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THE ZERO-SETS OF THE RADIAL-LIMIT FUNCTIONS OF INNER FUNCTIONS

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ABSTRACT. A set E on the unit circle is the zero-set of the radial-limit function of some inner function if and only if E is a countable intersection of F_σ-sets of measure 0.

1. INTRODUCTION

By Fatou's theorem [F] (or [CL, Theorem 2.1, p. 17]), the points ζ on the unit circle C where the radial limit $f^*(\zeta)$ of a bounded holomorphic function f in the unit disk D exists constitute a set of Lebesgue measure 2π. If in addition $f \not\equiv w_0$, then the set $E(f, w_0)$ of points ζ on C where $f^*(\zeta) = w_0$ has measure 0, by a theorem of F. Riesz and M. Riesz [RR, p. 41] (or [CL, Theorem 2.5, p. 22]). On the other hand, to each set E of measure 0 on C correspond nonconstant, bounded holomorphic functions f in D such that the ordinary limit of f (and therefore the radial limit of f) is 0 everywhere on E (see [P, p. 214] or [Z, Vol. I, end of p. 276]).

G. T. Cargo has pointed out that not every set of measure 0 on C is contained in the set $E(f, 0)$ of some *inner* function f, that is, of some bounded holomorphic function satisfying almost everywhere on C the condition $|f^*(\zeta)| = 1$. (We refer to [CL] for background concerning the class of inner functions and the subclasses of singular inner functions and Blaschke products that will be considered in the sequel.) Cargo showed [C, Theorems 1 and 4] that for each complex number w_0 and each nonconstant inner function f the set $E(f, w_0)$ is *meagre*; in other words, it is a set of first category.

In [B], the second author considered the problem of characterizing the sets $E(f, 0)$ for inner functions f. He showed that *if f is an inner function, then the set of points where f has the radial limit 0 is the intersection of countably many F_σ-sets of measure 0*. Theorem 2.4 in [B] gives some sufficient conditions for a set E on C to be the zero-set of the radial-limit function of an inner function. Example: *if E is the intersection of an F_σ-set of measure 0 with a G_δ-set, then the zero-set of the radial-limit function of some Blaschke product is equal to E.*

The present paper completes the characterization.

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Theorem. *In order that a set on the unit circle C be the set where some inner function has the radial limit 0, it is necessary and sufficient that it be a countable intersection of F_σ-sets of measure 0.*

Our proof of the sufficiency in the theorem requires a construction involving certain inner functions. In Section 2, we develop our elementary building blocks; in Section 3, we perform the final synthesis. Section 4 is devoted to some concluding remarks.

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2. Singular inner functions

We use functions of the form

$$f(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{z+e^{it}}{z-e^{it}}\, d\mu(t)\right), \tag{1}$$

where the symbol μ in the Stieltjes integral denotes a nondecreasing function that is defined on the real line and satisfies at each point t the three conditions

$\mu(0) = 0$	(initial standardization),
$\mu(t+2\pi) - \mu(t) = \mu(2\pi) = \|\mu\|$	(2π-periodicity of the distribution $d\mu$),
$\mu(t^+) + \mu(t^-) = 2\mu(t)$	(normalization at discontinuities).

At each point $z = re^{i\theta}$ in D, the real part of $\log[1/f(z)]$ is

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{(1-r)^2 + 4r\sin^2[(t-\theta)/2]}\, d\mu(t). \tag{2}$$

For each fixed point $z = re^{i\theta}$ $(r > 0)$ in D, the integrand in (2) is a decreasing function of $|t-\theta|$ $(0 \le |t-\theta| \le \pi)$. This implies that if we interpret $d\mu$ as a distribution on the unit circle C and shift all (or a portion) of the mass in $d\mu$ to a point nearer to $e^{i\theta}$, the change does not decrease the value of $u(z)$. Suppose, for example, that μ is constant on an arc

$$A = \{\zeta : \zeta = e^{it},\ t_1 < t < t_2\}$$

and that the point z lies in the sector S of D bounded by A and by the two line segments joining the origin to the endpoints of A. To be definite, suppose that $|z - e^{it_1}| \le |z - e^{it_2}|$. Shifting the entire mass of $d\mu$ to e^{it_1}, we obtain a new function

$$\mu^*(t) = \begin{cases} 0, & t_1 - 2\pi < t < t_1, \\ \|\mu\|, & t_1 < t < t_1 + 2\pi, \end{cases}$$

and we deduce that

$$\log\frac{1}{|f(z)|} \le \frac{\|\mu\|}{2\pi}\,\Re e\,\frac{e^{it_1}+z}{e^{it_1}-z}.$$

