

AN ACCESS THEOREM FOR ANALYTIC FUNCTIONS

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ABSTRACT. Suppose that $\mathcal{M}$ is an analytic manifold, $m_0 \in \mathcal{M}$, $f: \mathcal{M} \rightarrow \mathbb{R}$, and f is analytic. Then at least one of the following three statements is true: (1) m_0 is a local maximum of f . (2) There is a continuous path $\sigma: [0, 1] \rightarrow \mathcal{M}$ such that $\sigma(0) = m_0$, $f \circ \sigma$ is strictly increasing on $[0, 1]$, and $\sigma(1)$ is a local maximum of f . (3) There is a continuous path $\sigma: [0, 1] \rightarrow \mathcal{M}$ with these properties: $\sigma(0) = m_0$; $f \circ \sigma$ is strictly increasing on $[0, 1]$; whenever K is a compact subset of $\mathcal{M}$, there is a corresponding number $d(K) \in [0, 1]$ such that $\sigma(t) \notin K$ for all $t \in [d(K), 1]$.

1. INTRODUCTION

Professor W. K. Hayman proposed the main theorem of this paper, and we thank him for encouraging us to find a proof of it. This theorem is restated below, using the technical terminology that is subsequently required, and all of the relevant definitions are given in section 2.

Theorem 1.1. *Suppose that $\mathcal{M}$ is an analytic manifold, $m_0 \in \mathcal{M}$, $f: \mathcal{M} \rightarrow \mathbb{R}$, and f is analytic. Then at least one of the following statements is true:*

- (1) m_0 is a local maximum of f .
- (2) *There is an $(\mathcal{M}, f)$ path $\sigma: [0, 1] \rightarrow \mathcal{M}$ such that $\sigma(0) = m_0$ and $\sigma(1)$ is a local maximum of f .*
- (3) *There is an $(\mathcal{M}, f)$ path $\sigma: [0, 1] \rightarrow \mathcal{M}$ such that $\sigma(0) = m_0$ and $\text{Cluster}(\sigma) = \emptyset$.*

Theorem 1.1 has the following corollary in potential theory:

Suppose that $\mathcal{M}$ is an open subset of $\mathbb{R}^n$; $m_0 \in \mathcal{M}$; $f: \mathcal{M} \rightarrow \mathbb{R}$; and f is nonconstant, analytic, and subharmonic. Then there is an $(\mathcal{M}, f)$ path $\sigma: [0, 1] \rightarrow \mathcal{M}$ such that $\sigma(0) = m_0$ and $\text{Cluster}(\sigma) = \emptyset$. (Note that, by definition, $\text{Cluster}(\sigma)$ is a closed subset of $\mathcal{M}$.)

This statement, which was previously at issue, is a sharpening of the *access theorem* of Hornblower and Thomas for the special case of analytic functions [H-T; H, pp. 779–784]. These results provide topological information pertaining to the behavior of subharmonic functions near the boundary of their domains.

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Theorem 1.1 also distinguishes between the *topological* features of C^∞ functions and those of analytic functions. For a local distinction, note that there is a C^∞ function $f : \mathbb{R} \rightarrow \mathbb{R}$ with these two properties: (1) zero is not a local maximum of f . (2) No $(\mathbb{R}, f)$ path begins at 0. For a distinction involving global behavior, consider the following more complicated construction:

There is a C^∞ function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that has the following two properties: (1) Whenever $m_0 \in \mathbb{R}^2$, there is an $(\mathbb{R}^2, f)$ path $\sigma : [0, 1) \rightarrow \mathbb{R}^2$ with $\sigma(0) = m_0$, (2) If τ is any $(\mathbb{R}^2, f)$ path, then $\text{Cluster}(\tau) \neq \emptyset$.

The details of this construction will appear in [O].

The proof of Theorem 1.1 is organized as the eight statements of Theorem 3.1 (section 3). The first two of these statements concern local problems, including the creation of an $(\mathcal{M}, f)$ curve originating at a point which is not a local maximum of f . The remaining statements concern the selection of a succession of $(\mathcal{M}, f)$ curves whose concatenation will exit the manifold $\mathcal{M}$. The local problems are easily solved by reference to the theory of singularities. The selection process, however, is complicated by the fact that the cluster set of an $(\mathcal{M}, f)$ curve may contain more than one point. Because of this, $(\mathcal{M}, f)$ curves (which are usually constructed from integral curves of $\text{grad } f$) must be modified before they can be joined. Moreover, in order to guarantee that the resulting concatenation will actually exit $\mathcal{M}$, the selection process must be based on a certain minimum principle.

We acknowledge, with thanks, the referee's contributions to this work, which led to several improvements. In particular, the referee gave us the precise references (in the theory of semianalytic sets) which lead to the short proofs of statements (1) and (2) in Theorem 3.1; our original proofs of these two statements were based on the seminal paper [B-C].

2. NOTATION AND TERMINOLOGY

$\mathbb{Z}$ is the set of integers, $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$, $\mathbb{R}$ is the set of real numbers, and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. An *interval* is a connected set of real numbers containing more than one point. If f is a function, then $\text{dmn } f = \{x : (x, y) \in f\}$ is the *domain* of f and $\text{im } f = \{y : (x, y) \in f\}$ is the *image* of f . If f is a function and S is a set, then $f|S = \{(s, f(s)) : s \in S \cap \text{dmn } f\}$ denotes the *restriction of f to S* . The symbols $f : A \rightarrow B$ signify that f is a function, $\text{dmn } f = A$, and $\text{im } f \subset B$.

We use the following terminology when (S, τ) is a topological space, $A \subset S$, $s_0 \in S$, and $f : S \rightarrow \mathbb{R}$: The symbols $\text{Clos } A$, $\text{Int } A$, and $\text{Bdry } A$ denote, respectively, the *closure*, *interior*, and *boundary* of A (with respect to τ). An open set containing s_0 is called a *neighborhood* of s_0 . The statement that s_0 is a *local maximum* of f means that $f(u) \leq f(s_0)$ for all u in a neighborhood of s_0 .

A full discussion of *analytic* mappings between open subsets of *real* normed vector spaces is given in [Fed]. The term *analytic manifold* is defined in [S, pp. 32–33]; thus, we explicitly assume that manifolds are second countable. If $\mathcal{M}$ is an analytic manifold, then $\mathcal{M}$ is a countable union of relatively compact open subsets of $\mathcal{M}$; also, the *dimension* of $\mathcal{M}$ is denoted $\dim \mathcal{M}$ and we allow the case $\dim \mathcal{M} = 0$ (hence $\dim \mathcal{M} \in \mathbb{N}$). For the definition and general properties of analytic functions and tensors on analytic manifolds, consult [He] and [S].

