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BANACH SPACES WITH THE 2-SUMMING PROPERTY

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ABSTRACT. A Banach space X has the 2-summing property if the norm of every linear operator from X to a Hilbert space is equal to the 2-summing norm of the operator. Up to a point, the theory of spaces which have this property is independent of the scalar field: the property is self-dual and any space with the property is a finite dimensional space of maximal distance to the Hilbert space of the same dimension. In the case of real scalars only the real line and real ℓ_∞^2 have the 2-summing property. In the complex case there are more examples; e.g., all subspaces of complex ℓ_∞^3 and their duals.

0. INTRODUCTION

Some important classical Banach spaces; in particular, $C(K)$ spaces, L_1 spaces, the disk algebra; as well as some other spaces (such as quotients of L_1 spaces by reflexive subspaces [K], [Pi]), have the property that every (bounded, linear) operator from the space into a Hilbert space is 2-summing. (Later we review equivalent formulations of the definition of 2-summing operator. Here we mention only that an operator $T : X \to \ell_2$ is 2-summing provided that for all operators $u : \ell_2 \to X$ the composition Tu is a Hilbert-Schmidt operator; moreover, the 2-summing norm $\pi_2(T)$ of T is the supremum of the Hilbert-Schmidt norm of Tu as u ranges over all norm one operators $u : \ell_2 \to X$.) In this paper we investigate the isometric version of this property: say that a Banach space X has the *2-summing property* provided that $\pi_2(T) = \|T\|$ for all operators $T : X \to \ell_2$.

While the 2-summing property is a purely Banach space concept and our investigation lies purely in the realm of Banach space theory, part of the motivation for studying the 2-summing property comes from operator spaces. In [Pa], Paulsen defines for a Banach space X the parameter $\alpha(X)$ to be the supremum of the completely bounded norm of T as T ranges over all norm one operators from X into the space $B(\ell_2)$ of all bounded linear operators on ℓ_2 and asks which spaces X have the property that $\alpha(X) = 1$. Paulsen's problem and study of $\alpha(X)$ is motivated by old results of von Neumann, Sz.-Nagy, Arveson, and Parrott as well as more recent research of Misra and Sastry. The connection between Paulsen's problem and the present paper is Blecher's

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result [B] that $\alpha(X) = 1$ implies that X has the 2-summing property. Another connection is through the *property* (P) introduced by Bagchi and Misra [BM], which Pisier noticed is equivalent to the 2-summing property. However, since we shall not investigate here $\alpha(X)$ or property (P) directly and do not require results from operator theory, we refer the interested reader to [Pa] and [BM] for definitions, history, and references. On the other hand, since the topic we treat here is relevant for operator theorists, we repeat standard background in Banach space theory used herein for their benefit.

In [Pa] Paulsen asks whether $\alpha(X) = 1$ only if X is a 1- or 2-dimensional $C(K)$ or L_1 space; in other words, ignoring the trivial 1-dimensional case, whether $\alpha(X) = 1$ implies that X is 2-dimensional, and among 2-dimensional spaces, whether only ℓ_∞^2 and ℓ_1^2 satisfy this identity. He proves that $\alpha(X) = 1$ implies that $\dim(X)$ is at most 4, that $\alpha(X) = \alpha(X^*)$, and he gives another proof of Haagerup's theorem that $\alpha(\ell_\infty^2) = 1$. Paulsen, interested in operator theory, is referring to *complex* Banach spaces, so ℓ_∞^2 is not the same space as ℓ_1^2.

From the point of view of Banach space theory, it is natural to ask which Banach spaces have the 2-summing property both in the real and the complex cases, and here we investigate both questions. Up to a point, the theory is independent of the scalar field: In section 2 we show that the 2-summing property is self-dual, that only spaces of sufficiently small (finite) dimension can have the property, and that a space with the property is a maximal distance space—that is, it has maximal Banach-Mazur distance to the Hilbert space of the same dimension. The main result in section 2, Proposition 2.6, gives a useful condition for checking whether a space has the 2-summing property which takes a particularly simple form when the space is 2-dimensional (Corollary 2.7.a).

The analysis in section 3 yields that the situation is very simple in the case of real scalars; namely, $\mathbb{R}$ and ℓ_∞^2 are the only spaces which have the 2-summing property. Two ingredients for proving this are Proposition 3.1, which says that there are many norm one operators from real ℓ_1^3 into ℓ_2^2 which have 2-summing norm larger than one, and a geometrical argument, which together with a recent lemma of Maurey implies that a maximal distance real space of dimension at least three has a 2-dimensional quotient whose unit ball is a regular hexagon.

The complex case is *a priori* more complicated, since ℓ_∞^2 and ℓ_1^2 both have the 2-summing property but are not isometrically isomorphic. In fact, in section 4 we show that there are many other examples of complex spaces which have the 2-summing property; in particular, ℓ_∞^3 and all of its subspaces. The simplest way to prove that these spaces have the 2-summing property is to apply Proposition 2.6, but we also give direct proofs for ℓ_∞^3 in section 4 and for its 2-dimensional subspaces in the appendix. The case of ℓ_∞^3 itself reduces via a simple but slightly strange "abstract nonsense" argument to a calculus lemma, which, while easy, does not look familiar. (In [BM] the authors give an argument that ℓ_∞^3 satisfies their property (P) which uses a variation of the calculus lemma but replaces the "abstract nonsense" with a reduction to self-adjoint matrices.) We also give in Proposition 4.5 an inequality which is equivalent to the assertion that all 2-dimensional subspaces of complex ℓ_∞^3 have the 2-summing property. While we do not see a simple direct proof of this inequality, we give a very simple proof of a weaker inequality which is equivalent to the assertion that every 2-dimensional subspace of complex ℓ_∞^3 is of maximal distance.

In section 5 we make some additional observations.

1. PRELIMINARIES

Standard Banach space theory language and results can be found in [LT1], [LT2], while basic results in the local theory of Banach spaces are contained in [T-J2]. However, we recall here that part of the theory and language which we think may not be well known to specialists in operator theory.

Spaces are always Banach spaces, and subspaces are assumed to be closed. Operators are always bounded and linear. The [Banach-Mazur] distance between spaces X and Y is the isomorphism constant, defined as the infimum of $\|T\|\,\|T^{-1}\|$ as T runs over all invertible operators from X onto Y. The closed unit ball of X is denoted by Ball (X). "Local theory" is loosely defined as the study of properties of infinite-dimensional spaces which depend only on their finite-dimensional spaces, as well as the study of numerical parameters associated with finite-dimensional spaces. Basic for our study and most other investigations in local theory is the fact (see [T-J2, p. 54]) that the distance from an n-dimensional space to ℓ_2^n is at most $\sqrt{n}$. One proves this by using the following consequence of F. John's theorem ([T-J2, p. 123]): If X is n-dimensional and $\mathscr{E}$ is the ellipsoid of minimal volume containing Ball (X), then $n^{-\frac{1}{2}}\mathscr{E} \subset$ Ball (X). This statement perhaps should be elaborated: Since $\dim(X) < \infty$, we can regard X as $\mathbb{R}^n$ or $\mathbb{C}^n$ with some norm. Among all norm-increasing operators u from ℓ_2^n into X, there is by compactness one which minimizes the volume of $u(\text{Ball }(\ell_2^n))$; the distance assertion says that $\|u^{-1}\| \le \sqrt{n}$. Alternatively, if one chooses from among all norm one operators from X into ℓ_2^n one which maximizes the volume of the image of Ball (X), then the norm of the inverse of this operator is at most $\sqrt{n}$. If complex ℓ_2^n is considered as a real space, then it is isometrically isomorphic to real ℓ_2^{2n}. Thus the distance statement for *complex* spaces says that a complex space of dimension n, when considered as a *real* space of dimension $2n$, has (real) distance to (real) ℓ_2^{2n} at most $\sqrt{n}$.

