\varphi(R_2) - \varphi(R_1)$ reflects the number of zeros of u in $[R_1, R_2]$. Therefore we have only to calculate the value of $\varphi(r)$ instead of the number of zeros.

Lemma 5.5. (i) *Let $u \in S_k$ and $\{r_j\}_{j=1}^k$ ($0 < r_1 < r_2 < \dots < r_k$) denote its zeros. Then we have*

$$\varphi(r_j) = j\pi \quad \text{for } 1 \leq j \leq k,$$

$$(j-1)\pi < \varphi(r) < j\pi \quad \text{for } r_{j-1} < r < r_j, \quad 1 \leq j \leq k+1,$$

where we set $r_0 = 0$ and $r_{k+1} = \infty$.

(ii) Let $u \not\equiv 0$ satisfy (2.1) and $u'(0) = 0$. Then it holds that $u \in S_k$ if and only if $\varphi(r)$ tends to $(k+1)\pi$ as $r \rightarrow \infty$.

Proof. Notice by (5.2) that $u(\tau) = 0$ if and only if $\varphi(\tau) = 0 \pmod{\pi}$. Consider the first zero r_1 of $u(r)$. Then it holds that $u(r) \neq 0$ ($0 < r < r_1$) and $u(r_1) = 0$, in other words, $\varphi(r) \neq 0 \pmod{\pi}$ for $0 < r < r_1$ and $\varphi(r_1) = 0 \pmod{\pi}$. Since $\varphi(0) = \pi/2$ by (5.3), it follows that $0 < \varphi(r) < \pi$ for $0 < r < r_1$ and $\varphi(r_1) = 0$ or π . Then Lemma 5.4 implies that $\varphi(r_1) = \pi$ and $\varphi(r) > \pi$ for r slightly larger than r_1 . Since $u(r) \neq 0$ for $r_1 < r < r_2$, it holds that $\pi < \varphi(r) < 2\pi$ for $r_1 < r < r_2$. Proceeding with this argument inductively, we obtain (i).

We prove assertion (ii). The definitions (5.1) and (5.2) imply

$$(5.8) \quad \tan \varphi = \frac{Y(u)}{u'}.$$

Let $u \in S_k$. Then assertion (i) gives

$$(5.9) \quad k\pi < \varphi(r) < (k+1)\pi \quad \text{for } r \in (r_k, \infty).$$

Observing Lemma 4.6, we see that

$$(5.10) \quad u(r)u'(r) < 0 \quad \text{for } r \in (t_k, \infty),$$

where t_k denotes the largest zero of $u'(r)$. In case of $k = 0$, we set $r_0 = 0$ in (5.9) and $t_0 = 0$ in (5.10). Since $sY(s) > 0$ for $s \neq 0$, it follows from (5.8) and (5.10) that

$$(5.11) \quad \tan \varphi(r) < 0 \quad \text{for } r \in (t_k, \infty).$$

Combining (5.9) and (5.11), we get

$$(5.12) \quad (k+1/2)\pi < \varphi(r) < (k+1)\pi \quad \text{for } r \in (t_k, \infty).$$

On the other hand, there is a constant $C > 0$ by (3.9) such that $G(u) \leq C|u|^{\mu+1}$ for $|u|$ sufficiently small. Then (5.8) together with Lemma 4.15 implies that

$$|\tan \varphi| = \frac{\sqrt{G(u)}}{|u'|} \leq C_1|u|^{(\mu-q)/2} \quad \text{for } r \text{ large enough.}$$

Since $\mu > 2q - 1 \geq q$ by (3.7), $\tan \varphi(r)$ tends to 0 as $r \rightarrow \infty$. This together with (5.12) means

$$(5.13) \quad \lim_{r \rightarrow \infty} \varphi(r) = (k+1)\pi.$$

Conversely, suppose that $\varphi(r)$ satisfies (5.13). Then $\tan \varphi(r)$ tends to 0 as $r \rightarrow \infty$. Since $|u'(r)|$ is bounded by Lemma 4.2, we have

$$|Y(u)| = |u' \tan \varphi| \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

This implies $\lim_{r \rightarrow \infty} u(r) = 0$, i.e., $u \in S$. Then we have (5.10) for r large enough, and hence $\tan \varphi(r) < 0$ by (5.8). This fact with (5.13) implies that

$$(k+1/2)\pi < \varphi(r) < (k+1)\pi \quad \text{for all } r \text{ large enough.}$$

Therefore assertion (i) proves $u \in S_k$. Thus the proof is complete.

We now explain our plan to prove Proposition 5.1 and Theorem 1. Using (5.4) and dividing (5.5) by ρ^2 , we obtain

$$\varphi' = Z(u, u', r),$$

where Z is a function of variables u , u' and r , which can be written explicitly by (5.5). Integrating both sides over $[0, \infty)$, we have

$$(5.14) \quad \varphi(\infty) - \varphi(0) = \int_0^\infty Z(u, u', r) dr.$$

The left-hand side represents the number of zeros of u in $[0, \infty)$ by Lemma 5.5. The right-hand side will be estimated from above or below via the L^2 -norm of ∇u . More precisely, we will prove that

$$(5.15) \quad C_1 \|\nabla u\|_{L^2}^{2/n} \leq \int_0^\infty Z(u, u', r) dr \leq C_2 \|\nabla u\|_{L^2}^{2/n}.$$

To make a success of the computation above, we adopted the nonlinear transformation (5.2). Consequently, we can connect the number of zeros and the Sobolev norm by using (5.14) and (5.15).

In what follows, we prove (5.15). To do so, we alter (5.5) into a simpler form as below.

Lemma 5.6. *There exist positive constants C_1 and C_2 such that*

$$(5.16) \quad \varphi' \leq C_1 (g(u)/u)^{1/2} + (n-1) \frac{u' Y(u)}{r \rho^2},$$

$$(5.17) \quad \varphi' \geq C_2 (g(u)/u)^{1/2} + (n-1) \frac{u' Y(u)}{r \rho^2}.$$

Proof. From the definition (3.9) of $g(s)$ together with (f5), there exist constants $C_1, C_2 > 0$ such that

$$C_1 s g(s) \leq G(s) \leq C_2 s g(s) \quad \text{for } s \in \mathbf{R}.$$

Using this inequality in the relation $Y'(s) = (1/2)|g(s)|G(s)^{-1/2}$, we have

$$(5.18) \quad C_3 (g(s)/s)^{1/2} \leq Y'(s) \leq C_4 (g(s)/s)^{1/2} \quad \text{for some } C_3, C_4 > 0.$$

On the other hand, by Lemma 3.4 (iii) there exists a positive constant C such that $f(s)/g(s) \leq C$ for all $s \in \mathbf{R}$. Then we have

$$(5.19) \quad |u'|^2 + 2f(u) \frac{G(u)}{g(u)} \leq |u'|^2 + 2CG(u) \leq \max(1, 2C) \rho^2,$$

where we have used the relation $\rho^2 = |u'|^2 + G(u)$ by (5.4). Combining (5.18), (5.19) and (5.5), we obtain (5.16).

We prove (5.17). Using $f(u) = g(u) - h(u)$ and Lemma 3.4 (iv), we get

$$|u'|^2 + 2f(u) \frac{G(u)}{g(u)} = |u'|^2 + 2G(u) - 2h(u) \frac{G(u)}{g(u)} \geq |u'|^2 + 2G(u) - H(u).$$

Now, recall Lemma 4.4 (ii), that is,

$$E(r) \equiv \frac{1}{2}|u'|^2 + G(u) - H(u) > 0.$$

Then the two inequalities above yield

$$(5.20) \quad |u'|^2 + 2f(u) \frac{G(u)}{g(u)} \geq \frac{1}{2}|u'|^2 + G(u) \geq \frac{1}{2}\rho^2.$$

By (5.18), (5.20) and (5.5) we have (5.17). The proof is complete.

