

LOCAL UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT

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ABSTRACT. We prove local uniqueness of a domain D entering the conductivity equation $\operatorname{div}((1 + \chi(D))\nabla u) = 0$ in a bounded planar domain Ω given the Cauchy data for u on a part of $\partial\Omega$. The main assumption is that ∇u has zero index on $\partial\Omega$ which is easy to guarantee by choosing special boundary data for u . To achieve our goals we study index of critical points of u on $\partial\Omega$.

We are interested in finding a simply connected domain D entering the conductivity equation $\operatorname{div}((1 + \chi(D))\nabla u) = 0$ from one exterior boundary measurement (the Cauchy data for u on a piece of boundary of a given planar domain Ω containing D). This problem is fundamental for electrical and seismical prospection, but not much is known about it at present. The only available uniqueness results are for convex polyhedrons [FI] or cylinders [IP]. There are more general results when D is assumed to be close to a given domain D_0 . In [CV], a local uniqueness theorem is proved, under the technical assumption which involves a condition on the interior values of ∇u . In [BFI], [P], under assumptions on the boundary data which guarantee that ∇u has zero index on $\partial\Omega$, and under the restrictive assumption that ∂D_0 , ∂D are (piecewise) analytic, local uniqueness results have been obtained.

In this paper we remove this last assumption and we show that solutions $D \in C^{1+\lambda}$ of the inverse problem are isolated.

Notation. $B(a; r)$ is the ball of radius r centered at a ; ν is the (exterior) unit normal to the boundary, $\partial/\partial z = 1/2(\partial/\partial x - i\partial/\partial y)$, $\partial/\partial \bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$, $\|\cdot\|_{k+\lambda}(\Omega)$ is the norm in the Hölder space $C^{k+\lambda}(\Omega)$ of functions whose partial derivatives up to order k are Hölder continuous of exponent λ , $0 < \lambda < 1$. C denote positive constants which depend only on Ω , D_0 and M .

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1. MAIN RESULTS

Let Ω be a bounded domain in \mathbb{R}^2 with $C^{2+\lambda}$ -boundary $\partial\Omega$, $0 < \lambda < 1$. For a domain D , $\overline{D} \subset \Omega$, we consider the refraction problem

$$(1) \quad \operatorname{div}((1 + \chi(D))\nabla u) = 0 \text{ in } \Omega, \quad \partial u / \partial \nu = g \text{ on } \partial\Omega$$

where $g \in C^{1+\lambda}(\partial\Omega)$ and $\int_{\partial\Omega} g = 0$. When $\partial D \in C^{1+\lambda}$, this problem has a unique solution $u \in H^{(1)}(\Omega)$ such that $u \in C^{1+\lambda}(\overline{D}^e)$, $u \in C^{1+\lambda}(\overline{D})$ and $\int_{\partial\Omega} u = 0$ (see [DEF, Appendix]). The differential equation (1) is equivalent to the following relations:

$$(1_d) \quad \Delta u^e = 0 \text{ in } D^e, \quad \Delta u^i = 0 \text{ in } D, \quad u^e = u^i, \quad \partial u^e / \partial \nu = 2\partial u^i / \partial \nu \text{ on } \partial D$$

where $D^e = \Omega \setminus \overline{D}$, $u^e = u$ on D^e , $u^i = u$ on D .

The inverse conductivity problem with one boundary measurement is to find D given

$$(2) \quad u = h \quad \text{on } \Gamma \subset \partial\Omega.$$

To formulate the main result of this paper we consider a family D_ψ of domains close to a domain $D_0 \in C^{1+\lambda}$ which contains the origin 0 in \mathbb{R}^2 . Let $z_0(t)$ be the conformal mapping of the unit disk B onto D_0 normalized by the conditions

$$(3) \quad z_0(0) = 0, \quad z'_0(0) > 0.$$

Consider the family Ψ_M of function ψ analytic in B and satisfying the following conditions:

$$(4) \quad \psi(0) = 0, \quad \operatorname{Im} \psi'(0) = 0, \quad |\psi|_{1+\lambda}(B) \leq M$$

and such that $z_0 + \psi$ is a conformal mapping. Let D_ψ be the image of B under this mapping.