The mapping $w(z) = (e^{it_1}+z)/(e^{it_1}-z)$ carries each circle Γ internally tangent to C at e^{it_1} onto a vertical line; moreover, if δ denotes the diameter of Γ, the image of Γ passes through the point $w = (2-\delta)/\delta = 2/\delta - 1$. Dealing similarly with points z in the half of S whose boundary passes through e^{it_2}, we see that *if μ is constant on A, the inequality*

$$\log\frac{1}{|f(z)|} \le \frac{\|\mu\|}{2\pi}\left(\frac{2}{\delta} - 1\right) \tag{3}$$

holds at each point z *in* S *lying outside the union of the two disks of diameter* δ *that are internally tangent to* C *at* e^{it_1} *and* e^{it_2}, *respectively.*

In connection with (1), we also recall that if the function μ in (1) has an infinite derivative at a point $t = \theta$, then, by virtue of a standard proof of Fatou's radial-limit theorem, the function f has the radial limit 0 at $e^{i\theta}$.

In the sequel, we shall use a lemma that is proved in passing (but not stated) in the upper half of page 9 of [LP].

Lemma LP. *If* E *is a closed subset of* C *of measure* 0, *then there exists a function* μ *that meets the specifications stated in connection with the formula* (1) *and has the property*

$$\mu'(t) = \begin{cases} +\infty, & e^{it} \in E, \\ 0, & e^{it} \in C \setminus E. \end{cases}$$

To prove the lemma, we point out that if E is a closed set of measure 0 in C, then each point of E is either a bilateral limit point of E or the endpoint of a component of $C \setminus E$. In the first case, the function μ constructed in the second paragraph of page 9 of [LP] has infinite right- and left-hand derivatives at t, and therefore $\mu'(t) = \infty$. In the second case, μ has a saltus at t, and therefore $\mu'(t) = \infty$ by virtue of the normalization.

Corresponding to each number δ in the interval (0,2), let $T = T(E, \delta)$ denote the union of all circular disks that have diameter δ and are internally tangent to C at some point of E. We note that the number of components of T is less than $2\pi/\delta$. In our application of Lemma LP to a function f of the form (1), it will be useful to consider the auxiliary mental picture consisting of the graph W of the real-valued function $|f|$ over the unit disk D in zh-space ($h \geq 0$). The following informal proposition may be visualized in terms of the topography encountered by an imaginary traveler on the surface W.

Proposition. *Let* f *be a function given by* (1), *and denote by* W *the surface described above. Then, for each point* z *in* $D \setminus T$, *the point* $(z, |f(z)|)$ *on* W *lies above the plane* $h = \exp(-\|\mu\|/\delta\pi)$. *If the function* μ *has an infinite derivative at the point* t, *then the closure of* D *is* (*in a reasonable sense*) *tangent at* $(e^{it}, 0)$ *to the image on* W *of each triangular domain in* D *that has a vertex at* e^{it}.

Consequently, any traveler on W who strays onto the image of the territory T incurs the risk of sliding to the bottom of a steep-walled pit.

A subset E of C satisfies the condition in our theorem if and only if it has a representation

$$E = \bigcap_{j=1}^{\infty} E_j,$$

where each set E_j is the union of closed sets E_{jk} ($k = 1, 2, \dots$) of measure 0. Because every closed set of measure 0 on C is nowhere dense, we can assume that for each index j the sets E_{jk} ($k = 1, 2, \dots$) are pairwise disjoint ([S], see also [B, Theorem 3.2 and Corollary 3.4]). Also, since a point e^{it} lies in E if and only if it lies in each of the sets E_j ($j = 1, 2, \dots$), and since, moreover, the intersection of two sets of type F_σ is again of type F_σ, we can assume that the relation $E_j \supset E_{j+1}$ holds for each index j. Finally, to eliminate certain trivialities from our exposition, we assume that E is nonempty and if one of the sets E_{jk} is empty, then E_{mn} is empty whenever $m = j$ and $n \geq k$.

