

INTERPRETATION OF LAVRENTIEV PHENOMENON BY RELAXATION: THE HIGHER ORDER CASE

MARINO BELLONI

ABSTRACT. We consider integral functionals of the calculus of variations of the form

$$F(u) = \int_0^1 f(x, u, u', \dots, u^{(n)}) dx$$

defined for $u \in W^{n, \infty}(0, 1)$, and we show that the relaxed functional $\bar{F}$ with respect to the weak $W_{\text{loc}}^{n, 1}(0, 1)$ convergence can be written as

$$\bar{F}(u) = \int_0^1 f(x, u, u', \dots, u^{(n)}) dx + L(u),$$

where the additional term $L(u)$, the Lavrentiev Gap, is explicitly identified in terms of F .

1. INTRODUCTION

In 1926 M. Lavrentiev (see [L]) first demonstrated this surprising result: given a variational integral of a two-point Lagrange problem, which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, its infimum on the dense subclass of C^1 admissible functions may be *strictly greater* than its minimum value on the full admissible class. Some years later Manià (see [M]) gave an example of this phenomenon with a polynomial integrand. In recent years there have been additional works by several authors; for further bibliographical references the reader can see for instance [BuM].

In this paper we follow the Buttazzo and Mizel [BuM] approach which consists in studying the *Lavrentiev Phenomenon* from the point of view of relaxation theory. More precisely let X be a topological space, $Y \subset X$ a dense subset, $F : X \rightarrow [0, +\infty]$ a given functional, and define

$$\begin{aligned} \bar{F}_X &= \sup\{G : X \rightarrow [0, +\infty] : G \text{ l.s.c., } G \leq F \text{ on } X\}, \\ \bar{F}_Y &= \sup\{G : X \rightarrow [0, +\infty] : G \text{ l.s.c., } G \leq F \text{ on } Y\}, \\ L(u) &= \begin{cases} \bar{F}_Y(u) - \bar{F}_X(u) & \text{if } \bar{F}_X(u) < +\infty, \\ 0 & \text{otherwise,} \end{cases} \quad u \in X. \end{aligned}$$

We call this nonnegative functional L (notice that $\bar{F}_X \leq \bar{F}_Y$) the “*Lavrentiev Gap*” associated to F , X and Y . In their paper Buttazzo and Mizel [BuM]

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considered integral functionals of the form

$$G(u) = \int_0^1 f(x, u(x), u'(x)) dx$$

with $X = W^{1,1}(0, 1)$ and $Y = W^{1,+\infty}(0, 1)$, and gave a characterization of L in term of the “Value Function” V (see §2 below). Then they obtained an explicit representation of L for a large class of integrands.

In this paper we extend the results of [BuM] to integral functionals depending on higher order derivatives, of the form

$$G(u) = \int_0^1 f(x, u(x), \dots, u^{(n)}(x)) dx$$

with $X = W^{n,1}(0, 1)$ and $Y = W^{n,+\infty}(0, 1)$. More precisely, in §2 we obtain a characterization of L in terms of the “Value Function” V ; in §3 we provide an explicit representation of L for some integrands which satisfy a “homogeneity condition”, and an integrand of Manià type (see [M], [BM1], [BM2]) is analyzed in detail by following this approach. Our results deal with regular integrands (in a sense to be specified), but we want to point out an interesting result involving autonomous second order integrands (see Cheng [C], Cheng and Mizel [CM]) showing the nonoccurrence of the gap phenomenon when the integrand satisfies some continuity assumptions, with an example of a nonvanishing gap when a constraint of the form $\{u \geq 0\}$ is added.

2. THE REPRESENTATION THEOREM

Let Ω be the interval $(0, 1)$; we consider the following spaces:

- $W^{n,1}(0, 1)$ the space of all functions $u : \Omega \rightarrow \mathbf{R}$ which are absolutely continuous together with their $(n - 1)$ derivatives;
- $W^{n,\infty}[0, 1]$ the space of all functions $u : \Omega \rightarrow \mathbf{R}$ which are Lipschitz continuous together with their $(n - 1)$ derivatives;
- $W_{\text{loc}}^{n,\infty}]0, 1]$ the space of all functions $u : \Omega \rightarrow \mathbf{R}$ which are Lipschitz continuous together with their $(n - 1)$ derivatives on every interval $[\delta, 1]$, with $\delta > 0$;
- $\mathcal{A}_\infty$ the space of all function $u \in W^{n,1}(0, 1) \cap W_{\text{loc}}^{n,\infty}]0, 1]$ such that $u^{(i)}(0) = 0$ for $i = 0, \dots, (n - 1)$.

