HADAMARD CONVEXITY AND MULTIPLICITY AND LOCATION OF ZEROS

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ABSTRACT. We consider certain questions arising from a paper of Hayman concerning quantitative versions of the Hadamard three-circle theorem for entire functions. If $b(r)$ denotes the second derivative of $\log M(r)$ with respect to $\log r$, the principal contributions of this work are (i) a characterization of those entire f with nonnegative Maclaurin coefficients for which $\limsup b(r) = \frac{1}{4}$ and (ii) some exploration of the relationship between multiple zeros of f and the growth of $b(r)$.

1. INTRODUCTION

Let f be an entire function and put

$$
(1) \qquad b(r) = b(r; f) = d^2 \log M(r)/d(\log r)^2
$$

where $M(r) = M(r; f) = \sup_{|z|=r} |f(z)|$ is the maximum modulus of f. The simplest result of this paper is the following:

Theorem. *If f has nonnegative Maclaurin coefficients and satisfies*

$$
(2) \qquad \limsup_{r \to \infty} b(r) = \frac{1}{4}
$$

then all but a finite number of the zeros of f are simple and lie on the negative real axis. Furthermore, if r_n is the modulus of the nth zero then

$$
(3) \qquad r_{n+1}/r_n \to \infty \qquad as \ n \to \infty.
$$

This theorem puts in sharp form an earlier result [1] where the zeros were shown to lie in arbitrarily small angles containing the negative real axis.

The study of the growth of $b(r)$ was initiated in 1966 by Hayman [3] who showed that the classical estimate $b(r) \geq 0$ obtained from Hadamard's three-circle theorem could be improved under certain conditions. Hayman showed that there exists a positive absolute constant $A_0 > 0.180$ such that $\limsup_{r \to \infty} b(r) \geq A_0$ for every transcendental entire function f. In 1973, Kjelleberg [4] improved the inequality on A_0 to $0.24 < A_0 < 0.25$ and this was later improved to $0.24 < A_0 < 0.247$ by Hilditch [5]. In the special case where f

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has nonnegative Maclaurin coefficients, Boichuck and Gol'dberg [2] obtained $A_0 = 0.25$ which is a sharp result for this class of functions. The best possible value of A_0 in the class of all transcendental entire functions is as yet unknown. The fact that the growth of $b(r)$ is related to the distribution of values of f was first pointed out in [1] where, also, another proof of the Boichuck-Gol'dberg result was obtained. Here we go a little further by investigating the relationship between the growth of $b(r)$ and the *multiple* zeros of f or, more generally, zeros having equal moduli. It turns out that there is an absolute constant $2A_1$, a sort of an "impact parameter", such that if $\limsup_{r\to\infty} b(r) < 2A_1$ and f has only real coefficients, then all but a finite number of the zeros of f must be real and simple. Since $2A_1 > 0.36$, the theorem stated above follows from this and the positivity of the coefficients. The exact value 0.25 in (2) is needed to get (3).

The most concise way of stating our results requires the use of Jacobian elliptic functions, and we start by summarizing what is needed from their theory [6].

We set $q = e^{\pi i \tau}$ where $\text{Im}\,\tau > 0$ and define the theta function θ by

$$(4) \qquad \theta(z) = \theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inz}.$$

The other theta functions θ_1, θ_2 and θ_3 are defined in terms of θ_4 and we have the product representations [6, p.470]

$$\theta_1(z) = 2Gq^{1/4}\sin z \prod_{n=1}^{\infty}(1 - q^{2n}e^{2iz})(1 - q^{2n}e^{-2iz}),$$

(5)

$$\theta_2(z) = 2Gq^{1/4}\cos z \prod_{n=1}^{\infty}(1 + q^{2n}e^{2iz})(1 + q^{2n}e^{-2iz}),$$

where G is a constant whose precise value need not concern us at the moment. The Jacobian elliptic functions may be defined [6, p. 492] by

$$sn(z, k) = \frac{\theta_3(0)}{\theta_2(0)} \cdot \frac{\theta_1(z/\theta_3^2(0))}{\theta_4(z/\theta_3^2(0))},$$

