TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 8, August 1995

COMPARISON OF CERTAIN H^∞-DOMAINS OF HOLOMORPHY

ULF BACKLUND

ABSTRACT. We study open sets defined by certain global plurisubharmonic functions in $\mathbb{C}^N$. We examine how the fact that the connected components of the sets are H^∞-domains of holomorphy is related to the structure of the set of discontinuity points of the global defining functions and to polynomial convexity.

0. INTRODUCTION

In this paper we study certain properties of some open sets defined by global plurisubharmonic functions in $\mathbb{C}^N$. More precisely, we consider the sets

$$D_u = \{z \in \mathbb{C}^N : u(z) < 0\},$$
$$E_h = \{(z, w) \in \mathbb{C}^N \times \mathbb{C} : h(z, w) < 1\}$$

where u is a plurisubharmonic function of minimal growth and $h \not\equiv 0$ is a nonnegative homogeneous plurisubharmonic function. (That is, the functions u and h belong to the classes $L(\mathbb{C}^N)$ and $H_+(\mathbb{C}^N \times \mathbb{C})$ respectively. For definitions, see §1.) We examine how the fact that E_h and the connected components of D_u are H^∞-domains of holomorphy is related to the structure of the set of discontinuity points of the global defining functions and to polynomial convexity. One of our results is that if D_u is bounded and if the set of discontinuity points of u is pluripolar, then D_u is of type H^∞ (Theorem 9). We also examine whether these notions are preserved under a certain bijective mapping between $L(\mathbb{C}^N)$ and $H_+(\mathbb{C}^N \times \mathbb{C})$. In particular, we give two counterexamples (Theorem 11 and Theorem 12) which show that polynomial convexity is not preserved under this bijection.

1. DEFINITIONS

If Ω is an open subset of $\mathbb{C}^N$, $N \geq 1$, we denote by $\mathrm{PSH}(\Omega)$ the family of plurisubharmonic functions on Ω.

Received by the editors February 3, 1994.
1991 *Mathematics Subject Classification*. Primary 32A10, 32D20; Secondary 32E05.
Key words and phrases. Bounded holomorphic function, domain of holomorphy, plurisubharmonic function, pluripolar set, polynomially convex set.

We consider the following subsets of $\mathrm{PSH}(\mathbb{C}^N)$:

$$L(\mathbb{C}^N) = \left\{\varphi \in \mathrm{PSH}(\mathbb{C}^N) : \sup_{z\in\mathbb{C}^N}\{\varphi(z) - \log(1+|z|)\} < +\infty\right\},$$

$$L^+(\mathbb{C}^N) = \left\{\varphi \in L(\mathbb{C}^N) : \inf_{z\in\mathbb{C}^N}\{\varphi(z) - \log(1+|z|)\} > -\infty\right\},$$

$$H_+(\mathbb{C}^N) = \{\varphi \in \mathrm{PSH}(\mathbb{C}^N) : \varphi \not\equiv 0,\ \varphi(\lambda z) = |\lambda|\varphi(z)\ \forall\lambda \in \mathbb{C},\ \forall z \in \mathbb{C}^N\}.$$

We will study the following two mappings:

$$S : L(\mathbb{C}^N) \to H_+(\mathbb{C}^N \times \mathbb{C}) \quad \text{defined by}$$

$$(Su)(z, w) := |w| \exp u(z/w), \qquad (z, w) \in \mathbb{C}^N \times (\mathbb{C}\backslash\{0\}),$$

$$:= \limsup_{(z,\zeta)\to(z,0)} |\zeta| \exp u(z/\zeta), \qquad (z, w) \in \mathbb{C}^N \times \{0\},$$

and

$$T : H_+(\mathbb{C}^N \times \mathbb{C}) \to L(\mathbb{C}^N) \quad \text{defined by}$$

$$(Th)(z) := \log h(z, 1), \qquad z \in \mathbb{C}^N.$$

Su is clearly nonnegative homogeneous. The fact that $u \in L(\mathbb{C}^N)$ means that there is a constant β_u such that

$$u(z) \le \beta_u + \log(1+|z|), \qquad z \in \mathbb{C}^N.$$

Hence

$$|w| \exp u(z/w) \le e^{\beta_u}(|z| + |w|)$$

which implies that the plurisubharmonic function $|w|\exp u(z/w)$ is locally bounded from above near the analytic set $\{(z, w) \in \mathbb{C}^N \times \mathbb{C} : |w| = 0\}$. From this the plurisubharmonicity of Su follows. That Th belongs to $L(\mathbb{C}^N)$ follows from [4].