Let $f : \Omega \times \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$ be a function such that:

- (i) $f(x, s, z)$ is of Carathéodory type (i.e. measurable in X and continuous in (s, z));
- (ii) $f(x, s, \cdot)$ is convex on $\mathbf{R}$ for every $(x, s) \in \Omega \times \mathbf{R}^n$;
- (iii) there exists a function $\omega : \Omega \times \mathbf{R} \times \mathbf{R} \rightarrow [0, +\infty[$, with $\omega(x, t, \tau)$ integrable in x and increasing in t, τ , such that

$$0 \leq f(x, s, z) \leq \omega(x, |s|, |z|) \quad \forall (x, s, z) \in \Omega \times \mathbf{R}^n \times \mathbf{R}.$$

For every $u \in \mathcal{A}_\infty$, define

$$F(u) = \int_0^1 f(x, u, \dots, u^{(n)}) dx,$$

$$G(u) = \begin{cases} F(u) & \text{if } u \in W^{n,\infty}[0, 1], \\ +\infty & \text{otherwise,} \end{cases}$$

and denote by $\bar{G}$ the functional

$$\bar{G} = \max \left\{ H : \mathcal{A}_\infty \rightarrow [0, +\infty] : H \text{ seq. w-} W_{\text{loc}}^{n,1}\text{-l.s.c., } H \leq G \right\}.$$

Our goal is to give a representation formula for $\bar{G}$ over $\mathcal{A}_\infty$.

Since F is sequentially weakly lower semicontinuous on $W_{\text{loc}}^{n,1}(0, 1)$ (briefly seq. w- $W_{\text{loc}}^{n,1}(0, 1)$ -l.s.c.) (see [B]), we have

$$\bar{G}(u) \geq F(u) \quad \forall u \in \mathcal{A}_\infty,$$

and then

$$\bar{G}(u) = F(u) + L(u) \quad \forall u \in \mathcal{A}_\infty$$

for a suitable functional $L \geq 0$. We call the functional L the “Lavrentiev Gap” relative to G over the space $\mathcal{A}_\infty$. Obviously we have that $\bar{G} \leq G$. Then

$$\bar{G}(u) = F(u) \quad \forall u \in W^{n,\infty}[0, 1];$$

i.e. $L(u) = 0$ for every $u \in W^{n,\infty}[0, 1]$. In order to identify the functional L we introduce the “Value Function” $V(x, s)$ defined for every $(x, s) \in \Omega \times \mathbb{R}^n$ by:

$$V(x, s) = \inf \left\{ \int_0^x f(t, u, \dots, u^{(n)}) dt : u \in W^{n,\infty}[0, 1], u^{(i)}(0) = 0, \right. \\ \left. u^{(i)}(x) = s_i, \quad i = 0, \dots, (n-1) \right\}$$

and its lower semicontinuous envelope with respect to $s = (s_0, \dots, s_{(n-1)})$, given by

$$W(x, s) = \liminf_{\xi \rightarrow s} V(x, \xi).$$

We now state a representation result for the Lavrentiev Gap L .

Theorem 2.1. *If the integrand $f(x, s, z)$ satisfies the hypotheses above, then*

$$L(u) = \liminf_{x \rightarrow 0^+} W(x, u(x), \dots, u^{(n-1)}(x)) \quad \text{for every } u \in \mathcal{A}_\infty.$$

In order to achieve the proof of Theorem 2.1 we need some lemmas. For the sake of simplicity in the following we set

$$M(u) = \liminf_{x \rightarrow 0^+} W(x, u(x), \dots, u^{(n-1)}(x))$$

and, when no confusion is possible, we use the notation $\bar{u}(x)$ to indicate the vector $(u^{(i)}(x))_{i=0}^{n-1}$.

Lemma 2.2. Take $u, u_h \in \mathcal{A}_\infty$ with $u_h \in W^{n,\infty}[0, 1]$ and assume that $u_h \rightarrow u$ weakly in $W_{\text{loc}}^{n,1}(0, 1)$. Then

$$F(u) + M(u) \leq \liminf_{h \rightarrow +\infty} G(u_h).$$

Proof. Take $\delta > 0$; for every $h \in \mathbf{N}$, by the definitions of $V(x, s)$ and $W(x, s)$ we get

$$\begin{aligned} G(u_h) &= \int_0^\delta f(x, u_h, \dots, u_h^{(n)}) dx + \int_\delta^1 f(x, u_h, \dots, u_h^{(n)}) dx \\ &\geq V(\delta, \bar{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \dots, u_h^{(n)}) dx \\ &\geq W(\delta, \bar{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \dots, u_h^{(n)}) dx. \end{aligned}$$