We integrate (5.16) over $[0, \infty)$ and estimate the integral of the first term on the right-hand side. To this end, we divide the interval $[0, \infty)$ into two domains I and J in the following way.

Definition 5.7. Let $u \in S$. Let a_0 be defined by Lemma 4.15. For $\delta \in (0, a_0)$ we set

$$I = I(\delta, u) = \{r \in (0, \infty) : |u(r)| \geq \delta\},$$

$$J = J(\delta, u) = \{r \in (0, \infty) : |u(r)| < \delta\}.$$

Lemma 5.8. For any $\nu > (q-1)/2$ there exists a positive constant C_ν such that

$$\int_J |u|^\nu dr \leq C_\nu \delta^{(2\nu-q+1)/2} (k+1) \quad \text{for } u \in S_k \text{ and } \delta \in (0, a_0).$$

Proof. Let $u \in S_k$ and $\{r_i\}_{i=1}^k$ ($0 < r_1 < r_2 < \dots < r_k$) denote the zeros of u . Recalling the outline of the graph of $u(r)$, that is, noting Lemma 4.6, we see that J is represented as

$$J = \bigcup_{i=1}^{k+1} (\alpha_i, \beta_i),$$

where

$$\alpha_i < r_i < \beta_i \quad (1 \leq i \leq k), \quad \alpha_{k+1} < \beta_{k+1} = \infty,$$

$$|u(\alpha_i)| = |u(\beta_i)| = \delta, \quad u(\alpha_i)u(\beta_i) < 0 \quad (1 \leq i \leq k),$$

$$|u(\alpha_{k+1})| = \delta, \quad \text{and} \quad u(\beta_{k+1}) = u(\infty) = 0.$$

We consider the interval (α_i, r_i) with $i \leq k$ in which $u(r) > 0$ and $u'(r) < 0$. Hence $u(\alpha_i) = \delta$ and $u(r_i) = 0$. Let $\sigma > 0$. Then Lemma 4.15 gives

$$-(u^\sigma)' = -\sigma u^{\sigma-1} u' = \sigma |u|^{\sigma-1} |u'| \geq C_\sigma |u|^{(2\sigma+q-1)/2}$$

for some $C_\sigma > 0$. We integrate both sides over $[\alpha_i, r_i]$ to obtain

$$C_\sigma \int_{\alpha_i}^{r_i} |u|^{(2\sigma+q-1)/2} dr \leq -u(r_i)^\sigma + u(\alpha_i)^\sigma = \delta^\sigma.$$

In the case where $u(r) < 0$ for $r \in (\alpha_i, r_i)$, moreover, in the interval (r_i, β_i) or $(\alpha_{k+1}, \beta_{k+1})$ even, the above estimate remains still valid. Summing up these inequalities on i from 1 to $k+1$, we have

$$C_\sigma \int_J |u|^{(2\sigma+q-1)/2} dr \leq (2k+1)\delta^\sigma.$$

Substituting $\sigma = (2\nu-q+1)/2 > 0$ into this inequality, we obtain the conclusion of Lemma 5.8.

By (3.9) there exists a constant $C > 0$ such that

$$(g(s)/s)^{1/2} \leq C|s|^{(\mu-1)/2} \quad \text{for } |s| \text{ sufficiently small.}$$

Using this inequality in Lemma 5.8, we have the following lemma.

Lemma 5.9. There exists a positive constant C independent of δ and k such that

$$\int_J (g(u)/u)^{1/2} dr \leq C\delta^{(\mu-q)/2} (k+1) \quad \text{for } u \in S_k \text{ and } \delta \in (0, a_0).$$

The above lemma gives the estimate for the integral over J . Next we need to estimate the integral over $I(\delta, u)$ of $(g(u)/u)^{1/2}$. To this end, we present a brief lemma on the Lebesgue integral of nonnegative and monotone functions.

Lemma 5.10. *Let f be a nonnegative function defined on $(0, \infty)$. Suppose that I is a measurable subset of $(0, \infty)$. Then the following assertions (i) and (ii) hold:*

(i) *If f is nondecreasing, then we have*

$$\int_0^{|I|} f(r) dr \leq \int_I f(r) dr.$$

(ii) *If f is nonincreasing, then we have*

$$\int_I f(r) dr \leq \int_0^{|I|} f(r) dr.$$

Here $|I|$ denotes the measure of I . Some of these integrals may be $+\infty$. But this lemma asserts that when the right-hand side is finite, so is the left-hand side, and hence the inequality is valid.

Proof. We prove (i) only, but the assertion (ii) can be treated in the same way. If $f(r) = 0$ a.e. on $(0, \infty)$, then the inequality is trivial. Suppose that there exists a $\delta > 0$ such that $f(r) > \delta$ for r large enough since f is nondecreasing and nonnegative. If $|I| = \infty$, both sides of the inequality (i) are equal to $+\infty$. Suppose that $|I| < \infty$. We first deal with the case where I is a bounded open set. Hence it is represented as $I = \bigcup_{i=1}^{\infty} (a_i, b_i)$. We set $I_k = \bigcup_{i=1}^k (a_i, b_i)$. Rearranging it, we have $I_k = \bigcup_{i=1}^k (\alpha_i, \beta_i)$ where $0 \leq \alpha_1 < \beta_1 \leq \alpha_2 < \dots < \beta_k < \infty$. Set $T_j = \sum_{i=1}^j (\beta_i - \alpha_i)$ for $1 \leq j \leq k$ and $T_0 = 0$. Then since f is nondecreasing, we have

$$\int_{\alpha_i}^{\beta_i} f(r) dr = \int_{T_{i-1}}^{T_i} f(r + \alpha_i - T_{i-1}) dr \geq \int_{T_{i-1}}^{T_i} f(r) dr.$$

The summation on i gives

$$\int_{I_k} f(r) dr \geq \int_0^{T_k} f(r) dr = \int_0^{|I_k|} f(r) dr.$$

Since $I_k \subset I$ and $f \geq 0$, we get

$$\int_I f(r) dr \geq \int_0^{|I_k|} f(r) dr.$$

Letting $k \rightarrow \infty$, we obtain (i). If I is an unbounded open set, then we use an approximate sequence of bounded open sets to get the conclusion.

Let I be a measurable set. Then there exists a sequence $\{I_k\}_{k=1}^{\infty}$ of open sets such that $I \subset I_k$ and $\lim_{k \rightarrow \infty} |I_k - I| = 0$. Therefore we have

$$\int_{I_k} f(r) dr \geq \int_0^{|I_k|} f(r) dr \geq \int_0^{|I|} f(r) dr.$$

Letting $k \rightarrow \infty$, we obtain assertion (i) and the proof is complete.

Applying the above lemma, we can estimate the integral of $(g(u)/u)^{1/2}$ over $I(\delta, u)$.

Lemma 5.11. *For any $\delta \in (0, a_0)$ there exists a constant C_δ depending on δ such that*

$$(5.21) \quad \int_I (g(u)/u)^{1/2} dr \leq C_\delta \left(\int_I u g(u) r^{n-1} dr \right)^{1/n}$$

for $u \in S$ and $I = I(\delta, u)$.

Proof. Let $\delta \in (0, a_0)$. Then there exist constants $m(> n)$ and $C_\delta(> 0)$ such that

$$(5.22) \quad (g(s)/s)^{m/2} \leq C_\delta s g(s) \quad \text{for } |s| \geq \delta.$$

Indeed, by Lemma 3.3 and Lemma 3.4 (i) we have a constant $m(> n)$ such that

$$p+1 < \frac{2m}{m-2} < \infty \quad \text{if } n=2, \quad p+1 < \frac{2m}{m-2} < \frac{2n}{n-2} \quad \text{if } n \geq 3,$$

$$s g(s) \leq C |s|^{2m/(m-2)} \quad \text{for } s \in (-\infty, -B_1] \cup [A_1, \infty).$$

The last inequality remains still valid for $|s| \geq \delta$ if we replace C by a larger constant C_δ . Then the above inequality is equivalent to (5.22).

