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CONVERGENCE OF DIAGONAL PADÉ APPROXIMANTS FOR FUNCTIONS ANALYTIC NEAR 0

D. S. LUBINSKY

ABSTRACT. For functions analytic in a neighbourhood of 0, we show that at least for a subsequence of the diagonal Padé approximants, the point 0 attracts a zero proportion of the poles. The same is true for every "sufficiently dense" diagonal subsequence. Consequently these subsequences have a convergence in capacity type property, which is possibly the correct analogue of the Nuttall-Pommerenke theorem in this setting.

1. INTRODUCTION

Recall that if f is analytic near 0, then for $m, n \geq 0$, the m, n Padé approximant to f is a rational function $[m/n](z) = (P/Q)(z)$, where P, Q have degree $\leq m, n$ respectively, Q is not identically zero, and

$$(fQ - P)(z) = O(z^{m+n+1}), \qquad z \to 0.$$

For functions meromorphic in $\mathbb{C}$, or even with singularities of capacity 0, it is known that the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ converges in capacity and in measure [11, 14]. Similar results are available in more general circumstances [3, 6, 16, 17, 19].

By contrast, for functions analytic only near 0, the full diagonal sequence of Padé approximants need not converge in capacity in any neighbourhood of zero [7, 8, 15], and moreover, at least for infinitely many n, $[n/n]$ may have at least $n - \log n$ poles arbitrarily near 0 [18]. (We could replace $\log n$ by any sequence increasing to ∞.) The 1961 Baker-Gammel-Wills conjecture [1, 2] asserts that a subsequence of $\{[n/n]\}$ converges uniformly near 0, but at present it is not even known if a subsequence converges in capacity.

In this paper we show that, at least for a subsequence of $\{[n/n]\}$, the proportion of poles of $[n/n]$ near 0 shrinks to 0, in a certain sense. This result also holds for subsequences of $\{[n_j/n_j]\}$ provided $n_{j+1}/n_j \to 1$ as $j \to \infty$. Then we deduce a convergence in capacity type property. Since by a variable scaling $z \to rz$ any function analytic near 0 can be scaled to a function analytic in $|z| < 1$, the transformation properties of Padé approximants permit us to consider only the latter:

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Theorem 1.1. *Let f be analytic in $\{z : |z| < 1\}$. Let $\{n_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers with*

$$\lim_{j\to\infty} n_{j+1}/n_j = 1. \tag{1.1}$$

Let $0 < \delta < 1$. Then there exists an infinite sequence of positive integers $\mathscr{S}$ with the following property: For $j \in \mathscr{S}$, the total multiplicity of poles of $[n_j/n_j]$ in $\{z : |z| \le e^{-19/\delta}\}$ is at most δn_j.

Under a regularity assumption on the errors of best rational approximation, we can say the same for full sequences of Padé approximants: For $0 < \rho < 1$, we let

$$E_{nn}(f;\rho) := \inf\{\|f - R\|_{L_\infty(|z|\le\rho)} : R \text{ is a rational function of type } (n, n)\}.$$

Theorem 1.2. *Assume that f is analytic in $\{z : |z| < 1\}$, and that*

$$0 < \limsup_{n\to\infty} E_{nn}(f;\rho)^{1/n} \le \kappa(\rho) \liminf_{n\to\infty} E_{nn}(f;\rho)^{1/n}, \tag{1.2}$$

where $\kappa(\rho)$ is finite for $\rho \in (0, 1)$. Then for large enough n, the total multiplicity of poles of $[n/n](z)$ in $\{z : |z| \le \rho\}$ is at most δn, provided

$$\rho\kappa(\rho)^{1/\delta} \le \exp(-19/\delta). \tag{1.3}$$

In particular, if $\lim_{n\to\infty} E_{nn}(f;\rho)^{1/n}$ exists for $0 < \rho < 1$, then for n large enough, the total multiplicity of poles of $[n/n](z)$ in $\{z : |z| \le \exp(-19/\delta)\}$ is at most δn.

Remarks. (i) Similar results hold if we consider sectorial sequences of Padé approximants of the form $\{[m_j/n_j]\}$, where $\{m_j\}$, $\{n_j\}$ satisfy (1.1) and, for some fixed λ,

$$1/\lambda \le m_j/n_j \le \lambda, \qquad j \ge 1.$$

The formulation will be more complicated and the proofs will be messier, but we hope to attend to this in a subsequent paper. (See [9], where similar results were proved for functions analytic in $\mathbb{C}$ except for singularities of capacity 0 and for general sectorial sequences $\{[m_j/n_j]\}$.)