Actually, we need more than just the distance consequence of John's theorem. The theorem itself [T-J2, p. 122] says that if $\mathscr{E}$ is the ellipsoid of minimal volume containing Ball (X), then there exist points of contact $y_1, \dots, y_m$ between the unit sphere of X and the boundary of $\mathscr{E}$, and there exist positive real numbers $\mu_1, \dots, \mu_m$ summing to $\dim X$ so that for each x in X, $x = \sum_{i=1}^m \mu_i \langle x, y_i \rangle y_i$, where "$\langle \cdot, \cdot \rangle$" is the scalar product which generates the ellipsoid $\mathscr{E}$. The existence of many contact points between Ball (X) and $\mathscr{E}$ is important for the proof of Theorem 3.3.

The dual concept to minimal volume ellipsoid is maximal volume ellipsoid. More precisely, an n-dimensional space can be regarded as $\mathbb{R}^n$ or $\mathbb{C}^n$ under some norm $\|\cdot\|$ in such a way that Ball $(X) \subset \mathscr{E}$, where $\mathscr{E}$ is the usual Euclidean ball and is also the ellipsoid of minimal volume containing Ball (X). Then X^* is naturally represented as $\mathbb{R}^n$ or $\mathbb{C}^n$ under some norm, and the action of X^* on X is given by the usual inner product. Then $\mathscr{E}$ is the ellipsoid of maximal volume contained in Ball (X^*).

John's theorem gives many points of contact between Ball (X) and the boundary of the ellipsoid of minimal volume containing Ball (X), and many points of contact between the boundary of Ball (X) and the ellipsoid of maximal volume contained in Ball (X). It is a nuisance that these two ellipsoids are not generally homothetic (two ellipsoids are homothetic if one of them is a multiple of the other); however, the situation is better when X has the 2-summing property

Lemma 1.1. *Assume that the real or complex n-dimensional space X has the 2-summing property and let $\mathscr{E}_1$ be the ellipsoid of minimal volume containing* Ball (X). *Then $n^{-\frac{1}{2}}\mathscr{E}_1$ is the ellipsoid of maximal volume contained in* Ball (X).

Proof. Let $\mathscr{E}_2$ be the ellipsoid of maximal volume contained in Ball (X) and for $i = 1, 2$ let $|\cdot|_i$ be the Euclidean norm which has for its unit ball $\mathscr{E}_i$. Let u_1 be the formal identity map from X to the Euclidean space $(X, |\cdot|_1)$, u_2 the identity map from $(X, |\cdot|_2)$ to X, and let $\lambda_1, \dots, \lambda_n$ be the s-numbers of the Hilbert space operator u_1u_2 (i.e., the square roots of the eigenvalues of $(u_1u_2)^*u_1u_2$). Since $\pi_2(u_1) = \|u_1\| = 1$ and $\|u_2\| = 1$ we have that $\pi_2(u_1u_2) \le 1$. This implies that

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 \le 1. \tag{1.1}$$

On the other hand $\mathrm{vol}(\mathscr{E}_2) \ge \mathrm{vol}(\mathscr{E}_1/\sqrt{n})$ because $n^{-\frac{1}{2}}\mathscr{E}_1 \subset$ Ball (X), so that in the case of real scalars we get,

$$\lambda_1\lambda_2\cdots\lambda_n \ge \left(\frac{1}{\sqrt{n}}\right)^n, \tag{1.2.R}$$

and in the complex case

$$\lambda_1^2\lambda_2^2\cdots\lambda_n^2 \ge \left(\frac{1}{\sqrt{n}}\right)^{2n}. \tag{1.2.C}$$

The only way that (1.1) and (1.2) are true is if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1/\sqrt{n}$. But this implies that $\mathscr{E}_2 = \mathscr{E}_1/\sqrt{n}$.

Remark 1.2. B. Maurey has proved a far reaching generalization of Lemma 1.1; namely, that if a space X does not have a unique (up to homothety) distance ellipsoid, then there is a subspace which has the same distance to a Hilbert space as the whole space and which has a unique distance ellipsoid. This implies an unpublished result due to Tomczak-Jaegermann which is stronger than Lemma 1.1; namely, that when the distance is maximal, the minimal and maximal volume ellipsoids must be homothetic.

Basic facts about 2-summing operators, and, more generally, p-summing operators, can be found in [LT1] and [T-J2]. The 2-summing norm $\pi_2(T)$ of an operator from a space X to a space Y is defined to be the supremum of $\left(\sum_1^n \|TUe_i\|^2\right)^{1/2}$ where the sup is over all norm one operators U from ℓ_2^n, $n = 1, 2, \dots$, into X and $\{e_i\}_{i=1}^n$ is the unit vector basis for ℓ_2^n. When Y is a Hilbert space, this reduces to the definition given in the first paragraph of the introduction, and when X is also a Hilbert space, $\pi_2(T)$ is the Hilbert-Schmidt norm of T. Note that if U is an operator from ℓ_2^n to a subspace X of ℓ_∞, then $\|U\|^2 = \left\|\sum_{i=1}^n |Ue_i|^2\right\|$ [the absolute value is interpreted coordinatewise in ℓ_∞]. So if T goes from X into a space Y, $\pi_2(T)$ can be defined intrinsically by

$$\pi_2(T)^2 = \sup\{\sum_{i=1}^{n} \|Tx_i\|^2 : \left\|\sum_{i=1}^{n} |x_i|^2\right\| \le 1; \ x_i \in X; \ n = 1, 2, 3, ...\};$$

but when Y is an N-dimensional Hilbert space, the "sup" is already achieved for $n = N$. (Not relevant for this paper but worth noting is that when Y is a general N-dimensional space, the "sup" is achieved for $n \le N^2$ [FLM], [T-J2, p. 141] and is estimated up to the multiplicative constant $\sqrt{2}$ for $n = N$ [T-J1], [T-J2, p. 143].)