By the imbedding theorem of Morrey [Mor], it is possible to construct an *analytic Riemannian metric* on any analytic manifold $\mathcal{M}$. Suppose that $\mathcal{M}$ is an analytic manifold and $m \rightarrow \langle \cdot, \cdot \rangle_m$ is an analytic Riemannian metric on $\mathcal{M}$. Then we write $\langle \cdot, \cdot \rangle$ in place of $\langle \cdot, \cdot \rangle_m$, and we write $\text{dist}(\cdot, \cdot)$ to denote the corresponding *distance function* on $\mathcal{M}$ (with the underlying Riemannian metric determined by context). If $f: \mathcal{M} \rightarrow \mathbb{R}$ is analytic, then there is a corresponding analytic vector field, $\text{grad } f$, on $\mathcal{M}$ such that $\langle \text{grad } f(m), v \rangle = df(m) \cdot v$ whenever $m \in \mathcal{M}$ and v is in the tangent space of $\mathcal{M}$ at m .

Suppose that $\mathcal{M}$ is an analytic manifold and V is an analytic vector field on $\mathcal{M}$. If I is an interval of $\mathbb{R}$, $\gamma: I \rightarrow \mathcal{M}$ is continuous, and $\dot{\gamma}(t) = V(\gamma(t))$ for all $t \in \text{Int } I$, then we say that $\gamma: I \rightarrow \mathcal{M}$ is an *integral curve* of V ; under these circumstances, γ is analytic on $\text{Int } I$. We say that γ is a *maximal integral curve* of V if γ has the following two properties: (1) γ is an integral curve of V ; (2) If $\tilde{\gamma}$ is any integral curve of V and $\tilde{\gamma}(t_0) = \gamma(t_0)$ for some $t_0 \in \mathbb{R}$, then there is an interval J such that $\tilde{\gamma} = \gamma|_J$. If $m_0 \in \mathcal{M}$ and $t_0 \in \mathbb{R}$, then there is one and only one γ such that γ is a maximal integral curve of V , $t_0 \in \text{dmn } \gamma$, and $\gamma(t_0) = m_0$. If γ is a maximal integral curve of V , then $\text{dmn } \gamma$ is an open interval. If $\mathcal{M}$ is compact and γ is a maximal integral curve of V , then $\text{dmn } \gamma = \mathbb{R}$ (see [L, Theorem 4, p. 65]). We say that σ is a *terminal integral curve* of V if there exist a, b, c, γ such that $-\infty \leq a < b < c \leq +\infty$, $\gamma: (a, c) \rightarrow \mathcal{M}$ is a maximal integral curve of V , and $\sigma = \gamma|_{[b, c)}$.

Assume that $\mathcal{M}$ is an analytic manifold of dimension K . We say that $\mathcal{B}$ is an $\mathcal{M}$ -ball if there exists $m_0 \in \mathcal{M}$, $\epsilon \in \mathbb{R}^+$, and a coordinate system, $\phi: U \rightarrow \mathbb{R}^K$, such that $m_0 \in U$, $\phi(m_0) = 0$, $\{x \in \mathbb{R}^K: |x| \leq \epsilon\} \subset \phi[U]$, and $\mathcal{B} = \{m \in \mathcal{M}: |\phi(m)| < \epsilon\}$. If Ω is an open subset of $\mathcal{M}$, $m_0, m_1 \in \Omega$, and $m_0 \neq m_1$, then there is an $\mathcal{M}$ -ball $\mathcal{B}$ such that $m_0 \in \mathcal{B} \subset \text{Clos } \mathcal{B} \subset \Omega$ and $m_1 \notin \text{Clos } \mathcal{B}$; moreover, if $\mathcal{S} = \text{Bdry } \mathcal{B}$, then every continuous path from m_0 to m_1 passes through $\mathcal{S}$. If $K = \dim \mathcal{M} > 0$, $f: \mathcal{M} \rightarrow \mathbb{R}$ is analytic, and $\mathcal{B}$ is an $\mathcal{M}$ -ball, then $\mathcal{S} = \text{Bdry } \mathcal{B}$ is a *properly imbedded submanifold* of $\mathcal{M}$ with $\dim \mathcal{S} = K - 1$ [S, p. 43]; moreover, $g = f|_{\mathcal{S}}$ is an analytic function and $dg(s) = 0$ if $s \in \mathcal{S}$ and $df(s) = 0$.

Finally, in the following two definitions, we introduce the special technical terms that appear in the statement and proof of Theorem 1.1.

Definition 2.1. We use the following terminology when $\mathcal{M}$ is an analytic manifold and $f: \mathcal{M} \rightarrow \mathbb{R}$ is analytic.

(1) We say that $\gamma: I \rightarrow \mathcal{M}$ is an $(\mathcal{M}, f)$ path if I is an interval, $\gamma: I \rightarrow \mathcal{M}$ is a continuous mapping, and $f \circ \gamma$ is strictly increasing on I .

(2) If $a, b \in \mathbb{R} \cup \{\infty\}$, $a < b$, and $\gamma: [a, b) \rightarrow \mathcal{M}$, we define

$$\text{Cluster}(\gamma) = \bigcap_{s \in [a, b)} \text{Clos } \{\gamma(t) : t \in [s, b)\}.$$

(Thus, $\text{Cluster}(\gamma)$ is a closed subset of $\mathcal{M}$.)

(3) If $A \subset \mathcal{M}$, we define the following sets of *critical points* and *critical values*:

$$\text{Critpt}(A, f) = \{a \in A : df(a) = 0\};$$

$$\text{Critval}(A, f) = \{f(a) : a \in A \text{ and } df(a) = 0\}.$$

(4) We say that $(K_n)_{\mathbb{Z}}$ is an exhaustion of $\mathcal{M}$ if $(K_n)_{\mathbb{Z}}$ is a family of compact subsets of $\mathcal{M}$, $K_n = \emptyset$ if $n < 0$, $\emptyset \neq K_n \subset \text{Int } K_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{Z}} K_n = \mathcal{M}$.

The initial empty sets appearing in an exhaustion are a convenience. Every analytic manifold possesses an exhaustion.