Corresponding to each of our sets E_j, we shall construct a singular inner function f_j such that $E(f_j, 0) = E_j$. The construction of f_j will be strongly affected by a certain sequence (δ_{jk}) of positive numbers; our choice of each number δ_{1k} will depend on the geometric relations among the sets E_{1m} $(m \leq k)$; if $j > 1$, the choice of δ_{jk} will reflect also the relations between the sets E_{j-1} and E_j.

For the sake of notational consistency with what will follow, we introduce the alternate symbol F_{1k} for the set E_{1k} $(k = 1, 2, \ldots)$. The intersections $E_{2m} \cap F_{1n}$ $(m, n = 1, 2, \ldots)$ form a countable collection of disjoint closed sets whose union is the set E_2. We order the nonempty elements of this collection into a simple sequence (F_{2k}) $(k = 1, 2, \ldots)$. More generally, once the sequence $(F_{j-1,k})_{k=1}^{\infty}$ is defined, we order the family of all nonempty closed sets $E_{jm} \cap F_{j-1,n}$ $(m, n = 1, 2, \ldots)$ into a simple sequence $(F_{jk})_{k=1}^{\infty}$ of disjoint closed sets.

For δ_{11} we choose any number in the interval $(0, 1)$, and we construct the corresponding territory $T_{11} = T(F_{11}, \delta_{11})$. After choosing constants δ_{1m} and constructing the corresponding territories $T_{1m} = T(F_{1m}, \delta_{1m})$ $(m = 1, \ldots, j-1)$, we choose a number δ_{1j} lying in $(0, \delta_{11})$ and small enough so that the territory $T_{1j} = T(F_{1j}, \delta_{1j})$ lies at a positive distance from each of the territories $T(E_{1m}, \delta_{1m})$ $(m < j)$.

Suppose we have chosen the numbers $\delta_{m\nu}$ for $m = 1, \ldots, j-1$ and $\nu = 1, 2, \ldots$ and for $m = j$ and $\nu < k$. Let n be the unique positive integer for which the set F_{jk} is a subset of $F_{j-1,n}$ $(n = 1, 2, \ldots)$. Clearly, we can choose δ_{jk} small enough so that the territory $T_{jk} = T(F_{jk}, \delta_{jk})$ is a subset of $T_{j-1,n}$ and lies at a positive distance from each of the territories T_{jh} $(h = 1, 2, \ldots, k-1)$.

Corresponding to each index j, we denote the union of the territories T_{jk} $(k = 1, 2, \ldots)$ by T_j. It is easy to see that *if* $e^{it} \in C \setminus E_j$ *and the radius* $R_t = [0, e^{it})$ *meets a component of a territory* T_{jk}, *then the endpoints of the segment* $R_t \cap T_{jk}$ *lie in none of the territories* T_{jh} $(h \neq k)$ *and in none of the sets* $T_{j+1}, T_{j+2}, \ldots$.

If c_{jk} is a positive constant and μ_{jk}^* is a function satisfying with regard to the set F_{jk} the condition described in Lemma LP, then the function $c_{jk}\mu_{jk}^* = \mu_{jk}$ also satisfies that condition. By virtue of the inequality (3), we can choose the constant c_{jk} small enough so that the function f_{jk} generated by μ_{jk} in accordance with the formula (1) has at each point of F_{jk} the radial limit 0 and satisfies everywhere in $D \setminus T_{jk}$ the inequality

$$\log \frac{1}{|f_{jk}(z)|} < \frac{1}{2^{j+k}}. \tag{4}$$

If $e^{it} \in F_{jk}$, then f_{jk} has at e^{it} the radial limit 0; if $e^{it} \in C \setminus F_{jk}$, then the function μ_{jk} is constant on some open segment with the midpoint t, and therefore f_{jk} is holomorphic at e^{it} and $|f_{jk}(e^{it})| = 1$. Clearly, the function $\mu_j = \sum_k \mu_{jk}$ has finite norm, and by a classical differentiation theorem of

Fubini, $\mu_j' = 0$ a.e. Also, the product

$$f_j = \prod_{k=1}^{\infty} f_{jk}$$

converges uniformly on compact sets, satisfies everywhere in $D \setminus T_j$ the inequality

$$\log \frac{1}{|f_j(z)|} < \frac{1}{2^j},$$

and has at each point of E_j the radial limit 0.