(ii) Note that the size of the neighbourhood in which there are at most δn poles is a function of δ only, not of f. However, the factor e^{-19} is not optimal.

(iii) Note that if

$$\liminf_{n\to\infty} E_{nn}(f;\rho)^{1/n} = 0,$$

then it is easy to see that a subsequence of the $[n/n]$ Padé approximants actually converges in capacity on compact subsets of $\{z : |z| < \rho\}$. Note too that if

$$\lim_{n\to\infty} E_{nn}(f;\rho)^{1/n} = 0,$$

then f belongs to the Gončar-Walsh class [3, 10], the $[n/n]$ Padé approximants converge in capacity in $\{z : |z| < 1\}$ [3, 19], and for $j \in \mathscr{S}$ all but $o(n_j)$ poles of $[n_j/n_j]$ leave every compact subset of $\{z : |z| < 1\}$ [9, 10].

Now we turn to *convergence in capacity*. Recall that, for a compact set K, the logarithmic capacity $\operatorname{cap}(K)$ is defined by

$$\operatorname{cap}(K) := \lim_{n\to\infty} \left(\min_{P_n} \|P_n\|_{L_\infty(K)} \right)^{1/n},$$

where the minimum is taken over all monic polynomials P_n of degree n. For arbitrary S, the inner logarithmic capacity $\mathrm{cap}(S)$ is defined by

$$\mathrm{cap}(S) := \sup\{\mathrm{cap}(K) : K \subset S,\, K \text{ compact}\}.$$

Convergence in capacity is essentially the same as convergence in measure. We say $f_n \to f$ in capacity in $\{z : |z| \le r\}$ if $\forall \varepsilon > 0$,

$$\mathrm{cap}\{z : |z| \le r \text{ and } |f - f_n|(z) \le \varepsilon\} \to 0, \qquad n \to \infty.$$

The Nuttall-Pommerenke theorem [11, 14] and its extensions actually prove *geometric convergence in capacity* under suitable hypotheses on f:

$$\mathrm{cap}\{z : |z| \le r \text{ and } |f - [n/n]|(z) \ge \varepsilon^n\} \to 0, \qquad n \to \infty.$$

Here we shall show that Theorem 1.1 implies a weak convergence in capacity property:

Given $0 < \Delta < \frac{1}{2}$, $A > 1$, there exists $\rho = \rho(A) < 1$ (independent of f) such that, for all n in a subsequence,

$$\mathrm{cap}\{z : |z| \le \rho : |f - [n/n]|(z) \ge \rho^{n\Delta}\} \le \mathrm{cap}\{z : |z| \le \rho\}^A = \rho^A.$$

The same estimate holds if we replace cap by planar Lebesgue measure or one-dimensional Hausdorff content.

The point is that, in most of $\{z : |z| \le \rho\}$, $[n/n]$ is geometrically close to f, and we have a weak convergence in capacity property: The capacity (or area or one-dimensional Hausdorff content) of the set on which $[n/n]$ does not approximate is an arbitrarily small proportion of the total capacity (or area or content). I believe that in the setting of the following theorem the conclusion of Theorem 1.3 may possibly be the correct analogue of the Nuttall-Pommerenke theorem: Nothing more can be said of subsequences of $\{[n_j/n_j]\}$, other than this weak convergence in capacity type property near 0. Of subsequences of the full diagonal sequence $\{[n/n]\}$, the Baker-Gammel-Wills conjecture may well be true.

Theorem 1.3. *Let f be analytic in $\{z : |z| < 1\}$. Let $0 < \Delta < \frac{1}{2}$, $A > 1$ and $\rho := \frac{1}{2}\exp(-19A/(\frac{1}{2} - \Delta))$. Let $\{n_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers satisfying* (1.1). *Then there exists an infinite sequence of positive integers $\mathscr{S}$ with the following property: For $j \in \mathscr{S}$,*

$$\mathrm{cap}\left\{ z : |z| \le \rho \text{ and } |f - [n_j/n_j]|(z) > \left(\frac{|z|}{\rho} \cdot \rho^{\Delta}\right)^{2n_j} \right\} \le \mathrm{cap}\{z : |z| \le \rho\}^A.$$