Definition 2.2. When $\mathcal{M}$ is an analytic manifold, $f : \mathcal{M} \rightarrow \mathbb{R}$ is analytic, and $(K_n)_{\mathbb{Z}}$ is an exhaustion of $\mathcal{M}$, we say that γ is a (j, k) path for $(K_n)_{\mathbb{Z}}$ if the following five conditions obtain:

- (1) There exist $a, b \in \mathbb{R}$ such that $a < b$ and $\gamma : [a, b] \rightarrow \mathcal{M}$ is an $(\mathcal{M}, f)$ path.
- (2) $j, k \in \mathbb{N}$.
- (3) $\gamma(a) \in (\text{Int } K_j) \sim (\text{Int } K_{j-1})$ and $\gamma(b) \in (\text{Int } K_k) \sim (\text{Int } K_{k-1})$.
- (4) $f(\gamma(b)) \in \text{Critval}(K_k, f)$.
- (5) If $p \in \mathbb{N}$, $p \leq \min\{j, k\} - 3$, $a < \alpha < \beta < b$, and $\{\gamma(t) : t \in (\alpha, \beta)\} \subset \text{Int } K_p$, then $\gamma|(\alpha, \beta)$ is an integral curve of $\text{grad } f$.

For each $n \in \mathbb{Z}$, let us refer to the set $(\text{Int } K_n) \sim (\text{Int } K_{n-1})$ as level n . Thus, a (j, k) path γ starts in level j and terminates in level k . Moreover, f increases along γ and $f(\gamma(b))$ is a critical value of f (although $\gamma(b)$ need not be a critical point of f). Clause (5) of the definition refers to an arc of γ residing at least three levels below the initial and terminal levels of γ : such an arc is required to be an integral curve of $\text{grad } f$.

3. PROOF OF THEOREM 1.1

Since every analytic manifold admits an analytic Riemannian metric, Theorem 1.1 follows from statement (8) of the following Theorem.

Theorem 3.1. Suppose that $\mathcal{M}$ is an analytic manifold, $f : \mathcal{M} \rightarrow \mathbb{R}$ is analytic, and $m \rightarrow \langle \cdot, \cdot \rangle$ is an analytic Riemannian metric on $\mathcal{M}$ with corresponding distance function $\text{dist}(\cdot, \cdot)$. Then the following statements are true.

- (1) If $m_0 \in \mathcal{M}$ and m_0 is not a local maximum of f , then there is an $(\mathcal{M}, f)$ path $\gamma : [0, 1] \rightarrow \mathcal{M}$ with $\gamma(0) = m_0$.
- (2) If E is a compact subset of $\mathcal{M}$, then $\text{Critval}(E, f)$ is a finite set. Consequently, $\text{Critval}(\mathcal{M}, f)$ is a countable set.
- (3) Suppose that $m \in \mathcal{M}$, $m \notin \text{Critpt}(\mathcal{M}, f)$, and $\epsilon > 0$. Then there exists $\delta \in (0, \epsilon)$ such that the following statement is true: If $\gamma : (A, B) \rightarrow \mathcal{M}$ is an integral curve of $\text{grad } f$, $A < a < b < B$, $\text{dist}(\gamma(a), m) < \delta$, and $\epsilon < \text{dist}(\gamma(b), m)$, then $f(\gamma(b)) > f(m)$.
- (4) If $\gamma : [a, b) \rightarrow \mathcal{M}$ is a terminal integral curve of $\text{grad } f$ and $m^* \in \text{Cluster}(\gamma)$, then $m^* \in \text{Critpt}(\mathcal{M}, f)$.
- (5) Suppose that $\epsilon \in \mathbb{R}^+$ and $m^* \in \mathcal{M}$. For each $n \in \mathbb{N}$ suppose that $A_n < \alpha_n < s(n) < \beta_n < B_n$, $\gamma_n : (A_n, B_n) \rightarrow \mathcal{M}$ is an integral curve of $\text{grad } f$, $\text{dist}(\gamma_n(\alpha_n), m^*) > \epsilon$, $\text{dist}(\gamma_n(\beta_n), m^*) > \epsilon$, and $f(\gamma_n(\beta_n)) \leq f(\gamma_{n+1}(\alpha_{n+1}))$. In addition, suppose that $\lim_{n \uparrow \infty} \gamma_n(s(n)) = m^*$. Then $m^* \in \text{Critpt}(\mathcal{M}, f)$.
- (6) Suppose that $\mathcal{W}$ is a compact analytic manifold, $g : \mathcal{W} \rightarrow \mathbb{R}$ is analytic, $w_0 \in \mathcal{W}$, and $w_0 \notin \text{Critpt}(\mathcal{W}, g)$. Let y^* be the smallest critical value of g which is greater than $g(w_0)$ (y^* always exists). Then there is a $(\mathcal{W}, g)$ path $\mu : [0, 1] \rightarrow \mathcal{W}$ such that $\mu(0) = w_0$ and $g(\mu(1)) = y^*$.

(7) Suppose that $a \in \mathbb{R}$, $(K_n)_Z$ is an exhaustion of $\mathcal{M}$, $j \in \mathbb{N}$, $m \in (\text{Int } K_j) \sim (\text{Int } K_{j-1})$, m is not a local maximum of f , and there is no $(\mathcal{M}, f)$ path $\sigma : [0, 1) \rightarrow \mathcal{M}$ with $\sigma(0) = m$ and $\text{Cluster}(\sigma) = \emptyset$. Then there exists $b \in (a, \infty)$, $k \in \mathbb{N}$, and $\gamma : [a, b] \rightarrow \mathcal{M}$ such that $\gamma(a) = m$ and γ is a (j, k) path for $(K_n)_Z$.

(8) Suppose that $m_0 \in \mathcal{M}$, m_0 is not a local maximum of f , and there is no $(\mathcal{M}, f)$ path $\sigma : [0, 1] \rightarrow \mathcal{M}$ for which $\sigma(0) = m_0$ and $\sigma(1)$ is a local maximum of f . Then there exists an $(\mathcal{M}, f)$ path $\gamma : [0, 1) \rightarrow \mathcal{M}$ such that $\gamma(0) = m_0$ and $\text{Cluster}(\gamma) = \emptyset$.

Proof. (1) It suffices to prove the assertion in the case that $n \in \{1, 2, \dots\}$, $\mathcal{M}$ is a connected open subset of $\mathbb{R}^n$, $\{x \in \mathbb{R}^n : |x| \leq 1\} \subset \mathcal{M}$, $m_0 = 0$, and $f(0) = 0$.