It is easy to see that π_2 is a complete norm on the space of all 2-summing operators from X to Y and that π_2 has the ideal property; that is, for any defined composition $T_1T_2T_3$ of operators, $\pi_2(T_1T_2T_3) \le \|T_1\|\,\pi_2(T_2)\,\|T_3\|$. The typical 2-summing operator is the formal identity mapping $I_{\infty,2}$ from $L_\infty(\Omega, \mu)$ to $L_2(\Omega, \mu)$ when μ is a finite measure. In this case one gets easily that $\pi_2(I_{\infty,2}) = \mu(\Omega)^{\frac{1}{2}}$. That such operators are typical is a consequence of the Pietsch factorization theorem ([LT, p. 64], [T-J2, p. 47]), which says that if the space X is isometrically included in a $C(K)$ space, and $T: X \to Y$ is 2-summing, then there is a probability measure μ on K and an operator S from $L_2(K, \mu)$ into Y so that T is the restriction of $SI_{\infty,2}$ to X and $\|S\| = \pi_2(T)$. That is, there is a probability measure ν on K so that for each x in X,

$$\|Tx\|^2 \le \pi_2(T)^2 \int |x^*(x)|^2 \, d\nu(x^*). \tag{1.3}$$

Of course, the converse to the Pietsch factorization theorem follows from the ideal property for 2-summing operators.

The qualitative version of Dvoretzky's theorem [T-J2, p. 26] says that every infinite dimensional space X contains for every n and $\epsilon > 0$ a subspace whose distance to ℓ_2^n is less than $1+\epsilon$. In fact, for a fixed n and ϵ, the same conclusion is true if $\dim(X) \ge N(n, \epsilon)$, and the known estimates for $N(n, \epsilon)$ are rather good.

2. General results

Here we mention some simple results about spaces which have the 2-summing property, present some motivating examples and then find a characterization of spaces with that property. Let us say that X satisfies the k-dimensional 2-summing property if $\pi_2(T) = \|T\|$ for every operator T from X into ℓ_2^k. Thus every space has the 1-dimensional 2-summing property, and X has the 2-summing property if X has the k-dimensional 2-summing property for every positive integer k. We introduce this definition because our techniques suggest that a space with the 2-dimensional 2-summing property has the 2-summing property, but we cannot prove this even in the case of real scalars.

Throughout this section the scalars can be either $\mathbb{R}$ or $\mathbb{C}$ unless explicitly stated otherwise.

Proposition 2.1.

(a) *If X has the 2-dimensional 2-summing property, then X is finite dimensional.*

(b) *If X has the k-dimensional 2-summing property for some k, then so does X^*.*

(c) *If X has the 2-summing property, then X is a maximal distance space.*

Proof. For (a), we use the fact that ℓ_1^m fails the 2-dimensional 2-summing property for some integer m. In fact, in the real case, m can be taken to be 3 (Example 2.3), while in the complex case, $m = 4$ suffices (remark after example 2.3). Alternatively, one can check that a quotient mapping from ℓ_1 onto ℓ_2^2 has 2-summing norm larger than one, which implies that ℓ_1^m fails the 2-dimensional 2-summing property if m is sufficiently large. So fix a norm one operator u from ℓ_1^m into ℓ_2^2 for which $\pi_2(u) > 1$. By Dvoretzky's theorem, ℓ_2^2 is almost a quotient of every infinite-dimensional space, so if $\dim X$ is sufficiently large, then there is an operator Q from X into ℓ_2^2 with Ball $(\ell_2^2) \subset Q[\text{Ball }(X)]$ but $\|Q\| < \pi_2(u)$. Pick $z_1, \dots, z_m$ in Ball (X) with $Qz_i = ue_i$ for $i = 1, 2, \dots, m$ and define T from ℓ_1^m into X by $Te_i = z_i$, $i = 1, 2, \dots, m$. Then $u = QT$ and $\pi_2(u) \leq \pi_2(Q)$ but $\|Q\| < \pi_2(u)$.

For (b), assume that X has the k-dimensional 2-summing property and let T be any norm one operator from X^* into ℓ_2^k. It is enough to show that $\pi_2(Tu) \leq 1$ when u is a norm one operator from ℓ_2^k into X^*. This brings us back to the familiar setting of Hilbert-Schmidt operators:

$$\pi_2(Tu) = \pi_2(u^*T^*) \leq \|T\|\pi_2(u^*) = \|T\|\|u^*\| = 1;$$

the last equality following from the hypothesis that X has the k-dimensional 2-summing property and the fact that, by (a), X is reflexive.

For (c), let $T : X \to \ell_2^n$ be such that $\|T\|\|T^{-1}\| = d(X, \ell_2^n)$; then $\sqrt{n} = \pi_2(T^{-1}T) \leq \|T^{-1}\|\pi_2(T) = \|T^{-1}\|\|T\| = d(X, \ell_n^2) \leq \sqrt{n}$. Therefore, $d(X, \ell_2^n) = \sqrt{n}$.

Remark. To make the proof of (b) as simple as possible, we used (a) to reduce to the case of reflexive spaces. Actually, it is well known that if $\pi_2(T) \leq C\|T\|$ for every operator from X into a Hilbert space H, then X^{**} has the same property. Indeed, it is easy to see that it is enough to consider finite rank operators from X^{**} into H and then use local reflexivity (see [LT, p. 33]) and a weak* approximation argument.

Example 2.2. *ℓ_∞^2 has the 2-summing property.*

Proof. Let $u : \ell_\infty^2 \to \ell_2^2$, $\|u\| = 1$. We can assume that

$$u = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

and that $(|a| + |b|)^2 + |d|^2 \leq 1$; or equivalently, $|a|^2 + |b|^2 + |d|^2 + 2|ab| \leq 1$.

For $x = (c_1, c_2) \in \ell_\infty^2$ we have that

$$\begin{aligned} \|ux\|^2 &= |ac_1|^2 + (|b|^2 + |d|^2)|c_2|^2 + 2\Re(ac_1\overline{bc_2}) \\ &\leq |ac_1|^2 + (|b|^2 + |d|^2)|c_2|^2 + |a||b|(|c_1|^2 + |c_2|^2). \end{aligned} \tag{2.1}$$

Set $\lambda = |a|^2 + |ab|$, so that $1 - \lambda \geq (|b|^2 + |d|^2) + |ab|$. Thus (2.1) gives

$$\|ux\|^2 \leq |c_1|^2\lambda + |c_2|^2(1 - \lambda). \tag{2.2}$$

Then since (2.2) is in the form of (1.3) with constant 1, we get that $\pi_2(u) \leq 1$.

Remark. At least in the complex case, Example 2.2 follows from the fact that $\alpha(\ell_\infty^2) = 1$, but we thought it desirable to give a direct proof. Another proof is given in [BM].

Proposition 2.6 provides a useful criterion for determining whether a space has the 2-summing property. All of the intuition behind Proposition 2.6 is already contained in Example 2.3.