Let $\partial\Omega$ be the union of two disjoint arcs Γ_1, Γ_2 and assume that

$$(5) \quad 0 \leq g \text{ on } \Gamma_1, \quad g \leq 0 \text{ on } \Gamma_2, \quad g \neq 0.$$

Theorem 1. *If the condition (5) is satisfied, then for a domain D_0 there is a number $\varepsilon_0(M)$ such that if $|\psi|_0(B) < \varepsilon_0$ and D_ψ, D_0 are solutions to the same inverse conductivity problem (1), (2), then $\psi = 0$.*

In other words, $C^{1+\lambda}$ -solutions D_ψ are isolated. By applying a standard compactness argument, we can claim that the number of solutions is finite.

Corollary 1*. *Let Ψ_M^* be a subfamily of Ψ_M which is closed in $C^{1+\lambda}(B)$. Then the number of solutions D_ψ with $\psi \in \Psi_M^*$ is finite.*

Proof. We observe that by Theorem 1 for any $\psi \in \Psi_M^*$ there is $\varepsilon(\psi, M)$ such that in the ε -neighborhood of ψ in $C(B)$ there is no other solution of the inverse problem. These ε -neighborhoods form an open covering of Ψ_M^* in $C(B)$; since Ψ_M^* is compact in $C(B)$ we can find a finite subcovering. Since in any neighborhood there is no more than one solution, the number of all solutions in Ψ_M^* is finite.

2. INDEX OF ∇u^ε

Theorem 2. Let $u \in H^{1,2}(\Omega)$ be a nonconstant (weak) solution to the equation $\operatorname{div}(a\nabla u) = 0$, $0 < \varepsilon < a$, $a \in L_\infty(\Omega)$, in Ω .

Then there is a quasiconformal mapping g of Ω onto B and a harmonic function h on B such that $u = h \circ g$ on Ω .

Observe, as an immediate consequence, that u cannot vanish of infinite order at any point of Ω .

We call $z^0 \in \Omega$ a *geometric critical point* of u if in a neighborhood of z^0 the level set $\{u = u(z^0)\}$ consists of $N+1$, $N > 0$, simple arcs whose pairwise intersection is $\{z^0\}$. We call N the *geometric index* of u at z^0 . If z^0 is not a geometrical critical point, we let $N = 0$.

Theorem 3. Let $g \in C^{1+\lambda}(\partial\Omega)$. Assume that $\partial\Omega$ is the union of two disjoint arcs Γ_1, Γ_2 and let $g \geq 0$ on Γ_1 and $g \leq 0$ on Γ_2 . Let u be a solution to (1).

Then u has no geometrical critical points in Ω .

The proof of Theorems 2, 3 (in a more general situation) is given in [AM].

Now we return to the equation (1) and relate the geometrical index at a point of $\partial D \in C^{1+\lambda}$ to the traditional index. Theorem 4 below is stated in the general case when $D \subset \Omega \subset \mathbb{R}^n$, $n \geq 2$.

We assume that $0 \in \partial D$ and D near 0 is given as $\{x_n > f(x_1, \dots, x_{n-1})\}$, $f \in C^{1+\lambda}$, $f(0) = |\nabla f(0)| = 0$. Let χ^+ be the characteristic function of the half-space $\{x_n > 0\}$.

Theorem 4. Let u be a nonconstant solution to the equation (1) near $0 \in \partial D$.

Then there is a homogeneous function H_N of degree N such that

$$(6) \quad \operatorname{div}((1 + \chi^+)\nabla H_N) = 0 \quad \text{in } \mathbb{R}^n$$

and

$$(7) \quad u(x) - u(0) = H_N(x) + O(|x|^{N+\lambda}).$$

Before proving we discuss this result and obtain some corollaries.

Any homogeneous solution to the equation (6) admits the representation

$$H_N = (1 - \tfrac{1}{2}\chi^+)O_N + E_N$$

where O_N, E_N are homogeneous harmonic polynomials of degree N , O_N is odd and E_N is even with respect to x_N . In particular, when $n = 2$, we have

$$(8) \quad H_N(r \cos \theta, r \sin \theta) = (1 - \tfrac{1}{2}\chi^+)Ar^N \cos N\theta + Br^N \sin N\theta$$

with $A^2 + B^2 \neq 0$.