The set $W \equiv \{x \in \mathcal{M} : f(x) \neq 0, |x| < 1\}$ is semianalytic and relatively compact in $\mathcal{M}$. Therefore, W has only a finite number of connected components (by [Loj, Proposition 2, p. 76]) and each of these components is semianalytic (by [Loj, Theorem 2, p. 69]). Therefore, since 0 is not a local maximum of f , there is a set Ω that has the following properties: Ω is a connected component of W ; $0 \in \text{Bdry } \Omega$; $f(x) > 0$ whenever $x \in \Omega$; Ω is a semianalytic set.

By the Bruhat-Cartan-Wallace lemma ([Loj, Proposition 2, p. 103]), there is a function ξ , a number $\delta > 0$, and a set λ with the following properties: $\xi : (0, 1 + \delta) \rightarrow \mathcal{M}$ is analytic; $\lim_{t \downarrow 0} \xi(t) = 0$; $\lambda \subset \Omega$; λ is semianalytic; and $\xi|_{(0, 1]}$ is a homeomorphism of $(0, 1]$ onto λ . Then, by a version of the desingularization theorem ([Sus, statement (b), p. 443]), there exists a number $\epsilon > 0$ and a finite sequence $(\psi_j)_{j=1}^J$ of functions such that $\psi_j : (-1 - \epsilon, 1 + \epsilon) \rightarrow \mathcal{M}$ is analytic whenever $j \in \{1, 2, \dots, J\}$ and $\lambda = \bigcup_{j=1}^J \{\psi_j(t) : -1 < t < 1\}$.

It follows that $0 \in \bigcup_{j=1}^J \{\psi_j(1), \psi_j(-1)\}$; therefore, we may assume that $0 = \psi_1(1)$. Now define $\sigma(t) = \psi_1(1 - t^2)$ whenever $t \in (-1, 1)$. Then $\sigma : (-1, 1) \rightarrow \mathcal{M}$ is analytic, $\sigma(0) = 0$, and $\sigma(t) \in \Omega$ whenever $0 < |t| < 1$. Thus $f \circ \sigma$ is analytic on $(-1, 1)$, $f \circ \sigma(0) = 0$, $f \circ \sigma(t) > 0$ whenever $t \in (0, 1)$, and there exists $\beta \in (0, 1)$ such that $(f \circ \sigma)'(t) > 0$ whenever $t \in (0, \beta]$. The required $(\mathcal{M}, f)$ path is defined by $\gamma(s) \equiv \sigma(s\beta)$ whenever $s \in [0, 1]$.

(2) Since $\text{Critpt}(\mathcal{M}, f)$ is a semianalytic subset of $\mathcal{M}$, it is a locally finite union of cubic images ([Sus, statement (b), p. 443]). It is immediate that f is constant on each of these cubic images. Therefore, both parts of statement (2) are valid.

(3) Fix m and ϵ as in statement (3). For $\rho \in \mathbb{R}^+$ define

$$\begin{aligned} G(\rho) &\equiv \sup \{ |\text{grad } f(m')| : m' \in \mathcal{M}, \text{dist}(m, m') < \rho \}, \\ g(\rho) &\equiv \inf \{ |\text{grad } f(m')| : m' \in \mathcal{M}, \text{dist}(m, m') < \rho \}, \\ \Delta(\rho) &\equiv \sup \{ |f(m') - f(m)| : m' \in \mathcal{M}, \text{dist}(m, m') < \rho \}. \end{aligned}$$

Choose $\epsilon_1 \in (0, \epsilon)$ and $\delta \in (0, \epsilon_1)$ such that

$$G(\epsilon_1) < \infty, \quad g(\epsilon_1) > 0, \quad g(\epsilon_1)^2 G(\epsilon_1)^{-1} (\epsilon_1 - \delta) - \Delta(\delta) > 0.$$

Now suppose that $\gamma : (A, B) \rightarrow \mathcal{M}$ is an integral curve of $\text{grad } f$ and

$$A < a < b < B, \quad \text{dist}(\gamma(a), m) < \delta, \quad \epsilon < \text{dist}(\gamma(b), m),$$

and define

$$t^* = \min \{t \in (a, b) : \text{dist}(\gamma(t), m) = \epsilon_1\}.$$

Then

$$\epsilon_1 - \delta \leq \text{dist}(\gamma(a), \gamma(t^*)) \leq \int_a^{t^*} |\dot{\gamma}(t)| dt = \int_a^{t^*} |\text{grad } f(\gamma(t))| dt \leq (t^* - a)G(\epsilon_1)$$

and

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &\geq f(\gamma(t^*)) - f(\gamma(a)) = \int_a^{t^*} \frac{d}{dt} f(\gamma(t)) dt \\ &= \int_a^{t^*} \langle \text{grad } f(\gamma(t)), \dot{\gamma}(t) \rangle dt = \int_a^{t^*} |\text{grad } f(\gamma(t))|^2 dt \\ &\geq g^2(\epsilon_1)(t^* - a) \geq g^2(\epsilon_1)G(\epsilon_1)^{-1}(\epsilon_1 - \delta). \end{aligned}$$

Therefore,

$$\begin{aligned} f(\gamma(b)) - f(m) &\geq [f(\gamma(b)) - f(\gamma(a))] + [f(\gamma(a)) - f(m)] \\ &\geq [f(\gamma(b)) - f(\gamma(a))] - \Delta(\delta) \geq g^2(\epsilon_1)G(\epsilon_1)^{-1}(\epsilon_1 - \delta) - \Delta(\delta) > 0. \end{aligned}$$

(4) We assume that $m^* \notin \text{Critpt}(\mathcal{M}, f)$ and argue to a contradiction.

Since $\text{Cluster}(\gamma)$ is not empty, the local existence theorem for integral curves implies $b = \infty$ [L, Theorem 4, p. 65]. Since $f \circ \gamma$ is strictly increasing on $[a, \infty)$, it follows that $f(\gamma(t)) \uparrow f(m^*)$ as $t \uparrow \infty$.

Now we prove that $\lim_{t \uparrow \infty} \gamma(t) = m^*$. If this is not the case, we can construct $\epsilon \in \mathbb{R}^+$ and sequences $(a_n)_{\mathbb{N}}, (b_n)_{\mathbb{N}}$ that has the following three properties: (1) $a_n < b_n < a_{n+1} < b_{n+1}$ for all $n \in \mathbb{N}$; (2) $\lim_{n \uparrow \infty} \gamma(a_n) = m^*$; (3) $\epsilon < \text{dist}(\gamma(b_n), m^*)$ for all $n \in \mathbb{N}$. Therefore, since $m^* \notin \text{Critpt}(\mathcal{M}, f)$, Theorem 3.1 (3) implies that $f(\gamma(b_n)) > f(m^*)$ for all suitably large $n \in \mathbb{N}$. But this contradicts the conclusion of the preceding paragraph.