The mapping S is a bijection and T is the inverse mapping of S.

Let $u \in L(\mathbb{C}^N)$ and $h \in H_+(\mathbb{C}^N \times \mathbb{C})$. We define two open sets.

$$D_u = \{z \in \mathbb{C}^N : u(z) < 0\},$$

$$E_h = \{(z, w) \in \mathbb{C}^N \times \mathbb{C} : h(z, w) < 1\}.$$

Remark. The connected components of D_u are domains of holomorphy and E_h is a balanced domain of holomorphy.

2. Lemmas

In this section we state and prove some lemmas.

Lemma 1. *If $u \in L^+(\mathbb{C}^N)$, then D_u and E_{Su} are bounded sets.*

Proof. Since $u \in L^+(\mathbb{C}^N)$, there exists a constant α_u such that

$$u(z) \ge \alpha_u + \log(1+|z|), \qquad z \in \mathbb{C}^N.$$

Hence D_u is bounded. Moreover

$$(Su)(z, w) \ge |w| \exp(\alpha_u + \log(1 + |z|/|w|)) = e^{\alpha_u}(|z| + |w|).$$

This gives that E_{Su} is bounded. □

Lemma 2. *If* $h \in H_+(\mathbb{C}^N \times \mathbb{C})$ *and* E_h *is bounded, then* D_{Th} *is bounded.*

Proof. Since E_h is bounded, we have $h(z, w) \geq 1$ for $|z|, |w|$ sufficiently large.

In particular, $h(z, 1) \geq 1$ and hence $(Th)(z) = \log h(z, 1) \geq 0$ for $|z|$ sufficiently large.

Hence D_{Th} is bounded. □

Notation. If φ is a subharmonic function on $\mathbb{C}^N$, then by $N(\varphi)$ we denote the set

$$N(\varphi) = \{z \in \mathbb{C}^N : \varphi \text{ is discontinuous at } z\}.$$

Lemma 3. *If* $u \in \mathrm{PSH}(\mathbb{C}^N)$ *and* D_u *is bounded, then there exists a function* $u^+ \in L^+(\mathbb{C}^N)$ *such that* $D_u = D_{u^+}$. *Moreover* $N(u^+)$ *is contained in* $N(u)$.

Proof. Since D_u is bounded, there is a number $R > 1$ such that D_u is contained in the ball of radius $R - 1$ centered at 0.

We define the function u^+ by

$$u^+(z) := \max\left(\frac{u(z)\log((1+R)/R)}{\sup_{|\xi|\leq R+1} u(\xi)}, -1, \log\frac{1+|z|}{R}\right), \qquad |z| \leq R,$$

$$:= \log\frac{1+|z|}{R}, \qquad |z| > R.$$

Since

$$\limsup_{\zeta\to z} \frac{u(\zeta)\log((1+R)/R)}{\sup_{|\xi|\leq R+1} u(\xi)} \leq \log\frac{1+|z|}{R}$$

for all z such that $|z| = R$, we see that u^+ is plurisubharmonic on $\mathbb{C}^N$. Obviously, u^+ belongs to $L^+(\mathbb{C}^N)$ and $D_u = D_{u^+}$.

It is easy to see that if u is continuous at a point z, then u^+ is also continuous at z and this proves the lemma. □

Lemma 4. *Let* v *be a subharmonic function on* $\mathbb{C}^N$. *If* $N(v)$ *is closed and has Lebesgue measure zero, then* $\operatorname{int}\overline{\{z \in \mathbb{C}^N : v(z) < 0\}} = \{z \in \mathbb{C}^N : v(z) < 0\}$.

Proof. Assume that v is nonconstant and suppose there is point

$$z^0 \in \operatorname{int}\overline{\{z \in \mathbb{C}^N : v(z) < 0\}} \setminus \{z \in \mathbb{C}^N : v(z) < 0\}.$$

Then $v(z^0) \geq 0$ and there is an $r > 0$ such that the ball $B(z^0, r)$ is contained in $\operatorname{int}\overline{\{z \in \mathbb{C}^N : v(z) < 0\}}$. By the maximum principle there is a point $z' \in B(z^0, r)$ with $v(z') > 0$ but points in $B(z^0, r)$ where $v > 0$ must belong to $B(z^0, r) \cap N(v)$ and, since $N(v)$ has Lebesgue measure zero, the subharmonicity of v implies that $v \leq 0$ on $B(z^0, r)$. This gives a contradiction. □

3. The main results

We recall some definitions.