Remarks. (i) The restriction $\Delta < \frac{1}{2}$ is related to the exponent $\frac{1}{2}$ in the right-hand side of

$$\limsup_{n\to\infty} E_{nn}(f;\, \rho)^{1/(2n)} \le \rho^{1/2}.$$

It is now known [12] that

$$\liminf_{n\to\infty} E_{nn}(f;\, \rho)^{1/(2n)} \le \rho.$$

Consequently, if we assume that $\lim_{n\to\infty} E_{nn}(f;\, \rho)^{1/(2n)}$ exists for $0 < \rho < 1$, then our proof allows us to replace $0 < \Delta < \frac{1}{2}$ by $0 < \Delta < 1$ and $\rho = \frac{1}{2}\exp(-19A/(\frac{1}{2} - \Delta))$ by $\rho = \frac{1}{2}\exp(-19A/(1 - \Delta))$ in the above result. In that case also, the weak convergence in capacity will hold for the full diagonal sequence, and not just a subsequence.

(ii) The subsequence $\mathcal{S}$ in Theorem 1.3 is the same sequence as in Theorem 1.1, with a suitable choice of $\delta = \delta(A)$.

We prove Theorems 1.1 and 1.2 in §2 and Theorem 1.3 in §3.

2. Proof of Theorems 1.1 and 1.2

We shall use the notion of one-dimensional Hausdorff content:

$$m(E) := \inf \left\{ \sum_j \operatorname{diam}(B_j) : E \subset \bigcup_j B_j \right\},$$

where the inf is taken over all countable collections of balls $\{B_j\}$ of diameters $\{\operatorname{diam} B_j\}$ covering E. We first present four lemmas (at least two of which are standard), and then prove Theorems 1.2 and 1.1. Throughout $\mathcal{P}_n$ denotes the polynomials of degree $\le n$, and $C, C_1, C_2, \ldots$ denote constants independent of n, P, and z. The same symbol does not necessarily denote the same constant in different occurrences. In the sequel, $[n/n] = p_n/q_n$.

Lemma 2.1. *Let $U \in \mathcal{P}_l \backslash \{0\}$ and $0 < \varepsilon \le \rho$. Then there exists a set $\mathcal{E} \subset [0, \rho]$ such that $m(\mathcal{E}) \le \varepsilon$ and, for $\sigma \in [0, \rho] \backslash \mathcal{E}$,*

$$\max\{|U(t)/U(z)| : |t| = \rho, |z| = \sigma\} \le (12e\rho/\varepsilon)^l. \tag{2.1}$$

Proof. Split $U = cVW$, where $c \ne 0$, and V, W are monic polynomials of degree ν, ω, respectively, with zeros outside $|z| \le 2\rho$, inside $|z| \le 2\rho$, respectively. Now for $|a| \ge 2\rho$, $|t| = \rho$, $|z| \le \rho$,

$$\left| \frac{t-a}{z-a} \right| \le \frac{1 + |t/a|}{1 - |z/a|} \le 3.$$

We deduce that

$$|V(t)/V(z)| \le 3^\nu, \qquad |t| = \rho, \ |z| \le \rho. \tag{2.2}$$

Next, by Cartan's lemma [1, p. 174],

$$|W(z)| \ge (\varepsilon/4e)^\omega, \qquad z \in \mathbb{C} \backslash \mathcal{F},$$

where $m(\mathcal{F}) \le \varepsilon$. Then using an easy covering argument, we see that $\mathcal{E} := \{|z| : z \in \mathcal{F}\}$ also has $m(\mathcal{E}) \le \varepsilon$. Moreover for $|t| = \rho$,

$$|W(t)| \le (3\rho)^\omega.$$

These last two inequalities and (2.2) give (2.1). □

Lemma 2.2. *Let f be analytic in $\{z : |z| < 1\}$. Let $0 < \varepsilon \le \rho < 1$. There exists $\mathcal{E}_n$ with $m(\mathcal{E}_n) \le \varepsilon$, such that, for $\sigma \in [0, \rho] \backslash \mathcal{E}_n$,*

$$\max_{|z|=\sigma} |f - [n/n](z)| \le E_{nn}(f; \rho) \left(\frac{12e\sigma}{\varepsilon} \right)^{2n} \frac{\sigma}{\rho - \sigma}. \tag{2.3}$$

