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# IRREDUCIBLE SEMIGROUPS OF FUNCTIONALLY POSITIVE NILPOTENT OPERATORS

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ABSTRACT. For each irrational number $\theta \in (0, 1)$, we construct a semigroup $\mathscr{S}_\theta$ of nilpotent operators on $\mathscr{L}^2([0, 1])$ that are also partial isometries and positive in the sense that the operator maps nonnegative functions to nonnegative functions. We prove that each semigroup $\mathscr{S}_\theta$ is discrete in the norm topology and hence norm-closed and that the weak closure of $\mathscr{S}_\theta$ is independent of $\theta$. We show that each semigroup $\mathscr{S}_\theta$ has no nontrivial invariant subspaces.

Consider the Hilbert space $\mathscr{L}^2([0, 1])$ with the Lebesgue measure $m$ on $[0, 1]$. An operator $T$ on $\mathscr{L}^2([0, 1])$ is called *functionally positive* (or simply, positive) if $T$ maps nonnegative functions to nonnegative functions. It has been proven in [5] that certain multiplicative semigroups of positive quasinilpotent operators are reducible, that is, all operators in the semigroup have a common nontrivial invariant subspace. One may ask: Is every semigroup of positive quasinilpotent operators reducible?

In [1, Theorem 1], Hadwin et al. constructed an irreducible semigroup of nilpotent operators on a Hilbert space such that every operator in the semigroup has nilpotency two but is not positive in any sense. And in [3], Schaefer provided a method of constructing a positive quasinilpotent operator on the Hilbert space of $L^2$-functions of the unit circle such that the operator does not have any nontrivial invariant subspaces corresponding to projections that are multiplication operators induced by characteristic functions on the unit circle. It is easy to see that neither of the two examples answers the above question. In this paper, we answer the question negatively by constructing an irreducible semigroup of positive nilpotent operators that are also partial isometries.

For every $\alpha \in [0, 1]$, we define $S_\alpha$ and $T_\alpha$ as follows:

$$(S_\alpha f)(t) = \begin{cases} f(t+\alpha) & \text{if } t \in [0, 1-\alpha], \\ 0 & \text{if } t \in (1-\alpha, 1], \end{cases} \quad f \in \mathscr{L}^2([0, 1]),$$

$$(T_\alpha f)(t) = \begin{cases} 0 & \text{if } t \in [0, \alpha), \\ f(t-\alpha) & \text{if } t \in [\alpha, 1], \end{cases} \quad f \in \mathscr{L}^2([0, 1]).$$

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Clearly, $S_\alpha$ and $T_\alpha$ are well-defined bounded linear operators on $\mathscr{L}^2([0, 1])$. For convenience, we define $S_\alpha = T_\alpha = 0$ for all $\alpha > 1$. Still, we denote by $M_\phi$ the multiplication operator corresponding to $\phi \in \mathscr{L}^\infty([0, 1], m)$.

**Lemma 1.** *For any* $\alpha \in [0, 1]$,

(i) $S_\alpha$ *and* $T_\alpha$ *are positive operators.*
(ii) $S_\alpha^* = T_\alpha$, $S_0 = T_0 = I$, $S_1 = T_1 = 0$.
(iii) $S_\alpha T_\alpha = M_{\chi_{[0,1-\alpha]}}$, $T_\alpha S_\alpha = M_{\chi_{[\alpha,1]}}$, *and therefore,* $S_\alpha$ *and* $T_\alpha$ *are partial isometries.*
(iv) *If* $\alpha \neq 1$, *then* $\|S_\alpha\| = \|T_\alpha\| = 1$.

*Proof.* (i) It is obvious that $S_\alpha$ and $T_\alpha$ are positive operators.
(ii) For any $f, g \in \mathscr{L}^2([0, 1])$,

$$\begin{aligned} \langle S_\alpha f, g\rangle &= \int_0^1 (S_\alpha f)(t)\bar{g}(t)\,dt \\ &= \int_0^{1-\alpha} f(t+\alpha)\bar{g}(t)\,dt \\ &= \int_\alpha^1 f(s)\bar{g}(s-\alpha)\,ds \\ &= \langle f, T_\alpha g\rangle. \end{aligned}$$