Corollary 5. Let u be a solution to the Neumann problem (1) where g satisfies the condition (5); then $|\nabla u^\varepsilon| > 0$, $|\nabla u^i| > 0$ on ∂D and $\operatorname{ind}(\nabla u^\varepsilon, \partial D)$ and $\operatorname{ind}(\nabla u^i, \partial D)$ are 0.

Proof of Corollary 5. Let $z^0 \in \partial D$. We may assume $z^0 = 0$. By Theorem 2 the point 0 is not a zero of infinite order; by Theorem 4 we have the representation (7). If $N > 1$, then from (7) and (8) it follows that the set $\{u - u(0) > 0\} \cap B_\varepsilon$ for small ε consists of at least 2 connected components. Therefore the geometric

index $I(u, z^0) \geq 1$ which contradicts Theorem 3. So $N = 1$ and $|\nabla u^e(z^0)| > 0$ as well as $|\nabla u^i|$.

Now, $\text{ind}(\nabla u^i, \partial D)$ is well defined. It is known that it is equal to the number of zeros of ∇u^i inside D which is zero by Theorem 3. We have $\nabla u^i = u_\nu^i \nu + u_\tau^i \tau$ where τ is the (anticlockwise) tangential direction on ∂D . A similar representation is valid for ∇u^e . Due to the refraction conditions (1_d) we have $2u_\nu^i = u_\nu^e$, $u_\tau^i = u_\tau^e$. To show that both interior and exterior gradients have the same index on ∂D we introduce the homotopy $v(t) = (1+t)u_\nu^i \nu + u_\tau^e \tau$ between these vector fields. We have $v(0) = \nabla u^i$, $v(1) = \nabla u^e$ and $|v(t)| > 0$ on ∂D when $0 \leq t \leq 1$. The index is a homotopic invariant so $\text{ind}(v(1), \partial D) = \text{ind}(v(0), \partial D) = 0$.

The proof is complete.

Proof of Theorem 4. Our proof will be modeled on arguments originally due to Bers [Be]. By the change of variables $y_j = x_j$, $j < n$, $y_n = x_n - f(x_1, \dots, x_{n-1})$ we can reduce the case to the equation

$$(1') \quad \text{div}((1 + \chi^+)a\nabla u) = 0$$

where a is a positive symmetric matrix with C^λ -coefficients and $a(0) = I$. Let $x^* = (x_1, \dots, x_{n-1}, -x_n)$. We shall use the standard fundamental solution $E(x - y)$ to the Laplace equation which is

$$-\frac{1}{2\pi} \log|x - y| \text{ when } n = 2 \quad \text{and} \quad \frac{1}{(n-2)\omega_n} |x - y|^{2-n} \text{ when } n > 2.$$

Denoting $\Delta_+ = \text{div}((1 + \chi^+)\nabla)$ we observe that a fundamental solution E_+ for this operator is given by $E_+(x, y) = \Theta E(x, y)$ where we define

$$\Theta f(x, y) = \begin{cases} \frac{1}{2}f(x, y) + \frac{1}{6}f(x, y^*) & \text{when } 0 < x_n, 0 < y_n, \\ \frac{2}{3}f(x, y) & \text{when } 0 < x_n, y_n < 0 \text{ or } x_n < 0, 0 < y_n, \\ f(x, y) - \frac{1}{3}f(x, y^*) & \text{when } x_n < 0, y_n < 0. \end{cases}$$

The Taylor series expansion of $E(x, y)$ gives

$$E(x, y) = E(0, y) + \sum_{j=1}^{+\infty} Q_j(x, y)$$

when $0 \leq |x| < |y|$. Here $Q_j(x, y)$ are homogeneous of degree j in x and of degree $2 - n - j$ in y . We have

$$Q_j(x, y) = \begin{cases} |x|^j |y|^{-j} j^{-1} T_j(x \cdot y / (|x| |y|)) & \text{when } n = 2, \\ |x|^j |y|^{2-n-j} C_j^{n/2-1}(x \cdot y / (|x| |y|)) & \text{when } n \geq 3. \end{cases}$$