(6)

$$cn(z, k) = \frac{\theta_4(0)}{\theta_2(0)} \cdot \frac{\theta_2(z/\theta_3^2(0))}{\theta_4(z/\theta_3^2(0))}.$$

Here k is a constant (called the modulus) defined in terms of q and confined, when q is real, to the interval $(0, 1)$. Associated to k are constants K, k', K' that may conveniently be defined, at least when $k \in (0, 1)$, by

$$(7) \qquad K = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2\sin^2 t}}, \qquad k'^2 + k^2 = 1\ (k' > 0),$$

$$K' = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k'^2\sin^2 t}},$$

and one has the relations [6, p. 479]

$$(8) \quad K' = \frac{K}{\pi} \log\left(\frac{1}{q}\right), \quad k'^{1/2} = \frac{\theta_4(0)}{\theta_3(0)}, \quad \theta_3^2(0) = \frac{2K}{\pi}, \quad sn(K', k') = 1.$$

From (6) and (8) one obtains

$$(9) \quad \frac{\theta_1(\pi z/2K)}{\theta_2(\pi z/2K)} = k'^{1/2} \cdot \frac{sn(z, k)}{cn(z, k)}.$$

Simple explicit values for k' may be obtained for certain given values of q or K/K'. Thus one has, for example, the following table [6, p. 525]

K/K'	1	$\sqrt{2}$	$\sqrt{3}$	2
k'	$\dfrac{1}{\sqrt{2}}$	$\sqrt{2} - 1$	$\dfrac{1}{\sqrt{2}(\sqrt{3}+1)}$	$(\sqrt{2}-1)^2$

In general, however, we have [6, p. 479]

$$(10) \quad k'^{1/2} = \frac{1 - 2q + 2q^4 + 2q^9 + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots} = \prod_{n=1}^{\infty}\left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}}\right)^2$$

and the middle expression in (10) is convenient for the calculation of k'.

Let $\alpha > 0$ and put $q = e^{-\pi^2/\alpha}$, so that k' becomes a function of α and $K'/K = \pi/\alpha$. Define

$$(11) \quad A_1(\alpha) = \frac{-8}{\alpha^2} \log k'^{1/2} = \frac{-8}{\alpha^2} \log k'(\alpha)^{1/2}.$$

Then $A_1(\alpha) > 0$ and it can be shown that $A_1(\alpha) \to 0$ as $\alpha \to 0+$ or ∞. So $A_1(\alpha)$ has an absolute maximum on $(0, \infty)$ at some point α_0. The table gives $A_1(\pi\sqrt{2}) = \frac{-2}{\pi^2} \log(\sqrt{2}-1) = A_1(2\pi)$ and so a critical value of A_1 exists between $\pi\sqrt{2}$ and 2π. At any rate we have $A_1(\alpha_0) \geq A_1(\pi\sqrt{3}) = \frac{4}{3\pi^2} \log(\sqrt{6} + \sqrt{2}) \simeq 0.183$.

Throughout this paper we shall take for q the value $e^{-\pi^2/\alpha_0}$ where, as mentioned above,

$$(12) \quad A_1(\alpha_0) = \max_{(0, \infty)} A_1(\alpha) = A_1.$$

The "impact parameter" referred to earlier in this introduction will be the constant $2A_1$, and the above discussion shows that $2A_1 > 0.365$.

2. STATEMENT OF RESULTS

For $r_0 > 0$ and α_0 defined by (12) we shall use the notation

$$D(r_0, \alpha_0) = \{z : r_0 \exp(-\tfrac{1}{2}\alpha_0) < |z| < r_0 \exp(\tfrac{1}{2}\alpha_0)\}$$

and

$$I(r_0, \alpha_0) = \{r : r_0 \exp(-\tfrac{1}{2}\alpha_0) < r < r_0 \exp(\tfrac{1}{2}\alpha_0)\}.$$

Theorem 1. *Let $f(z)$ be regular in $D(r_0, \alpha_0)$.*

(i) *If $f(z_1) = 0$ for some point z_1 satisfying $|z_1| = r_0$, then there exists r in $I(r_0, \alpha_0)$ such that $b(r) \geq A_1$ where A_1 is defined in (12).*