Thus, $S_\alpha^* = T_\alpha$. Clearly, $S_0 = T_0 = I$, $S_1 = T_1 = 0$.
(iii) For any $f, g \in \mathscr{L}^2([0, 1])$,

$$\begin{aligned} \langle S_\alpha T_\alpha f, g\rangle &= \langle T_\alpha f, T_\alpha g\rangle \\ &= \int_\alpha^1 f(t-\alpha)\bar{g}(t-\alpha)\,dt \\ &= \int_0^{1-\alpha} f(s)\bar{g}(s)\,ds \\ &= \langle M_{\chi_{[0,1-\alpha]}} f, g\rangle. \end{aligned}$$

Therefore, $S_\alpha T_\alpha = M_{\chi_{[0,1-\alpha]}}$. Similarly, $T_\alpha S_\alpha = M_{\chi_{[\alpha,1]}}$.
Since

$$S_\alpha^* S_\alpha = T_\alpha S_\alpha = M_{\chi_{[\alpha,1]}}, \quad T_\alpha^* T_\alpha = S_\alpha T_\alpha = M_{\chi_{[0,1-\alpha]}}$$

are projections, we have that $S_\alpha$ and $T_\alpha$ are partial isometries.
(iv) It follows immediately from (iii). □

**Lemma 2.** *For any* $\alpha \in [0, 1)$ *and any* $\phi \in \mathscr{L}^\infty([0, 1])$,

(i) $S_\alpha M_\phi = M_{S_\alpha \phi} S_\alpha$, $T_\alpha M_\phi = M_{T_\alpha \phi} T_\alpha$.
(ii) $M_\phi S_\alpha$, $S_\alpha M_\phi$, $M_\phi T_\alpha$, *and* $T_\alpha M_\phi$ *are all nilpotent operators.*

*Proof.* (i) For any $f \in \mathscr{L}^2([0, 1])$,

$$S_\alpha(\phi f) = (S_\alpha \phi)(S_\alpha f).$$

Therefore,

$$(S_\alpha M_\phi) f = S_\alpha(\phi f) = (S_\alpha \phi)(S_\alpha f) = (M_{S_\alpha \phi} S_\alpha) f.$$

Hence, $S_\alpha M_\phi = M_{S_\alpha \phi} S_\alpha$. Similarly, $T_\alpha M_\phi = M_{T_\alpha \phi} T_\alpha$.

(ii) It is obvious that $(S_\alpha)^p = S_{p\alpha}$ for any positive integer $p$. Therefore, it follows from (i) that

$$(M_\phi S_\alpha)^p = M_\phi M_{S_\alpha \phi} \cdots M_{S_{(p-1)\alpha}\phi} S_{p\alpha}$$

for any positive integer $p$. Hence, $(M_\phi S_\alpha)^p = 0$ for $p$ large enough to satisfy $p\alpha > 1$. Thus, $M_\phi S_\alpha$ is a nilpotent operator. Similarly, $S_\alpha M_\phi$, $M_\phi T_\alpha$, and $T_\alpha M_\phi$ are all nilpotent operators. □

**Lemma 3.** *If* $\alpha, \beta \in [0, 1]$, *then*

(i) $S_\alpha S_\beta = S_{\alpha+\beta}$ *and* $T_\alpha T_\beta = T_{\alpha+\beta}$.

(ii)

$$S_\alpha T_\beta = \begin{cases} M_{\chi_{[0,1-\alpha]}} T_{\beta-\alpha} & \text{if } \alpha \le \beta, \\ M_{\chi_{[0,1-\alpha]}} S_{\alpha-\beta} & \text{if } \alpha > \beta. \end{cases}$$

(iii)

$$T_\beta S_\alpha = \begin{cases} M_{\chi_{[\beta,1]}} T_{\beta-\alpha} & \text{if } \alpha \le \beta, \\ M_{\chi_{[\beta,1]}} S_{\alpha-\beta} & \text{if } \alpha > \beta. \end{cases}$$

*Proof*. (i) It is easy to check.