Definition 1. If K is a compact set in $\mathbb{C}^N$, then the set

$$\widehat{K} = \left\{z \in \mathbb{C}^N : |P(z)| \leq \sup_{\zeta\in K}|P(\zeta)| \text{ for every polynomial } P\right\}$$

is called the polynomially convex hull of K. A compact set K in $\mathbb{C}^N$ is said to be polynomially convex if $K = \widehat{K}$.

Definition 2. A domain Ω in $\mathbb{C}^N$ is said to be an H^∞-domain of holomorphy if Ω is biholomorphic to its H^∞-envelope of holomorphy.

Definition 3. An open set Ω in $\mathbb{C}^N$ is said to be of type H^∞ if the connected components of Ω are H^∞-domains of holomorphy.

Definition 4. A set E in $\mathbb{C}^N$ is said to be pluripolar if for each point $z \in \overline{E}$ there exists a neighbourhood U of z and a function $\varphi \in \mathrm{PSH}(U)$ such that $\varphi = -\infty$ on $E \cap U$.

We will examine how these notions are related for D_u and E_h. We also study whether they are preserved under the mappings S and T.

We prove that if h is a function in $H_+(\mathbb{C}^N \times \mathbb{C})$ such that E_h is bounded and if u is a function in $L^+(\mathbb{C}^N)$, then the following diagram holds for the mappings S and T.

$$
\begin{array}{llcccl}
 & & \text{Th1} & & \text{Th2} & \\
 & & & & \Leftarrow & \widehat{\overline{E}}_h = \overline{E}_h \ \& \\
 & N(h) \text{ is pluripolar} & \Leftrightarrow & E_h \text{ is of type } H^\infty & \not\Rightarrow & \operatorname{int}\overline{E}_h = E_h \\
 & & & & \text{Th3} & \\
T \downarrow\uparrow S & \text{Th4} \quad \Updownarrow & & \text{Th7} \ \not\Uparrow \Downarrow \ \text{Th10} & & \text{Th11} \ \not\Uparrow \not\Downarrow \ \text{Th12} \\
 & & \text{Th8} & & \text{Th5} & \\
 & & \not\Leftarrow & & \Leftarrow & \widehat{\overline{D}}_u = \overline{D}_u \ \& \\
 & N(u) \text{ is pluripolar} & & D_u \text{ is of type } H^\infty & & \\
 & & \Rightarrow & & \not\Rightarrow & \operatorname{int}\overline{D}_u = D_u \\
 & & \text{Th9} & & \text{Th6} &
\end{array}
$$

Remark. It follows from Lemma 1 and Lemma 2 that all sets being considered in the diagram are bounded.

Theorem 1 is due to J. Siciak [5]:

Theorem 1. *Let $h \in H_+(\mathbb{C}^N \times \mathbb{C})$ such that E_h is bounded. Then E_h is of type H^∞ if and only if $N(h)$ is pluripolar.*

Theorem 2. *Let $h \in H_+(\mathbb{C}^N \times \mathbb{C})$ such that E_h is bounded. Assume that $\overline{E}_h$ is polynomially convex and $\operatorname{int}\overline{E}_h = E_h$. Then E_h is of type H^∞.*

Proof. Suppose that E_h is not of type H^∞.

Then every function in $H^\infty(E_h)$ can be holomorphically continued to a strictly larger open set U in $\mathbb{C}^N \times \mathbb{C}$ such that $U \backslash E_h$ has nonempty interior. The polynomials belong to $H^\infty(E_h)$ and since the supremum norm of a function in $H^\infty(E_h)$ cannot increase under holomorphic extension to U, we get a contradiction to the fact that $\overline{E}_h$ is polynomially convex. □

Remark. The proof uses only the fact that the set Ω is a bounded open set in $\mathbb{C}^N \times \mathbb{C}$ with $\widehat{\overline{\Omega}} = \overline{\Omega}$ and $\operatorname{int}\overline{\Omega} = \Omega$.

The converse of Theorem 2 is not true as the following result of U. Cegrell [1] shows:

Theorem 3. *There exists a function* $h \in H_+(\mathbb{C}^3)$ *such that* E_h *is a bounded domain of type* H^∞ *with* $\operatorname{int}\overline{E}_h = E_h$ *but* $\overline{E}_h$ *is not polynomially convex.*

Proof.

$$h(z_1, z_2, w) = \exp\left(\sum_{j=1}^{\infty} \alpha_j \max(\log|z_1 - a_j z_2|, \log|w|)\right) + \max(|z_1|, |z_2|, |w|)$$

where $(a_j)_{j=1}^{\infty}$ is dense in the unit circle and $(\alpha_j)_{j=1}^{\infty}$ is a sequence of positive numbers with $\sum_{j=1}^{\infty} \alpha_j = 1$. For the details, see [1]. □

Remark. In [6] it was proved that the converse of Theorem 2 is true when $N = 1$.