As $h \rightarrow +\infty$, taking into account that W is seq. w- $W_{\text{loc}}^{n,1}$ -l.s.c. and that the assumptions on the integrand f provide the seq. w- $W_{\text{loc}}^{n,1}$ -l.s.c. of the integral term, we get

$$\begin{aligned} \liminf_{h \rightarrow +\infty} G(u_h) &\geq \liminf_{h \rightarrow +\infty} \left[W(\delta, \bar{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \dots, u_h^{(n)}) dx \right] \\ &\geq W(\delta, \bar{u}(\delta)) + \int_\delta^1 f(x, u, \dots, u^{(n)}) dx. \end{aligned}$$

Finally, as $\delta \rightarrow 0$ we obtain

$$\begin{aligned} \liminf_{h \rightarrow +\infty} G(u_h) &\geq \liminf_{\delta \rightarrow 0} \left[W(\delta, \bar{u}(\delta)) + \int_\delta^1 f(x, u, \dots, u^{(n)}) dx \right] \\ &\geq M(u) + \int_0^1 f(x, u, \dots, u^{(n)}) dx \\ &= M(u) + F(u), \end{aligned}$$

and the lemma is proved. $\square$

Lemma 2.3. The functional $F + M$ is seq. w- $W_{\text{loc}}^{n,1}$ -l.s.c. on $\mathcal{A}_\infty$.

Proof. Taking $u, u_h \in \mathcal{A}_\infty$ with $u_h \rightarrow u$ weakly in $W_{\text{loc}}^{n,1}$, we have to show that

$$F(u) + M(u) \leq \liminf_{h \rightarrow +\infty} [F(u_h) + M(u_h)].$$

Assume that the right-hand side is finite (otherwise there is nothing to prove), and consider a sequence (x_h) in Ω with $x_h \rightarrow 0$ such that

$$(2.1) \quad W(x_h, \bar{u}_h(x_h)) \geq M(u_h) + \frac{1}{h} \quad \forall h \in \mathbf{N}.$$

It is now possible to find a sequence (s_h) in $\mathbf{R}^n$, with $s_h \rightarrow 0$ such that

$$(2.2) \quad |s_h - \bar{u}_h(x_h)| \leq \frac{1}{h};$$

$$(2.3) \quad V(x_h, s_h) \leq W(x_h, \bar{u}_h(x_h)) + \frac{1}{h}.$$

Moreover, denoting by P_{n-1} the polynomial of degree $n-1$ such that $\bar{P}_{n-1}(x_h) = s_h - \bar{u}_h(x_h)$, it is easy to see that, since f is of Carathéodory type, the sequence (s_h) can be taken such that

$$(2.4) \quad \int_{x_h}^1 f(x, \bar{u}_h + \bar{P}_{n-1}, u_h^{(n)}) dx \leq \int_{x_h}^1 f(x, \bar{u}_h, u_h^{(n)}) dx + \frac{1}{h}.$$

Finally, let $v_h \in W^{n,\infty}[0, x_h]$ be such that $\bar{v}_h(0) = 0$, $\bar{v}_h(x_h) = s_h$ and

$$(2.5) \quad \int_0^{x_h} f(x, \bar{v}_h, v_h^{(n)}) dx \leq V(x_h, s_h) + \frac{1}{h}.$$

Setting

$$w_h(x) = \begin{cases} u_h(x) + P_{n-1}(x) & \text{if } x > x_h, \\ v_h(x) & \text{if } 0 \leq x \leq x_h, \end{cases}$$

we have $w_h \in W^{n,\infty}[0, 1]$, $\bar{w}_h(0) = 0$, and

$$(2.6) \quad w_h \xrightarrow{h \rightarrow +\infty} u \text{ w-}W_{\text{loc}}^{n-1}(0, 1).$$

Therefore, by using Lemma 2.2 and (2.1)–(2.6), we obtain

$$\begin{aligned} F(u) + M(u) &\leq \liminf_{h \rightarrow +\infty} F(w_h) \\ &= \liminf_{h \rightarrow +\infty} \left[\int_0^{x_h} f(x, \bar{v}_h, v_h^{(n)}) dx + \int_{x_h}^1 f(x, \bar{u}_h + \bar{P}_{n-1}(x_h), u_h^{(n)}) dx \right] \\ &\leq \liminf_{h \rightarrow +\infty} \left[\left(V(x_h, s_h) + \frac{1}{h} \right) + \left(\int_{x_h}^1 f(x, \bar{u}_h, u_h^{(n)}) dx + \frac{1}{h} \right) \right] \\ &\leq \liminf_{h \rightarrow +\infty} \left[\left(W(x_h, \bar{u}_h(x_h)) + \frac{1}{h} \right) + F(u_h) + \frac{2}{h} \right] \\ &\leq \liminf_{h \rightarrow +\infty} \left[\left(M(u_h) + \frac{1}{h} \right) + F(u_h) + \frac{3}{h} \right] \\ &= \liminf_{h \rightarrow +\infty} [M(u_h) + F(u_h)], \end{aligned}$$