(ii) *If $f(z_1) = f(z_2) = 0$ where $|z_1| = |z_2| = r_0$ there exists $r \in I(r_0, \alpha_0)$ such that $b(r) \geq 2A_1$ where A_1 is defined by (12).*

Corollary 1. *If f is a transcendental entire function then*

$$(2.1) \qquad\qquad \limsup_{r \to \infty} b(r) \geq A_1.$$

Corollary 2. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a transcendental entire function and assume that*

$$(2.2) \qquad\qquad \limsup_{r \to \infty} b(r) < 2A_1.$$

Then all but a finite number of the zeros of f have distinct moduli, in particular,

(i) *if all the coefficients of f are real, then all but a finite number of the zeros of f are simple and real;*

(ii) *if all the coefficients of f are nonnegative, then all but a finite number of the zeros of f are simple and lie on the negative real axis.*

Theorem 2. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a transcendental entire function whose coefficients a_k are nonnegative. If $b(r)$ is bounded, then $\limsup_{r \to \infty} b(r) = 0.25$ if and only if all but a finite number of zeros $\{z_n\}$ of f are simple, negative and satisfy $r_{n+1}/r_n \to \infty$ as $n \to \infty$ where $r_n = |z_n|$.*

We do not know whether equality can hold in relation (2.1), but we can have $\limsup b(r) = 2A_1$. This raises the question of determining the extremal functions. In order to do so let us write $\omega(r)$ for any argument at which the maximum modulus is achieved. That is $M(r) = |f(re^{i\omega(r)})|$. If z_n is the nth zero of f we write $z_n = r_n e^{i\theta_n}$.

Theorem 3. *Let f be an entire function and let δ, $0 < \delta \leq \pi$, be given.*

(i) *If there are infinitely many zeros $z_n = r_n e^{i\theta_n}$ satisfying $|\theta_n - \omega(r_n)| \leq \delta$ then*

$$(2.3) \qquad \limsup_{r \to \infty} b(r) \geq A_1 - \frac{8}{\alpha_0^2} \log \left| sn \left(\frac{K'\delta}{\pi}, k' \right) \right|.$$

(ii) *If there are infinitely many pairs of zeros $z_n = r_n e^{i\theta_n}$, $w_n = r_n e^{i\lambda_n}$ satisfying $|\theta_n - \omega(r_n)| \leq \delta$ and $|\lambda_n - \omega(r_n)| \leq \delta$ then*

$$(2.4) \qquad \limsup_{r \to \infty} b(r) \geq 2 \left(A_1 - \frac{8}{\alpha_0^2} \log \left| sn \left(\frac{K'\delta}{\pi}, k' \right) \right| \right).$$

(iii) *If $\limsup_{r \to \infty} b(r) = 2A_1$ then all but a finite number of the multiple zeros of f must satisfy $|\theta_n - \omega(r_n)| = \pi$.*

3. AN AUXILIARY LEMMA

The principle of our proofs is the same as that of Hayman's [3], but instead of using the cotangent function we use theta functions. This leads to the fol-

lowing lemma which is fundamental to our proofs and which may also be of independent interest.

Lemma. *Let $f(z)$ be regular in the annulus $D(r_0, \alpha_0)$ and let*
$$L(0) = \log \sqrt{M(r_0 e^{-\alpha_0/2}) M(r_0 e^{\alpha_0/2})}.$$
(i) If $f(z_1) = 0$ for some point z_1 satisfying $|z_1| = r_0$, then

$$(3.1) \qquad |f(z_1 e^{i\tau})| \le \exp(L(0)) \cdot k'^{\frac{1}{2}} \left| sn\left(\frac{K'\tau}{\pi}, k'\right) \right|$$

for all real τ.