Theorem 4. *If* $u \in L(\mathbb{C}^N)$ *and* $N(u)$ *is pluripolar, then* $N(Su)$ *is pluripolar. If* $h \in H_+(\mathbb{C}^N \times \mathbb{C})$ *and* $N(h)$ *is pluripolar, then* $N(Th)$ *is pluripolar.*

Proof. Assume that $u \in L(\mathbb{C}^N)$ and that $N(u)$ is pluripolar. By Josefson's theorem [2] and [3], there exists a function $\Phi_u \in L(\mathbb{C}^N)$ such that $\Phi_u \equiv -\infty$ on $N(u)$. Consider the plurisubharmonic function $\log|w| + \Phi_u(z/w)$ for $w \neq 0$. Since $\Phi_u \in L(\mathbb{C}^N)$, we have

$$\log|w| + \Phi_u(z/w) \leq \log|w| + \beta_u + \log(1 + |z|/|w|) = \beta_u + \log(|z| + |w|)$$

for some constant β_u. Hence $\log|w| + \Phi_u(z/w)$ is locally bounded from above near $\{(z, w) \in \mathbb{C}^N \times \mathbb{C} : |w| = 0\}$.

This means that the function

$$\begin{aligned} \Phi_{Su}(z, w) &:= \log|w| + \Phi_u(z/w), \qquad (z, w) \in \mathbb{C}^N \times (\mathbb{C}\backslash\{0\}), \\ &:= \limsup_{(z,\zeta)\to(z,0)} \log|\zeta| + \Phi_u(z/\zeta), \qquad (z, w) \in \mathbb{C}^N \times \{0\}, \end{aligned}$$

is plurisubharmonic on $\mathbb{C}^N \times \mathbb{C}$. We have that $N(Su)$ is contained in the set

$$\mathbb{C}^N \times \{0\} \bigcup_{0 \neq c \in \mathbb{C}} [cN(u) \times \{c\}]$$

and from this it follows that $\Phi_{Su} \equiv -\infty$ on $N(Su)$. Consequently, $N(Su)$ is pluripolar.

Suppose now that $h \in H_+(\mathbb{C}^N \times \mathbb{C})$ and that $N(h)$ is pluripolar. Then there exists a function $\Phi_h \in L(\mathbb{C}^N \times \mathbb{C})$ such that $\Phi_h \equiv -\infty$ on $N(h)$. Since $N(Th) = N(\log h(z, 1)) = N(h(z, 1))$, we see that $\Phi_h(z, 1) \equiv -\infty$ on $N(Th)$. Thus $N(Th)$ is pluripolar. □

Theorem 5. *Let* $u \in L(\mathbb{C}^N)$ *such that* D_u *is bounded. Assume that* $\overline{D}_u$ *is polynomially convex and* $\operatorname{int}\overline{D}_u = D_u$. *Then* D_u *is of type* H^∞.

Proof. The argument is the same as in Theorem 2. □

Theorem 6. *There exists a function* $u \in L^+(\mathbb{C}^3)$ *such that* D_u *is of type* H^∞ *with* $\operatorname{int}\overline{D}_u = D_u$ *but* $\overline{D}_u$ *is not polynomially convex.*

Proof. Consider the function h in the proof of Theorem 3. Since $h \in H_+(\mathbb{C}^3)$ the function $u = \log h$ belongs to $L(\mathbb{C}^3)$ (see e.g. [4]). The result then follows from Lemma 3. □

Theorem 7. *There exists a function $u \in L^+(\mathbb{C})$ such that D_u is of type H^∞ but E_{Su} is not of type H^∞.*

Proof. Consider the function

$$h(z, w) = \frac{1}{3}\exp\left(\sum_{j=1}^{\infty} \alpha_j \log|z - a_j w|\right) + \max(|z|, |w|)$$

where $(a_j)_{j=1}^{\infty}$ is dense in the unit circle and $(\alpha_j)_{j=1}^{\infty}$ is a sequence of positive numbers with $\sum_{j=1}^{\infty} \alpha_j = 1$. Then $H_+(\mathbb{C}^2)$ and we have

$$u(z) = (Th)(z) = \log\left(\exp\left(\sum_{j=1}^{\infty} \alpha_j \log|z - a_j|\right) + \max(|z|, 1)\right) - \log 3.$$

In [1] it is proved that D_u is of type H^∞ but E_h is not of type H^∞. To show that $u \in L^+(\mathbb{C})$ we observe that

$$u(z) \geq \log(\max(|z|, 1)) - \log 3 \geq \log\left(\frac{|z| + 1}{2}\right) - \log 3.$$