Example 2.3. *Real ℓ_∞^3 and complex ℓ_∞^4 do not have the 2-dimensional 2-summing property.*

Proof. Let $x_1 = (1, 0, \frac{1}{\sqrt{2}})$; $x_2 = (0, 1, \frac{1}{\sqrt{2}})$ and $X = \operatorname{span}\{x_1, x_2\}$ in real ℓ_∞^3. We denote X by X_∞ when considered as a subspace of L_∞^3 and by X_2 when considered as a subspace of L_2^3. Also denote by $I_{\infty,2}^X$ the restriction to X of the identity $I_{\infty,2}$ from L_∞^3 to L_2^3 (we use the standard convention $L_p^n = L_p^n(\mu)$ where μ is the probability space assigning mass $\frac{1}{n}$ to every point).

We claim that $\|I_{\infty,2}^X\| < 1$ and that $\pi_2(I_{\infty,2}^X) = 1$.

For every $\|x\|_\infty = 1$, $\|I_{\infty,2}x\|_2 \le 1$ and we have equality if and only if $|x|$ is *flat*; i.e., x is an extreme point of the unit ball of L_∞^3. Then we verify that $\|I_{\infty,2}^X\| < 1$ by checking that X does not contain any one of those vectors. For the second one, define $v : \ell_2^2 \to X_\infty$ by $ve_i = x_i$ for $i = 1, 2$. Then notice that $\|v\|^2 = \| |x_1|^2 + |x_2|^2\|_\infty = 1$, where $|x_1|^2 + |x_2|^2$ is taken coordinatewise in L_∞^3, and $\pi_2(I_{\infty,2}^X)^2 \ge \pi_2(I_{\infty,2}^X v)^2 = \|x_1\|_2^2 + \|x_2\|_2^2 = 1$. The equality follows, since $\pi_2(I_{\infty,2}^X) \le \pi_2(I_{\infty,2}) = 1$. We have thus proved that X does not have the 2-dimensional 2-summing property.

To conclude, define $u : L_\infty^3 \to X_2$ by $u = PI_{\infty,2}$, where P is the orthogonal projection from L_2^3 onto X_2. We claim that $\|u\| < 1$ and that $\pi_2(u) = 1$.

If $\|x\|_\infty = 1$, then $\|I_{\infty,2}x\|_2 = 1$ iff x is flat, and $\|Px\|_2 = \|x\|_2$ iff $x \in X$. Since these conditions are mutually exclusive we conclude that $\|u\| < 1$. But $1 = \|P\|^2\pi_2(I_{\infty,2})^2 \ge \pi_2(u)^2 \ge \pi_2(uv)^2 = \|Px_1\|_2^2 + \|Px_2\|_2^2 = 1$.

The proof for complex ℓ_∞^4 is similar: Let $x_1 = (1, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $x_2 = (1, 0, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $X = \operatorname{span}\{x_1, x_2\}$. It is easily checked that X does not contain any flat vectors and that $|x_1|^2 + |x_2|^2 \equiv 1$ coordinatewise.

Remark. We shall see in section 4 that complex ℓ_∞^3 has the 2-summing property.

Proposition 2.4. *Let X be an n-dimensional subspace of $C(K)$, K compact; $u : X \to \ell_2^k$ a map satisfying $\pi_2(u) = 1$ and $v : \ell_2^k \to X$ satisfying $\|v\| = 1$ and $\pi_2(u) = \pi_2(uv)$. Pietsch's factorization theorem gives the following diagram for some probability μ on K and some norm one operator $\alpha : X_2 \to \ell_2^k$:*

$$\begin{tikzcd}
 & C(K) \arrow[r, "I_{\infty,2}"] & L_2(K,\mu) \arrow[d, "P"] \\
\ell_2^k \arrow[r, "v"] & X_\infty \arrow[u, "i"] \arrow[r, "I_{\infty,2}^X"] \arrow[dr, "u"'] & X_2 \arrow[d, "\alpha"] \\
 & & \ell_2^k
\end{tikzcd}$$

Let $Y = v(\ell_2^k)$. Then α is an isometry on Y_2.

Proof. We have

$$1=\pi_2(uv)^2=\sum_{j=1}^{k}\|uve_j\|^2=\sum_{j=1}^{k}\|\alpha I_{\infty,2}ve_j\|^2$$
$$\leq\sum_{j=1}^{k}\|I_{\infty,2}ve_j\|^2\,\|\alpha\|^2\leq\pi_2(I_{\infty,2})^2=1\,,$$

so $\|\alpha I_{\infty,2}ve_j\|=\|I_{\infty,2}ve_j\|$ for each $1\leq j\leq k$. Recalling the elementary fact that if S is an operator between Hilbert spaces, then $\{x:\ \|Sx\|=\|S\|\,\|x\|\}$ is a linear subspace of the domain of S, we conclude that α is an isometry on Y_2.

Remark. Let X be an n-dimensional space; $v:\ell_2^n\to X$ the maximum volume ellipsoid map and set $u=\frac{1}{\sqrt{n}}v^{-1}$. It is well known that $\pi_2(u)=1$, and since $\pi_2(uv)=\frac{1}{\sqrt{n}}\pi_2(I)=1$ we conclude that $\alpha:X_2\to\ell_2^n$ is an isometry. Moreover, if X is of maximal distance, u is the minimal volume ellipsoid map (see Remark 1.2).

Corollary 2.5. *Let X be an n-dimensional subspace of $C(K)$, K compact, and $u:X\to\ell_2^n$ be an onto map satisfying $\pi_2(u)=1$. Suppose that for every orthogonal projection P from ℓ_2^n onto a proper subspace we have $\pi_2(Pu)<1$. Then $\alpha:X_2\to\ell_2^n$ is an isometry (α is the map appearing in Proposition 2.4).*

Proof. Let $v:\ell_2^n\to X_\infty$ be such that $\|v\|=1$ and $\pi_2(u)=\pi_2(uv)$ and let P be the orthogonal projection from ℓ_2^n onto $u(v(\ell_2^n))=\alpha(Y_2)$. We clearly have that $\pi_2(Pu)\geq\pi_2(Puv)=\pi_2(uv)=1$. Hence, the range of P cannot be a proper subspace of X_2 and therefore α is an isometry on X_2.

In the next proposition interpret $\frac{0}{0}$ as 0.

Proposition 2.6. *Let X be an n-dimensional subspace of $C(K)$, K compact, and $k\leq n$. Then*

$$\sup\left\{\frac{\pi_2(u)}{\|u\|}:\ u\colon X_\infty\to\ell_2^k\right\}=\sup\left\{\frac{1}{\|PI^X_{\infty,2}\|}:\mu\in\mathscr{P}(K),\quad P^2=P,\right.$$
$$\left.\|P\|=1\,,\ \mathrm{rk}\,P\leq k\,,\ \pi_2(PI^X_{\infty,2})=1\right\},$$

where $\mathscr{P}(K)$ consists of all the probability measures on K, and $I_{\infty,2}$ is the canonical identity from $C(K)$ to $L_2(K,\mu)$.