Since $\lim_{t \uparrow \infty} \gamma(t) = m^*$, we have

$$\lim_{t \uparrow \infty} |\text{grad } f(\gamma(t))| = |\text{grad } f(m^*)|$$

and

$$\lim_{t \uparrow \infty} f(\gamma(t)) - f(\gamma(a)) = \lim_{t \uparrow \infty} \int_a^t |\text{grad } f(\gamma(s))|^2 ds = f(m^*) - f(\gamma(a)) < \infty.$$

But this is impossible unless $\text{grad } f(m^*) = 0$.

(5) We assume that $m^* \notin \text{Critpt}(\mathcal{M}, f)$ and argue to a contradiction.

Since $m^* \notin \text{Critpt}(\mathcal{M}, f)$, we can choose $\delta \in (0, \epsilon)$ corresponding to $m^* = m$ as indicated in Theorem 3.1 (3). There exists $N \in \mathbb{N}$ such that $\text{dist}(\gamma_n(s(n)), m^*) < \delta$ whenever $n \geq N$. Therefore, by Theorem 3.1 (3), $f(\gamma_n(\beta_n)) > f(m^*)$ whenever $n \geq N$. But for each $n \in \mathbb{N}$ we have

$$\begin{aligned} f(\gamma_n(\beta_n)) &\leq f(\gamma_{n+1}(\alpha_{n+1})) < f(\gamma_{n+1}(s(n+1))) < f(\gamma_{n+1}(\beta_{n+1})) \\ &< \lim_{k \uparrow \infty} f(\gamma_k(s(k))) = f(m^*). \end{aligned}$$

(6) The number y^* exists because $\mathcal{W}$ is compact: indeed, the global maximum value of g is a critical value of g greater than $g(w_0)$ and g has only a finite number of critical values (by Theorem 3.1 (2)).

The proof of the main statement is by induction on $J = \dim(\mathscr{W})$. The statement is true when $J = 0$ because, in that case, every point of $\mathscr{W}$ is a critical point of g . We assume that the statement is true when $J = J_0 \in \mathbb{N}$ and that $\mathscr{W}$, g , w_0 , and y^* meet the hypotheses of the statement in the case $J = J_0 + 1$. Since $\mathscr{W}$ is compact, there is an analytic map $\gamma : (-\infty, \infty) \rightarrow \mathscr{W}$ which is a maximal integral curve of $\text{grad } g$ with $\gamma(0) = w_0$. Then $\text{Cluster}(\gamma) \neq \emptyset$ (since $\mathscr{W}$ is compact). Choose $m^* \in \text{Cluster}(\gamma)$. Then $m^* \in \text{Critpt}(\mathscr{W}, g)$ (by Theorem 3.1 (4)).

There are two possibilities: (1) $\lim_{t \uparrow \infty} \gamma(t) = m^*$; (2) $\lim_{t \uparrow \infty} \gamma(t)$ does not exist.

If $\lim_{t \uparrow \infty} \gamma(t) = m^*$, define $\sigma : [0, 1] \rightarrow \mathscr{W}$ by the rules $\sigma(t) = \gamma(t/1-t)$, for $t \in [0, 1)$, and $\sigma(1) = m^*$. Then $g(w_0) = g(\sigma(0)) < g(\sigma(1)) \in \text{Critval}(\mathscr{W}, g)$. Therefore, $g(w_0) < y^* \leq g(\sigma(1))$, and we may form μ by shortening σ .

If $\lim_{t \uparrow \infty} \gamma(t)$ does not exist, we may choose $m^{**} \in \text{Cluster}(\gamma)$ with $m^* \neq m^{**}$. Let $\mathscr{B}$ be a $\mathscr{W}$ -ball with $m^* \in \mathscr{B}$ and $m^{**} \notin \text{Clos } \mathscr{B}$. Set $\mathscr{S} = \text{Bdry } \mathscr{B}$, and define $g = f|_{\mathscr{S}}$. Then $\mathscr{S}$ is an analytic manifold of dimension J_0 and $g : \mathscr{S} \rightarrow \mathbb{R}$ is analytic. Moreover, there is a monotone sequence $(t_n)_{n \in \mathbb{N}}$ such that $\lim_{n \uparrow \infty} t_n = \infty$, $\gamma(t_n) \in \mathscr{S}$ for all $n \in \mathbb{N}$, and $\lim_{n \uparrow \infty} \gamma(t_n) = s^* \in \mathscr{S}$. Hence, $J_0 > 0$ and $s^* \in \text{Critpt}(\mathscr{S}, g)$ (since $s^* \in \text{Critpt}(\mathscr{W}, g)$). Also $g(s^*) = \lim_{n \uparrow \infty} g(\gamma(t_n)) = \lim_{n \uparrow \infty} f(\gamma(t_n)) = f(m^*)$, and $(g(\gamma(t_n)))_{n \in \mathbb{N}}$ is an increasing sequence. Therefore, there exists $N \in \mathbb{N}$ such that $\gamma(t_N) \notin \text{Critpt}(\mathscr{S}, g)$, and $g(\gamma(t_N)) > \max \{y \in \text{Critval}(\mathscr{S}, g) : y < g(s^*)\}$. By the inductive assumption, there is an $(\mathscr{S}, g)$ path $\sigma : [0, 1] \rightarrow \mathscr{S}$ with $\sigma(0) = \gamma(t_N)$ and $g(\sigma(1)) = \min \{y \in \text{Critval}(\mathscr{S}, g) : y > g(\gamma(t_N))\} = g(s^*) = f(m^*)$. Now define $\mu(t) = \gamma(2tt_N)$ for $t \in [0, 1/2]$ and $\mu(t) = \sigma(2t-1)$ for $t \in [1/2, 1]$. The path μ has the required properties.