(ii) If $\alpha \le \beta$, then by (i) and Lemma 1

$$S_\alpha T_\beta = S_\alpha T_\alpha T_{\beta-\alpha} = M_{\chi_{[0,1-\alpha]}} T_{\beta-\alpha}.$$

If $\alpha > \beta$, then by (i), Lemma 1 and Lemma 2,

$$\begin{aligned} S_\alpha T_\beta &= S_{\alpha-\beta} S_\beta T_\beta = S_{\alpha-\beta} M_{\chi_{[0,1-\beta]}} \\ &= M_{S_{\alpha-\beta}\chi_{[0,1-\beta]}} S_{\alpha-\beta} = M_{\chi_{[0,1-\alpha]}} S_{\alpha-\beta}. \end{aligned}$$

(iii) The proof is similar to that of (ii). □

In [4] it was proved that every positive operator $S$ on $\mathscr{L}^2([0, 1])$ is a pseudo-integral operator and that $S$ is determined by a positive finite Borel measure $\mu_S$ on $[0, 1] \times [0, 1]$ by the equation

$$\langle Sf, g\rangle = \int_{[0,1]\times[0,1]} f(y)\bar{g}(x)\mu_S\,(dx, dy).$$

For any $\alpha \in [0, 1]$, let

$$G_\alpha = \{(x, y) \in [0, 1] \times [0, 1] : y = x + \alpha\}$$

and

$$F_\alpha = \{(x, y) \in [0, 1] \times [0, 1] : y = x - \alpha\}.$$

It is easy to check that $S_\alpha$ is a pseudo-integral operator determined by $\mu_\alpha$, where $\mu_\alpha$ is the positive finite Borel measure defined by the equation

$$\mu_\alpha(E) = m(\{x \in [0, 1] : (x, y) \in E \cap G_\alpha \text{ for some } y \in [0, 1]\}).$$

Similarly, $T_\alpha$ is a pseudo-integral operator determined by $\nu_\alpha$, where $\nu_\alpha$ is defined by the equation

$$\nu_\alpha(E) = m(\{x \in [0, 1] : (x, y) \in E \cap F_\alpha \text{ for some } y \in [0, 1]\}).$$

Next we construct a multiplicative semigroup of positive nilpotent operators that are also partial isometries and prove that the semigroup is irreducible.

Choose an arbitrary irrational number $\theta \in (0, 1)$. Let $\mathcal{S}_\theta$ be the multiplicative semigroup generated by the set

$$\{S_a, T_{b\theta} : a, b \in (0, 1) \text{ are rational numbers}\}.$$

**Theorem 4.** *Suppose $\theta \in (0, 1)$ is irrational. Then the semigroup $\mathcal{S}_\theta$ consists of positive nilpotent operators that are also partial isometries.*

*Proof.* Easy to see that every operator in $\mathcal{S}_\theta$ is a product of positive operators and, therefore, positive itself.

By Lemma 3 (i), any 'word' in $\mathcal{S}_\theta$ looks like

$$W = S_{a_1}^{p_1} T_{b_1\theta}^{q_1} S_{a_2}^{p_2} T_{b_2\theta}^{q_2} \cdots S_{a_n}^{p_n} T_{b_n\theta}^{q_n}$$

for some integer $n \geq 1$, where $a_j, b_j \in (0, 1)$ are rational numbers and $p_j$, $q_j$ are nonnegative integers for all $j = 1, 2, \ldots, n$ and at least one of $p_j$, $q_j$ is nonzero $(j = 1, 2, \ldots, n)$.

By Lemma 3, $W$ is either $0$, or $M_\phi S_{a-b}$ with $a - b > 0$, or $M_\psi T_{b-a}$ with $a - b < 0$, where $a = \sum_{j=1}^n p_j a_j$, $b = \theta \sum_{j=1}^n q_j b_j$, and $\phi$ and $\psi$ are characteristic functions of some intervals. Hence $W$ is a partial isometry. Clearly, $a \neq b$ since $\theta$ is irrational, and thus, by Lemma 2 (ii), $W$ is a nilpotent operator. □

Now we prove that $\mathcal{S}_\theta$ is discrete and irreducible. To do this, we need the following lemmas.

**Lemma 5.** *Let $\alpha \in [0, 1]$ and $[a, b]$ be an interval in $[0, 1]$. Then $\|M_{\chi_{[a,b]}} S_\alpha\| = 1$ if $M_{\chi_{[a,b]}} S_\alpha \neq 0$, and $\|M_{\chi_{[a,b]}} T_\alpha\| = 1$ if $M_{\chi_{[a,b]}} T_\alpha \neq 0$.*

*Proof.* We only provide here the proof of the first part of this lemma. Since the range of $S_\alpha$ is $\chi_{[0, 1-\alpha]} \mathcal{L}^2([0, 1])$, the interval $[a', b'] = [a, b] \cap [0, 1-\alpha]$ has length $b' - a' > 0$ if $M_{\chi_{[a,b]}} S_\alpha \neq 0$ and

$$M_{\chi_{[a', b']}} S_\alpha = M_{\chi_{[a,b]}} S_\alpha.$$

Clearly, $\|M_{\chi_{[a,b]}} S_\alpha\| \leq \|M_{\chi_{[a,b]}}\| \, \|S_\alpha\| = 1$.