Proof. It is clear that the left-hand side dominates the right one. To prove the other inequality let $u:X_\infty\to\ell_2^k$ be such that $\pi_2(u)=1$. Then find $v:\ell_2^k\to X_\infty$ such that $\|v\|=1$ and $\pi_2(uv)=1$. Let Q be the orthogonal projection from ℓ_2^k onto $uv(\ell_2^k)=\alpha(Y_2)$ (with the notation of Proposition 2.4), and P be the orthogonal projection from X_2 onto Y_2. Notice that $Qu=\alpha PI^X_{\infty,2}$. Since α is an isometry on Y_2 we have that $\pi_2(PI^X_{\infty,2})=\pi_2(Qu)=1$ and that $\|PI^X_{\infty,2}\|=\|Qu\|\leq\|u\|$. Therefore

$$\frac{1}{\|PI^X_{\infty,2}\|}\geq\frac{\pi_2(u)}{\|u\|}.$$

Proposition 2.6 has a nice form when X is 2-dimensional because then we do not need to take the orthogonal projection on X_2. Indeed, if P has rank one then it is clear that $\pi_2(PI^X_{\infty,2}) = \|PI^X_{\infty,2}\|$. If μ is a probability measure on K with support K_0, then $\|I^X_{\infty,2}\| < 1$ iff Ball(X) does not contain any "flat" vector on K_0; i.e., whenever $x \in X$ and $\|x\| = 1$, then we have that $|x|_{|K_0} \not\equiv 1$. On the other hand, $\pi_2(I^X_{\infty,2}) = 1$ iff there exist vectors x_1, x_2 in X such that $|x_1|^2 + |x_2|^2 \leq 1$ on K and $|x_1|^2 + |x_2|^2 \equiv 1$ on K_0. To see why the second statement is true, find $v : \ell_2^2 \to X_\infty$ satisfying $\|v\| = 1$ and $\pi_2(I^X_{\infty,2}v) = 1$. Then let $x_i = ve_i$ for $i = 1, 2$ and the result is easily checked for these vectors. Since every closed subset of a compact metric space is the support of some probability measure, this discussion proves:

Corollary 2.7.a. *Let X be a 2-dimensional subspace of $C(K)$, K compact metric. Then X does not have the 2-summing property if and only if there exist vectors x_1, x_2 in X and a closed set $K_0 \subset K$ with $1_{K_0} \leq |x_1|^2 + |x_2|^2 \leq 1$ such that for every $x \in X$ with $\|x\|_\infty = 1$, we have that $|x|_{|K_0} \not\equiv 1$.*

If $X \subset C(K)$ contains a vector $|x| \equiv 1$, then for every probability measure μ on K we have $\|I_{\infty,2}x\| = \|x\|$; hence, $\|I^X_{\infty,2}\| = 1$ and we have

Corollary 2.8. *Let X be a 2-dimensional subspace of $C(K)$, K compact. If X contains a vector x, $|x| \equiv 1$ on K then X has the 2-summing property.*

This applies immediately to ℓ_∞^2 (both real and complex) and also to ℓ_1^2 (again real and complex) if embedded in a canonical way. It also implies that there are a continuum of pairwise nonisometric two dimensional complex spaces which have the 2-summing property. We shall see later that the real 2-summing property is quite different from the complex version. For the moment, take $X = \text{span}\,\{(1, 0, \frac{1}{\sqrt{2}}), (0, 1, \frac{1}{\sqrt{2}})\}$ inside ℓ_∞^3. We proved in Example 2.3 that real X does not have the 2-summing property. However, complex X has it. To see this, notice that $(1, 0, \frac{1}{\sqrt{2}}) + i(0, 1, \frac{1}{\sqrt{2}})$ is "flat" and hence Corollary 2.8 implies the result. The difference can be explained by saying that it is easier to get "flat" vectors in the complex setting (see Proposition 4.4).

Let us return to the discussion following Proposition 2.6. Suppose that X is an n-dimensional subspace of $C(K)$, μ is a probability measure on K with support $K_0 \subset K$, and P is an orthogonal projection from $X_2 \subset L_2(\mu)$ onto a subspace Y_2. Notice that $\pi_2(PI^X_{\infty,2}) = 1$ if and only if there exist vectors $x_1, x_2, \ldots, x_n$ in Y with $1_{K_0} \leq \sum_{j=1}^n |x_j|^2 \leq 1$ and each vector $I_{\infty,2}x_j$ is in Y_2. (Keep in mind that Y_2 is relatively $L_2(\mu)$–closed in X_2, hence if $y \in Y$, $z \in X$, and $1_{K_0}y = 1_{K_0}z$, then also z is in Y.) Similarly, $\left\|PI^X_{\infty,2}\right\| = 1$ if and only if there exists a *single* vector x in Y with $1_{K_0} \leq |x| \leq 1$. Thus we get a version of Corollary 2.7.a for all finite-dimensional spaces:

Corollary 2.7.b. *Let X be a finite-dimensional subspace of $C(K)$, K compact metric. Then X does not have the 2-summing property if and only if there exist vectors $x_1, x_2, \ldots, x_n$ in X and a closed set $K_0 \subset K$ with $1_{K_0} \leq \sum_{j=1}^n |x_j|^2 \leq 1$ such that for every $x \in X$ with $1_{K_0} \leq |x| \leq 1$, we have that $1_{K_0}x$ is not in* span$\{1_{K_0}x_1, 1_{K_0}x_2, \ldots, 1_{K_0}x_n\}$.

In the complex case, the 2-summing property is not hereditary, since complex ℓ_1^3 has the 2-summing property but ℓ_1^2 is the only 2-dimensional subspace of it which has the 2-summing property (see Theorem 4.2 and Proposition 5.7.) Nevertheless:

Proposition 2.9. *Let X be a subspace of ℓ_∞^N which has the 2-summing property. Then every subspace of X has the 2-summing property.*

Proof. Assume that X has a subspace which fails the 2-summing property. Write $K = \{1, 2, \dots, N\}$. Since $\ell_\infty^N = C(K)$, we can apply Corollary 2.7.b. There exists a subset $K_0 \subset K$ for which we can find vectors $x_1, x_2, \dots, x_n$ in X with $1_{K_0} \le \sum_{j=1}^n |x_j|^2 \le 1$ such that no norm one vector in $Y \equiv \operatorname{span}\{x_1, \dots, x_n\}$ is unimodular on K_0. We can also assume that K_0 is maximal with respect to this property; in particular, $\sum_{j=1}^n |x_j|^2$ is strictly less than one off K_0 and hence $\sum_{j=1}^n 1_{\sim K_0}|x_j|^2 < 1 - \epsilon$ for some $\epsilon > 0$.

On the other hand, since X has the 2-summing property, there exists a vector $y \in Y$ which is unimodular on K_0 (and whose restriction to K_0 agrees with the restriction to K_0 of some unit vector in X). Evidently $\|y\| > 1$. Thus there exists $1 > \tau > 0$ so that $z = z_\tau \equiv |\tau^{\frac{1}{2}} y|^2 + \sum_{j=1}^n |(1-\tau)^{\frac{1}{2}} x_j|^2$ satisfies $\|1_{\sim K_0} z\|_\infty = 1$. But then $z \le 1$, a square function of a system from Y, is unimodular on a set which properly contains K_0; this contradicts the maximality of K_0.