In particular, for some $\rho_1 \in [\frac{1}{3}\rho, \frac{2}{3}\rho]$,

$$\max_{|z|=\rho_1} |f - [n/n]|(z) \le 2E_{nn}(f; \rho)(32e)^{2n}. \tag{2.4}$$

Proof. Let $r_n^* := p_n^*/q_n^*$ be the best approximant of type (n, n) to f on $|z| \le \rho$. Then for $|z| < \rho$,

$$q_n^*(z)(fq_n - p_n)(z)/z^{2n+1} = \frac{1}{2\pi i}\int_{|t|=\rho}[q_n(t)(fq_n^* - p_n^*)(t)/t^{2n+1}]\frac{dt}{t-z}.$$

This is an easy consequence of Cauchy's integral formula and the fact that, for any $\Pi \in \mathscr{P}_{2n}$,

$$\frac{1}{2\pi i}\int_{|t|=\rho}[\Pi(t)/t^{2n+1}]\frac{dt}{t-z} = 0.$$

(We chose $\Pi := p_n^* q_n - p_n q_n^*$.) We deduce that, for $\sigma < \rho$,

$$\begin{aligned}
&\max_{|z|=\sigma}|f - [n/n](z)| \\
&\quad\le \left(\frac{\sigma}{\rho}\right)^{2n+1}\max\left\{\left|\frac{(q_n q_n^*)(t)}{(q_n q_n^*)(z)}\right| : |t| = \rho,\ |z| = \sigma\right\}\frac{\rho}{\rho-\sigma}E_{nn}(f;\rho) \\
&\quad\le \left(\frac{12e\sigma}{\varepsilon}\right)^{2n}\frac{\sigma}{\rho-\sigma}E_{nn}(f;\rho),
\end{aligned}$$

by Lemma 2.1, provided $\sigma \notin \mathscr{E}$, where $m(\mathscr{E}) \le \varepsilon$. In particular, if $\varepsilon = \rho/4$, we can choose such a $\rho_1 := \sigma \in [\frac{1}{3}\rho, \frac{2}{3}\rho]\backslash\mathscr{E}$. □

We shall need a lemma of Gončar and Grigorjan:

Lemma 2.3. *If g is analytic in $\{z : |z| \le \rho\}$ except for poles of total multiplicity m, none lying on $|z| = \rho$, and if $\mathscr{A}_\rho(g)$ denotes the analytic part of g in $\{z : |z| \le \rho\}$ (that is, g minus its principal parts in $|z| < \rho$), then*

$$\|\mathscr{A}_\rho(g)\|_{L_\infty(|z|\le\rho)} \le 7m^2\|g\|_{L_\infty(|z|=\rho)}.$$

Proof. See [4]. For more precise results and references, see [5, 12, 13]. □

Following is our main lemma:

Lemma 2.4. *Let f be analytic in $\{z : |z| < 1\}$, and $0 < \rho < 1$, $K \ge 1$, with $3K\rho < 1$. If $[n/n]$ has $\tau = \tau(n)$ poles counting multiplicity in $\{z : |z| \le K\rho\}$, then, for large enough n,*

$$E_{n-\tau,n-\tau}(f; K\rho) \le [e^{16}K]^n E_{nn}(f;\rho)^{1+(\log 6K)/(\log 3K\rho)}. \tag{2.5}$$

Proof. Let S_m be the mth partial sum of the Maclaurin series of f. Let $\varepsilon := 3K\rho$. We have, for large enough m,

$$\|f - S_m\|_{L_\infty(|z|\le 2K\rho)} \le \varepsilon^m.$$

Let $\langle x\rangle$ denote the largest integer $\le x$. We let

$$m := \left\langle\frac{\log E_{nn}(f;\rho)}{\log\varepsilon}\right\rangle + 1,$$

so that $\varepsilon^m \le E_{nn}(f;\rho)$. Let $\rho_1 \in [\frac{1}{3}\rho, \frac{2}{3}\rho]$ be as in Lemma 2.2. We deduce from (2.4) and our choice of m that, for n large enough,

$$\|S_m - [n/n]\|_{L_\infty(|z|=\rho_1)} \le E_{nn}(f;\rho)\{1 + 2(32e)^{2n}\},$$

so

$$\|S_m q_n - p_n\|_{L_\infty(|z|=\rho_1)} \le E_{nn}(f;\rho)3(32e)^{2n}\|q_n\|_{L_\infty(|z|=\rho_1)}.$$