and the lemma is proved. $\square$

Proof of Theorem 2.1. It is easy to see that

$$M(u) = 0 \quad \text{for every } u \in \mathcal{A}_\infty \cap W^{n,\infty}[0, 1];$$

hence we have $F + M \leq G$ on $\mathcal{A}_\infty$. By Lemma 2.3 we have $F + M \leq \bar{G}$ on $\mathcal{A}_\infty$, so it remains to prove that

$$\bar{G} \leq F(u) + M(u) \quad \text{for every } u \in \mathcal{A}_\infty.$$

To this aim, fix $u \in \mathcal{A}_\infty$ and take a sequence (x_h) in Ω , $x_h \rightarrow 0$, such that

$$(2.7) \quad M(u) = \lim_{h \rightarrow +\infty} W(x_h, \bar{u}(x_h)).$$

By definition of W and the assumptions on the integrand f we may find a sequence (s_h) in $\mathbf{R}^n$, $s_h \rightarrow 0$, such that for every $h \in \mathbf{N}$

$$(2.8) \quad |s_h - \bar{u}(x_h)| \leq \frac{1}{h},$$

$$(2.9) \quad V(x_h, s_h) \leq W(x_h, \bar{u}(x_h)) + \frac{1}{h},$$

$$(2.10) \quad \int_{x_h}^1 f(x, \bar{u} + \bar{P}_{n-1}, u^{(n)}) dx \leq \int_{x_h}^1 f(x, \bar{u}, u^{(n)}) dx + \frac{1}{h},$$

where P_{n-1} is as in the proof of Lemma 2.3. Finally, let $v_h \in W^{n,\infty}[0, x_h]$ be a sequence such that $\bar{v}_h(0) = 0$, $\bar{v}_h(x_h) = s_h$ and

$$(2.11) \quad \int_0^{x_h} f(x, \bar{v}_h, v_h^{(n)}) dx \leq V(x_h, s_h) + \frac{1}{h}.$$

As in the proof of Lemma 2.3, we define the sequence

$$w_h(x) = \begin{cases} u_h(x) + P_{n-1}(x) & \text{if } x > x_h, \\ v_h(x) & \text{if } 0 \leq x \leq x_h. \end{cases}$$

Then $w_h \in W^{n,\infty}[0, 1]$, $\bar{w}_h(0) = 0$ and

$$w_h \xrightarrow{h \rightarrow +\infty} u \text{ strongly } W_{\text{loc}}^{n,1}(0, 1).$$

Hence, by using (2.7)–(2.11), we obtain

$$\begin{aligned} \bar{G}(u) &\leq \liminf_{h \rightarrow +\infty} G(w_h) \\ &= \liminf_{h \rightarrow +\infty} \left[\int_0^{x_h} f(x, \bar{v}_h, v_h^{(n)}) dx + \int_{x_h}^1 f(x, \bar{u} + \bar{P}_{n-1}(x_h), u^{(n)}) dx \right] \\ &\leq \liminf_{h \rightarrow +\infty} \left[\left(V(x_h, s_h) + \frac{1}{h} \right) + \left(\int_{x_h}^1 f(x, \bar{u}, u^{(n)}) dx + \frac{1}{h} \right) \right] \\ &\leq \liminf_{h \rightarrow +\infty} \left[\left(W(x_h, \bar{u}(x_h)) + \frac{1}{h} \right) + F(u) + \frac{2}{h} \right] \\ &= M(u) + F(u); \end{aligned}$$

so $M = L$, and the theorem is completely proved. $\square$

Remark 2.4. Fix a subset β of $\{0, 1, \dots, n-1\}$ and consider the class $\mathcal{A}_\infty^\beta$ of all functions $u \in W^{n,1}(0, 1) \cap W_{\text{loc}}^{n,\infty}[0, 1]$ such that $u^{(i)}(0) = 0$ for $i \in \beta$. We denote by $\bar{G}_\beta$ the functional

$$\bar{G}_\beta = \max\{H : \mathcal{A}_\infty^\beta \rightarrow [0, +\infty] : H \text{ seq. w-} W_{\text{loc}}^{n,1}\text{-l.s.c., } H \leq G\}.$$

As in the previous case, we have

$$\bar{G}_\beta(u) = F(u) + L_\beta(u) \quad \forall u \in \mathcal{A}_\infty$$

for a suitable functional $L_\beta \geq 0$, the “Lavrentiev Gap” relative to G over the space $\mathcal{A}_\infty^\beta$. In order to identify the functional L_β we introduce the Value