For any real Banach space F, let $F_{\mathbb{C}}$ denote the linear space $F \oplus F$ with complex structure defined by means of the formula $(a+bi)(f_1 \oplus f_2) = (af_1 - bf_2) \oplus (af_2 + bf_1)$. There is a natural topology on $F_{\mathbb{C}}$, namely $F_{\mathbb{C}}$ is homeomorphic with the direct sum of two real Banach spaces $F \oplus F$. In two special cases we shall define a norm on $F_{\mathbb{C}}$ which will make it a complex Banach space. First, if E is a linear subspace of some real ℓ_∞^k, we endow $E_{\mathbb{C}}$ with the norm induced from complex ℓ_∞^k by means of the obvious embedding. Secondly, if $E = H$ is a Hilbert space, then $H_{\mathbb{C}}$ is normed by means of the formula $\|h_1 \oplus h_2\| = (\|h_1\|^2 + \|h_2\|^2)^{1/2}$. These two definitions are consistent, because H is isometric to a subspace of real ℓ_∞^k only if $\dim_{\mathbb{R}} H \le 1$. Now, if $T : F \to G$ is a linear operator, we define $T_{\mathbb{C}} : F_{\mathbb{C}} \to G_{\mathbb{C}}$ by the formula $T_{\mathbb{C}}(f_1 \oplus f_2) = (Tf_1 \oplus Tf_2)$.

Proposition 2.10. *Let E be a subspace of real ℓ_∞^k and $S : E \to \ell_2^2$ be a real-linear mapping from E into a 2-dimensional real Hilbert space. Then*

$$\pi_2(S_{\mathbb{C}}) = \pi_2(S) = \|S_{\mathbb{C}}\|.$$

Proof. From the discussion in section 1 we see that there are vectors x, y in E such that $|x|^2 + |y|^2 \le 1$ (interpreted coordinatewise) and

$$\pi_2(S) = \left(\|Sx\|^2 + \|Sy\|^2\right)^{1/2}.$$

Observe that, by our definition, $\|x \oplus y\|_{E_{\mathbb{C}}} = \|x + iy\|_{l_\infty^k(\mathbb{C})} \le 1$. It follows that

$$\pi_2(S_{\mathbb{C}}) \ge \|S_{\mathbb{C}}\| \ge \|S_{\mathbb{C}}(x \oplus y)\|_{H_{\mathbb{C}}} = \left(\|Sx\|^2 + \|Sy\|^2\right)^{1/2} = \pi_2(S),$$

hence it remains to verify that $\pi_2(S_{\mathbb{C}}) \le \pi_2(S)$.

Take u, w in $E_{\mathbb{C}}$ with $|u|^2 + |w|^2 \le 1$ and $\pi_2(S_{\mathbb{C}})^2 = \|S_{\mathbb{C}}(u)\|^2 + \|S_{\mathbb{C}}(w)\|^2$. Interpreting real and imaginary parts of vectors in ℓ_∞^k coordinatewise, we see that $\Re u, \Im u$ are in E and similarly for w. Moreover,

$$\|S_{\mathbb{C}}(u)\|^2 + \|S_{\mathbb{C}}(w)\|^2 = \|S\Re u\|^2 + \|S\Im u\|^2 + \|S\Re w\|^2 + \|S\Im w\|^2.$$

This last quantity is at most $\pi_2(S)$ since $|\Re u|^2 + |\Im u|^2 + |\Re w|^2 + |\Im w|^2 = |u|^2 + |w|^2 \leq 1$.

It is easy to determine when a complex-linear operator is the complexification of a real-linear operator:

Proposition 2.11. *Let E be a real Banach space and let G be a complex Hilbert space. Let $T : E_{\mathbb{C}} \to G$ be a complex-linear continuous operator. The following conditions are equivalent:*

(i) *There is a real Hilbert space H and continuous linear operators $S : E \to H$, $U : H_{\mathbb{C}} \to G$, such that $T = U \circ S_{\mathbb{C}}$ and U is an isometric embedding.*
(ii) *For each $e, e' \in E$ one has $\|T(e + ie')\| = \|T(e - ie')\|$.*
(iii) *For each $e, e' \in E$ one has $\Im(Te, Te') = 0$.*
(iv) *There is a subset E_0 of E such that the linear span of E_0 is dense in E and for each e, e' in E_0, $\Im(Te, Te') = 0$.*

Proof. (i) implies (ii), because

$$\|T(e \pm ie')\| = \|S_{\mathbb{C}}(e \pm ie')\| = (\|Se\|^2 + \|Se'\|^2)^{1/2}.$$

To see that (ii) implies (i) we let H denote the closure of $T(E)$ in G. Observe that, if $x = e \oplus e' \in E_{\mathbb{C}}$, then using the parallelogram identity we obtain

$$\|Te + iTe'\|^2 = \frac{1}{2}(\|Te + iTe'\|^2 + \|Te - iTe'\|^2) = \|Te\|^2 + \|Te'\|^2.$$

Hence $H \oplus iH$ is linearly isometric to $H_{\mathbb{C}}$ and, if U denotes the natural embedding and $S = T|_E$, then we have $T = U \circ S_{\mathbb{C}}$.

It is clear that (iii) and (iv) are equivalent. On the other hand, the identity

$$\begin{aligned} &\|T(e + ie')\|^2 - \|T(e - ie')\|^2 \\ &\quad = (Te + iTe', Te + iTe') - (Te - iTe', Te - iTe') \\ &\quad = 2i((Te', Te) - (Te, Te')) = 4\Im(Te, Te'), \end{aligned}$$

makes it obvious that (ii) and (iii) are equivalent.

3. The real case

Throughout this section we deal with spaces over the reals. Example 2.3 and Proposition 2.1 imply that there are many norm one operators from real ℓ_1^3 into ℓ_2^2 whose 2-summing norm is larger than one.

Proposition 3.1. *Let u be an operator from real ℓ_1^3 to ℓ_2^2 such that ue_1, ue_2, ue_3 have norm one and every two of them are linearly independent. Then $\pi_2(u) > 1$.*

Proof. Let $u : \ell_1^3 \to \ell_2^2$ satisfy the assumptions of Proposition 3.1, and set $x_i = ue_i$, $i = 1, 2, 3$. Notice that ℓ_1^3 embeds isometrically via the natural evaluation mapping into $C(K)$, where $K = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$, regarded as a subset of $\ell_\infty^3 = (\ell_1^3)^*$. Assume that $\pi_2(u) = 1$ and consider the Pietsch factorization diagram:

$$\begin{array}{ccc} C(K) & \xrightarrow{I_{\infty,2}} & L_2(K,\lambda) \\ i\uparrow & & \downarrow P \\ X_\infty & \xrightarrow{I^X_{\infty,2}} & X_2 \\ & u\searrow \qquad \swarrow\alpha & \\ & \ell_2^2 & \end{array}$$

So $\|\alpha\| = \pi_2(u) = 1$. We claim that $\dim X_2 = 3$. (This is not automatic since the support K_0 of λ may not be all of K.) Indeed, any two of $\{f_1, f_2, f_3\}$ are linearly independent, since this is true of $\{\alpha f_1, \alpha f_2, \alpha f_3\}$, so the cardinality of K_0 is at least three. But $\{f_{i|L}\}_{i=1}^3$ is linearly independent if L is any three (or four) element subset of K.