We use Hölder's inequality to get

$$(5.23) \quad \int_I (g(u)/u)^{1/2} dr = \int_I (g(u)/u)^{1/2} r^{(n-1)/m} r^{-(n-1)/m} dr$$

$$\leq \left(\int_I (g(u)/u)^{m/2} r^{n-1} dr \right)^{1/m} \left(\int_I r^{-m'(n-1)/m} dr \right)^{1/m'},$$

where $1/m + 1/m' = 1$. By Lemma 5.10 (ii) the last term is estimated as

$$(5.24) \quad \int_I r^{-m'(n-1)/m} dr = \int_I r^{-(n-1)/(m-1)} dr$$

$$\leq \int_0^{|I|} r^{-(n-1)/(m-1)} dr = \frac{m-1}{m-n} |I|^{(m-n)/(m-1)}.$$

Here we note that $|I| < \infty$. Indeed, I is a bounded set because $u(r)$ tends to 0 as $r \rightarrow \infty$. Combining (5.22), (5.23) and (5.24), we obtain

$$(5.25) \quad \int_I (g(u)/u)^{1/2} dr \leq C_\delta \left(\int_I u g(u) r^{n-1} dr \right)^{1/m} |I|^{(m-n)/m}.$$

On the other hand, the application of Lemma 5.10 (i) yields

$$\int_I u g(u) r^{n-1} dr \geq D_\delta \int_I r^{n-1} dr \geq D_\delta \int_0^{|I|} r^{n-1} dr = (D_\delta/n) |I|^n,$$

where D_δ denotes the infimum of $s g(s)$ for $|s| \geq \delta$. The above inequality is rewritten as

$$(5.26) \quad |I| \leq D'_\delta \left(\int_I u g(u) r^{n-1} dr \right)^{1/n}.$$

By (5.25) and (5.26) we obtain the conclusion (5.21) and the proof is complete.

Summing up the inequalities in the assertions of Lemmas 5.9 and 5.11, we get the lemma below.

Lemma 5.12. *There exist constants $C, C_\delta > 0$ such that*

$$\int_0^\infty (g(u)/u)^{1/2} dr \leq C_\delta \left(\int_0^\infty u g(u) r^{n-1} dr \right)^{1/n} + C \delta^{(\mu-q)/2} (k+1)$$

for $u \in S_k$ and $\delta \in (0, a_0)$. Here C_δ depends only on δ , but C does not depend on anything.

Recall Definition 4.11. We modify the notation $t(\lambda)$ slightly, which has been defined for the initial data λ . But, in what follows we define $t(u)$ for any solution u ($\neq 0$) of (2.1) with (2.2).

Definition 5.13. Observing Lemma 4.8 and Definition 4.11, we define $t(u)$ by

$$t(u) = \min\{r > 0 : u(r) = \theta u(0)\} \quad \text{for } u \in S.$$

We now proceed to estimate the integral of the second term on the right-hand side of (5.16). The interval $[0, \infty)$ of integration is divided into three subintervals $[0, t(u))$, $[t(u), R)$ and $[R, \infty)$. Here R will be determined later on. First, in the interval $[0, t(u))$ the function $u'Y(u)/r\rho^2$ is negative. Indeed, Lemma 4.6 implies that $u'(r)u(r) < 0$ for $r \in (0, t(u))$ because $t(u)$ is less than the first zero r_1 of u . Since $sY(s) > 0$ for $s \neq 0$, it holds that $u'Y(u) < 0$ for $r \in (0, t(u))$. Next, we estimate the integral of $u'Y(u)/r\rho^2$ over $[t(u), R)$.

Lemma 5.14. For any $\varepsilon > 0$ and any $R \geq 1$ there exists a constant $C_{R,\varepsilon} > 0$ depending only on ε and R such that

$$\int_{t(u)}^R \frac{|u'Y(u)|}{r\rho^2} dr \leq C_{R,\varepsilon} \left(\int_0^{t(u)} F(u(r))r^{n-1} dr \right)^\varepsilon$$

for any $u \in S$.

Proof. Let $u \in S^+$. For simplicity, we use the notations $\lambda = u(0)$ and $t = t(u)$. We recall (4.25), i.e.,

$$(5.27) \quad \int_0^t F(u)r^{n-1} dr \geq \frac{1}{2n} F(\lambda)t^n.$$

On the other hand, from (5.4) it follows that

$$\frac{|u'Y(u)|}{\rho^2} \leq \frac{|u'|^2 + Y(u)^2}{2\rho^2} \leq \frac{1}{2},$$

and therefore

$$(5.28) \quad \int_t^R \frac{|u'Y(u)|}{r\rho^2} dr \leq \frac{1}{2} \int_t^R r^{-1} dr \leq \frac{1}{2} t^{-\varepsilon} \int_t^R r^{-1+\varepsilon} dr \leq \frac{1}{2\varepsilon} R^\varepsilon t^{-\varepsilon}.$$

We now show that there exist constants $\alpha > 0$ and $C_0 > 0$ such that

$$(5.29) \quad C_0 F(\lambda)t^n \geq t^{-\alpha}.$$

From Lemmas 4.13 and 3.3, it follows that

$$F(\lambda)t^{n+\alpha} \geq C_1 \lambda^{n+\alpha} F(\lambda)^{-(n+\alpha-2)/2} \geq C_2 \lambda^{n+\alpha-(p+1)(n+\alpha-2)/2},$$

where $C_1, C_2 > 0$ are independent of λ . We fix α so small that

$$2(p+1)/(p-1) - n > \alpha > 0,$$

and therefore $n + \alpha - (p + 1)(n + \alpha - 2)/2 > 0$. Then we obtain the inequality (5.29) by using $\lambda \geq A_*$. Combining (5.28), (5.29) and (5.27), we obtain

$$\begin{aligned} \int_t^R \frac{|u' Y(u)|}{r \rho^2} dr &\leq \frac{1}{2\varepsilon} R^\varepsilon t^{-\varepsilon} = C_1 (t^{-\alpha})^{\varepsilon/\alpha} \\ &\leq C_1 (C_0 F(\lambda) t^n)^{\varepsilon/\alpha} \leq C_2 \left(\int_0^t F(u) r^{n-1} dr \right)^{\varepsilon/\alpha}, \end{aligned}$$

where each C_i ($0 \leq i \leq 2$) depends only on R , α and ε . Since ε is an arbitrary positive number and α is a positive constant, we can replace ε/α by ε . This completes the proof.

The lemma above gives the estimate for the integral over $[t(u), R)$. In the next lemma, we estimate the integral over $[R, R_1)$ or $[R, \infty)$ of $|u' Y(u)|/r \rho^2$.

Lemma 5.15. *For any $\sigma > 0$ there exist constants $C, C_\sigma > 0$ such that*

$$\int_R^{R_1} \frac{|u' Y(u)|}{r \rho^2} dr \leq \frac{C_\sigma}{R} \int_R^{R_1} (g(u)/u)^{1/2} dr + C \sigma^{(\mu-2q+1)/2} (k+1)$$

for any $1 \leq R < R_1 \leq \infty$ and $u \in S_k$. Here C_σ depends only on σ , but C does not on anything.