Here T_j are the Chebyshev polynomials and $C_j^{n/2-1}$ are Gegenbauer (ultraspherical) polynomial; see [E, formulas (30), 10.11, (29), 10.9]. We obtain

$$E_+(x, y) = \frac{2}{3} E(0, y) + \sum_{j \geq 1} \Theta Q_j(x, y)$$

when $|x| < |y|$. From [E, formula (7), 10.18], and using harmonicity and homogeneity, we have

$$|\nabla_y Q_j(x, y)| \leq C_n j^{n-3} |x|^j |y|^{1-n-j}$$

and, consequently,

$$(9) \quad |\nabla_y \Theta Q_j(x, y)| \leq C_n j^{n-3} |x|^j |y|^{1-n-j}.$$

Lemma 6. *Let a vector function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy the condition*

$$(10) \quad |F|(x) \leq |x|^{N+\varepsilon-1}$$

for some $\varepsilon > 0$, $N = 1, 2, \dots$. Define

$$\psi(x) = - \int_{|y| < R} \nabla_y E_+(x, y) \cdot F(y) dy.$$

Then

$$(11) \quad \psi(x) = p_N(x) + O(|x|^{N+\varepsilon})$$

as $|x| \rightarrow 0$ where $p_N(x)$ is the sum of homogeneous solutions H_k to $\Delta_+ H_k = 0$ of degree $k \leq N$ (a Δ_+ -harmonic polynomial of degree N).

Proof. Let us set

$$\begin{aligned} E_+^N(x, y) &= \frac{2}{3} E(0, y) + \sum_{j \geq 1} \Theta Q_j(x, y), \\ R^N(x, y) &= E_+(x, y) - E_+^N(x, y). \end{aligned}$$

Define

$$p_N(x) = - \int_{|y| < R} \nabla_y E_+^N(x, y) \cdot F(y) dy.$$

To complete the proof it is sufficient to show that

$$(12) \quad \psi(x) - p_N(x) = O(|x|^{N+\varepsilon}).$$

We will do it by splitting $\psi - p_N$ into the sum of the three integrals

$$\begin{aligned} I_1 &= - \int_{|y| < 2|x|} \nabla_y E_+(x, y) \cdot F(y) dy, \\ I_2 &= \int_{|y| < 2|x|} \nabla_y E_+^N(x, y) \cdot F(y) dy, \\ I_3 &= - \int_{2|x| < |y| < R} \sum_{N+1 \leq j} \nabla_y (\Theta Q_j(x, y)) \cdot F(y) dy \end{aligned}$$

and bounding I_j by $C|x|^{N+\varepsilon}$.

We have $|\nabla_y E_+| \leq C|x-y|^{1-n}$, so by using the hypothesis (10) and Hölder's inequality we obtain

$$|I_1| \leq C \left(\int_{|y| < 2|x|} |x-y|^{(1-n)} dy \right) |x|^{\frac{n}{2}+N+\varepsilon}.$$

Observe that on the integration domain $|x-y| \leq |x| + |y| \leq 3|x|$, so

$$|I_1| \leq C \int_{0 < r < 3|x|} dr |x|^{N+\varepsilon-1} \leq C|x|^{N+\varepsilon}.$$

By using (9) and homogeneity we get

$$\begin{aligned}
 |I_2| &\leq \int_{|y|<2|x|} |\nabla_y \Theta_{\frac{2}{3}} E(0, y)| |F(y)| dy \\
 &\quad + C \sum_{1 \leq j \leq N} \int_{|y|<2|x|} |\nabla_y \Theta Q_j(x, y)| |F(y)| dy \\
 &\leq C \sum_{j \leq N} j^{n-3} |x|^j \int_{|y|<2|x|} |y|^{1-n-j} |F(y)| dy \\
 &\leq C \sum_{j \leq N} j^{n-3} |x|^j \int_{|y|<2|x|} |y|^{N-n-j+\varepsilon} dy \leq CN^n |x|^{N+\varepsilon}.
 \end{aligned}$$

Finally, we bound

$$\begin{aligned}
 |I_3| &\leq C \sum_{N+1 \leq j} j^{n-3} |x|^j \int_{2|x|<|y|<R} |y|^{1-n-j} |F(y)| dy \\
 &\leq C \sum_{N+1 \leq j} j^{n-3} |x|^j \int_{2|x|<|y|} |y|^{N-n-j+\varepsilon} dy \\
 &\leq C |x|^{N+\varepsilon} \sum_{N+1 \leq j} j^{n-3} (j - N - \varepsilon)^{-1} 2^{-j+N+\varepsilon} \leq C |x|^{N+\varepsilon}.
 \end{aligned}$$

Lemma 7. Suppose that $\Delta_+ u = \operatorname{div}(F)$ in $B(0; R)$ with F satisfying the condition (10).