To complete the proof, just recall that no norm one linear operator β from ℓ_2^3 into ℓ_2^2 achieves its norm at three linearly independent vectors since, e.g., the set $\{x \in \ell_2^3 : \|\beta x\| = \|\beta\|\,\|x\|\}$ is a subspace of ℓ_2^3.

Having treated the "worst" case of ℓ_1^n, it is easy to formulate a version of Proposition 3.1 for general spaces.

Corollary 3.2. *Let X be a real space.*

(a) *If T is a norm one operator from X into ℓ_2^2 such that for some z_1, z_2, z_3 in* Ball (X), *Tz_1, Tz_2, Tz_3 have norm one and every two of them are linearly independent, then $\pi_2(T) > 1$.*

(b) *If there exist $u : \ell_2^2 \to X$ and $x_1, x_2, x_3 \in$* Ball (ℓ_2^2) *such that $\|u\| = 1 = \|ux_i\| = 1$ for $i = 1, 2, 3$ and every two of x_1, x_2, x_3 are linearly independent, then X does not have the 2-dimensional 2-summing property.*

Proof. For (a), define $w : \ell_1^3 \to X$ by $we_i = z_i$, $i \le 3$. Then Tw satisfies the hypothesis of Proposition 3.1. Therefore $\pi_2(Tw) > 1$; and since $\|w\| = 1$ we have that $\pi_2(T) > 1$.

For (b) it is enough to prove that X^* does not have the 2-dimensional 2-summing property. If $x_i^* \in$ Ball (X^*), $i = 1, 2, 3$, satisfy $\langle x_i^*, ux_i\rangle = 1$, then $1 \ge \|u^*x_i^*\| \ge \langle u^*x_i^*, x_i\rangle = \langle x_i^*, ux_i\rangle = 1$; therefore, $u^*x_i^* = x_i$ for $i = 1, 2, 3$ and the previous part gives us that $\pi_2(u^*) > 1$.

We are now ready for the main result of this section:

Theorem 3.3. *If X is a real space of dimension at least three, then X does not have the 2-summing property. Consequently, the only real spaces which have the 2-summing property are $\mathbb{R}$ and ℓ_∞^2.*

Proof. The "consequently" follows from the first statement and Proposition 2.1(c) because in the real case ℓ_∞^2 is the only 2-dimensional maximal distance space. This is an unpublished result of Davis and the second author; for a proof see Lewis' paper [L].

So assume that X is $\mathbb{R}^n$, $n \ge 3$, under some norm and has the 2-summing property. We can also assume that the usual Euclidean ball $\mathscr{E}$ is the ellipsoid of minimal volume containing Ball (X), and we use $|\cdot|$ to denote the Euclidean norm.

By John's theorem, there exist $\mu_1, \dots, \mu_m > 0$ such that $\sum_{i=1}^m \mu_i = n$ and $y_1, \dots, y_m \in X$ outside contact points (i.e., $\|y_i\| = |y_i| = 1$ for $i =$

$1, 2, \ldots, m$) such that every $x \in X$ satisfies $x = \sum_{i=1}^m \mu_i \langle x, y_i \rangle y_i$. Recall that $\mathscr{E}/\sqrt{n} \subset \text{Ball}(X)$, in fact, by Lemma 1.1, $\mathscr{E}/\sqrt{n}$ is the ellipsoid of maximal volume contained in Ball (X).

If x is an inside contact point (i.e., $\|x\| = 1$ and $|x| = 1/\sqrt{n}$), Milman and Wolfson [MW] proved that

$$|\langle x, y_i \rangle| = \frac{1}{n} \quad \text{for every } i = 1, 2, \ldots, m. \tag{3.1}$$

To see this, observe that $\{z \in X : \langle z, x \rangle = 1/n\}$ supports $\mathscr{E}/\sqrt{n}$ at the inside contact point x, hence—draw a picture—the norm of $\langle \cdot, x \rangle$ in X^* is $1/n$. Thus

$$\frac{1}{n} = \langle x, x \rangle = \sum_{i=1}^m \mu_i \langle x, y_i \rangle^2 \leq \sum_{i=1}^m \mu_i \, \|\langle \cdot, x \rangle\|_{X^*}^2 \, \|y_i\|^2 = \frac{1}{n}.$$

This implies that $|\langle x, y_i \rangle| = \frac{1}{n}$ for $i = 1, 2, \ldots, m$ and proves (3.1).

In other words, the norm one (in X^*) functional $\langle \cdot, nx \rangle$ norms all of the y_i 's as well as x. This implies that both conv $\{y_i : \langle y_i, nx \rangle = 1\}$ and conv $\{y_i : \langle y_i, nx \rangle = -1\}$ are subsets of the unit sphere of X. Now we know that X has maximal distance, hence at least one inside contact point exists, whence Ball (X) has at least two "flat" faces.

The next step is to observe that we can find n linearly independent outside contact points $y_1, \ldots, y_n$ and n linearly independent inside contact points $x_1, \ldots, x_n$ satisfying (3.1). This will give us enough faces on Ball (X) so that a 2-dimensional section will be a hexagon and we can apply Corollary 3.2 (b). Now the John representation of the identity gives the existence of the outside contact points, and since $\mathscr{E}/\sqrt{n}$ is the ellipsoid of maximal volume contained in Ball (X), another application of John's theorem gives the inside contact points.

So let $y_1, \ldots, y_n$ be linearly independent outside contact points, and let $x_1, \ldots, x_n$ linearly independent inside contact points satisfying (3.1), such that for the first three of them we have

$$\begin{array}{lll} \langle nx_1, y_1 \rangle = 1, & \langle nx_1, y_2 \rangle = 1, & \langle nx_1, y_3 \rangle = 1, \\ \langle nx_2, y_1 \rangle = 1, & \langle nx_2, y_2 \rangle = 1, & \langle nx_2, y_3 \rangle = -1, \\ \langle nx_3, y_1 \rangle = 1, & \langle nx_3, y_2 \rangle = -1, & \langle nx_3, y_3 \rangle = 1. \end{array}$$

(We are allowed to change signs and renumber the contact points.) Let v denote the linear map from span $\{y_1, y_2, y_3\}$ into l_1^3 which takes y_i to e_i. Then $\|vy\|_1 = \|y\|$ if $y = \sum_1^3 \alpha_i y_i$ and either $\alpha_1\alpha_2 \geq 0$ or $\alpha_1\alpha_3 \geq 0$, hence the restriction of v to $F = \text{span}\,\{\frac{y_1+y_2}{2}, \frac{y_1+y_3}{2}\}$ is an isometry. But then $v[\text{Ball}(F)]$ is a regular hexagon, which implies that the maximum volume ellipsoid for Ball (F) touches the unit sphere of F at six points. We finish by applying Corollary 3.2 (b).