The Bernstein-Walsh lemma gives

$$\|S_m q_n - p_n\|_{L_\infty(|z|\le 2K\rho)} \le E_{nn}(f;\,\rho)3(32e)^{2n}\|q_n\|_{L_\infty(|z|=\rho_1)}(6K)^{m+n}.$$

We deduce that, for $\sigma \le 2K\rho$,

$$\begin{aligned}\|f-[n/n]\|_{L_\infty(|z|=\sigma)} &\le \|f-S_m\|_{L_\infty(|z|=\sigma)} + \|S_m-[n/n]\|_{L_\infty(|z|=\sigma)}\\ &\le E_{nn}(f;\,\rho)\left[1+3(32e)^{2n}(6K)^{m+n}\max\left\{\left|\frac{q_n(t)}{q_n(z)}\right| : |t|=\rho_1,\,|z|=\sigma\right\}\right],\end{aligned}$$

provided, of course, that the right-hand side is finite. By Lemma 2.1 (with ρ there replaced by $2K\rho$, and $\varepsilon = \frac{3}{4}K\rho$), we can choose $\sigma_1 \in (K\rho,\,2K\rho)$ such that

$$\begin{aligned}\max\left\{\left|\frac{q_n(t)}{q_n(z)}\right| : |t|=\rho_1,\,|z|=\sigma_1\right\} &\le \max\left\{\left|\frac{q_n(t)}{q_n(z)}\right| : |t|=2K\rho,\,|z|=\sigma_1\right\}\\ &\le \left(\frac{12e\cdot 2K\rho}{3K\rho/4}\right)^n = (32e)^n.\end{aligned}$$

Since f is analytic, $\mathscr{A}_{\sigma_1}(f-[n/n]) = f - \mathscr{A}_{\sigma_1}([n/n])$. Also $\sigma_1 \ge K\rho$. Then Lemma 2.3 gives

$$\begin{aligned}\|f-\mathscr{A}_{\sigma_1}([n/n])\|_{L_\infty(|z|\le K\rho)} &\le \|f-\mathscr{A}_{\sigma_1}([n/n])\|_{L_\infty(|z|=\sigma_1)}\\ &\le 7n^2\|f-[n/n]\|_{L_\infty(|z|=\sigma_1)} \le 28n^2[6K(32e)^3]^n(6K)^m E_{nn}(f;\,\rho)\\ &\le E_{nn}(f;\,\rho)^{1+\log 6K/\log 3K\rho}(e^{16}K)^n,\end{aligned}$$

for n large enough, by our choice of m and of $\varepsilon = 3K\rho$. Since $[n/n]$ has at least τ poles in $|z| \le K\rho < \sigma_1$, $\mathscr{A}_{\sigma_1}([n/n])$ is a rational function of type $(n-\tau,\, n-\tau)$, and the result follows. □

We turn to the proofs of the theorems. To indicate the ideas, we first prove the simpler Theorem 1.2. In the sequel, we let

$$A(\rho) := \limsup_{n\to\infty} E_{nn}(f;\,\rho)^{1/n}.$$

Recall (as in the Introduction) that if $A(\rho) = 0$ for some $\rho \in (0,\,1)$, then, by a result of Gončar [3], $A(\rho) = 0$ for all $0 < \rho < 1$, and then stronger results are available [9]. So we assume that $A(\rho) > 0$ for all $\rho > 0$ in the sequel.

Proof of Theorem 1.2. Assume that, for some $\delta \in (0,\,1)$, $\rho \in (0,\,\frac{1}{2})$, and for n belonging to some infinite sequence of integers $\mathscr{N}$, $[n/n]$ has poles of total multiplicity $\ge \delta n$ in $\{z : |z| \le \rho\}$. We show that ρ cannot be too small assuming that $\mathscr{N}$ is an infinite set. Applying Lemma 2.4 with $K = 1$ gives

$$E_{\langle n(1-\delta)\rangle,\,\langle n(1-\delta)\rangle}(f;\,\rho) \le e^{16n}E_{nn}(f;\,\rho)^{1+\log 6/\log 3\rho}.$$

Taking nth roots, letting $n \to \infty$ through $\mathscr{N}$, and using (1.2) gives

$$[\kappa(\rho)^{-1}A(\rho)]^{1-\delta} \le e^{16}A(\rho)^{1+\log 6/\log 3\rho}.$$