Then u admits the representation (11).

Proof. We extend F outside $B(0; R)$ as zero. Consider the potential $\psi(x)$ from Lemma 6; then $v = u - \psi$ satisfies the homogeneous equation $\Delta_+ v = 0$ in $B(0; R)$.

By using reflections we conclude that $v = (1 - \chi^+/2)h^o + h^e$ where h^o , h^e are harmonic functions, h^o is odd and h^e is even with respect to x_n . Expanding h^o , h^e in Taylor series around 0 we represent v as the sum of the series of homogeneous Δ_+ -harmonic polynomials H_k which is convergent in $B(0; R/2)$. In particular $v = H_N^* + O(|x|^{N+1})$, where H_N^* is the sum of H_k over $k \leq N$. By using Lemma 6 we complete the proof.

Lemma 8. Let u be a solution to the equation (1') in $B(0; R)$. If for some $A > 0$ we have $|u(x)| \leq |x|^A$, then $|\nabla u(x)| \leq C|x|^{A-1}$.

Proof. The proof is an elementary consequence of the Caccioppoli's inequality

$$\int_{B(0; r)} |\nabla u|^2 \leq Cr^{-2} \int_{B(0; 2r)} u^2$$

and of the piecewise $C^{1+\lambda}$ -regularity estimate of [DEF] which, in particular, gives us

$$\sup_{B(0; r)} |\nabla u| \leq Cr^{-n/2} \left(\int_{B(0; 2r)} |\nabla u|^2 + u^2 \right)^{1/2}.$$

Proof of Theorem 4 (end). If u is a nonconstant solution, then there is an integer N such that

$$(14) \quad |u(x) - u(0)| \leq C|x|^N$$

and

$$(15) \quad \limsup |x|^{-N-1} |u(x) - u(0)| = +\infty$$

as $x \rightarrow 0$.

Let $F = (1 + \chi^+)(I - a)\nabla u$. Apparently $\Delta_+ u = \operatorname{div} F$. Since $a \in C^\lambda$ near 0 we have

$$|F(x)| \leq C|x|r^{N-1+\lambda}$$

by Lemma 8. Therefore, by Lemma 7, we have $u(x) - u(0) = p_N(x) + O(|x|^{N+\lambda})$ where p_N is a Δ_+ -harmonic polynomial of degree $\leq N$. By (14) the polynomial p_N is either identically zero or homogeneous of degree N . In the latter case the proof is complete. To finish the proof we show that the first case is impossible. If not we can assume that (14) is valid for $N + \alpha$, $0 < \alpha < \lambda$, α is irrational, and by repeating the above procedure we can conclude that $u(x) - u(0)$ is $O(|x|^{N+k\alpha})$ when $k \leq [1/\alpha]$ and $p_{N+1}(x) + O(|x|^{N+k\alpha})$ when $k = [1/\alpha] + 1$, so $u(x) - u(0)$ is $O(|x|^{N+1})$ which contradicts (15).

The proof is complete.

3. PROOF OF THE MAIN THEOREM

We will develop some arguments already used in [C] and [P].

Let D_0, D_ψ be two solutions to the inverse problem (1), (2). Let u_0, u_ψ be the corresponding solutions to the direct problem (1). We will pick ε_0 so small that $\Omega \setminus (\overline{D_0} \cup D_\psi)$ and $\Omega \setminus (\overline{D_0} \cap \overline{D_\psi})$ are connected and $\partial(D_0 \cap D_\psi)$ satisfies a Lipschitz condition with a prescribed constant. Since $u_0^e = u_\psi^e$ there, henceforth the following definition is correct: $u^e = u_0^e$ on $\partial/\overline{D_0}$ and $u^e = u_\psi^e$ on $\Omega/\overline{D_\psi}$. Moreover u^e is harmonic outside $\overline{D_0} \cap \overline{D_\psi}$. From Schauder estimates for transmission problems [DEF, Appendix] and from the a priori constraints $\partial D_0 \in C^{1+\lambda}$, $|\psi|_{1+\lambda} \leq M$, we conclude that $|u^e|_{1+\lambda}(\Omega \setminus (D_0 \cap D_\psi)) \leq C$.