Proof. Since $g(s)/s > 0$ for $s \neq 0$ and $\lim_{s \rightarrow \pm\infty} g(s)/s = \infty$, we set

$$C_\sigma = \sup\{(1/2)(s/g(s))^{1/2} : |s| \geq \sigma\}.$$

It suffices to prove the lemma for $\sigma \in (0, a_0)$ only. We use the notations $I = I(\sigma, u)$ and $J = J(\sigma, u)$ defined by Definition 5.7 with $\delta = \sigma$. Then for $r \in [R, R_1) \cap I(\sigma, u)$ it holds that

$$\frac{|u' Y(u)|}{r \rho^2} \leq \frac{|u'|^2 + Y(u)^2}{2r \rho^2} \leq \frac{1}{2R} \leq \frac{C_\sigma}{R} (g(u)/u)^{1/2},$$

and therefore

$$(5.30) \quad \int_{[R, R_1) \cap I} \frac{|u' Y(u)|}{r \rho^2} dr \leq \frac{C_\sigma}{R} \int_R^{R_1} (g(u)/u)^{1/2} dr.$$

Next we use that $r \geq R \geq 1$ for $r \in [R, R_1)$ and employ the inequality that $G(u) \leq C|u|^{\mu+1}$ for $|u|$ small enough because of (3.9). Then Lemma 4.15 gives for $r \in [R, R_1) \cap J$,

$$\frac{|u' Y(u)|}{r \rho^2} \leq \frac{|u'| G(u)^{1/2}}{|u'|^2} = \frac{G(u)^{1/2}}{|u'|} \leq C|u|^{(\mu-q)/2}.$$

Hence Lemma 5.8 with $J = J(\sigma, u)$ and $\nu = (\mu - q)/2$ implies that

$$(5.31) \quad \int_{[R, R_1) \cap J} \frac{|u' Y(u)|}{r \rho^2} dr \leq C \int_J |u|^{(\mu-q)/2} dr \leq C \sigma^{(\mu-2q+1)/2} (k+1).$$

Here we note that the assumption of Lemma 5.8 is fulfilled since $\nu = (\mu - q)/2 > (q - 1)/2$ by (3.7). Summing up (5.30) and (5.31), we obtain the conclusion.

We connect the two integrals of $ug(u)$ and $u'(r)^2$ as below.

Lemma 5.16. *There exists a constant $C > 0$ such that*

$$(5.32) \quad \int_0^\infty u g(u) r^{n-1} dr \leq C \int_0^\infty |u'|^2 r^{n-1} dr \quad \text{for } u \in S.$$

Proof. Let $u \in S$. We recall Lemma 4.4 (iv), i.e.,

$$\sup_{0 \leq r < \infty} |u'(r)| r^{n-1} < \infty.$$

Therefore, letting $R \rightarrow \infty$ in (4.17), we have

$$(5.33) \quad \int_0^\infty |u'|^2 r^{n-1} dr = \int_0^\infty u f(u) r^{n-1} dr.$$

We now set

$$K = \{r \in (0, \infty) : -b_1 < u(r) < a_1\} \quad \text{and} \quad L = (0, \infty) \setminus K,$$

where a_1 and b_1 are the constants fixed in Lemma 3.4, and so $a_1 \in (a, A)$ and $-b_1 \in (-B, -b)$. Then there exists a $C > 0$ independent of u such that

$$(5.34) \quad C \int_0^\infty |u'|^2 r^{n-1} dr \geq \int_K (-u f(u) + u g(u)) r^{n-1} dr.$$

This will be proved soon. Summing up (5.33) and (5.34), we have

$$(1+C) \int_0^\infty |u'|^2 r^{n-1} dr \geq \int_L u f(u) r^{n-1} dr + \int_K u g(u) r^{n-1} dr = \int_0^\infty u g(u) r^{n-1} dr,$$

where we have used that $f(u) = g(u)$ for $r \in L$ by Lemma 3.4 (i). Therefore we get (5.32).

We show (5.34). To this end, it suffices to prove the stronger assertion

$$(5.35) \quad C|u'|^2 \geq |u f(u)| + u g(u) \quad \text{for } r \in K.$$

Since $\limsup_{s \rightarrow 0} f(s)/|s|^{q-1} s < 0$ and $\limsup_{s \rightarrow 0} |s f(s)/F(s)| < \infty$, there exist constants $\delta_0 > 0$ and $C_i > 0$ ($1 \leq i \leq 4$) such that

$$(5.36) \quad |s f(s)| \leq C_1 |F(s)| = -C_1 F(s),$$

$$(5.37) \quad s g(s) \leq C_2 |s|^{\mu+1} \leq C_3 |s|^{q+1} \leq -C_4 F(s)$$

for $|s| \leq \delta_0$. We note that $-F(s) > 0$ for $s \in [-b_1, -\delta_0] \cup [\delta_0, a_1]$. Hence the inequalities (5.36) and (5.37) remain valid for $s \in [-b_1, a_1]$ if we replace each C_i by a larger constant. We sum up (5.36) and (5.37) to get

$$(5.38) \quad -C F(u) \geq |u f(u)| + u g(u) \quad \text{for } r \in K,$$

where C is independent of u . Since $E(r) = |u'|^2/2 + F(u) > 0$ by Lemma 4.4 (ii), we have

$$(5.39) \quad \frac{1}{2} u'(r)^2 \geq -F(u).$$

By (5.38) and (5.39) we obtain (5.35) and the proof is complete.

We are now in a position to give the proof of Proposition 5.1.

Proof of Proposition 5.1. Let $u \in S_k$. We define $t = t(u)$ by Definition 5.13. Integrating (5.16) over $[0, \infty)$ and using Lemma 5.5, we have

$$(5.40) \quad (k + 1/2)\pi \leq C_1 \int_0^\infty (g(u)/u)^{1/2} dr + (n-1) \int_0^\infty \frac{u' Y(u)}{r \rho^2} dr.$$

We estimate the second term on the right-hand side. In view of the outline of the graph of $u(r)$ (recall Lemma 4.6) we see that $u'(r)u(r) < 0$ for $0 < r < t(u)$, and hence

$$(5.41) \quad \int_0^t \frac{u' Y(u)}{r \rho^2} dr < 0.$$

Next, it follows from Lemmas 5.14 and 5.15 with $R = 1$ and $R_1 = \infty$ that

$$(5.42) \quad \left(\int_t^1 + \int_1^\infty \right) \frac{|u' Y(u)|}{r \rho^2} dr \leq C_\varepsilon \left(\int_0^t F(u) r^{n-1} dr \right)^\varepsilon \\ + C_\sigma \int_1^\infty (g(u)/u)^{1/2} dr + C \sigma^{(\mu-2q+1)/2} (k+1).$$

Here and in what follows, C_ε and C_σ denote various positive constants depending on ε and σ , respectively. However, C , C_1 and C_2 denote positive constants independent of ε and δ . We set $\varepsilon = 1/n$ and combine (5.40), (5.41) and (5.42) to get

$$\frac{1}{2}(k+1) \leq k + \frac{1}{2} \leq C_\sigma \int_0^\infty (g(u)/u)^{1/2} dr \\ + C_1 \left(\int_0^t F(u) r^{n-1} dr \right)^{1/n} + C_2 \sigma^{(\mu-2q+1)/2} (k+1).$$

We choose $\sigma > 0$ so small that $C_2 \sigma^{(\mu-2q+1)/2} < 1/4$ to get

$$\frac{1}{4}(k+1) \leq C_1 \int_0^\infty (g(u)/u)^{1/2} dr + C_2 \left(\int_0^t F(u) r^{n-1} dr \right)^{1/n}.$$

Using Lemma 5.12, we have

$$\frac{1}{4}(k+1) \leq C_\delta \left(\int_0^\infty u g(u) r^{n-1} dr \right)^{1/n} + C_1 \delta^{(\mu-q)/2} (k+1) \\ + C_2 \left(\int_0^t F(u) r^{n-1} dr \right)^{1/n}.$$

Fix $\delta > 0$ so small that $C_1 \delta^{(\mu-q)/2} < 1/8$. Then it follows that

$$(5.43) \quad ((k+1)/8)^n \leq C_1 \int_0^\infty u g(u) r^{n-1} dr + C_2 \int_0^t F(u) r^{n-1} dr.$$

We here recall Lemma 4.8, (4.13) and (4.14). Then it holds that

$$u(r) \geq A_1 \quad \text{when } r \in [0, t(u)] \text{ and } u \in S^+, \\ u(r) \leq -B_1 \quad \text{when } r \in [0, t(u)] \text{ and } u \in S^-.$$

Moreover, by (f5) and Lemma 3.4 there exists a constant $C > 0$ such that

$$F(s) \leq C s f(s) = C s g(s) \quad \text{for } s \in (-\infty, -B_1] \cup [A_1, \infty),$$

and so

$$F(u(r)) \leq C u(r) g(u(r)) \quad \text{for } r \in [0, t] \text{ and } u \in S.$$

Therefore it follows from (5.43) that

$$(k+1)^n \leq C \int_0^\infty u g(u) r^{n-1} dr,$$

which together with Lemma 5.16 yields

$$(k+1)^n \leq C \int_0^\infty |u'|^2 r^{n-1} dr.$$

This completes the proof of Proposition 5.1.