Function $V_\beta(x, s)$ defined for every $(x, s) \in \Omega \times \mathbf{R}^k$ by:

$$V_\beta(x, s) = \inf \left\{ \int_0^x f(t, u, \dots, u^{(n)}) dt : u \in W^{n, \infty}[0, 1], u^{(i_j)}(0) = 0, \right. \\ \left. u^{(i_j)}(x) = s_{i_j}, j = 0, 1, \dots, k-1 \right\}$$

and its lower semicontinuous envelope with respect to $s = (s_{i_0}, \dots, s_{i_{k-1}})$, given by

$$W_\beta(x, s) = \liminf_{\xi \rightarrow s} V_\beta(x, \xi).$$

By repeating step by step the proof of Theorem 2.1, we obtain the following result:

Theorem 2.5. *If the integrand $f(x, s, z)$ satisfies the assumptions of Theorem 2.1, then*

$$L_\beta(u) = \liminf_{x \rightarrow 0^+} W_\beta(x, u^{(i_0)}(x), \dots, u^{(i_{k-1})}(x)) \quad \text{for every } u \in \mathcal{A}_\infty^\beta.$$

Note that the polynomial P_{n-1} may be chosen, in this case, such that

$$P_{n-1}^{(r)} = \begin{cases} u_h^{(r)}(x_h) - (s_h)_r & \text{if } r \in \beta, \\ 0 & \text{otherwise.} \end{cases}$$

3. SOME EXAMPLES

In this section we give an explicit representation formula for a class of second order integrands f (we mean that f is a function depending on x, u, u', u''). We introduce the so-called “invariance property” for second order integrands (analogous to the one introduced in [HM1] for first order integrands, and to the one of [CM] for second order autonomous integrands):

there exists $\gamma \in]1, 2[$ such that for every $t > 0$ and $(x, s, z, w) \in \Omega \times \mathbf{R}^3$

$$tf(tx, t^\gamma s, t^{\gamma-1} z, t^{\gamma-2} w) = f(x, s, z, w).$$

We want to analyze a class of second order integrands $f(x, s, z, w)$ that satisfies this invariance property only in an asymptotic sense near the relevant singular abscissa. Let us take $\delta > 1, \tau \in [1, \delta[$; we suppose the integrand $f : \Omega \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ has the form

$$f(x, s, z, w) = x^{\tau-1} a(x, s) b(x, z) |w|^\delta,$$

with $a(x, s), b(x, z)$ nonnegative, continuous functions such that, setting $\gamma = 2 - \frac{\tau}{\delta}$, for every $y \in \Omega$ the functions $m_y, n_y, M_y, N_y : \Omega \rightarrow \mathbf{R}$ defined by

$$m_y(s) = \inf\{a(x, x^\gamma s) : x \leq y\}, \quad n_y(s) = \inf\{b(x, x^{\gamma-1} s) : x \leq y\}, \\ M_y(s) = \sup\{a(x, x^\gamma s) : x \leq y\}, \quad N_y(s) = \sup\{b(x, x^{\gamma-1} s) : x \leq y\}$$

are locally bounded. Take now $x, y \in \Omega, x \leq y$, and consider the following functionals:

$$F_x(u) = \int_0^x f(t, u, u', u'') dt, \\ F_{*,x,y}(u) = \int_0^x t^{\tau-1} m_y(t^{-\gamma} u) n_y(t^{1-\gamma} u') |u''|^\delta dt, \\ F_{x,y}^*(u) = \int_0^x t^{\tau-1} M_y(t^{-\gamma} u) N_y(t^{1-\gamma} u') |u''|^\delta dt.$$

We suppose that there exists $\bar{y} \in \Omega$ such that, for every $x \in \{x \in \Omega : x \leq \bar{y}\}$, we have

$$(3.1) \quad F_{x,\bar{y}}^*(u) < +\infty \quad \text{whenever } F_x(u) < +\infty.$$

Obviously for every $x, y \in \Omega$ with $x \leq y$

$$F_{*,x,y}^*(u) \leq F_x(u) \leq F_{x,y}^*(u) \quad \forall u \in \mathcal{A}_\infty^1;$$

then for every $x \in \Omega$

$$\sup_{\substack{y \in \Omega \\ y \leq \bar{y}}} F_{*,x,y}^*(u) \leq F_x(u) \leq \inf_{\substack{y \in \Omega \\ y \leq \bar{y}}} F_{x,y}^*(u) \quad \forall u \in \mathcal{A}_\infty^1,$$

say

$$(3.2) \quad \lim_{y \rightarrow 0^+} F_{*,x,y}^*(u) \leq F_x(u) \leq \lim_{y \rightarrow 0^+} F_{x,y}^*(u) \quad \forall u \in \mathcal{A}_\infty^1.$$