From the condition (1) on g and from Green's formula it follows that $\int_\gamma \partial u^e / \partial N d\gamma = 0$ for any cycle γ in $\Omega \setminus (\overline{D_0} \cap \overline{D_\psi})$, so there is a harmonic conjugate v^e to u^e . We will normalize it by the condition $v^e(z^o) = 0$ for some $z^o \in \partial\Omega$. Let $U^e = u^e + iv^e$. From the estimate in the previous item we have

$$(16) \quad |U^e|_{1+\lambda}(\Omega \setminus (\overline{D_0} \cap \overline{D_\psi})) \leq C.$$

We will show that the inverse problem with respect to the unknown domain D_ψ is equivalent to the following nonlinear boundary value problem for analytic functions ϕ, ψ in the unit disk B :

$$(17) \quad A(\phi, \psi)(t) = B\psi(t) \quad \text{when } |t| = 1$$

where $A(\phi, \psi) = \phi - 3a\psi - \overline{a\psi}$, $a(t) = u_z^e(z_0(t))$ and $B\psi = 3B_1\psi + \overline{B_1\psi}$, $B_1\psi = U^e(z_0 + \psi) - U^e(z_0) - u_z^e(z_0)\psi$. Here A is considered as an operator from $\Phi \times \Psi$ into $C^\lambda(\partial B)$. We define Φ as the space of functions $\phi \in C^\lambda(\overline{B})$ analytic in B and Ψ as the space of functions $\psi \in \Phi$ satisfying the conditions (4). The operator A is continuous from $\Phi \times \Psi$ onto its range $\mathfrak{R} \subset C^\lambda(\partial B)$.

To derive (17) we need the following form of the Cauchy-Riemann system for the real and imaginary parts u, v of a complex analytic function

$$(CR) \quad \partial u / \partial \tau = \partial v / \partial \nu, \quad \partial v / \partial \tau = -\partial u / \partial \nu$$

where τ is the unit tangent to ∂D such that the pair (τ, ν) is oriented as the coordinate vectors of the x - and y -axes.

Let D_ψ be a solution to the inverse conductivity problem. From the conditions (CR) and from the boundary conditions (1_d) we conclude that $\partial v^e / \partial \tau = 2\partial v^i / \partial \tau$. Therefore, $v^e = 2v^i + C$. This relation and the continuity of u yield

$$U^e + \overline{U^e} = U^i + \overline{U^i}, \quad U^e - \overline{U^e} = 2(U^i - \overline{U^i}) + iC, \quad C \in \mathbb{R}, \text{ on } \partial D.$$

Substituting U^i from the first equality into the second one and letting $z = z(t)$, $\phi(t) = 4U^i(z(t)) + iC$ we obtain the boundary condition

$$(17_a) \quad 3U^e(z(t)) + \overline{U^e(z(t))} = \phi(t) \quad \text{when } |t| = 1$$

where ϕ is complex analytic in the unit disc and is contained in $C^{1+\lambda}(|t| \leq 1)$.

On other hand, let (17_a) hold. Let U_D^e be the exterior function constructed from the exterior part of the solution to the direct conductivity problem and $z(t)$ be the normalized conformal map of the unit disk onto D . Then we have the relation (17_a) with U^e replaced by U_D^e , and ϕ replaced by $4U_D^i(z(t)) + iC_1$. Subtracting the relations (17_a) for U and for U_D , defining $U^i(z) = \phi(t(z))/4$ and letting $V = U - U_D$ we obtain $3V^e + V^e = 4V^i + iC$ on ∂D . Since $\Re U^e$ and $\Re U_D^e$ have the same Dirichlet data, we have $\Re V = 0$ on $\partial \Omega$. Subtracting the relations on ∂D for V and for \overline{V} and adding these relations yield

$$2(V^e - \overline{V^e}) = 4(V^i - \overline{V^i}) + iC, \quad 4(V^e + \overline{V^e}) = 4(V^i + \overline{V^i}) \quad \text{on } \partial D.$$

Letting $u^* = \Re V$, $v^* = \Im V$, we will have

$$u^{*e} = u^{*i}, \quad v^{*e} = 2v^{*i} + C \quad \text{on } \partial D.$$

Differentiating the second equality in the tangential direction and using (CR) we conclude that $\partial u^{*e} / \partial \nu = 2\partial u^{*i} / \partial \nu$ on ∂D . So u^* solves the direct conductivity problem with zero Dirichlet data on $\partial \Omega$. By the maximum principle $u^* = 0$, and hence $u^e = u_D^e$.