6. PROOF OF THE UPPER ESTIMATE

The purpose of this section is to prove the following proposition and to complete the proof of Theorem 1.

Proposition 6.1. *There exists a constant $C > 0$ such that*

$$\int_0^\infty |u'|^2 r^{n-1} dr \leq C(k+1)^n \quad \text{for } u \in S_k.$$

We slightly modify the notation $E(r, \lambda)$, which has been defined for the initial data λ . However, from now on the energy is defined for any solution u of (2.1) with (2.2).

Definition 6.2. We define $E(r, u)$ by

$$E(r, u) = \frac{1}{2} u'(r)^2 + F(u(r)) \quad \text{for } u \in S.$$

Recall $E(r, u) > 0$, which has been proved in Lemma 4.4. We consider the infimum of the energy at $r = 1$.

Lemma 6.3. *It holds that $\inf\{E(1, u) : u \in S\} > 0$.*

Proof. Suppose that the assertion is false. Then there exists a sequence $\{u_j\} \subset S$ such that $E(1, u_j)$ tends to 0 as $j \rightarrow \infty$. Therefore Lemma 4.14 implies that $\{u_j(0)\}$ is bounded. Set $\lambda_j = u_j(0)$. We choose a convergent subsequence (denoted again by λ_j) of $\{\lambda_j\}$ and denote its limit by λ_0 . We set $u_0(r) = u(r, \lambda_0)$. Then it follows that $E(1, u_0) = 0$. The function $u_0(r)$ satisfies equation (2.1) but may not tend to 0 as $r \rightarrow \infty$, and hence $u_0(r)$ does not need to satisfy $E(r, u_0) > 0$. On the other hand, since $\lambda_j \geq A_*$ or $\lambda_j \leq -B_*$ by Lemma 4.8, it holds that $\lambda_0 \neq -b, 0, a$. Therefore $E(r, u_0)$ is strictly decreasing, and in particular $E(2, u_0) < 0$. The convergence of $\{u_j\}$ to u_0 in $C^2[0, 2]$ implies that $E(2, u_j) < 0$ for j large enough. But this result contradicts the fact that $E(r, u) > 0$ for $r \geq 0$ and $u \in S$. This contradiction completes the proof.

Definition 6.4. We define the constant e_0 by

$$e_0 = \inf\{E(1, u) : u \in S\} > 0.$$

We note that for each $u \in S$, the energy $E(r, u)$ is strictly decreasing and converges to 0 as $r \rightarrow \infty$. For $u \in S$ and $e \in (0, e_0)$, we denote by $T(u, e)$ the unique point r which satisfies $E(r, u) = e$. From the definition of e_0 , it follows that $T(u, e) > 1$.

The next proposition plays the most important role in the proof of Proposition 6.1.

Proposition 6.5. (i) If $e \in (0, e_0)$ is sufficiently small, then we have

$$\int_T^\infty F(u)r^{n-1}dr < 0 \quad \text{for } T = T(u, e) \quad \text{and } u \in S.$$

(ii) For any $e > 0$ sufficiently small, there exists a constant $C(e) > 0$ depending only on e such that

$$\int_T^\infty uf(u)r^{n-1}dr + C(e) \int_T^\infty F(u)r^{n-1}dr < 0 \quad \text{for } T = T(u, e) \quad \text{and } u \in S.$$

This proposition will be proved later on. In what follows, we always fix e so small that $0 < e < \min(e_0, 1)$ and that the conclusion of Proposition 6.5 is valid. The application of Proposition 6.5 gives the next a priori estimate related to the integrals of $u'(r)^2$ and $G(u)$.

Lemma 6.6. There is a constant $C > 0$ such that

$$(6.1) \quad \int_0^\infty |u'|^2 r^{n-1} dr \leq C \int_0^T G(u)r^{n-1} dr \quad \text{for } T = T(u, e) \quad \text{and } u \in S.$$

Proof. We recall that $E(r)r^n$ tends to 0 as $r \rightarrow \infty$ by Lemma 4.4 (iv). Hence letting $R \rightarrow \infty$ in (4.18), we obtain

$$(6.2) \quad (n-2) \int_0^\infty |u'|^2 r^{n-1} dr = 2n \int_0^\infty F(u)r^{n-1} dr.$$

We first deal with the case of $n = 2$. Then the left-hand side of (6.2) vanishes, and so we have

$$\int_0^\infty F(u)rdr = 0,$$

or equivalently

$$\int_T^\infty F(u)rdr = - \int_0^T F(u)rdr.$$

We substitute this relation into Proposition 6.5 (ii) with $n = 2$ to obtain

$$(6.3) \quad \int_T^\infty uf(u)rdr \leq C(e) \int_0^T F(u)rdr \leq C(e) \int_0^T G(u)rdr,$$

where we have used that $F(s) = G(s) - H(s) \leq G(s)$ for $s \in \mathbf{R}$ by Lemma 3.4 (ii).

We here prove that there is a constant $C_1 > 0$ such that

$$(6.4) \quad sf(s) \leq C_1 G(s) \quad \text{for } s \in \mathbf{R}.$$

Using (f5) and noting that $F(s) = G(s) - H(s) \leq G(s)$, we get a constant $C > 0$ satisfying

$$sf(s) \leq CF(s) \leq CG(s) \quad \text{for } |s| \text{ large enough.}$$

Next, it follows from (f2) that

$$sf(s) \leq 0 \leq G(s) \quad \text{for } s \in [-b, a].$$

Hence the two inequalities above prove (6.4) since $G(s) > 0$ for any $s \neq 0$. In view of (6.4) we have

$$(6.5) \quad \int_0^T u f(u) r dr \leq C_1 \int_0^T G(u) r dr.$$

Summing up (6.3) and (6.5), we obtain

$$(6.6) \quad \int_0^\infty u f(u) r dr \leq C \int_0^T G(u) r dr.$$

We recall (5.33) with $n = 2$, i.e.,

$$\int_0^\infty |u'|^2 r dr = \int_0^\infty u f(u) r dr.$$

This together with (6.6) gives (6.1) in case of $n = 2$.

We treat the case $n \geq 3$. Using (6.2) together with Proposition 6.5 (i), we have

$$\int_0^\infty |u'|^2 r^{n-1} dr \leq \frac{2n}{n-2} \int_0^T F(u) r^{n-1} dr \leq \frac{2n}{n-2} \int_0^T G(u) r^{n-1} dr,$$

where we have used that $F(s) \leq G(s)$ for $s \in \mathbf{R}$. The proof is now complete.

In view of the above lemma, we want to estimate the integral of $G(u)r^{n-1}$ over $[0, T]$ from above.

Lemma 6.7. *There exists a constant $C > 0$ such that*

$$(6.7) \quad \int_0^T G(u) r^{n-1} dr \leq C T^n \quad \text{for } T = T(u, e) \quad \text{and } u \in S.$$

Proof. We substitute $R = T$ into (4.16) and use the inequality (4.19). Then it follows that

$$2E(T)T^n + (n-2)u'(T)u(T)T^{n-1} \geq \nu \int_0^T F(u) r^{n-1} dr - \frac{C}{n} T^n.$$

We see that $|u'(T)|$ and $|u(T)|$ are bounded since $E(T) = e < 1$. Therefore we have

$$C T^n + C T^{n-1} \geq \nu \int_0^T F(u) r^{n-1} dr.$$

We find a constant $C_0 > 0$ such that $F(s) = G(s) - H(s) \geq G(s) - C_0$ for $s \in \mathbf{R}$ because $H(s)$ is bounded on $\mathbf{R}$. Since $T \geq 1$, we have $T^n \geq T^{n-1}$, and so the above inequality yields (6.7). This completes the proof.