Define, for $y \leq \bar{y}$,

$$\begin{aligned} \lim_{y \rightarrow 0^+} m_y(s) &= m_0(s), & \lim_{y \rightarrow 0^+} M_y(s) &= M_0(s), \\ \lim_{y \rightarrow 0^+} n_y(s) &= n_0(s), & \lim_{y \rightarrow 0^+} N_y(s) &= N_0(s); \end{aligned}$$

by the assumptions (3.1) we apply the Monotone and Lebesgue Convergence Theorems to (3.2) obtaining

$$(3.3) \quad F_{0,x}(u) \leq F_x(u) \leq F_x^0(u) \quad \forall u \in \mathcal{A}_\infty^1$$

where

$$\begin{aligned} F_{0,x}(u) &= \int_0^x t^{\tau-1} m_0(t^{-\gamma} u) n_0(t^{1-\gamma} u') |u''|^\delta dt, \\ F_x^0(u) &= \int_0^x t^{\tau-1} M_0(t^{-\gamma} u) N_0(t^{1-\gamma} u') |u''|^\delta dt. \end{aligned}$$

We suppose also that $m_0(s) = P \leq Q = M_0(s)$, with $P, Q \in [0, +\infty[$.

Theorem 3.1. *Under the previous assumptions, for every*

$$u \in \mathcal{A}_\infty^1 = \{u \in W^{2,1}(0,1) \cap W_{\text{loc}}^{2,\infty}[0,1] : u'(0) = 0\}$$

we have

$$(3.4) \quad \begin{aligned} P \delta k^{\delta-1} \left| \int_0^{\liminf_{x \rightarrow 0} u'(x) x^{1-\gamma}} n_0(\xi) |\xi|^{\delta-1} d\xi \right| \\ \leq L(u) \leq Q \delta k^{\delta-1} \left| \int_0^{\liminf_{x \rightarrow 0} u'(x) x^{1-\gamma}} N_0(\xi) |\xi|^{\delta-1} d\xi \right|, \end{aligned}$$

where $k = \frac{\delta(\gamma-1)}{\delta-1}$.

In order to achieve the proof of Theorem 3.1, we need a lemma.

Lemma 3.2. Let $h(Z)$ be the solution of the minimum problem

$$\inf\{G(u) : u \in W^{2,\infty}(0, 1), u'(0) = 0, u'(1) = Z\},$$

where

$$G(u) = \int_0^1 x^{\tau-1} n(x^{1-\gamma} u'(x)) |u''(x)|^\delta dx.$$

We have

$$h(Z) = \delta k^{\delta-1} \left| \int_0^Z n(\xi) |\xi|^{\delta-1} d\xi \right|,$$

where $k = \frac{\delta(\gamma-1)}{\delta-1}$ and the function $h(Z)$ is the solution of the equation

$$(3.5) \quad \begin{cases} (\gamma - 1)Zh'(Z) = \sup\{Qh'(Z) - n(Z)|Q|^\delta : Q \in \mathbf{R}\}, \\ h(0) = 0. \end{cases}$$

Proof. By explicitly carrying out the maximization, the equation (3.5) becomes

$$h'(Z) = \delta k^{\delta-1} n(Z) |Z|^{\delta-2}$$

and by direct integration

$$h(Z) = \delta k^{\delta-1} \left| \int_0^Z n(\xi) |\xi|^{\delta-1} d\xi \right|.$$

Let us take $u \in \mathcal{A}(x, z) = \{u \in W^{2,\infty} : u'(0) = 0, u'(x) = z\}$; from (3.5), setting $Z(t) = t^{1-\gamma} u'(t)$ and $Q(t) = t^{2-\gamma} u''(t)$ we obtain

$$(\gamma - 1)Z(t)h'(Z(t)) \geq |Q(t)|h'(Z(t)) - n(Z(t))|Q(t)|^\delta.$$

Then

$$\begin{aligned} t^{-1}n(Z(t))|Q(t)|^\delta &\geq t^{-1}h'(Z(t))[Q(t) + (1 - \gamma)Z(t)] \\ &= h'(Z(t))Z'(t) = (h \circ Z)'(t) \end{aligned}$$

(for the last equality see [MM]). Integrating on $]0, x[$ yields

$$\begin{aligned} I(u) &= \int_0^x t^{\tau-1} n(t^{1-\gamma} u'(t)) |u''(t)|^\delta dt \\ &= \int_0^x t^{-1} n(t^{1-\gamma} u'(t)) |t^{\tau/\delta} u''(t)|^\delta dt \\ &= \int_0^x t^{-1} n(Z(t)) |Q(t)|^\delta dt \\ &\geq \int_0^x (h \circ Z)'(t) dt \\ &= h(Z(x)) - \lim_{t \rightarrow 0^+} h(Z(t)) \\ &= h(Z(x)) \end{aligned}$$