Thus $u \in L^+(\mathbb{C})$. □

Theorem 8. *There exists a function $u \in L^+(\mathbb{C})$ such that D_u is of type H^∞ but $N(u)$ is a nonpluripolar set.*

Proof. As a consequence of Theorem 1 and Theorem 4, the function in Theorem 7 gives the result. □

Theorem 9. *Let $u \in L(\mathbb{C}^N)$. Assume that D_u is bounded and $N(u)$ is pluripolar. Then D_u is of type H^∞.*

Proof. Since $N(u)$ is pluripolar, there exists a function $\varphi \in L(\mathbb{C}^N)$ ([2], [3]) such that $N(u)$ is contained in

$$\{z \in \mathbb{C}^N : \varphi(z) = -\infty\}$$

and we can assume that $\varphi \leq 0$ on $\overline{D}_u$.

If z is a boundary point of D_u, then $u(z) \geq 0$ and if $z \in \partial D_u \backslash N(u)$, then equality holds.

Let z^0 be an arbitrary boundary point of D_u and let $r > 0$ be an arbitrary real number. Then u is not constant on the ball $B(z^0, r)$ since $B(z^0, r) \cap D_u \neq \varnothing$ and it follows from the plurisubharmonicity of u that $u > 0$ on a set of positive Lebesgue measure in $B(z^0, r)$.

Since $\{z \in \mathbb{C}^N : \varphi = -\infty\}$ have Lebesgue measure zero, there is a set of positive Lebesgue measure in $B(z^0, r)$ such that $\varphi > -\infty$ and $u > 0$ on that set.

We define the function g by

$$g(z) = \alpha u(z) + (1 - \alpha)\varphi(z)$$

where $0 < \alpha < 1$. By choosing α sufficiently close to 1 we have that $g > 0$ on a set of positive Lebesgue measure in $B(z^0, r)$. We also have that

$$\sup_{\zeta \in \overline{D}_u} g(\zeta) \leq 0.$$

Since $g \in L(\mathbb{C}^N)$ there exists a sequence $\{P_j\}$ of complex polynomials ($\deg P_j \le j$) on $\mathbb{C}^N$ such that

$$g(z) = \left(\limsup_{j\to\infty} \frac{1}{\deg P_j} \log|P_j(z)|\right)^* .$$

(Here the asterisk denotes upper regularization; i.e. $\psi^*(z) = \limsup_{z'\to z} \psi(z')$.) Hence

$$\sup_{\zeta\in\widehat{\overline{D}}_u} g(\zeta) \le 0$$

except possibly for a pluripolar set. This implies that there is a point z' in $B(z^0, r)$ such that $z' \notin \widehat{\overline{D}}_u$.

Thus we have shown that in an arbitrary neighbourhood of each boundary point of D_u there is a point which does not lie in the polynomially convex hull of $\overline{D}_u$. It follows that D_u is of type H^∞. □

Theorem 10. *Let $h \in H_+(\mathbb{C}^N \times \mathbb{C})$ such that E_h is bounded. If E_h is of type H^∞, then D_{Th} is of type H^∞.*

Proof. This follows from Theorem 1, Theorem 4, and Theorem 9. □

Theorem 11. *There exists a function $u \in L^+(\mathbb{C}^2)$ such that $\overline{D}_u$ is polynomially convex and $\operatorname{int}\overline{D}_u = D_u$ but $\overline{E}_{Su}$ is not polynomially convex.*

Proof. Let h be the function in the proof of Theorem 3. By applying the mapping T to $h/3$ we get

$$u(z) = \log\left(\exp\left(\sum_{j=1}^{\infty} \alpha_j \max(\log|z_1 - a_j z_2|, 0)\right) + \max(|z_1|, |z_2|, 1)\right) - \log 3$$

where $(a_j)_{j=1}^\infty$ is dense in the unit circle and $(\alpha_j)_{j=1}^\infty$ is a sequence of positive numbers with $\sum_{j=1}^\infty \alpha_j = 1$. We prove that $\overline{D}_u$ is polynomially convex. Observe that $\overline{D}_u$ is the intersection of a decreasing sequence of compact sets $\overline{D}_{u_m}$; that is,

$$\overline{D}_u = \bigcap_{m=1}^{\infty} \overline{D}_{u_m}$$

for

$$u_m(z) = \log\left(\exp\left(\sum_{j=1}^{m} \alpha_j \max(\log|z_1 - a_j z_2|, 0)\right) + \max(|z_1|, |z_2|, 1)\right) - \log 3$$

where $(a_j)_{j=1}^\infty$ and $(\alpha_j)_{j=1}^\infty$ are the same sequences as for u. It is enough to prove that each $\overline{D}_{u_m}$ is polynomially convex.