The inequality (6.7) provides a clue to establish a sharper estimate for the integral of $G(u)r^{n-1}$. In fact, from (6.7) we derive the following estimate.

Lemma 6.8. *There exists a positive constant C such that*

$$\left(\int_0^T G(u) r^{n-1} dr \right)^{1/n} \leq C(k+1) + C \int_t^T (g(u)/u)^{1/2} dr$$

for $t = t(u)$, $T = T(u, e)$ and $u \in S_k$, where $t(u)$ is defined by Definition 5.13.

Proof. We recall that $0 < t < 1 < T$ by Lemma 4.12. Let $u \in S_k$. We set

$$I = \{r \in (t, T) : -b < u(r) < a\}.$$

Since $F(u) \leq 0$ for $r \in I$, we have

$$\frac{1}{2}|u'|^2 \geq \frac{1}{2}|u'|^2 + F(u) = E(r) \geq E(T) = e \quad \text{for } r \in I,$$

and therefore

$$(6.8) \quad |u'(r)| \geq \sqrt{2e} \quad \text{for } r \in I.$$

Considering the outline of the graph of $u(r)$, we can write I in the form of $I = \bigcup_{i=1}^j (\alpha_i, \beta_i)$ with some $j \leq k+1$. Here α_i and β_i satisfy the following conditions:

(i) $u(\alpha_i) = a$ and $u(\beta_i) = -b$, or $u(\alpha_i) = -b$ and $u(\beta_i) = a$ for $1 \leq i \leq j-1$;

(ii) $u(\alpha_j) = a$, $-b$ and $u(\beta_j) \in [-b, a]$.

Fix $i \in [1, j]$ arbitrarily. By the mean-value theorem, there exists a point $\xi_i \in (\alpha_i, \beta_i)$ such that

$$u(\beta_i) - u(\alpha_i) = u'(\xi_i)(\beta_i - \alpha_i).$$

In any case of (i) and (ii) we have

$$a + b \geq |u(\beta_i) - u(\alpha_i)| = |u'(\xi_i)|(\beta_i - \alpha_i).$$

This together with (6.8) yields

$$\beta_i - \alpha_i \leq (a + b)/\sqrt{2e}.$$

Thus we estimate the measure of I as

$$(6.9) \quad |I| = \sum_{i=1}^j (\beta_i - \alpha_i) \leq C_1(k+1),$$

where $C_1 = (a + b)/\sqrt{2e}$.

On the other hand, we set $I^c = [t, T] \setminus I$ to get

$$(6.10) \quad \int_t^T (g(u)/u)^{1/2} dr \geq \int_{I^c} (g(u)/u)^{1/2} dr \geq \delta_1 |I^c|,$$

where

$$\delta_1 = \min\{(g(s)/s)^{1/2} : s \in (-\infty, -b] \cup [a, \infty)\} > 0.$$

Combining (6.9) and (6.10), we obtain

$$\begin{aligned} C_1(k+1) + \delta_1^{-1} \int_t^T (g(u)/u)^{1/2} dr \\ \geq |I| + |I^c| = T - t \geq T - 1 \geq C \left(\int_0^T G(u)r^{n-1} dr \right)^{1/n} - 1, \end{aligned}$$

where we have used Lemma 6.7 in the last inequality and employed $t(u) \leq 1$. The above inequality completes the proof.

Combining Lemmas 6.6, 6.8 and using several a priori estimates in Section 5, we prove Proposition 6.1 in the following.

Proof of Proposition 6.1. Let $u \in S_k$. Integrating (5.17) over $[t, T]$ and using Lemma 5.5, we get

$$(k+1)\pi \geq \varphi(T) - \varphi(t) \geq C_2 \int_t^T (g(u)/u)^{1/2} dr + (n-1) \int_t^T \frac{u' Y(u)}{r \rho^2} dr,$$

which is rewritten as

$$(6.11) \quad \int_t^T (g(u)/u)^{1/2} dr \leq C_1(k+1) + C_2 \int_t^T \frac{|u' Y(u)|}{r \rho^2} dr.$$

For any $R > 1$, we set

$$R_1 = \max(R, T).$$

Using Lemmas 5.14 and 5.15 and noting that $F(s) \leq G(s)$ for $s \in \mathbf{R}$, we have

$$(6.12) \quad \begin{aligned} \int_t^{R_1} \frac{|u' Y(u)|}{r \rho^2} dr &= \left(\int_t^R + \int_R^{R_1} \right) \frac{|u' Y(u)|}{r \rho^2} dr \\ &\leq C_{R,\varepsilon} \left(\int_0^t G(u) r^{n-1} dr \right)^\varepsilon + \frac{C_\sigma}{R} \int_R^{R_1} (g(u)/u)^{1/2} dr \\ &\quad + C \sigma^{(\mu-2q+1)/2} (k+1), \end{aligned}$$

where $C_{R,\varepsilon}$ and C_σ depend only on R , ε and σ , respectively, but C does not depend on anything. First, we set $\sigma = 1$. Then (6.11) and (6.12) yield

$$(6.13) \quad \begin{aligned} \int_t^T (g(u)/u)^{1/2} dr &\leq C_0(k+1) + C_{R,\varepsilon} \left(\int_0^t G(u) r^{n-1} dr \right)^\varepsilon \\ &\quad + \frac{C_1}{R} \int_R^{R_1} (g(u)/u)^{1/2} dr. \end{aligned}$$

Secondly, we fix $R > 1$ so large that $C_1/R < 1/2$. For such R , if $R \geq T$, then $R_1 = R$ and the last term in (6.13) vanishes. If $R < T$, then $R_1 = T$, hence the last term is estimated from above by

$$\frac{1}{2} \int_t^T (g(u)/u)^{1/2} dr.$$

Then (6.13) yields

$$\int_t^T (g(u)/u)^{1/2} dr \leq C(k+1) + C_\varepsilon \left(\int_0^t G(u) r^{n-1} dr \right)^\varepsilon.$$

This together with Lemma 6.8 gives

$$\left(\int_0^T G(u) r^{n-1} dr \right)^{1/n} \leq C(k+1) + C_\varepsilon \left(\int_0^t G(u) r^{n-1} dr \right)^\varepsilon.$$

Lastly, noting that $t < 1 < T$ and setting $\varepsilon = 1/2n$, we obtain

$$\left(\int_0^T G(u) r^{n-1} dr \right)^{1/n} \leq C(k+1).$$

This inequality together with Lemma 6.6 yields

$$\int_0^\infty |u'|^2 r^{n-1} dr \leq C(k+1)^n,$$

which proves Proposition 6.1.

To complete the proof of Proposition 6.1, we must prove Proposition 6.5. To this end, we introduce some notations.

Definition 6.9. For $\delta > 0$, $e > 0$, $u \in S$ and $T = T(u, e)$, we set

$$\begin{aligned} I^+ &= \{r \in (T, \infty) : 0 \leq u(r) < a\}, \\ I^- &= \{r \in (T, \infty) : -b < u(r) < 0\}, \\ J^+ &= \{r \in (T, \infty) : u(r) > A - \delta\}, \\ J^- &= \{r \in (T, \infty) : u(r) < -B + \delta\}. \end{aligned}$$

Since T depends on u and e , the sets I^+ and I^- depend only on u and e , however, J^+ and J^- depend on u , e and δ as well. The proof of Proposition 6.5 is based on the next lemma.