Suppose now that $e^{it} \in C \setminus E_j$. Because $\delta_{jk} \to 0$ as $k \to \infty$ and

$$\operatorname{dist}(T_{jh}, T_{jk}) > 0$$

whenever $h \neq k$, each of the radial segments

$$\{ z : z = re^{it}, 1 - 1/m \leq r < 1 \}$$

contains a point z_m in $D \setminus T_j$. The inequality (4) implies that the relation

$$\prod_{k>K} |f_{jk}(z_m)| > \exp(-2^{-j-K})$$

holds for all positive integers K and m. Since the right-hand side tends to 1 as $K \to \infty$, and since each of the K factors $f_{j1}, \ldots, f_{jK}$ is holomorphic and has modulus 1 at e^{it}, it follows that the cluster set at e^{it} of f_j contains at least one point of modulus 1.

3. Celebration after the grubby work

It remains only to combine the functions f_j into a function f whose radial limit at each point of the set $E = \bigcap E_j$ is 0 and whose radial cluster set at each of the points of $C \setminus E$ includes a point other than 0.

Corresponding to each index j, let

$$g_j = \frac{1/2 + f_j}{1 + f_j/2}. \tag{5}$$

Clearly, g_j is an inner function. At each point where the radial limit of f_j is 0, the radial limit of g_j is $1/2$, and at each point where the radial cluster set of f_j includes a point of C, the radial cluster set of g_j also includes a point of C. Moreover, because $-1/2$ is not the radial limit of f_j at any point of C, the function g_j does not have the radial limit 0 at any point, and therefore g_j has no singular factor; in other words, g_j is a Blaschke product, that is, a function of the form

$$B_j(z) = e^{i\gamma_j} z^{\nu_j} \prod_{n=1}^{\infty} |a_{jn}| \frac{1 - z/a_{jn}}{1 - \overline{a}_{jn} z},$$

where γ_j and ν_j represent a real constant and a nonnegative integer, respectively, and where the points a_{jn} lie in $D \setminus \{0\}$ and satisfy the condition $\sum_n (1 - |a_{jn}|) < \infty$.

Except in the trivial case where $E = \varnothing$, formula (1) guarantees that the value of $\log 1/f_j(0)$ has a positive value, and therefore $B_j(0)$ is positive. This in turn implies that the factors $e^{i\gamma_j}$ and ν_j in our formula for $B_j(z)$ have the values 1 and 0, respectively.

The final stage of our synthesis is the creation of the formal product $f = \prod g_j$. That the symbol f represents a genuine Blaschke product is equivalent to the convergence of the double series

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(1-|a_{jn}|).$$

This follows from the inequality $\log r < r-1$ $(r>0)$, the equality $1/B_j(0) = \prod_{n=1}^{\infty}|a_{jn}|^{-1}$, and the absolute convergence of the series $\sum_{j=1}^{\infty}\log 1/B_j(0)$.

To see that the Blaschke product f has the desired property, suppose first that $e^{it}\in E$. Then $e^{it}\in E_j$ $(j=1, 2, \ldots)$, and because each of the functions g_j has at e^{it} the radial limit $1/2$, the radial limit of f is 0, at the point e^{it}.

Next, suppose that $e^{it}\in C\setminus E$, and let J denote the least nonnegative integer such that $e^{it}\in C\setminus E_{J+1}$. At the point e^{it}, the product of the first J factors g_j has the radial limit 2^{-J}. As we pointed out in Section 2, the radius $[0, e^{it})$ supports a sequence $\{z_m\}$ that converges to e^{it} and lies in $D\setminus T_{J+1}$. Because the sequence of sets T_j decreases, the sequence $\{z_m\}$ lies in each of the sets $D\setminus T_j$ $(j=J+1, J+2, \ldots)$. To study the behavior of the corresponding factors g_j $(j=J+1, J+2, \ldots)$ on the sequence of points z_m $(m=1, 2, \ldots)$, we point out that if $|f_j(z_m)|=\rho$, then, by virtue of formula (5), the function g_j carries the point z_m to a point on the circle having a diameter with the endpoints