(7) We may assume that $a = 0$. Using Theorem 3.1 (1) and (2), we construct an $(\mathscr{M}, f)$ path $\tau : [0, 1] \rightarrow \mathscr{M}$ such that $\tau(0) = m$, $\tau(1) \notin \text{Critpt}(\mathscr{M}, f)$, and $\{\tau(t) : t \in [0, 1]\} \subset (\text{Int } K_j) \sim K_{j-2}$. Let $\rho : [1, c) \rightarrow \mathscr{M}$ be a terminal integral curve of $\text{grad } f$ with $\rho(1) = \tau(1)$. Define $\sigma(t) = \tau(2t)$ for $t \in [0, 1/2]$ and $\sigma(t) = \rho(s(t))$ for $t \in [1/2, 1)$, where $s : [1/2, 1) \rightarrow [1, c)$ is any homeomorphism. By our present hypotheses, we conclude that $\emptyset \neq \text{Cluster}(\sigma) = \text{Cluster}(\rho)$. Therefore $c = \infty$ [L, Theorem 4, p. 65].

Choose $m^* \in \text{Cluster}(\rho)$. Let $k \in \mathbb{N}$ satisfy $m^* \in (\text{Int } K_k) \sim (\text{Int } K_{k-1})$. Then $m^* \in \text{Critpt}(K_k, f)$ by Theorem 3.1 (4). There are only two possible cases: (1) $\lim_{t \uparrow \infty} \rho(t) = m^*$; (2) $\lim_{t \uparrow \infty} \rho(t)$ does not exist. Our proof covers these cases separately.

Suppose that $\lim_{t \uparrow \infty} \rho(t) = m^*$. Let $\mathscr{B}$ be an $\mathscr{M}$ -ball such that $m \notin \text{Clos } \mathscr{B}$ and $m^* \in \mathscr{B} \subset \text{Clos } \mathscr{B} \subset (\text{Int } K_k) \sim K_{k-2}$. Set

$$t^* = \max \{t \in [1, \infty) : \rho(t) \in \text{Bdry } \mathscr{B}\}.$$

Define $\gamma : [0, t^* + 1] \rightarrow \mathscr{M}$ as follows: $\gamma(t) = \tau(t)$ for $t \in [0, 1]$; $\gamma(t) = \rho(t)$ for $t \in [1, t^*]$; $\gamma(t) = \rho(t^* + \frac{t-t^*}{t^*+1-t})$ for $t \in [t^*, t^* + 1]$; $\gamma(t^* + 1) = m^*$. Note the following three properties of γ :

- (i) $\{\gamma(t) : t \in [0, 1]\} \subset \{\tau(t) : t \in [0, 1]\} \subset (\text{Int } K_j) \sim K_{j-2}$;
- (ii) $\gamma|_{[1, t^*]} = \rho|_{[1, t^*]}$ is an integral curve of $\text{grad } f$;
- (iii) $\{\gamma(t) : t \in [t^*, t^* + 1]\} = \{m^*\} \cup \{\rho(t) : t \in [t^*, \infty)\} \subset \text{Clos } \mathscr{B} \subset (\text{Int } K_k) \sim K_{k-2}$.

With attention to the condition $p \leq \min\{j, k\} - 3$ in Definition 2.2 (5), we conclude that $\gamma : [0, t^* + 1] \rightarrow \mathcal{M}$ is a (j, k) path for $(K_n)_{\mathbb{Z}}$ with $\gamma(0) = m$.

Suppose that $\lim_{t \uparrow \infty} \rho(t)$ does not exist. Then we can choose $m^{**} \in \text{Cluster}(\rho)$ with $m^{**} \neq m^*$, and we can construct an $\mathcal{M}$ -ball $\mathcal{B}$ satisfying the following three conditions: $m \notin \text{Clos } \mathcal{B}$; $m^* \in \mathcal{B} \subset \text{Clos } \mathcal{B} \subset (\text{Int } K_k) \sim K_{k-2}$; $m^{**} \notin \text{Clos } \mathcal{B}$. Consequently, there is a sequence $(t_n)_{\mathbb{N}}$ such that $t_n \uparrow \infty$ as $n \uparrow \infty$, $\rho(t_n) \in \text{Bdry } \mathcal{B}$ for all $n \in \mathbb{N}$, and $\lim_{n \uparrow \infty} \rho(t_n) = s^* \in \text{Bdry } \mathcal{B}$. Therefore $f(s^*) = \lim_{n \uparrow \infty} f(\rho(t_n)) = \lim_{t \uparrow \infty} f(\rho(t)) = f(m^*)$. Also, because $s^* \in \text{Cluster}(\rho)$ and because of Theorem 3.1 (4), we conclude $s^* \in \text{Critpt}(\mathcal{M}, f)$.

Define $g = f|_{\text{Bdry } \mathcal{B}}$. Then $g : \text{Bdry } \mathcal{B} \rightarrow \mathbb{R}$ and g is analytic. Moreover, since $s^* \in \text{Critpt}(\mathcal{M}, f)$, we have $s^* \in \text{Critpt}(\text{Bdry } \mathcal{B}, g)$. The set $\text{Critval}(\text{Bdry } \mathcal{B}, g)$ is finite because $\text{Bdry } \mathcal{B}$ is compact. Therefore, because the sequence $(g(\rho(t_n)))_{\mathbb{N}}$ is strictly increasing, we may choose $N \in \mathbb{N}$ so that $\rho(t_N) \notin \text{Critpt}(\text{Bdry } \mathcal{B}, g)$ and

$$g(s^*) = \min\{y \in \text{Critval}(\text{Bdry } \mathcal{B}, g) : y > g(\rho(t_N))\}.$$

By Theorem 3.1 (6) there is a continuous mapping $\mu : [0, 1] \rightarrow \text{Bdry } \mathcal{B}$ such that $g \circ \mu$ is strictly increasing on $[0, 1]$, $\mu(0) = \rho(t_N)$, and $g(\mu(1)) = g(s^*) = f(s^*) = f(m^*)$. Now we define $\gamma : [0, t_N + 1] \rightarrow \mathcal{M}$ as follows: $\gamma(t) = \tau(t)$ for $t \in [0, 1]$; $\gamma(t) = \rho(t)$ for $t \in [1, t_N]$; $\gamma(t) = \mu(t - t_N)$ for $t \in [t_N, t_N + 1]$. Note the following three properties of γ :

- (i) $\{\gamma(t) : t \in [0, 1]\} = \{\tau(t) : t \in [0, 1]\} \subset (\text{Int } K_j) \sim K_{j-2}$;
- (ii) $\gamma|_{[1, t_N]} = \rho|_{[1, t_N]}$ is an integral curve of $\text{grad } f$;
- (iii) $\{\gamma(t) : t \in [t_N, t_N + 1]\} = \{\mu(t) : t \in [0, 1]\} \subset \text{Bdry } \mathcal{B} \subset (\text{Int } K_k) \sim K_{k-2}$.