That is,

$$A(\rho)^{-\delta-\log 6/\log 3\rho} \le e^{16}\kappa(\rho). \tag{2.6}$$

(Recall that (1.2) forces $\kappa(\rho) \ge 1$.) The exponent of $A(\rho)$ is negative if

$$\rho \le \frac{1}{3}\exp\left(-\frac{\log 6}{\delta}\right)\ \left(\le \frac{1}{18}\right). \tag{2.7}$$

Now $A(\rho) \le \rho$ by analyticity of f in $|z| < 1$, so, for ρ satisfying (2.7), it follows from (2.6) that

$$\delta |\log \rho| \le 16 + \log \kappa(\rho) + \log 6 \left| \frac{\log \rho}{\log 3\rho} \right| .$$

Here for $\rho \le \frac{1}{18}$,

$$\log 6 \left| \frac{\log \rho}{\log 3\rho} \right| \le \log 18 < 3 .$$

So we obtain

$$\rho \kappa(\rho)^{1/\delta} > \exp(-19/\delta) .$$

Therefore for large enough n, $[n/n]$ can have no more than δn poles in $\{z : |z| < \rho\}$ if $\rho\kappa(\rho)^{1/\delta} \le \exp(-19/\delta)$. $\square$

Proof of Theorem 1.1. The consequence of (1.1) that we shall use is

$$\limsup_{k\to\infty} E_{n_k, n_k}(f; \rho)^{1/n_k} = \limsup_{n\to\infty} E_{nn}(f; \rho)^{1/n} = A(\rho) . \tag{2.8}$$

This follows easily from the fact that $E_{nn}(f; \rho)$ is decreasing in n. Let $0 < \eta < \delta < 1$. For large enough k, we define $j = j(k)$ to be the largest integer j for which $n_k \ge n_j(1-\eta)$, so that

$$n_{j(k)}(1-\eta) \le n_k < n_{j(k)+1}(1-\eta) .$$

Let $\tau_k := n_{j(k)} - n_k$. We see from (1.1) and our choice of $j(k)$ that

$$\lim_{k\to\infty} \tau_k / n_{j(k)} = \eta ; \qquad \lim_{k\to\infty} n_k / n_{j(k)} = 1 - \eta . \tag{2.9}$$

Suppose that for some $0 < \rho < 1$ and for large enough k, $[n_k/n_k]$ has more than δn_k poles in $\{z : |z| \le \rho\}$. Then for large enough k, and $j = j(k)$, $[n_{j(k)}/n_{j(k)}]$ has $> \delta n_{j(k)} > \tau_k$ poles in $\{z : |z| \le \rho\}$ (recall that $\eta < \delta$). As $n_{j(k)} - \tau_k = n_k$, Lemma 2.4 (with $K = 1$) gives

$$E_{n_k, n_k}(f; \rho) \le e^{16 n_{j(k)}} E_{n_{j(k)}, n_{j(k)}}(f; \rho)^{1 + (\log 6)/(\log 3\rho)} .$$

Taking $n_{j(k)}$th roots in this last inequality, and then lim sups as $k \to \infty$, and using (2.8) and (2.9), give

$$A(\rho)^{1-\eta} \le e^{16} A(\rho)^{1 + (\log 6)/(\log 3\rho)} .$$

Since $\eta < \delta$ is arbitrary, we deduce that

$$A(\rho)^{1-\delta} \le e^{16} A(\rho)^{1 + (\log 6)/(\log 3\rho)} .$$

Then

$$A(\rho)^{-\delta - (\log 6)/(\log 3\rho)} \le e^{16} .$$

This is the exact same relation as (2.6) with $\kappa(\rho) \equiv 1$. Proceeding exactly as in the previous proof with $\kappa(\rho) \equiv 1$, we obtain

$$\rho > \exp(-19/\delta) . \quad \square$$

3. Proof of Theorem 1.3

Let $0 < \rho < \frac{1}{2}, 0 < \delta < 1, A > 1$, and assume that for n belonging to some infinite sequence of integers $\mathscr{N}$, $[n/n] = p_n/q_n$ has no more than δn

poles, counting multiplicity, in $\{z : |z| \leq 2\rho\}$. Let $r_n^* := p_n^*/q_n^*$ be a best approximation of type (n, n) to f on $\{z : |z| \leq 2\rho\}$. We begin with the identity from Lemma 2.2: For $|z| < 2\rho$,

$$(f - [n/n])(z) = \frac{1}{2\pi i} \int_{|t|=2\rho} \left(\frac{z}{t}\right)^{2n+1} \frac{(q_n^* q_n)(t)}{(q_n^* q_n)(z)} \frac{(f - r_n^*)(t)}{t - z}\, dt.$$