$$\frac{1-2\rho}{2-\rho} \quad \text{and} \quad \frac{1+2\rho}{2+\rho}$$

on the real line. Because the left-hand endpoint is nearer to the origin than the one on the right,

$$|g_j(z_m)| \geq \left|\frac{1-2\rho}{2-\rho}\right|.$$

The inequality $|f_j(z_m)|>1/2$ resolves the ambiguity of sign, and we see that

$$1-|g_j(z_m)| \leq 1-\frac{2\rho-1}{2-\rho} = \frac{3(1-\rho)}{2-\rho} < 3(1-|f_j(z_m)|).$$

Since the values of $g_j(z_m)$ are bounded away from zero, there exists a constant c such that

$$-\log|g_j(z_m)| < c(1-|g_j(z_m)|) < 3c(1-|f_j(z_m)|).$$

It follows that

$$\log\prod_{j>J}\frac{1}{|g_j(z_m)|} < 3c\sum_{j>J}(1-|f_j(z_m)|).$$

The inequality (4) implies that to each positive number ϵ there corresponds an integer N $(N\geq J)$ such that the inequality

$$\sum_{j>N}(1-|f_j(z_m)|) < \epsilon$$

holds for each index m. The argument at the end of Section 2 shows that for all sufficiently large values of the index m, the number ϵ is also an upper bound for the sum of the analogous terms with the indices $j=J+1, \ldots, N$.

It follows that the radial cluster set at e^{it} of the product $\prod_{j>J} g_j(z_m)$ contains at least one point of C, and we see immediately that the radial cluster set at e^{it} of the function f contains at least one point of the circle $|w| = 2^{-J}$. This concludes the proof that every subset of C that is a countable intersection of F_σ-sets of measure 0 is the set where an inner function has radial limit 0.

We point out that if some open subset A of C contains no points of the set E, then each component of A lies in a domain (cuspidate at both ends of the component) that contains no zeros of any of the Blaschke products g_j. Consequently, the Blaschke product f is holomorphic and has absolute value 1 on each component of A. Moreover, corresponding to each positive ϵ and each compact subset K of $D \cup A$, we can choose our set $\{\delta_{jk}\}$ $(j, k = 1, 2, \dots)$ so that the inequality $|f(z) - 1| < \epsilon$ holds everywhere in K.

4. The cases where $w \neq 0$

The following extension of our main theorem holds.

If $w \in D \setminus \{0\}$, then a set E on C is the set $E(f_w, w)$ for some Blaschke product f_w if and only if E is a countable intersection of F_σ-sets of measure 0.

The necessity of the condition is obvious. To prove sufficiency, suppose first that $|w| \neq 2^{-1}, 2^{-2}, 2^{-3}, \dots$. We construct the function f as in Sections 2 and 3 and define f_w by the formula

$$f_w = \frac{w + f}{1 + \overline{w} f}.$$

Then f_w has the radial limit w at all points e^{it} where f has the radial limit 0. Moreover, since $-w$ is not a radial limit value for f, the set $E(f_w, 0)$ is empty, and therefore f_w is a Blaschke product.

If $|w|$ has one of the values 2^{-m} $(m = 1, 2, \dots)$, we proceed in the same way, except that we now define the Blaschke products B_j by the formula

$$B_j = \frac{1/3 + f_j}{1 + f_j/3}.$$

In the case where f is an inner function and $|w| = 1$, the set $E(f, w)$ is still of type $F_{\sigma\delta}$; also, the theorems of F. and M. Riesz and of Cargo mentioned in our introduction are still applicable. But it has been shown (see [BN]) that $E(f, w)$ need not be contained in an F_σ-set of measure 0. In other words, the characterization of the sets $E(f, w)$ for inner functions depends on whether $w \in D$ or $w \in C$.

An interesting question—perhaps easier than the problem of complete characterization of the sets $E(f, 1)$—is whether every meagre $F_{\sigma\delta}$-set of measure 0 is contained in the set $E(f, 1)$ for some nonconstant inner function f.

Another natural extension of the radial-limit zero-set problem concerns the possibility of radial-limit interpolation. We refer to [BN] for results in this direction and a discussion of the literature.

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