With attention to the condition $p \leq \min\{j, k\} - 3$ in Definition 2.2 (5), we conclude that $\gamma : [0, t_N + 1] \rightarrow \mathcal{M}$ is a (j, k) path for $(K_n)_{\mathbb{Z}}$ with $\gamma(0) = m$. This completes the proof of Theorem 3.1 (7).

(8) We shall assume that statement (8) is false and argue to a contradiction. Therefore, we explicitly assume that the following three statements are valid:

- (8.1) m_0 is not a local maximum of f ,
- (8.2) Whenever $\gamma : [a, b] \rightarrow \mathcal{M}$ is an $(\mathcal{M}, f)$ path with $\gamma(a) = m_0$, then $\gamma(b)$ is not a local maximum of f ,
- (8.3) Whenever $\gamma : [a, b] \rightarrow \mathcal{M}$ is an $(\mathcal{M}, f)$ path with $\gamma(a) = m_0$, then $\text{Cluster}(\gamma) \neq \emptyset$.

Let $(K_n)_{\mathbb{Z}}$ be an exhaustion of $\mathcal{M}$; all (j, k) paths in this proof will be constructed for this exhaustion.

Now we construct three sequences, $(j_n)_{\mathbb{N}}$, $(k_n)_{\mathbb{N}}$, and $(\gamma_n : [a_n, b_n] \rightarrow \mathcal{M})_{\mathbb{N}}$, that have the following four properties (note that (iv) is a minimality condition):

- (i) $\gamma_n : [a_n, b_n] \rightarrow \mathcal{M}$ is a (j_n, k_n) path for all $n \in \mathbb{N}$.
- (ii) $\gamma_0(a_0) = m_0$.
- (iii) $b_n = a_{n+1}$ and $\gamma_n(b_n) = \gamma_{n+1}(a_{n+1})$ for all $n \in \mathbb{N}$.
- (iv) If $n \in \mathbb{N}$, $p \in \mathbb{N}$, and $p \leq k_n - 1$, then there is no (j_n, p) path whose initial point is $\gamma_n(a_n)$.

The construction is by induction. By (8.1), (8.3), and Theorem 3.1 (7), we can construct a (j_0, k_0) path $\gamma_0 : [a_0, b_0] \rightarrow \mathcal{M}$ with $\gamma_0(a_0) = m_0$ and with k_0 as small as possible. Suppose that the construction has been carried out for all $n \leq N \in \mathbb{N}$. Then $\gamma_N(b_N)$ is not a local maximum of f ; because,

otherwise, the concatenation of $\gamma_0, \gamma_1, \dots, \gamma_N$ would end at a local maximum of f , which would contradict (8.2). Also, every $(\mathcal{M}, f)$ path beginning at $\gamma(b_N)$ has a non-empty cluster set; because if $\tau : [b_N, b) \rightarrow \mathcal{M}$ is an $(\mathcal{M}, f)$ path with $\tau(b_N) = \gamma_N(b_N)$ and $\text{Cluster}(\tau) = \emptyset$, then the concatenation of $\gamma_0, \gamma_1, \dots, \gamma_N, \tau$ would also have an empty cluster set, which would contradict (8.3). Thus, by Theorem 3.1 (7), we can construct the required objects for $n = N + 1$ (with k_{N+1} as small as possible). The construction is complete.

Now set $[a, b) = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ and define $\gamma : [a, b) \rightarrow \mathcal{M}$ so that $\gamma|_{[a_n, b_n]} = \gamma_n$ for all $n \in \mathbb{N}$. Then γ is an $(\mathcal{M}, f)$ path. Also, $\text{Cluster}(\gamma) \neq \emptyset$ because γ is an $(\mathcal{M}, f)$ path with $\gamma(a_0) = m_0$. Choose $m^* \in \text{Cluster}(\gamma)$. Let $k^* \in \mathbb{N}$ be such that $m^* \in (\text{Int } K_{k^*}) \sim (\text{Int } K_{k^*-1})$. Let $\mathcal{B}$ be an $\mathcal{M}$ -ball such that $m^* \in \mathcal{B} \subset \text{Clos } \mathcal{B} \subset (\text{Int } K_{k^*}) \sim K_{k^*-2}$.

Since the set $\text{Critval}(K_{k^*+3}, f)$ is finite, since $f(\gamma(b_n)) \in \text{Critval}(K_{k_n}, f)$ for all $n \in \mathbb{N}$, and since $f(\gamma(b_n)) < f(\gamma(b_{n+1}))$ for all $n \in \mathbb{N}$, we conclude that only a finite number of terms from the sequence $(\gamma(b_n))_{n \in \mathbb{N}}$ lie in K_{k^*+3} . Therefore,

(v) $\gamma(a_n), \gamma(b_n) \notin K_{k^*+3}$ if n is sufficiently large.

Since $m^* \in \mathcal{B} \subset \text{Clos } \mathcal{B} \subset (\text{Int } K_{k^*}) \sim K_{k^*-2}$ and $m^* \in \text{Cluster}(\gamma)$, there exist sequences $(q(n))_{n \in \mathbb{N}}$ and $(s(n))_{n \in \mathbb{N}}$ that have the following properties:

- (vi) $q(n) \in \mathbb{N}$ and $q(n) < q(n+1)$ for all $n \in \mathbb{N}$.
- (vii) $s(n) \in (a_{q(n)}, b_{q(n)})$ for all $n \in \mathbb{N}$.
- (viii) $\gamma(s(n)) \in \mathcal{B} \subset (\text{Int } K_{k^*}) \sim K_{k^*-2}$ for all $n \in \mathbb{N}$.
- (ix) $\lim_{n \uparrow \infty} \gamma(s(n)) = m^*$.

Because of (v), we can also require that

(x) $\gamma(a_{q(n)}), \gamma(b_{q(n)}) \notin K_{k^*+3}$ and $j_{q(n)}, k_{q(n)} > k^* + 3$ for all $n \in \mathbb{N}$.