Let z^0 be an arbitrary point in $\widehat{\overline{D}}_{u_m}$. The function u_m is continuous and therefore $\overline{D}_{u_m}$ is contained in the set

$$A = \{z \in \mathbb{C}^2 : u_m(z) \le 0\}.$$

By the continuity of u_m, the point z^0 cannot lie in $\mathbb{C}^2\setminus A$. Thus we can assume that $u_m(z^0) = 0$.

Let U be an arbitrary neighbourhood of z^0 and let λ be a complex number with $|\lambda|$ sufficiently close to 1, $|\lambda| < 1$, so that $\lambda z^0 \in U$. Then $u_m(\lambda z^0) < 0$ which means that $z^0 \in \overline{D}_{u_m}$. Hence $\overline{D}_{u_m}$ is polynomially convex.

By Lemma 4, we see that $\operatorname{int}\overline{D}_u = D_u$ and it is clear, by Theorem 3, that $\overline{E}_{h/3}$ is not polynomially convex.

It remains to show that $u \in L^+(\mathbb{C}^2)$. This follows from the observation that

$$u(z) \geq \log(\max(|z_1|, |z_2|, 1)) - \log 3 \geq \log\left(\frac{(|z_1|^2 + |z_2|^2)^{\frac{1}{2}} + 1}{2\sqrt{2}}\right) - \log 3. \quad \square$$

Theorem 12. *There exists a function $h \in H_+(\mathbb{C}^2)$ such that $\overline{E}_h$ is a bounded polynomially convex set and $\operatorname{int}\overline{E}_h = E_h$ but $\overline{D}_{Th}$ is not polynomially convex.*

Proof. Suppose that we already have constructed a continuous function $u \in L^+(\mathbb{C})$ with the properties that $u = 0$ on the closed unit disk $\overline{D(0, 1)}$ in $\mathbb{C}$ and $u < 0$ on a bounded set consisting of open disks in $\mathbb{C}\backslash\overline{D(0, 1)}$ with centers clustering at the boundary of $D(0, 1)$. This means that $\partial D(0, 1)$ is contained in $\overline{D}_u$. Consequently, $\overline{D(0, 1)}$ is contained in $\widehat{\overline{D}}_u$ and since $D(0, 1)$ is not contained in $\overline{D}_u$, we see that $\overline{D}_u$ is not polynomially convex.

By applying the mapping S to u we obtain a function $h = Su$ in $H_+(\mathbb{C}^2)$ and it follows from the definition of S that h is continuous. E_h is bounded by Lemma 1. We have that $E_h = \operatorname{int}\widehat{\overline{E}}_h$ (see [6]) and this gives that $\overline{E}_h$ is polynomially convex. (Since the dimension is two, this could also be seen, alternatively, from Theorem 1 and the remark after Theorem 3.) Obviously, we have $\operatorname{int}\overline{E}_h = E_h$. Hence the theorem follows if we can construct such a function u. In order to do this we will inductively define a sequence $\{u_j\}_{j=1}^{\infty}$ of continuous functions in $L(\mathbb{C})$.

First recall that the L-extremal function of a set F in $\mathbb{C}^N$ is defined by

$$V_F(z) = \sup\{u(z) : u \in L(\mathbb{C}^N), u \leq 0 \text{ on } F\}, \qquad z \in \mathbb{C}^N.$$

For $\overline{D(z^0, r)} = \{z \in \mathbb{C} : |z - z^0| \leq r\}$ we have

$$V_{\overline{D(z^0, r)}}(z) = \max\left(\log\frac{|z - z^0|}{r}, 0\right).$$

Let $M > 3$ be a fixed number. Let $u_1 = V_{\overline{D(0,1)}}$ and let δ_1 be a number such that $0 < \delta_1 < 1$ and $0 < u_1 < 2^{-3}$ on $D(0, 1 + \delta_1)\backslash\overline{D(0, 1)}$. Furthermore let $\overline{D(c_1, r_1)}$ $(r_1 > 0)$ be a closed disk in $D(0, 1 + \delta_1)\backslash\overline{D(0, 1)}$.