Let $f = \chi_{[a'+\alpha, b'+\alpha]}$. Then $\|f\| = \|\chi_{[a', b']}\| \neq 0$ and $S_\alpha f = \chi_{[a', b']}$. Therefore,

$$\|(M_{\chi_{[a,b]}} S_\alpha) f\| = \|(M_{\chi_{[a',b']}} S_\alpha) f\| = \|M_{\chi_{[a',b']}} \chi_{[a',b']}\| = \|f\|,$$

and hence, $\|M_{\chi_{[a,b]}} S_\alpha\| \geq 1$. It follows that $\|M_{\chi_{[a,b]}} S_\alpha\| = 1$. □

**Lemma 6.** *Suppose $\alpha, \beta \in [0, 1]$ and $E$, $F$ are two intervals in $[0, 1]$ such that $M_{\chi_E} S_\alpha \neq M_{\chi_F} T_\beta$. Then $\|M_{\chi_E} S_\alpha - M_{\chi_F} T_\beta\| = 1$.*

*Proof.* If either $M_{\chi_E} S_\alpha$ or $M_{\chi_F} T_\beta$ is $0$, then we are done by Lemma 5.

Suppose $M_{\chi_E} S_\alpha \neq 0$ and $M_{\chi_F} T_\beta \neq 0$. Then both $E' = E \cap [0, 1-\alpha]$ and $F' = F \cap [\beta, 1]$ are intervals of length greater than $0$. If $\alpha = \beta = 0$, then $S_\alpha = T_\beta = I$, and therefore, the result is obviously true. Thus we may assume that $\alpha + \beta > 0$. By the definition of $E'$, we can choose an interval $[a, b]$ satisfying $0 < b - a < \alpha + \beta$ and

$$[a, b] \subseteq E' + \alpha \subseteq [\alpha, 1].$$

Hence

$$[a, b] - \alpha = [a - \alpha, b - \alpha] \subseteq E'$$

and, because $b-\alpha<a+\beta$,

$$([a,b]-\alpha)\cap([a,b]+\beta)=[a-\alpha,b-\alpha]\cap[a+\beta,b+\beta]=\varnothing.$$

Let $f=\chi_{[a,b]}$. Then $f\neq 0$ and

$$\begin{aligned}\|(M_{\chi_E}S_\alpha-M_{\chi_F}T_\beta)f\|^2&=\|(M_{\chi_{E'}}S_\alpha-M_{\chi_{F'}}T_\beta)f\|^2\\&=\|\chi_{E'}\chi_{[a,b]-\alpha}-\chi_{F'}\chi_{[a,b]+\beta}\|^2\\&=\|\chi_{[a,b]-\alpha}-\chi_{F'\cap([a,b]+\beta)}\|^2\\&=\|\chi_{[a,b]-\alpha}\|^2+\|\chi_{F'\cap([a,b]+\beta)}\|^2\\&\geq\|\chi_{[a,b]-\alpha}\|^2\\&=\|\chi_{[a,b]}\|^2\\&=\|f\|^2.\end{aligned}$$

It follows that $\|M_{\chi_E}S_\alpha-M_{\chi_F}T_\beta\|\geq 1$. □

**Lemma 7.** *Suppose* $\alpha,\beta\in[0,1]$ *and* $E$, $F$ *are two intervals in* $[0,1]$. *Then* $\|M_{\chi_E}S_\alpha-M_{\chi_F}S_\beta\|\geq 1$ *if* $M_{\chi_E}S_\alpha\neq M_{\chi_F}S_\beta$ *and* $\|M_{\chi_E}T_\alpha-M_{\chi_F}T_\beta\|\geq 1$ *if* $M_{\chi_E}T_\alpha\neq M_{\chi_F}T_\beta$.