(in fact $u \in W^{2,\infty}[0, x]$ with $u'(0) = 0$ implies

$$\lim_{t \rightarrow 0^+} t^{1-\gamma} u'(t) = 0 \quad \forall \gamma \in [1, 2[,$$

and hence $\lim_{t \rightarrow 0^+} h(Z(t)) = 0$). It follows that

$$(3.6) \quad W(x, z) = \inf\{I(u) : u \in \mathcal{A}(x, z)\} \geq h(x^{1-\gamma} z) = h(Z).$$

Consider now the sequence $(u_\varepsilon) \subset \mathcal{A}(x, z)$ defined by

$$u_\varepsilon(0) = 0, \quad u'_\varepsilon(t) = \begin{cases} (\frac{t}{x})^k z & \text{if } t \geq \varepsilon, \\ t^{\frac{k-1}{k}} z & \text{if } t < \varepsilon. \end{cases}$$

Taking ε sufficiently small we have

$$W(x, z) \leq I(u_\varepsilon) = \int_0^\varepsilon t^{\tau-1} n(t^{1-\gamma} u'_\varepsilon) |u''_\varepsilon|^\delta dt + \int_\varepsilon^x t^{\tau-1} n(t^{1-\gamma} u'_0) |u''_0|^\delta dt,$$

where $u'_0(t) = (\frac{t}{x})^k$, $u_0(0) = 0$; passing to the limit for $\varepsilon \rightarrow 0$ the first integral tends to 0, and hence

$$W(x, z) \leq I(u_0).$$

At this point we can easily verify that $I(u_0) = h(x^{1-\gamma} z)$, and the proof of the lemma is then complete. $\square$

Proof of Theorem 3.1. We fix $u \in \mathcal{A}_\infty^1$; by Theorem 2.5

$$L_1(u) = \liminf_{x \rightarrow 0} W_1(x, u'(x))$$

where $W_1(x, z) = \liminf_{q \rightarrow z} V_1(x, q)$ and

$$\begin{aligned} V_1(x, z) &= \inf\{F_x(u) : u \in W^{2,\infty}(0, 1), u'(0) = 0, u'(x) = z\} \\ &= \inf\{F_x(u) : u \in \mathcal{A}(x, z)\}, \end{aligned}$$

where $\mathcal{A}(x, z) = \{u \in W^{2,\infty}(0, x) : u'(0) = 0, u'(x) = z\}$.

Let us introduce the Value Functions relative to the functionals $F_{0,x}, F_x^0$ given by

$$\begin{aligned} (3.7) \quad V_0(x, z) &= \inf\{F_{0,x}(u) : u \in \mathcal{A}(x, z)\}, \\ V^0(x, z) &= \inf\{F_x^0(u) : u \in \mathcal{A}(x, z)\}; \end{aligned}$$

obviously, for every $x \in \Omega$ and for every $z \in \mathbf{R}$, we have by (3.3)

$$(3.8) \quad V_0(x, z) \leq V_1(x, z) \leq V^0(x, z).$$

Setting $S = sx^{-\gamma}$, $Z = zx^{1-\gamma}$ and

$$\begin{aligned} G_0(u) &= P \int_0^1 t^{\tau-1} n_0(t^{1-\gamma} u') |u''|^\delta dt, \\ G^0(u) &= Q \int_0^1 t^{\tau-1} N_0(t^{1-\gamma} u') |u''|^\delta dt, \\ \mathcal{A}(Z) &= \{u \in W^{2,\infty}(0, 1) : u'(0) = 0, u'(1) = Z\}, \end{aligned}$$

by the change of variable $t = xy$ we get

$$\begin{aligned} V_0(x, z) &= H_0(Z) = \inf\{G_0(u) : u \in \mathcal{A}(Z)\}, \\ V^0(x, z) &= H^0(Z) = \inf\{G^0(u) : u \in \mathcal{A}(Z)\}, \end{aligned}$$

so that inequalities (3.8) become

$$(3.9) \quad H_0(Z) \leq V_1(x, z) \leq H^0(Z)$$

for every $x \in \Omega$ and for every $z \in \mathbf{R}$.