(ii) *Suppose $f(z_1) = f(z_2) = 0$ where $z_1 = r_0 e^{i\theta_1}$, $z_2 = r_0 e^{i\theta_2}$. Set $z_0 = r_0 e^{i\gamma}$ where $\gamma = \frac{1}{2}(\theta_1 + \theta_2)$. Then, if $\beta = \frac{1}{2}(\theta_2 - \theta_1)$ and $-\pi \le \tau \le \pi$, we have*

$$(3.2) \quad |f(z_0 e^{i\tau})| \le \exp(L(0)) \cdot k' \left| sn\left(\frac{K'(\tau - \beta)}{\pi}, k'\right) sn\left(\frac{K'(\tau + \beta)}{\pi}, k'\right) \right|.$$

It will be convenient to omit the subscript in α_0 and write α instead. Let $S = S(\alpha)$ be the vertical strip in the z-plane defined by

$$(3.3) \qquad S = \left\{ z = \sigma + i\tau : -\frac{1}{2}\alpha < \sigma < \frac{1}{2}\alpha, -\infty < \tau < +\infty \right\}.$$

The maximum principle will be applied to a certain auxiliary function on S and this requires a study of factors of the form $(1 \pm q^{2n} e^{\pm i\pi z/\alpha})$ on the boundary of S. More precisely, if $q = e^{-\pi^2/\alpha}$ where $\alpha > 0$ and $z = \pm\frac{1}{2}\alpha + iy$ then, since $e^{\pm i\pi z/\alpha}$ is pure imaginary, we have

$$(3.4) \qquad \left| \frac{1 + q^{2n} e^{i\pi z/\alpha}}{1 - q^{2n} e^{i\pi z/\alpha}} \right| = \left| \frac{1 + q^{2n} e^{-i\pi z/\alpha}}{1 - q^{2n} e^{-i\pi z/\alpha}} \right| = 1 \qquad (\infty < y < +\infty).$$

Also, if $z = \pm\frac{1}{2}\alpha + iy$ then

$$(3.5) \qquad |\cot(\pi z/2\alpha)| = 1 \qquad (-\infty < y < +\infty).$$

It is clear for each fixed n that if $z \in S$ and $z \to \infty$ then

$$(3.6) \qquad \left| \frac{1 + q^{2n} e^{\pm\pi i z/\alpha}}{1 - q^{2n} e^{\pm\pi i z/\alpha}} \right| \to 1.$$

Finally, since

$$(3.7) \qquad \frac{\sinh^2 \pi y/2\alpha}{1 + \sinh^2 \pi y/2\alpha} \le \left| \cot\left(\frac{\pi x}{2\alpha} + i\frac{\pi y}{2\alpha}\right) \right|^2 \le \frac{1 + \sinh^2 \pi y/2\alpha}{\sinh^2 \pi y/2\alpha}$$

for $|x| \le \frac{1}{2}\alpha$ and $y \ne 0$, we see that $|\cot(\pi z/2\alpha)| \to 1$ as $z \to \infty$ inside S.

We are now ready to proceed with the proof of the lemma.

Proof. We shall only prove (3.2) since the proof of (3.1) is similar (and simpler). Let f satisfy the conditions in (ii) of the lemma and consider the function $f(z_0 e^z)$ on the strip S. Let $L(\sigma) = B\sigma + C$ where the real constants B and C are determined by the equations

$$\lim_{\sigma \to \pm\frac{1}{2}\alpha} (\log M(r_0 e^\sigma) - L(\sigma)) = 0.$$

Since $\log M(r_0 e^\sigma)$ is a bounded convex function of σ in the interval $I(r_0, \alpha)$, the limits exist and determine L uniquely. By the Hadamard three-circle theorem we also have

$$(3.8) \qquad |f(z_0 e^z)| \leq \exp(L(\sigma))$$

for all $z = \sigma + i\tau$ in S.

Fix the integer $N \geq 1$ and consider in S the functions

$$(3.9) \qquad \psi_N(z) = \frac{\cos(\pi z/2\alpha)}{\sin(\pi z/2\alpha)} \prod_{n=1}^{N} \frac{(1 + q^{2n} e^{i\pi z/\alpha})(1 + q^{2n} e^{-i\pi z/\alpha})}{(1 - q^{2n} e^{i\pi z/\alpha})(1 - q^{2n} e^{-i\pi z/\alpha})},$$

and

$$(3.10) \qquad \phi_N(z) = f(z_0 e^z) e^{-L(z)} \psi_N(z - i\beta) \psi_N(z + i\beta)$$