*Proof*. For the first part of the lemma, if either $\alpha$ or $\beta$ is $0$ or $1$, then we are done by Lemma 5, Lemma 6 and the fact that $S_0=T_0=I$. So we may assume that $0<\alpha\leq\beta<1$. Therefore, by (ii) of Lemma 3,

$$\begin{aligned}\|M_{\chi_E}S_\alpha-M_{\chi_F}S_\beta\|&=\|M_{\chi_E}S_\alpha-M_{\chi_F}S_\beta\|\,\|T_\alpha\|\\&\geq\|M_{\chi_E}S_\alpha T_\alpha-M_{\chi_F}S_\beta T_\alpha\|\\&=\|M_{\chi_E}M_{\chi_{[0,1-\alpha]}}-M_{\chi_F}M_{\chi_{[0,1-\beta]}}S_{\beta-\alpha}\|\\&=\|M_{\chi_{E\cap[0,1-\alpha]}}T_0-M_{\chi_{F\cap[0,1-\beta]}}S_{\beta-\alpha}\|.\end{aligned}$$

By Lemma 6, either

$$\|M_{\chi_{E\cap[0,1-\alpha]}}T_0-M_{\chi_{F\cap[0,1-\beta]}}S_{\beta-\alpha}\|\geq 1$$

or

$$M_{\chi_{E\cap[0,1-\alpha]}}T_0-M_{\chi_{F\cap[0,1-\beta]}}S_{\beta-\alpha}=0.$$

Thus, either

$$\|M_{\chi_E}S_\alpha-M_{\chi_F}S_\beta\|\geq 1$$

or

$$\begin{aligned}M_{\chi_E}S_\alpha-M_{\chi_F}S_\beta&=M_{\chi_{E\cap[0,1-\alpha]}}S_\alpha-M_{\chi_{F\cap[0,1-\beta]}}S_\beta\\&=(M_{\chi_{E\cap[0,1-\alpha]}}-M_{\chi_{F\cap[0,1-\beta]}}S_{\beta-\alpha})S_\alpha\\&=0.\end{aligned}$$

Since $T_\alpha=(S_\alpha)^*$ and $T_\beta=(S_\beta)^*$, we have

$$\begin{aligned}(M_{\chi_E}T_\alpha-M_{\chi_F}T_\beta)^*&=S_\alpha M_{\chi_E}-S_\beta M_{\chi_F}\\&=M_{S_\alpha\chi_E}S_\alpha-M_{S_\beta\chi_F}S_\beta\\&=M_{\chi_{E-\alpha}}S_\alpha-M_{\chi_{F-\beta}}S_\beta.\end{aligned}$$

Thus, the second part of the lemma follows directly from the first part. □

**Theorem 8.** *Suppose $\theta \in (0, 1)$ is irrational. Then the norm-distance between any two distinct elements of $\mathscr{S}_\theta$ is at least 1. Therefore, the semigroup $\mathscr{S}_\theta$ is discrete and, hence, norm-closed in $\mathscr{B}(\mathscr{L}^2([0, 1]))$.*

*Proof.* From the proof of Theorem 4, any element in $\mathscr{S}_\theta$ is of the form $M_{\chi_E} S_\alpha$ or $M_{\chi_F} T_\beta$ where $E$ and $F$ are intervals in $[0, 1]$ and $\alpha = a - b\theta$, $\beta = c\theta - d$ are in $[0, 1]$ for some rational numbers $a$, $b$, $c$, and $d$. Thus, the result follows immediately from Lemma 6 and Lemma 7. □

**Theorem 9.** *Suppose $\theta \in (0, 1)$ is irrational. Then for any $\alpha \in [0, 1]$, $S_\alpha$ and $T_\alpha$ are in the weak closure $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$ of $\mathscr{S}_\theta$. Consequently, $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$ is selfadjoint and independent of $\theta$.*

*Proof.* Clearly, $S_1 = T_1 = 0 \in \mathscr{S}_\theta$.

For any $\alpha \in [0, 1)$, choose a decreasing sequence $\{a_j\}$ of rational numbers in $(0, 1)$ such that $\lim a_j = \alpha$. We claim that $S_\alpha$ is the weak limit of the sequence $\{S_{a_j}\}$, and hence, $S_\alpha \in \overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$.

We need to show that

$$\langle S_{a_j} f, g\rangle \to \langle S_\alpha f, g\rangle \qquad (j \to \infty)$$

for all $f$ and $g$ in $\mathscr{L}^2([0, 1])$. Since $\|S_{a_j}\| = 1$ for all $j$ and since $C([0, 1])$ is dense in $\mathscr{L}^2([0, 1])$, it suffices to show that

$$\langle S_{a_j} f, g\rangle \to \langle S_\alpha f, g\rangle \qquad (j \to \infty)$$

for all $f$ and $g$ in $C([0, 1])$.