We deduce that, for $|z| \leq \rho$,

$$|f - [n/n]|(z) \leq 2 \left(\frac{|z|}{2\rho}\right)^{2n} E_{nn}(f; 2\rho) \max_{|t|=2\rho} \left|\frac{(q_n^* q_n)(t)}{(q_n^* q_n)(z)}\right|. \tag{3.1}$$

Now for $n \geq n_0(\rho)$,

$$E_{nn}(f; 2\rho) \leq (3\rho)^n. \tag{3.2}$$

Recall that q_n^* has all zeros outside $|z| \leq 2\rho$. We split $q_n = S_n U_n$, where S_n is monic of degree $s_n \leq \delta n$ and has zeros in $|z| \leq 2\rho$, and U_n has zeros in $|z| > 2\rho$. Exactly as in the proof of Lemma 2.1, we see that, for $|z| \leq \rho$,

$$\max_{|t|=\rho} \left|\frac{(q_n^* U_n)(t)}{(q_n^* U_n)(z)}\right| \leq 3^{2n}. \tag{3.3}$$

Next, as S_n is monic, the set $\mathscr{E}_n := \{z : |S_n(z)| \leq \rho^{A s_n}\}$ has $\operatorname{cap}(\mathscr{E}_n) = \rho^A$. Then as in Lemma 2.1, since S_n has all its zeros in $|u| \leq 2\rho$, we have, for $|z| \leq \rho$, $z \notin \mathscr{E}_n$,

$$\max_{|t|=\rho} \left|\frac{(S_n)(t)}{(S_n)(z)}\right| \leq \left(\frac{3\rho}{\rho^A}\right)^{s_n} \leq (3\rho^{-A})^{\delta n}. \tag{3.4}$$

Combining (3.1)–(3.4), we have, for $|z| \leq \rho$, $z \notin \mathscr{E}_n$,

$$|f - [n/n]|(z)^{1/(2n)} \leq 2^{1/2n} \frac{|z|}{2\rho} (3\rho)^{1/2} 3 (3\rho^{-A})^{\delta/2} \leq \frac{|z|}{\rho} 8 \rho^{(1-A\delta)/2} \leq \frac{|z|}{\rho} \rho^{\Delta},$$

provided

$$\rho^{1/2-\Delta-A\delta/2} \leq \frac{1}{8}. \tag{3.5}$$

In summary, we have shown that, for large enough $n \in \mathscr{N}$,

$$\begin{aligned} \operatorname{cap}\left\{z : |z| \leq \rho \text{ and } |f - [n/n]|(z)^{1/2n} > \frac{|z|}{\rho}\rho^{\Delta}\right\} &\leq \operatorname{cap}(\mathscr{E}_n) \\ = \rho^A = \operatorname{cap}\{z : |z| \leq \rho\}^A, \end{aligned}$$

provided (3.5) holds. Let us choose δ and ρ by

$$A\delta = \tfrac{1}{2} - \Delta, \qquad 2\rho = \exp(-19/\delta) = \exp(-19A/(\tfrac{1}{2} - \Delta)).$$

Then

$$\rho^{1/2-\Delta-A\delta/2} = \rho^{(1/2-\Delta)/2} \leq \exp(-19A/2) \leq \exp(-19/2),$$

as $A \geq 1$, so (3.5) is satisfied. Finally, with this choice of δ and ρ, Theorem 1.1 guarantees that, given $\{n_j\}$ satisfying (1.1), we can find infinitely many j such that for $n = n_j$, $j \in \mathscr{S}$; that is, $[n_j/n_j]$ has at most δn_j poles in $\{z : |z| \leq 2\rho\}$. □

We remark that when the limit

$$\lim_{n\to\infty} E_{nn}(f;\,\rho)^{1/n}$$

exists, then the aforementioned result of Parfenov guarantees that it is $\leq \rho^2$. Then we can replace (3.2) by

$$E_{nn}(f;\,2\rho) \leq (3\rho^2)^n\,.$$

Proceeding as before, we see that we can then choose $\rho = \frac{1}{2}\exp(-19A/(1-\Delta))$, for any $0 < \Delta < 1$.

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Department of Mathematics, University of The Witwatersrand, P.O. Wits 2050, Johannesburg, South Africa

E-mail address: `036dsl@cosmos.wits.ac.za`