Assume that we have, for $j \in \mathbb{Z}^+$, numbers $\delta_1 > \delta_2 > \cdots > \delta_j > 0$, pairwise disjoint closed disks $\overline{D(c_k, r_k)}$ $(r_k > 0)$ in $D(0, 1 + \delta_k)\backslash\overline{D(0, 1)}$ $(1 \leq k \leq j)$ and a continuous function $u_j \in L(\mathbb{C})$ satisfying

(i) $u_j = 0$ on $\overline{D(0, 1)}$,

(ii) $0 < u_j < 2^{-(j+2)}$ on $D(0, 1 + \delta_j)\backslash\overline{D(0, 1)}$,

(iii) $-\frac{3}{2^{k+2}} + 4^{-(j+2)} \leq u_j \leq -\frac{1}{2^{k+2}} - 4^{-(j+2)}$ on $\overline{D(c_k, r_k)}$ $(1 \leq k < j)$,

(iv) $|u_j - u_1| < 1 - 2^{-j}$ on $\mathbb{C}$.

We claim that under these assumptions we can find a number δ_{j+1} such that $0 < \delta_{j+1} < \delta_j$, a closed disk $\overline{D(c_{j+1}, r_{j+1})}$ $(r_{j+1} > 0)$ in $D(0, 1+\delta_{j+1})\backslash\overline{D(0, 1)}$ and a continuous function $u_{j+1} \in L(\mathbb{C})$ satisfying

(i′) $u_{j+1} = 0$ on $\overline{D(0, 1)}$,

(ii′) $0 < u_{j+1} < 2^{-(j+3)}$ on $D(0, 1+\delta_{j+1})\backslash\overline{D(0,1)}$,
(iii′) $-\frac{3}{2^{k+2}} + 4^{-(j+3)} \le u_{j+1} \le -\frac{1}{2^{k+2}} - 4^{-(j+3)}$ on $\overline{D(c_k, r_k)}$ $(1 \le k < j+1)$,
(iv′) $|u_j - u_{j+1}| < 2^{-(j+1)}$ on $\mathbb{C}$.

Observe that (iv) and (iv′) imply that $|u_{j+1} - u_1| < 1 - 2^{-(j+1)}$ on $\mathbb{C}$.

We now prove the claim.

Choose numbers $c_{j+1} \in \mathbb{C}$, $\tilde{r}_{j+1} > 0$ and $\tilde{\delta}_{j+1} > 0$ such that

$$(\dagger) \qquad \overline{D(c_{j+1}, \tilde{r}_{j+1})} \subset D(0, 1+\delta_j)\backslash\overline{D(0, 1+\tilde{\delta}_{j+1})}.$$

For a number $\varepsilon > 0$ we set

$$u_{j+1}(z) = \varepsilon u_1(z) + v_{j+1}(z)$$

where

$$\begin{aligned} v_{j+1}(z) &:= \frac{1-\varepsilon}{1+\varepsilon}(\max(u_j(z) + \varepsilon\log|z - c_{j+1}| - M\varepsilon, 0)) \quad \text{on } D(0, 1+\tilde{\delta}_{j+1}) \\ &:= \frac{1-\varepsilon}{1+\varepsilon}(\max(u_j(z) + \varepsilon\log|z - c_{j+1}| - M\varepsilon, -2^{-(j+2)})) \\ &\qquad\qquad \text{on } D(c_{j+1}, \tilde{r}_{j+1}) \\ &:= \frac{1-\varepsilon}{1+\varepsilon}(u_j(z) + \varepsilon\log|z - c_{j+1}| - M\varepsilon) \\ &\qquad\qquad \text{on } \mathbb{C}\backslash(D(0, 1+\tilde{\delta}_{j+1}) \cup D(c_{j+1}, \tilde{r}_{j+1})). \end{aligned}$$

If ε is chosen small enough, say $\varepsilon \le \varepsilon_1$ for some $\varepsilon_1 > 0$, then $u_j(z) > \varepsilon(M - \log|z - c_{j+1}|)$ for all z with $|z| = 1 + \tilde{\delta}_{j+1}$ by (ii) and (†). If ε is chosen small enough, say $\varepsilon < \varepsilon_2$, then $u_j(z) > \varepsilon(M - \log\tilde{r}_{j+1})$ for all z with $|z - c_{j+1}| = \tilde{r}_{j+1}$. This means that v_{j+1} is a well-defined subharmonic function in $L(\mathbb{C})$ if ε is chosen sufficiently small.

We verify that u_{j+1} has properties (i′)–(iv′).

(i′) Since $M > \log|z - c_{j+1}|$, we have that $v_{j+1} = 0$ on $\overline{D(0,1)}$ which implies that $u_{j+1} = 0$ on $\overline{D(0,1)}$.