Suppose $f$ and $g$ are in $C([0, 1])$. For any positive number $\varepsilon > 0$, by the continuity of $f$ we can find a number $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $x, y \in [0, 1]$ and $|x - y| < \delta$. Since $\lim a_j = \alpha$, we can find a positive integer $N$ such that

$$|a_j - \alpha| < \min(\varepsilon, \delta)$$

for all $j$ with $j \geq N$.

Therefore, for any $j$, $j \geq N$,

$$\begin{aligned}
&|\langle S_{a_j} f, g\rangle - \langle S_\alpha f, g\rangle| \\
&\quad = \left| \int_0^{1-a_j} f(x + a_j)\bar{g}(x)\,dx - \int_0^{1-\alpha} f(x+\alpha)\bar{g}(x)\,dx \right| \\
&\quad = \left| \int_0^{1-a_j} f(x + a_j)\bar{g}(x)\,dx - \int_0^{1-a_j} f(x+\alpha)\bar{g}(x)\,dx \right. \\
&\qquad\qquad \left. + \int_{1-a_j}^{1-\alpha} f(x+\alpha)\bar{g}(x)\,dx \right| \\
&\quad = \int_0^{1-a_j} |f(x+a_j) - f(x+\alpha)|\,|\bar{g}(x)|\,dx \\
&\qquad + \int_{1-a_j}^{1-\alpha} |f(x+\alpha)|\,|\bar{g}(x)|\,dx \\
&\quad \leq \varepsilon\|g\|_\infty + (a_j - \alpha)\|f\|_\infty\|g\|_\infty \\
&\quad \leq \varepsilon\|g\|_\infty(1 + \|f\|_\infty).
\end{aligned}$$

Thus,

$$\langle S_{a_j} f, g\rangle \to \langle S_\alpha f, g\rangle \qquad (j \to \infty).$$

Similarly, by choosing a decreasing sequence $\{b_j\}$ of rational numbers in $(0, 1)$ with $\lim b_j\theta = \alpha$, we can prove that $T_\alpha \in \overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$.

Since $S_\alpha$ and $T_\alpha$ are in $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$ and $S_\alpha^* = T_\alpha$ for any $\alpha \in [0, 1]$, we have that $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$ is selfadjoint.

We now prove that $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$ is independent of $\theta$. Let $\theta_1$ and $\theta_2$ be two irrational numbers in $(0, 1)$. For every $\alpha \in [0, 1]$, by what we just proved, $S_\alpha$ and $T_\alpha$ are the weak limits of sequences of operators in $\mathscr{S}_{\theta_1}$. Let $W$ be an arbitrary operator in $\mathscr{S}_{\theta_2}$. To prove that $W$ is in $\overline{\mathscr{S}_{\theta_1}}^{\mathrm{WOT}}$, we may assume that $W \neq 0$. From the proof of Theorem 4, $W$ is in the form of $M_{\chi_{[a,b]}}S_\alpha$ or $M_{\chi_{[a,b]}}T_\alpha$ for some interval $[a, b] \subseteq [0, 1]$ and some number $\alpha \in [0, 1)$. Choose a sequence $\{[a_j, b_j]\}$ of subintervals of $[a, b]$ with the property that $\lim a_j = a$ and $\lim b_j = b$. Then it is easy to check that $M_{\chi_{[a,b]}}$ is the strong limit of the sequence $\{M_{\chi_{[a_j,b_j]}}\}$. However, by (iii) of Lemma 1,

$$M_{\chi_{[a_j,b_j]}} = M_{\chi_{[a_j,1]}} M_{\chi_{[0,b_j]}} = T_{a_j} S_{a_j} S_{1-b_j} T_{1-b_j}$$

for every integer $j$. We can choose $\{[a_j, b_j]\}$ so that all $M_{\chi_{[a_j,b_j]}}$ are in $\mathscr{S}_{\theta_1}$ because $\mathscr{S}_{\theta_1}$ is a semigroup. Thus $M_{\chi_{[a,b]}}$ is the strong limit of a sequence of operators in $\mathscr{S}_{\theta_1}$. It follows that $W$ is the weak limit of a sequence of operators in $\mathscr{S}_{\theta_1}$, and hence $\mathscr{S}_{\theta_2} \subseteq \overline{\mathscr{S}_{\theta_1}}^{\mathrm{WOT}}$. Consequently, $\overline{\mathscr{S}_{\theta_2}}^{\mathrm{WOT}} \subseteq \overline{\mathscr{S}_{\theta_1}}^{\mathrm{WOT}}$.