It follows from (viii) and (x) that $\gamma(s(n)) \in \mathcal{B}$ and $\gamma(a_{q(n)}), \gamma(b_{q(n)}) \notin \text{Clos } \mathcal{B}$ for all $n \in \mathbb{N}$. Therefore, we can construct sequences $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}$ that have the following property:

(xi) Whenever $n \in \mathbb{N}$, we have $a_{q(n)} < \alpha_n < s(n) < \beta_n < b_{q(n)}$ as well as the inclusions $\{\gamma(t) : \alpha_n < t < \beta_n\} \subset \mathcal{B}$ and $\gamma(\alpha_n), \gamma(\beta_n) \in \text{Bdry } \mathcal{B}$.

Recall that $\gamma|_{[a_{q(n)}, b_{q(n)}]} = \gamma_{q(n)}$ is a $(j_{q(n)}, k_{q(n)})$ path for each $n \in \mathbb{N}$. Moreover, $k^* < \min\{j_{q(n)}, k_{q(n)}\} - 3$. Therefore, $\gamma|_{(A_n, B_n)}$ will be an integral curve of $\text{grad } f$ if $n \in \mathbb{N}$ and both $\alpha_n - A_n$ and $B_n - \beta_n$ are sufficiently small positive numbers. Therefore, we can construct sequences $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}$ that have the following properties:

(xii) $a_{q(n)} < A_n < \alpha_n < s(n) < \beta_n < B_n < b_{q(n)}$ for all $n \in \mathbb{N}$.

(xiii) $\gamma|_{(A_n, B_n)} = \gamma_{q(n)}|_{(A_n, B_n)}$ is an integral curve of $\text{grad } f$ for each $n \in \mathbb{N}$.

Set $\epsilon = (1/2)\{\text{dist}(m^*, \text{Bdry } \mathcal{B})\}$. By (ix) and (xiii), the hypotheses of Theorem 3.1 (5) hold with this choice of ϵ , and we conclude that

(xiv) $m^* \in \text{Critpt}(\mathcal{M}, f) \cap K_{k^*} = \text{Critpt}(K_{k^*}, f)$.

Set $\mathcal{W} = \text{Bdry } \mathcal{B}$ and $g = f|_{\mathcal{W}}$, and note that $\gamma(\beta_n) \in \mathcal{W}$ for all $n \in \mathbb{N}$. Since $g(\gamma(\beta_n)) < g(\gamma(\beta_{n+1}))$ for all $n \in \mathbb{N}$ and since $\text{Critval}(\mathcal{W}, g)$ is a finite set, there exists $N \in \mathbb{N}$ such that $g(\gamma(\beta_n)) \notin \text{Critval}(\mathcal{W}, g)$ for all $n \geq N$. If y^* is the smallest critical value of g greater than $g(\gamma(\beta_N))$, then $y^* \geq \lim_{n \uparrow \infty} g(\gamma(\beta_n)) = f(m^*)$. By Theorem 3.1 (6), there is a $(\mathcal{W}, g)$ path $\mu : [0, 1] \rightarrow \mathcal{W}$ such that $\mu(0) = \gamma(\beta_N)$ and $g(\mu(1)) = y^* \geq f(m^*)$. By shortening μ we may assume that $g(\mu(1)) = f(m^*)$. Also, μ is an $(\mathcal{M}, f)$

path. Therefore, the following statement is true:

(xv) There is $N \in \mathbb{N}$ and an $(\mathcal{M}, f)$ path $\mu : [0, 1] \rightarrow \mathcal{M}$ such that $\gamma(\beta_N) = \mu(0)$, $\{\mu(t) : t \in [0, 1]\} \subset \text{Bdry } \mathcal{B} \subset (\text{Int } K_{k^*}) \sim K_{k^*-2}$, and $f(\mu(1)) \in \text{Critval}(\mathcal{M}, f)$.

Finally, to produce the contradiction, let N and μ be as specified in (xv) and construct the path $\Gamma : [a_{q(N)}, \beta_N + 1] \rightarrow \mathcal{M}$ defined by

$$\Gamma(t) = \begin{cases} \gamma_{q(N)}(t) & \text{for } t \in [a_{q(N)}, \beta_N], \\ \mu(t - \beta_N) & \text{for } t \in [\beta_N, \beta_N + 1]. \end{cases}$$

Also, let $\kappa \in \mathbb{N}$ be the unique integer such that $\Gamma(\beta_N + 1) \in (\text{Int } K_\kappa) \sim (\text{Int } K_{\kappa-1})$. We prove the following three statements:

(xvi) $\kappa = k^*$ or $\kappa = k^* - 1$; $\kappa = \min\{j_{q(N)}, \kappa\} \leq k^*$; Γ is a $(j_{q(N)}, \kappa)$ path with initial point $\gamma_{q(N)}(a_{q(N)})$.

The first statement follows from the fact that $\text{Bdry } \mathcal{B} \subset (\text{Int } K_{k^*}) \sim K_{k^*-2}$ (see (xv)). The second statement now follows from (x). It is immediate that Γ meets Definition 2.2 (1)–(4). Thus, with attention to Definition 2.2 (5), suppose that

$$p \in \mathbb{N}, \quad p \leq \kappa - 3, \quad a_{q(N)} < \alpha < \beta < \beta_N + 1, \quad \{\Gamma(t) : t \in (\alpha, \beta)\} \subset \text{Int } K_p.$$

Since $\{\Gamma(t) : t \in [\beta_N, \beta_N + 1]\} \subset (\text{Int } K_{k^*}) \sim K_{k^*-2}$ and $p < \kappa - 3 \leq k^* - 3$, we conclude $a_{q(N)} < \alpha < \beta < \beta_N$. Therefore $\Gamma|(\alpha, \beta) = \gamma_{q(N)}|(\alpha, \beta)$. Since $p \leq \kappa - 3 \leq k^* - 3$, since $k^* - 3 < \min\{j_{q(N)}, k_{q(N)}\} - 6$ (by (x)), and since $\gamma_{q(N)}$ is a $(j_{q(N)}, k_{q(N)})$ path, it follows that $\Gamma|(\alpha, \beta)$ is an integral curve of $\text{grad } f$. So Γ meets Definition (2.2) (5) and the proof of (xvi) is complete.

By (iv), there is no $(j_{q(N)}, p)$ path with initial point $\gamma_{q(N)}(a_{q(N)})$ and $p < k_{q(N)}$. By (xvi), Γ is a $(j_{q(N)}, \kappa)$ path with initial point $\gamma_{q(N)}(a_{q(N)})$. Therefore,

$$k_{q(N)} \leq \kappa.$$

But, by (xvi) and (x), we also have

$$\kappa \leq k^* < k_{q(N)} - 3,$$

which is our contradiction.

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