(ii′) Since $v_{j+1} \ge 0$ on $D(0, 1+\tilde{\delta}_{j+1})$ and $u_1 > 0$ on $D(0, 1+\tilde{\delta}_{j+1})\backslash\overline{D(0,1)}$, we can, by continuity, find a number $\delta_{j+1} < \tilde{\delta}_{j+1}$ so that (ii′) is fulfilled.

(iii′) If ε is chosen small enough, $\varepsilon < \varepsilon_3 = 1/(4^{-(j+3)}(M - \log\tilde{r}_{j+1}))$, then (iii′) holds for $1 \le k \le j$.

We choose $r_{j+1} \le \tilde{r}_{j+1}$ so that $v_{j+1} = \frac{1-\varepsilon}{1+\varepsilon}(-2^{-(j+2)})$ on $\overline{D_{j+1}(c_{j+1}, r_{j+1})}$. Then (iii′) holds for $k = j+1$ if ε is chosen small enough, say $\varepsilon < \varepsilon_4$.

(iv′) For points not in $\overline{D(c_{j+1}, r_{j+1})}$ we have

$$\begin{aligned} |u_j - u_{j+1}| &\le \varepsilon|u_j - u_1| + (1-\varepsilon)|u_j - v_{j+1}/(1-\varepsilon)| \\ &\le \varepsilon|u_j - u_1| + \frac{\varepsilon(1-\varepsilon)}{1+\varepsilon}|u_j - \log|z - c_{j+1}| + M| \end{aligned}$$

By (iv), this is less than

$$\varepsilon(2^{-1} + M + \log 3 - \log r_{j+1}) \quad \text{on } D(0,3)\backslash\overline{D(c_{j+1}, r_{j+1})}$$

and less than

$$\varepsilon(2^{-1} + M + 1) \quad \text{on } \mathbb{C}\backslash\overline{D(0,3)}.$$

If ε is small enough, say $\varepsilon \leq \varepsilon_5$, we see that $|u_j - u_{j+1}| \leq 2^{-(j+1)}$ on $\mathbb{C}$. For points in $\overline{D(c_{j+1}, r_{j+1})}$ we have $u_j < 2^{-(j+2)}$ by (ii). Since $v_{j+1} = \frac{1-\varepsilon}{1+\varepsilon}(-2^{-(j+2)})$, we get $|u_j - u_{j+1}| \leq (2-\varepsilon)2^{-(j+2)} < 2^{-(j+1)}$.

We define

$$\tilde{u}(z) = \lim_{j\to\infty} u_j(z)$$

where $\varepsilon > 0$ has been chosen such that $\varepsilon < \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$.

Then $\{u_j\}_{j=1}^{\infty}$ is a uniformly convergent sequence and hence $\tilde{u}$ is a continuous function in $L(\mathbb{C})$.

In order to get a continuous function $u \in L^+(\mathbb{C})$ we can, since $D_{\tilde{u}}$ is bounded, use Lemma 3 to $\tilde{u}$.

By (iii) and (iii$'$) we see that $u < 0$ on $\bigcup_{j=1}^{\infty} D_j(c_j, r_j)$.

In the construction above we notice that the sequence of centers $(c_j)_{j=1}^{\infty}$ can be chosen so that each point in $\partial D(0, 1)$ is an accumulation point of $(c_j)_{j=1}^{\infty}$. This concludes the proof. □

Acknowledgment

The author wishes to thank Urban Cegrell (University of Umeå) and Sławomir Kołodziej (Jagiellonian University, Kraków) for helpful discussions.

References

1. U. Cegrell, *On the spectrum of $A(\Omega)$ and $H^\infty(\Omega)$*, Ann. Polon. Math. **58** (1993), 193–199.
2. B. Josefson, *On the equivalence between locally and globally polar sets for plurisubharmonic functions in $\mathbb{C}^n$*, Ark. för Mat. **16** (1978), 109–115.
3. J. Siciak, *Extremal plurisubharmonic functions in $\mathbb{C}^n$*, Ann. Polon. Math. **39** (1981), 175–211.
4. ——, *Extremal plurisubharmonic functions and capacities in $\mathbb{C}^n$*, Sophia Kokyuroku in Math. 14, Dept. of Math., Sophia Univ., Tokyo, 1982.
5. ——, *Circled domains of holomorphy of type H^∞*, Bull. Soc. Sci. Lett. Lódz. **34** (1984/1985), 1–20.
6. ——, *Balanced domains of holomorphy of type H^∞*, Mat. Vesnik **37** (1985), 134–144.

Department of Mathematics, University of Umeå, S-901 87 Umeå, Sweden
E-mail address: ulfb@mathdept.umu.se