Lemma 6.10. For any $e > 0$ sufficiently small there exists a $\delta > 0$ such that

$$\begin{aligned} \text{(i)} \quad & \int_{J^+} u f(u) r^{n-1} dr \leq C_0 \sqrt{e} \int_{I^+} |u f(u)| r^{n-1} dr \quad \text{for } u \in S, \\ \text{(ii)} \quad & \int_{J^-} u f(u) r^{n-1} dr \leq C_0 \sqrt{e} \int_{I^-} |u f(u)| r^{n-1} dr \quad \text{for } u \in S, \end{aligned}$$

where C_0 is a positive constant independent of u , e and δ .

Proof. We prove assertion (i) only since assertion (ii) can be proved in the same way. Let $u \in S$. In the case where $J^+ = \emptyset$, the assertion is trivial. Let $J^+ \neq \emptyset$. Then I^+ is also nonempty since $u(r)$ tends to 0 as $r \rightarrow \infty$. Let $0 < e < \min(1, e_0)$. Then we have

$$(6.14) \quad \frac{1}{2} u'(r)^2 + F(u(r)) = E(r) \leq E(T) = e < 1 \quad \text{for } r \geq T.$$

Note that $F(s)$ has local minima at $s = -b, a$, and we set

$$-F_0 = \min\{F(-b), F(a)\} = \inf\{F(s) : s \in \mathbf{R}\} < 0.$$

Then (6.14) implies that

$$(6.15) \quad |u'(r)| \leq C_1 \equiv \sqrt{2(1 + F_0)} \quad \text{for } r \geq T.$$

On the other hand, since $F(s)$ is increasing in (a, ∞) , it holds that $F(u(r)) \geq F(A - \delta)$ for $r \in J^+$. Hence (6.14) implies that

$$(6.16) \quad |u'(r)|^2 \leq 2e - 2F(A - \delta) \quad \text{for } r \in J^+.$$

Since $F(A) = 0$, we can choose $\delta > 0$ so small that

$$(6.17) \quad |F(A - \delta)| \leq e \quad \text{and} \quad (A + a)/2 < A - \delta < A.$$

Then (6.16) gives

$$(6.18) \quad |u'(r)| \leq 2\sqrt{e} \quad \text{for } r \in J^+.$$

We here observe (6.14) again, which gives directly

$$F(u(r)) < 1 \quad \text{for } r \geq T.$$

Hence we set $\bar{A} = \sup\{s : F(s) \leq 1\}$ to obtain

$$(6.19) \quad (A + a)/2 < A - \delta \leq u(r) \leq \bar{A} \quad \text{for } r \in J^+.$$

Since $f(s) > 0$ for $s > a$ by (f2), we set

$$m = \min\{f(s) : (A + a)/2 \leq s \leq \bar{A}\} > 0.$$

Using (6.18) and noting that $r \geq T \geq 1$, we obtain

$$(6.20) \quad -u'' = f(u) + \frac{n-1}{r}u' \geq m - 2(n-1)\sqrt{e} \geq \frac{m}{2} \quad \text{for } r \in J^+,$$

if $e > 0$ is so small that

$$(6.21) \quad \sqrt{e} \leq \frac{m}{4(n-1)}.$$

In view of Lemma 4.6, we can write J^+ as $J^+ = \bigcup_{i=1}^j (\alpha_i, \beta_i)$, where $u(\alpha_i) = u(\beta_i) = A - \delta$ for $2 \leq i \leq j$, $u(\alpha_1) \geq A - \delta$, $u(\beta_1) = A - \delta$. Integrating (6.20) over $[\alpha_i, \beta_i]$ and using (6.18), we get

$$4\sqrt{e} \geq -u'(\beta_i) + u'(\alpha_i) \geq \frac{m}{2}(\beta_i - \alpha_i),$$

or equivalently

$$(6.22) \quad 0 < \beta_i - \alpha_i \leq \frac{8}{m}\sqrt{e}.$$

In view of (6.19), we set $M = \max\{sf(s) : (A+a)/2 \leq s \leq \bar{A}\}$ to obtain

$$(6.23) \quad \int_{\alpha_i}^{\beta_i} uf(u)r^{n-1}dr \leq M\beta_i^{n-1}(\beta_i - \alpha_i) \leq \frac{8M}{m}\sqrt{e}\beta_i^{n-1}.$$

Let us estimate the integral of $|uf(u)|$ over I^+ . To do so, we define a subinterval (γ_i, δ_i) of I^+ . For $1 \leq i \leq j$, we set

$$\begin{aligned} \gamma_i &= \min\{r \in [\beta_i, \infty) : u(r) = a/2\}, \\ \delta_i &= \min\{r \in [\gamma_i, \infty) : u(r) = a/4\}. \end{aligned}$$

Therefore it holds that

$$\begin{aligned} \alpha_1 &< \beta_1 < \gamma_1 < \delta_1 < \alpha_2 < \cdots < \delta_j, \\ a/4 &< u(r) < a/2, \quad u'(r) < 0 \quad \text{for } r \in (\gamma_i, \delta_i). \end{aligned}$$

Using (6.15), we have

$$-u'(r) \leq C_1 \quad \text{for } r \in (\gamma_i, \delta_i).$$

Integrating both sides over $[\gamma_i, \delta_i]$, we obtain

$$\frac{a}{4} = -u(\delta_i) + u(\gamma_i) \leq C_1(\delta_i - \gamma_i),$$

that is,

$$(6.24) \quad C_2 \equiv \frac{a}{4C_1} \leq \delta_i - \gamma_i.$$

We set $m_1 = \min\{|sf(s)| : a/4 \leq s \leq a/2\} > 0$. Then (6.24) yields

$$(6.25) \quad \int_{\gamma_i}^{\delta_i} |uf(u)|r^{n-1}dr \geq m_1\gamma_i^{n-1}(\delta_i - \gamma_i) \geq m_1C_2\beta_i^{n-1}.$$

Combining (6.23) and (6.25), we obtain

$$\int_{\alpha_i}^{\beta_i} uf(u)r^{n-1}dr \leq C_3\sqrt{e} \int_{\gamma_i}^{\delta_i} |uf(u)|r^{n-1}dr,$$

where $C_3 = 8M/mm_1C_2$. We note that $(\gamma_i, \delta_i) \subset I^+$. Summing up both sides on i , we have

$$(6.26) \quad \int_{J^+} uf(u)r^{n-1}dr \leq C_3\sqrt{e} \int_{I^+} |uf(u)|r^{n-1}dr.$$

Note that C_3 is independent of u , e and δ . In view of (6.21), we set

$$e_1 = \min\{1, e_0, (m/4(n-1))^2\}.$$

For given $e \in (0, e_1)$ we choose a small $\delta > 0$ which satisfies (6.17). Then the choice of δ yields (6.26), which is the desired inequality (i).

To prove Proposition 6.5, we prepare the next notations and need another lemma.