By Lemma 3.2 we have that $H_0(Z)$ and $H^0(Z)$ are given by

$$(3.10) \quad \begin{aligned} H_0(Z) &= P\delta k^{\delta-1} \left| \int_0^Z n_0(\xi) |\xi|^{\delta-1} d\xi \right|, \\ H^0(Z) &= Q\delta k^{\delta-1} \left| \int_0^Z N_0(\xi) |\xi|^{\delta-1} d\xi \right|; \end{aligned}$$

and inserting (3.10) into (3.9) we obtain the inequality (3.4), that is the thesis. $\square$

Example 3.3. Consider the functional

$$F(u) = \int_0^1 f(x, u(x), u'(x), u''(x)) dx,$$

where the integrand f has the following form, with $1 < p < 2$ and $0 < q < 1$,

$$\begin{aligned} f(x, s, z, w) &= (s - x^p)^2 (z - x^q)^2 |w|^\delta \\ &= x^{2(p+q)} (sx^{-p} - 1)^2 (zx^{-q} - 1)^2 |w|^\delta \\ &= x^{2(p+q)} a(x, s) b(x, z) |w|^\delta, \end{aligned}$$

where we set

$$\begin{aligned} a(x, s) &= (sx^{-p} - 1)^2, \\ b(x, z) &= (zx^{-q} - 1)^2. \end{aligned}$$

If $\delta \leq 1$ we can easily verify that the Lavrentiev Gap $L(u)$ is identically equal to 0: for every $u \in W^{2,1}(0, 1)$ with $u'(0) = 0$ we construct the sequence in $W^{2,\infty}$

$$(3.11) \quad \begin{aligned} u'_\varepsilon(x) &= \begin{cases} u'(x), & \text{if } x_\varepsilon \leq x, \\ \frac{u'(x_\varepsilon)}{x_\varepsilon} x, & \text{if } 0 \leq x \leq x_\varepsilon, \end{cases} \\ u_\varepsilon(0) &= u(0), \end{aligned}$$

where $x_\varepsilon \in [0, 1]$ is a sequence with limit 0 as $\varepsilon \rightarrow 0$, and we verify that $F(u_\varepsilon) \rightarrow F(u)$ as $\varepsilon \rightarrow 0$. Here, for simplicity, we restrict our attention to the case $\delta > \frac{1+2(p+q)}{2-p}$. With the notation above

$$\tau = 1 + 2(p+q), \quad \gamma = 2 - \frac{\tau}{\delta} = 2 - \frac{1 + 2(p+q)}{\delta}.$$

This integrand f has as “zero cost curves” the functions $z_1(x) = x^p$, $z_2(x) = (q+1)^{-1}x^{q+1}$; by the assumption on p and q we obtain $z_1(x), z_2(x) \in W^{2,1}(0, 1) \setminus W^{2,\infty}[0, 1]$.

When $\delta > \frac{1+2(p+q)}{1-q}$, we have $\gamma > p$, $\gamma > q$ and then

$$m_0(s) = n_0(s) = M_0(s) = N_0(s) = 1;$$

hence for every fixed $u \in \mathcal{A}_\infty^1$ we obtain

$$L_1(u) = k^{\delta-1} \liminf_{x \rightarrow 0^+} \left| \frac{u'(x)}{x^{\gamma-1}} \right|^\delta;$$

this functional is not identically equal to 0: for instance, $L_1(x^p) = +\infty$ and $L_1((q+1)^{-1}x^{q+1}) = +\infty$.

When $\delta = \frac{1+2(p+q)}{1-q}$, by a computation similar to the previous case, we obtain

$$m_0(s) = M_0(s) = 1, \quad n_0(s) = N_0(s) = (s-1)^2.$$

Then, for every fixed $u \in \mathcal{A}_\infty^1$ we have

$$L_1(u) = \delta k^{\delta-1} \left| \liminf_{x \rightarrow 0^+} \int_0^{u'(x)x^{1-\gamma}} (\xi-1)^2 |\xi|^{\delta-1} d\xi \right|;$$

also in this case this functional is not identically equal to 0: for instance $L_1(x^p) = +\infty$, while $L_1((q+1)^{-1}x^{q+1}) = 2k^{\delta-1}/(\delta+1)(\delta+2)$.

Finally, when $\delta < \frac{1+2(p+q)}{1-q}$, Theorem 3.1 does not apply because the functions n_y, N_y are not locally bounded. However it is possible to show that in this case the gap phenomenon does not occur (see [Be]): for every $u \in \mathcal{A}_{W^{2,1}}^1$ we construct u_ε in $W^{2,\infty}(0,1)$ by (3.11) and we prove that, if $F(u) < +\infty$, then $F(u_\varepsilon) \rightarrow F(u)$ as $\varepsilon \rightarrow 0$, i.e.

$$\int_0^1 f(x, u, u', u'') dx < +\infty \Rightarrow L(u) = 0.$$

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PARMA, VIÁ MASSIMO D'AZEGLIO, 85/A, 63100
PARMA, ITALY

E-mail address: belloni@dm.unipi.it