where $\beta = \frac{1}{2}(\arg z_2 - \arg z_1)$ may be assumed to satisfy $0 \leq \beta < \pi$. Suppose $\beta \neq 0$. The zeros of the denominator of ϕ_N in S are at the zeros of $\sin(\pi(z - i\beta)/2\alpha)\sin(\pi(z + i\beta)/2\alpha)$ in S, which are $\pm i\beta$, and at the zeros of the factors $(1 - q^{2n} e^{i\pi(z \pm i\beta)/\alpha})(1 - q^{2n} e^{-i\pi(z \pm i\beta)/\alpha})$ in S, which are $\pm i\beta \pm 2n\pi i$. So the zeros of the denominator of ϕ_N in S occur at $\pm i\beta + 2n\pi i$ where $n = 0, \pm 1, \pm 2, \ldots, \pm N$. On the other hand, since $z_0 = r_0 e^{\frac{1}{2}(\theta_1 + \theta_2)}$ and $f(r_0 e^{i\theta_1}) = f(r_0 e^{i\theta_2}) = 0$, the function $f(z_0 e^z)$ has zeros at all the points $\pm i\beta + 2n\pi i$ where $n = 0, \pm 1, \pm 2, \ldots$. Thus the function ϕ_N is regular in S. If $\beta = 0$ then the zeros of the denominator of ϕ_N occur at $2n\pi i$ and f has double zeros there so that ϕ_N is again regular in S. The boundary behavior of ϕ_N follows immediately from that of ψ_N. Thus $\limsup |\phi_N(z)| \leq 1$ as z approaches any finite boundary point of S by (3.4) and (3.5). Also $\limsup |\phi_N(z)| \leq 1$ as $z \to \infty$ inside S by (3.6) and (3.7). It now follows by the maximum principle that $|\phi_N(z)| \leq 1$ for all $z \in S$ and so by (3.8)

$$(3.11) \qquad |f(z_0 e^z)| \leq \exp(L(\sigma)) |\psi_N(z - i\beta) \psi_N(z + i\beta)|^{-1}$$

where $z = \sigma + i\tau$, $|\sigma| \leq \frac{1}{2}\alpha$ and $-\infty < \tau < \infty$.

If we let $N \to \infty$ the product defining ψ_N becomes a ratio of theta functions given in (5). These can then be simplified using (6), (8) and (9) to give by (3.11)

$$(3.12) \quad |f(z_0 e^z)| \leq \exp(L(\sigma)) k' \left| \frac{sn}{cn} \left(\frac{K(z - i\beta)}{\alpha}, k \right) \right| \left| \frac{sn}{cn} \left(\frac{K(z + i\beta)}{\alpha}, k \right) \right|,$$

valid for all $z = \sigma + i\tau$, $|\sigma| \leq \frac{1}{2}\alpha$ and $-\infty < \tau < \infty$.

In particular, if we take $\sigma = 0$ and use the Jacobian imaginary transformation [6, p. 506] namely,

$$sn(iu, k) = i \frac{sn(u, k')}{cn(u, k')}, \quad cn(iu, k) = \frac{1}{cn(u, k')},$$

we obtain

$$(3.13) \quad |f(z_0 e^{i\tau})| \leq \exp(L(0)) \cdot k' |sn(K(\tau - \beta)/\alpha, k')| |sn(K(\tau + \beta)/\alpha, k')|.$$

If we now recall that $q = e^{-\pi^2/\alpha}$ and, from (8), $\frac{\pi K'}{K} = \log q^{-1}$, we obtain $\frac{K'}{K} = \frac{\pi}{\alpha}$ so that $\frac{K(\tau - \beta)}{\alpha} = \frac{K'(\tau - \beta)}{\pi}$ and $\frac{K(\tau + \beta)}{\alpha} = \frac{K'(\tau + \beta)}{\pi}$. Thus (3.13) takes the form

$$(3.14) \quad |f(z_0 e^{i\tau})| \leq \exp(L(0)) \cdot k' \left| sn \left(\frac{K'(\tau - \beta)}{\pi}, k' \right) sn \left(\frac{K'(\tau + \beta)}{\pi}, k' \right) \right|.$$

This finishes the proof of the lemma.