We proved that (17_a) is equivalent to our inverse problem. The relation (17) is a form of (17_a) obtained by letting $z = z_0 + \psi$ and using Taylor's Formula.

This equivalence is established in the papers [C] and [P]; we proved it only for the reader's convenience.

Theorem 9. *The operator A has the continuous inverse A^{-1} from \mathfrak{R} onto $\Phi \times \Psi$.*

To prove this result we transform the operator A to some canonical form, make use of the theory of index of one-dimensional singular integral equations by Mushelishvili [Mu] and of some known estimates for the Cauchy integral operators.

To transform A we will use the formula

$$(18) \quad 3u_z^e(z_0(t)) = 4u_z^i(z_0(t)) + \overline{u_z^e(z_0(t))z_0'(t)\bar{t}} / (z_0'(t)t).$$

To prove it we make use of the parametrization $z_0(t)$, $t = e^{i\theta}$ of ∂D_0 . By differentiating the composition we obtain $(\partial/\partial\theta)u^i = u_{z_0t}^i e^{i\theta}$ and the similar formula for u^e . From the refraction conditions (1_d) as in [P, (4.5)], we have the equality $4U^i(z_0) = 3U^e(z_0) + \overline{U^e(z_0)} + \text{const}$ when $z_0 \in \partial D$. Differentiate both parts with respect to θ , utilize the above formulae for U_θ^i , U_θ^e and multiply both parts by $|z_0'|/(it z_0')$ to obtain (18).

Substitute (18) into (17) to replace A by

$$(17_1) \quad A_*(\phi^*, \psi)(t) = \phi^*(t) - \overline{a(t)}(\overline{z_0(t)}\bar{t}/(z'_0(t)t))\psi(t) - \overline{a(t)} \overline{\psi(t)}$$

where $\phi^* = \phi - 4u_z^i(z_0)\psi$. We can prove Theorem 9 with A_* instead of A . From now on we drop $*$.

As in [P, section 7], the equation $A(\phi, \psi) = f$ is equivalent to the system of two Riemann-Hilbert boundary value problems

$$(19) \quad \begin{aligned} \phi(t) &= (\overline{ac}/(ac))(t)\overline{\phi(t)} + F(t), \\ F(t) &= f(t) - (\overline{ac}/ac)(t)\overline{f(t)}, \end{aligned}$$

and

$$(20) \quad \begin{aligned} \psi(t) &= -(c/\bar{c})(t)\bar{\psi}(t) + G(t), \\ G(t) &= (\psi(t) - f(t))c(t)/(\overline{ac})(t) \end{aligned}$$

where $c(t) = z'_0(t)t$. By Corollary 5 we have $|a| \neq 0$ on ∂B and $\text{ind}(u_z^e; \partial D_0) = 0$, so (19) and (20) are the Riemann-Hilbert problems of index $\lambda = -2$, $\mu = 2$ respectively (see [Mu, sec. 40]). Hence the homogeneous problem (19) with $F = 0$ has only the solution $\phi = 0$; then $G = 0$. This homogeneous problem (20) with index $\mu = 2$ has solutions $\psi(t) = X(t)(C_0t^2 + C_1t + C_2)$, $C_0 = \overline{C_2}$, $C_1 \in \mathbb{R}$, where $X(t)$ is the so-called fundamental function for the homogeneous Hilbert (20) (see [Mu, §3.5, pp. 103–104]). The condition $\psi(0) = 0$ gives $C_2 = 0 = C_0$. Now a solution to the homogeneous (20) is $\psi(t) = ic(t)$ which does not satisfy $\text{Im } \psi'(0) = 0$. Finally we conclude that the kernel of A is trivial, so A has the inverse on \mathfrak{R} .