Definition 6.11. For $u \in S$ we set

$$\begin{aligned} K^+ &= \{r \in (T, \infty) : 0 \leq u(r) < A\}, \\ K^- &= \{r \in (T, \infty) : -B < u(r) < 0\}, \\ L^+ &= \{r \in (T, \infty) : u(r) \geq A\}, \\ L^- &= \{r \in (T, \infty) : u(r) \leq -B\}. \end{aligned}$$

Lemma 6.12. If $e > 0$ is sufficiently small, then for any $u \in S$ we have

$$\begin{aligned} (i) \quad & \int_{L^+} F(u)r^{n-1}dr \leq \frac{1}{2} \int_{K^+} |F(u)|r^{n-1}dr = -\frac{1}{2} \int_{K^+} F(u)r^{n-1}dr, \\ (ii) \quad & \int_{L^-} F(u)r^{n-1}dr \leq \frac{1}{2} \int_{K^-} |F(u)|r^{n-1}dr = -\frac{1}{2} \int_{K^-} F(u)r^{n-1}dr. \end{aligned}$$

Proof. We prove assertion (i) only, however, (ii) can be treated in the same way. By assumptions (f2) and (f5), there exists a constant $C_1 > 0$ such that

$$(6.27) \quad F(s) \leq C_1sf(s) \quad \text{for } s \geq A.$$

Next, we note that $F(s) \neq 0$ for $0 < s \leq a$ and $F(0) = 0$. Then there exists a positive constant C_2 by (f3) such that

$$(6.28) \quad |sf(s)| \leq C_2|F(s)| \quad \text{for } 0 \leq s \leq a.$$

Compare Definitions 6.9 and 6.11. Then we see that $L^+ \subset J^+$ and $I^+ \subset K^+$. Hence Lemma 6.10, (6.27) and (6.28) imply that

$$\begin{aligned} \int_{L^+} F(u)r^{n-1}dr &\leq C_1 \int_{L^+} uf(u)r^{n-1}dr \leq C_1 \int_{J^+} uf(u)r^{n-1}dr \\ &\leq C_0C_1\sqrt{e} \int_{I^+} |uf(u)|r^{n-1}dr \leq C_0C_1C_2\sqrt{e} \int_{I^+} |F(u)|r^{n-1}dr \\ &\leq C_0C_1C_2\sqrt{e} \int_{K^+} |F(u)|r^{n-1}dr. \end{aligned}$$

We choose $e > 0$ so small that $C_0C_1C_2\sqrt{e} < 1/2$. This completes the proof.

We are now ready for proving Proposition 6.5.

Proof of Proposition 6.5. Summing up (i) and (ii) in Lemma 6.12, we have

$$(6.29) \quad \int_T^\infty F(u)r^{n-1}dr \leq \frac{1}{2} \int_K F(u)r^{n-1}dr,$$

where

$$K = K^+ \cup K^- = \{r \in (T, \infty) : -B < u(r) < A\}.$$

We note that $F(u(r)) < 0$ for $r \in K$. Hence the inequality (6.29) proves assertion (i) of Proposition 6.5.

We show assertion (ii). For small $\delta > 0$, $e > 0$ and any $u \in S$, we define

$$\begin{aligned} W &= W^+ \cup W^-, \\ W^+ &= \{r \in (T, \infty) : a \leq u(r) \leq A - \delta\}, \\ W^- &= \{r \in (T, \infty) : -B + \delta \leq u(r) \leq -b\}. \end{aligned}$$

We choose $e > 0$ so small that $C_0\sqrt{e} < 1$, and then $\delta > 0$ is determined by Lemma 6.10. Since $uf(u) \leq 0$ for $r \in I^+ \cup I^-$, the inequalities (i) and (ii) in Lemma 6.10 are rewritten as

$$\begin{aligned} \int_{J^+} uf(u)r^{n-1}dr &\leq - \int_{I^+} uf(u)r^{n-1}dr, \\ \int_{J^-} uf(u)r^{n-1}dr &\leq - \int_{I^-} uf(u)r^{n-1}dr. \end{aligned}$$

Note that $I^+ \cup I^- \cup J^+ \cup J^- = (T, \infty) \setminus W$. Then we sum up the two inequalities above to obtain

$$(6.30) \quad \int_T^\infty uf(u)r^{n-1}dr \leq \int_W uf(u)r^{n-1}dr.$$

On the other hand, since $F(s) < 0$ for $s \in (-B, A) \setminus \{0\}$ and $K \supset W$, the inequality (6.29) yields

$$\int_T^\infty F(u)r^{n-1}dr \leq -\frac{1}{2} \int_K |F(u)|r^{n-1}dr \leq -\frac{1}{2} \int_W |F(u)|r^{n-1}dr.$$

Summing up (6.30) and the above inequality multiplied by $C > 0$, we obtain

$$(6.31) \quad \int_T^\infty uf(u)r^{n-1}dr + C \int_T^\infty F(u)r^{n-1}dr \leq \int_W (uf(u) - \frac{C}{2}|F(u)|)r^{n-1}dr$$

for any $C > 0$. We set

$$\begin{aligned} M &= \max\{sf(s) : s \in [-B + \delta, -b] \cup [a, A - \delta]\} > 0, \\ m &= \min\{|F(s)| : s \in [-B + \delta, -b] \cup [a, A - \delta]\} > 0. \end{aligned}$$

We now choose $C > 0$ so large that $M - mC/2 < 0$. Then the right-hand side of (6.31) is negative. This proves Proposition 6.5(ii) and the proof is complete.

We conclude this paper by proving Theorem 1.

Proof of Theorem 1. By Propositions 5.1 and 6.1, there exist constants $C_1, C_2 > 0$ such that

$$(6.32) \quad C_1(k+1)^n \leq \int_0^\infty |u'|^2 r^{n-1}dr \leq C_2(k+1)^n$$

for $u \in S_k$ and $k \geq 0$. We estimate the Lagrangian $L(u)$ defined by (1.9), i.e.,

$$(6.33) \quad L(u) = \int_{\mathbf{R}^n} \left(\frac{1}{2} |\nabla u|^2 - F(u) \right) dx = \omega \int_0^\infty \left(\frac{1}{2} |u'|^2 - F(u) \right) r^{n-1}dr,$$

where ω means the surface area of the unit sphere $\{x \in \mathbf{R}^n : |x| = 1\}$. We show that

$$(6.34) \quad L(u) = \frac{\omega}{n} \int_0^\infty |u'|^2 r^{n-1} dr \quad \text{for } u \in S.$$

To this end, we recall the identities (5.33) and (6.2), i.e.,

$$(6.35) \quad \int_0^\infty |u'|^2 r^{n-1} dr = \int_0^\infty u f(u) r^{n-1} dr,$$

$$(6.36) \quad (n-2) \int_0^\infty |u'|^2 r^{n-1} dr = 2n \int_0^\infty F(u) r^{n-1} dr.$$

The substitution of (6.36) into (6.33) gives (6.34). By (6.32) and (6.34), we obtain the estimate

$$C_3(k+1)^n \leq L(u) \leq C_4(k+1)^n \quad \text{for } u \in S_k.$$

We now consider the special cases where $f(u) = |u|^{p-1}u - u$ and $f(u) = u \log |u|$. First, we treat $f(u) = |u|^{p-1}u - u$. Then $F(u) = |u|^{p+1}/(p+1) - u^2/2$. Therefore the inequalities (6.35) and (6.36) become

$$\begin{aligned} \int_0^\infty |u'|^2 r^{n-1} dr &= \int_0^\infty |u|^{p+1} r^{n-1} dr - \int_0^\infty u^2 r^{n-1} dr, \\ (n-2) \int_0^\infty |u'|^2 r^{n-1} dr &= \frac{2n}{p+1} \int_0^\infty |u|^{p+1} r^{n-1} dr - n \int_0^\infty u^2 r^{n-1} dr. \end{aligned}$$

These identities imply

$$\int_0^\infty |u'|^2 r^{n-1} dr = \frac{n(p-1)}{2(p+1)} \int_0^\infty |u|^{p+1} r^{n-1} dr = \frac{n(p-1)}{n+2-(n-2)p} \int_0^\infty u^2 r^{n-1} dr.$$

Each coefficient of the integrals is positive. This relation and (6.32) give

$$(6.37) \quad C_5(k+1)^{n/2} \leq \|u\|_{L^2} \leq \|u\|_{H^1} \leq C_6(k+1)^{n/2}.$$

Let $f(u) = u \log |u|$. Then $F(u) = (1/2)u^2 \log |u| - (1/4)u^2$. Substituting these functions into (6.35) and (6.36), we obtain

$$\int_0^\infty |u'|^2 r^{n-1} dr = \int_0^\infty u^2 \log |u| r^{n-1} dr = \frac{n}{4} \int_0^\infty u^2 r^{n-1} dr.$$

This identity together with (6.32) yields (6.37). The proof is thereby complete.

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