4. Proof of Theorem 3

Let us prove part (ii) of Theorem 3 since the proof of part (i) is similar. Suppose then that $0 < \delta \leq \pi$ and that f has infinitely many pairs of zeros $z_n = r_n e^{i\theta_n}$ and $w_n = r_n e^{i\lambda_n}$ having equal moduli and satisfying

$$(4.1) \qquad |\theta_n - \omega(r_n)| \leq \delta \quad \text{and} \quad |\lambda_n - \omega(r_n)| \leq \delta$$

where $|f(r_n e^{i\omega(r_n)})| = M(r_n)$.

Let $L_n(\sigma) = B_n \sigma + C_n$ where the real constants are, as in the proof of the lemma, determined by the equations

$$(4.2) \qquad \lim_{\sigma \to \pm \frac{1}{2}\alpha} (\log M(r_n e^{\sigma}) - L_n(\sigma)) = 0.$$

Applying part (ii) of the lemma with $z_0 = r_n e^{i(\theta_n + \lambda_n)/2}$, $z_1 = z_n$, $z_2 = w_n$ and $\tau = \omega(r_n) - (\theta_n + \lambda_n)/2$ we obtain

$$(4.3) \qquad M(r_n) \leq \exp(L_n(0))k' \left| sn\left(\frac{K'(\tau - \beta)}{\pi}, k'\right) sn\left(\frac{K'(\tau + \beta)}{\pi}, k'\right) \right|.$$

Here $\beta = \frac{1}{2}(\lambda_n - \theta_n)$ and so, in view of (4.1), we have

$$(4.4) \qquad |\tau - \beta| \leq \delta \quad \text{and} \quad |\tau + \beta| \leq \delta.$$

Since $sn(u, k')$ is an odd function of u that increases on $0 \leq u \leq K'$ we have

$$\left| sn\left(\frac{K'(\tau - \beta)}{\pi}, k'\right) sn\left(\frac{K'(\tau + \beta)}{\pi}, k'\right) \right| \leq \left| sn\left(\frac{K'\delta}{\pi}, k'\right) \right| \left| sn\left(\frac{K'\delta}{\pi}, k'\right) \right|,$$

and so

$$(4.5) \qquad \log M(r_n) \leq L_n(0) + \log k' + 2\log \left| sn\left(\frac{K'\delta}{\pi}, k'\right) \right|.$$

Now put $g(\sigma) = L_n(\sigma) - \log M(r_n e^{\sigma}) + \frac{A}{2}(\sigma^2 - \frac{1}{4}\alpha^2)$ and assume that $b(r) < A$ for all $r \in I(r_n, \alpha)$. Then $g''(\sigma) = -b(r_n e^{\sigma}) + A$ so that the assumption on b implies that g is convex on $\left[-\frac{1}{2}\alpha, \frac{1}{2}\alpha\right]$. As $g(-\frac{1}{2}\alpha) = g(\frac{1}{2}\alpha) = 0$, it follows that $g(\sigma) \leq 0$ for all $|\sigma| \leq \frac{1}{2}\alpha$. In particular $g(0) \leq 0$ and so

$$\log M(r_n) \geq L_n(0) - A\frac{\alpha^2}{8}.$$

If this last inequality is combined with (4.5) we obtain

$$(4.6) \qquad A \geq \frac{-8}{\alpha^2}\left(\log k' + 2\log\left| sn\left(\frac{K'\delta}{\pi}, k'\right) \right|\right).$$

It remains to recall, from (11) and (12), the definition $A_1 = \frac{-8}{\alpha_0^2}\log k'^{1/2}$ to see that inequality (2.4) of Theorem 3 follows from (4.6) and the assumption of an infinitude of pairs of zeros having equal moduli. This finishes the proof of part (ii) of Theorem 3 and we have, in passing, proved part (ii) of Theorem 1. Indeed, taking $\delta = \pi$ in (4.6) gives the required inequality in part (ii) of Theorem 1 and, of course, (4.1) is always satisfied in this case. The proofs of part (i) of Theorem 3 and part (i) of Theorem 1 are similar and use (3.1)

instead of (3.2). To prove part (iii) of Theorem 3, note that the assumption $\limsup b(r) = 2A_1$ implies that $sn\left(\frac{K'\delta}{\pi}, k'\right) = 1$ and so $\delta = \pi$. Thus, either f has only a finite number of multiple zeros, in which case (2.4) does not apply; or f has an infinite number of multiple zeros $z_n = r_n e^{i\theta_n}$ in which case (2.4) applies and gives $|\theta_n - \omega(r_n)| = \pi$ for all large n as required. This finishes the proof of Theorem 3.