Similarly, we have $\overline{\mathscr{S}_{\theta_1}}^{\mathrm{WOT}} \subseteq \overline{\mathscr{S}_{\theta_2}}^{\mathrm{WOT}}$. Therefore, $\overline{\mathscr{S}_{\theta_1}}^{\mathrm{WOT}} = \overline{\mathscr{S}_{\theta_2}}^{\mathrm{WOT}}$. □

**Theorem 10.** *Suppose $\theta \in (0, 1)$ is irrational. Then the algebra generated by the semigroup $\mathscr{S}_\theta$ is weakly dense in $\mathscr{B}(\mathscr{L}^2([0, 1]))$.*

*Proof.* Let $\mathscr{A}$ be the weakly closed algebra generated by the semigroup $\mathscr{S}_\theta$. Then $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}} \subseteq \mathscr{A}$. It follows that $\mathscr{A}$ is selfadjoint. To prove $\mathscr{A} = \mathscr{B}(\mathscr{L}^2([0, 1]))$, we only need to show that the commutant $\mathscr{A}'$ of $\mathscr{A}$ is trivial.

Since $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}} \subseteq \mathscr{A}$ and $S_\alpha$ and $T_\alpha$ are in $\overline{\mathscr{S}_\theta}^{\mathrm{WOT}}$ for any $\alpha \in [0, 1]$, we have that $\mathscr{A}$ contains all multiplication operators corresponding to characteristic functions of intervals in $[0, 1]$ by (iii) of Lemma 1. Thus $\mathscr{A}$ contains all multiplication operators $M_\phi$ with $\phi \in \mathscr{L}^\infty([0, 1])$. It follows that any projection in the commutant $\mathscr{A}'$ is of the form $M_{\chi_E}$ for some measurable set $E \subseteq [0, 1]$.

Let $E$ be a measurable set in $[0, 1]$ such that $M_{\chi_E}$ is a projection in the commutant $\mathscr{A}'$. Then for any $\alpha \in [0, 1]$, we have that

$$(S_\alpha + T_{1-\alpha})M_{\chi_E} = M_{\chi_E}(S_\alpha + T_{1-\alpha}).$$

Note that $(S_\alpha + T_{1-\alpha})\chi_{[0,1]} = \chi_{[0,1]}$. Therefore we have

$$(S_\alpha + T_{1-\alpha})\chi_E = \chi_E.$$

For all nonnegative integers $n$, calculating the Fourier coefficients

$$[(S_\alpha + T_{1-\alpha})M_{\chi_E}]\hat{\ }(n)$$

of $(S_\alpha + T_{1-\alpha})M_{\chi_E}$ directly, we have that

$$[(S_\alpha + T_{1-\alpha})M_{\chi_E}]\hat{}(n) = e^{2\pi n\alpha i}\widehat{\chi_E}(n).$$

By the fact that $(S_\alpha + T_{1-\alpha})\chi_E = \chi_E$ for all $\alpha \in [0, 1]$, we get that $\widehat{\chi_E}(n) = 0$ for all integers $n \neq 0$. Thus $\chi_E$ is a constant function, and hence, $M_{\chi_E}$ is either 0 or the identity operator. It follows that the commutant $\mathscr{A}'$ of $\mathscr{A}$ is trivial. □

**Corollary 11.** *Suppose $\theta \in (0, 1)$ is irrational. Then the semigroup $\mathscr{S}_\theta$ is irreducible.*

*Proof*. It follows directly from Theorem 10. □

*Remark.* The operators $S_\alpha$ and $T_\alpha$ $(\alpha \in [0, 1])$ are so-called *Bishop-type* operators. Some nice properties of the Bishop-type operators can be found in [2] and in the references at the end of [2].

It is easy to see that the index of nilpotence of operators in $\mathscr{S}_\theta$ is not bounded for any irrational $\theta \in (0, 1)$. Hadwin et al. [1, Theorem 6] proved that an algebra of nilpotent operators is simultaneously triangularizable if the index of nilpotence is bounded. Thus, it is natural to ask the following question:

*Question.* Is it true that any semigroup of positive nilpotent operators is reducible if the index of nilpotence is bounded?

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