To show that A^{-1} is continuous it suffices to be convinced that

$$(21) \quad |\phi|_\lambda(B) + |\psi|_\lambda(B) \leq C|f|_\lambda(\partial B).$$

To estimate ϕ we recall the formula [Mu, (40.20)]:

$$\phi(z) = \frac{1}{\pi i} X(z)S(z) \quad \text{where } S(z) = \int_{\partial B} h(t)/(t-z) dt$$

for $|z| < 1$. Here $h(t) = i(\text{Re}(-iacf))/(acX^+)(t)$ where X^+ is the limit of X at ∂B from inside B . The norms $|a|_\lambda(\partial B)_1$, $|c|_\lambda(\partial B)$ and $|X|_\lambda(B)$ (see [V, Theorem 4.1]) are bounded by C . In addition $|X| > 1/C$ on B . So the bound of ϕ follows from the well-known estimates

$$|S|_\lambda(B) \leq C|h|_\lambda(\partial B)$$

for the Cauchy integral operators (see [V, Theorem 1.10]).

To estimate ψ we make use of the representation

$$\begin{aligned} \psi(z) &= \frac{X(z)}{2\pi i} \left(\int_{\partial B} \frac{k(t) dt}{\overline{c(t)}X^+(t)(t-z)} + z^2 \int_{\partial B} \frac{t^{-2}k(t) dt}{\overline{c(t)}X^+(t)(t-z)} \right) \\ &\quad - \frac{z^2 X(z)}{2\pi i} \int_{\partial B} \frac{t^{-2}k(t) dt}{\overline{c(t)}X^+(t)t} + X(z)(C_0t^2 + C_1z + C_2) \end{aligned}$$

where $2k = (\phi - f)c/\bar{a}$ and the constants C_0, C_1, C_2 are chosen so that the conditions (4) are satisfied. By repeating the argument for ϕ we bound the

first three terms in the representation for ψ . To bound the fourth term it is sufficient to recall that

$$C_2 = -\frac{1}{2\pi i} \int_{\partial B} \frac{k(t)}{c(t)X^+(t)} \frac{dt}{t}, \quad C_1 = \frac{1}{2\pi} \frac{\operatorname{Re} X(0)}{\operatorname{Im} X(0)} \int_{\partial B} \frac{k(t)}{c(t)X^+(t)} \frac{dt}{t^2}.$$

The proof of Theorem 9 is complete.

Lemma 10. For any μ , $0 < \mu < \lambda^2/(1+\lambda)$ we have

$$(22) \quad |B\psi|_\mu \leq C|\psi|_\mu^{1+\delta}$$

with $\delta = \lambda - \mu(1+\lambda)/\lambda$.

Proof. It suffices to prove (22) when B is replaced with B_1 . Recalling that $U_z^e = 0$ in $\Omega \setminus (D_0 \cup D_\psi)$, we see that $U^e(z_0 + \psi) - B_1\psi$ equals the first order Taylor polynomial for $U^e(z_0 + \psi)$ centered at $\psi = 0$. Using the Lipschitz regularity of $\partial(\Omega \setminus (D_0 \cap D_\psi))$ and recalling (16) we obtain

$$|B_1\psi|_0 \leq C|\psi|_0^{1+\lambda}.$$

Notice that, again by (16) and the a priori bounds on z_0 , ψ , we have $|B_1(\psi)|_\lambda \leq C$; hence, by the standard interpolation inequality

$$|f|_\mu \leq 2|f|_0^{1-\mu/\lambda}|f|_\lambda^{\mu/\lambda}$$

we obtain (22).

End of the Proof of Theorem 1. Assume that there is a solution D_ψ then ψ satisfies the equation (17), so $B\psi \in \mathfrak{R}$. By using Theorem 9 we can write this equation as $(\phi, \psi) = A^{-1}B\psi$ in $\Phi \times \Psi$ with the norm $\|(\phi, \psi)\| = |\phi|_\mu(B) + |\psi|_\mu(B)$, $0 < \mu < \lambda$. By using Lemma 10 we obtain

$$\|A^{-1}B\psi\| \leq C|\psi|_\mu^{1+\delta} \leq \frac{1}{2}\|(\phi, \psi)\|$$

as soon as $|\psi|_\mu < \varepsilon_1$. By using an interpolation inequality ([I, Theorem 1.1.1]) and the a priori constraint $|\psi|_{1+\lambda} \leq M$ we conclude that $|\psi|_\mu < \varepsilon_1$ as soon as $|\psi|_0 < \varepsilon_0$ for some $\varepsilon_0(M)$. For such ε_0 we have $\|(\phi, \psi)\| \leq \frac{1}{2}\|(\phi, \psi)\|$, so $\psi = 0$.

The proof is complete.

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