5. PROOFS (CONTINUED)

The inequalities (2.1) and (2.2) are immediate consequences of parts (i) and (ii) of Theorem 1. To prove part (i) of Corollary 2 observe that if the coefficients of f are real then complex zeros, when they exist, occur in conjugate pairs. So if f has real coefficients and an infinite number of complex zeros then part (ii) of Theorem 1 or Theorem 3 implies that $\limsup_{r\to\infty} b(r) \geq 2A_1$, contradicting (2.2). Thus f can have at most a finite number of complex zeros and all other zeros must be real and simple. Part (ii) of Corollary 2 follows from part (i) and the fact that, since the coefficients of f are nonnegative, the real zeros of f when they occur cannot be positive.

Proof of Theorem 2. Let f be entire with nonnegative Maclaurin coefficients and suppse that $\limsup_{r\to\infty} b(r) = \frac{1}{4}$. Since $2A_1 > 0.365 > 0.25$, Corollary 2 implies that, except for a possible finite number, all the zeros of f are simple, real and negative. Thus we may write

$$(5.1) \qquad f(z) = p(z)\prod_{n=1}^{\infty}(1 + z/r_n)$$

where p is a polynomial carrying the nonreal zeros of f and the product carries all the negative real and simple zeros $\{-r_n\}$ of f. Since we are given that all coefficients of f are nonnegative, we have $M(r, f) = f(r)$ and so

$$(5.2) \qquad b(r, f) = d^2 \log p(r)/d(\log r)^2 + \sum_{n=1}^{\infty}\frac{rr_n}{(r + r_n)^2}.$$

The first term on the right-hand side of (5.2) tends to zero as $r \to \infty$ and so does not contribute to $\limsup_{r\to\infty} b(r)$. Thus we have

$$\frac{1}{4} = \limsup_{r\to\infty} b(r) \geq \limsup_{n\to\infty} b(r_n) \geq \limsup_{n\to\infty}\left(\frac{1}{4} + \frac{r_n r_{n+1}}{(r_n + r_{n+1})^2}\right) \geq \frac{1}{4}.$$

Therefore

$$(5.3) \qquad \limsup_{n\to\infty}\frac{r_{n+1}/r_n}{(1 + r_{n+1}/r_n)^2} = 0.$$

If a subsequence r_{n_k+1}/r_{n_k} of r_{n+1}/r_n were bounded (above), say by M, we would contradict (5.3). Hence $r_{n+1}/r_n \to \infty$ as $n \to \infty$. Conversely, if the real simple zeros of f satisfy $r_{n+1}/r_n \to \infty$ as $n \to \infty$ and $b(r)$ is bounded, then f factors as in (5.1) and $b(r, f)$ is given by (5.2) where, again, only the second term on the right-hand side matters. We can write

$$\sum_{n=1}^{\infty}\frac{rr_n}{(r + r_n)^2} \leq \frac{rr_n}{(r + r_n)^2} + \frac{1}{r}\sum_{k=1}^{n-1} r_k + r\sum_{k=n+1}^{\infty} 1/r_k.$$

We may assume that $r_{n+1}/r_n > M > 1$ for all $n \geq 1$. Then we have

$$\frac{rr_n}{(r+r_n)^2} \leq \sum_{n=1}^{\infty} \frac{rr_n}{(r+r_n)^2} \leq \frac{rr_n}{(r+r_n)^2} + \frac{M}{M-1} \left\{ \frac{r_{n-1}}{r} + \frac{r}{r_{n+1}} \right\}.$$

By considering $r \in ((r_{n-1}r_n)^{1/2}, (r_nr_{n+1})^{1/2})$, we conclude $\limsup_{r \to \infty} b(r) = \frac{1}{4}$ as required.

ACKNOWLEDGMENT

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