Lewis [L] proved that every real maximal distance space of dimension at least two contains a subspace isometrically isomorphic to ℓ_1^2; that is, a subspace whose unit ball is a parallelogram. In view of Remark 1.2, the proof of Theorem 3.3 yields:

Corollary 3.4. *If X is a real maximal distance space of dimension at least three, then X has a subspace whose unit ball is a regular hexagon.*

4. The complex case

As was mentioned in the introduction, the study of the 2-summing propertyin the complex case is *a priori* more complicated than in the real case even for two dimensional spaces simply because in the real case there is only one 2-dimensional maximal distance space, while in the complex space there are at least two, ℓ_∞^2 and its dual. In fact, it is not difficult to construct other complex 2-dimensional maximal distance spaces without using Corollary 2.8. One way is to use John's representation theorem; this was done independently by Gowers and Tomczak-Jaegermann [both unpublished] a couple of years ago in order to construct real 4-dimensional maximal distance spaces whose unit ball is not strictly convex; this approach also yields maximal distance 2-dimensional complex spaces different from ℓ_∞^2 and ℓ_1^2. However, a simpler way of seeing that there are many maximal distance 2-dimensional complex spaces is via Proposition 4.1:

Proposition 4.1. *Suppose that X is an n-dimensional complex space which, as a real space, contains a real n-dimensional subspace which has maximal distance to real ℓ_2^n. Then X has maximal distance to complex ℓ_2^n.*

Proof. Let Y be a real n-dimensional subspace of X which has maximal distance to real ℓ_2^n, and suppose that T is any complex linear isomorphism from X to complex ℓ_2^n. But considered as a real space, complex ℓ_2^n is just real ℓ_2^{2n}, and so the restriction of T to Y is a real linear isomorphism from Y to a real n-dimensional Hilbert space, hence by the assumption on Y,

$$\|T\| \left\|T^{-1}\right\| \geq \|T_{|Y}\| \left\|(T_{|Y})^{-1}\right\| \geq \sqrt{n},$$

so the desired conclusion follows.

Proposition 4.1 makes it easy to construct in an elementary manner 2-dimensional complex maximal distance spaces which are not isometric to either ℓ_∞^2 or ℓ_1^2. For example, in complex ℓ_∞^3, let $x = (1, 0, a)$ and $y = (0, 1, b)$ with $|a+b| \vee |a-b| \leq 1$ but $|a| + |b| > 1$, and set $X = \text{span}\,\{x, y\}$. In Proposition 4.4 we prove that *every* 2-dimensional subspace of complex ℓ_∞^3 is a maximal distance space, by proving that they all have the 2-summing property.

However, in the complex setting, the 2-summing property is not restricted to 2-dimensional spaces. In fact, we do not know a good bound for the dimension of complex spaces which have the 2-summing property, although we suspect that dimension three is the limit. In dimension three itself, we know of only two examples, ℓ_1^3 and its dual.

We can prove Theorem 4.2 using Proposition 2.6 and the proof of Proposition 4.4 (see Remark 4.6). The proof we give is of independent interest.

Theorem 4.2. *Complex ℓ_1^3 has the 2-summing property. Hence also ℓ_∞^3 has the 2-summing property.*

Proof. The proof reduces the theorem to the following calculus lemma, which we prove after giving the reduction:

Lemma 4.3. *Fix arbitrary complex numbers λ_1, λ_2, λ_3 and define a function f on the bidisk by*

$$f(z_1, z_2) = |1 + \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_1 \bar{z}_2| + |\lambda_3| \sqrt{1 - |z_1|^2} \sqrt{1 - |z_2|^2}.$$

Then the maximum of f is attained at some point on the 2-dimensional torus; that is, when $|z_1| = |z_2| = 1$.

Reduction to Lemma 4.3. Let $K = \{1\} \times \mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ is the unit circle, and regard ℓ_1^3 as the subspace of $C(K)$ spanned by the coordinate projections f_0, f_1, f_2. Let u be a norm one operator from ℓ_1^3 into ℓ_2 (complex scalars). We want to show that $\pi_2(u) = 1$. That is, we want a probability μ on K so that for each $x \in \ell_1^3$,

$$\|ux\|^2 \leq \int |x(k)|^2 \, d\mu(k).$$

Notice that if we add to the uf_j's mutually orthogonal vectors which are also orthogonal to the range of u, the π_2-norm of the resulting operator can only increase. Thus we assume, without loss of generality, that $\|uf_j\| = 1$ for $j = 0, 1, 2$, and that

$$uf_0 = \delta_0, \quad uf_1 = \alpha_1\delta_0 + \beta_1\delta_1, \quad uf_2 = \alpha_2\delta_0 + \beta_2\delta_1 + \gamma_2\delta_2,$$

where $\{\delta_0, \delta_1, \delta_2\}$ is an orthonormal set of ℓ_2. Define a linear functional F on $E = \operatorname{span}\{f_0, f_1, f_2, f_1\bar{f_2}\}$ by

$$Ff_0 = 1, \quad Ff_1 = \alpha_1, \quad Ff_2 = \alpha_2, \quad F(f_1\bar{f_2}) = \alpha_1\bar{\alpha_2} + \beta_1\bar{\beta_2}.$$

Claim. $\|F\| = 1$ *as a linear functional on* $(E, \|\cdot\|_{C(K)})$.

Assume the claim. Then by the Hahn-Banach theorem, F can be extended to a norm one linear functional, also denoted by F, on $C(K)$. Since $\|F\| = 1 = Ff_0$, F is given by integration against a probability, say, μ. Now by the definition of F, the mapping $v : \{0, f_0, f_1, f_2\} \to \ell_2$ defined by $v0 = 0$, $vf_j = uf_j$ for $j = 0, 1, 2$ is $L_2(\mu)$-to-ℓ_2 inner product preserving, hence an L_2-isometry, whence extends to a linear isometry from $(\operatorname{span}\{f_0, f_1, f_2\}, \|\cdot\|_{L_2(\mu)})$ into ℓ_2. This shows that $\pi_2(u) \leq 1$, as desired.

Proof of claim. The claim says that for all $\{\lambda_j\}_{j=0}^3$ in $\mathbb{C}^4$,

$$|\lambda_0 + \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3(\alpha_1\bar{\alpha_2} + \beta_1\bar{\beta_2})| \leq \sup_{(z_1, z_2) \in T \times T} |\lambda_0 + \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_1 \bar{z_2}|.$$

But the left-hand side is dominated by

$$|\lambda_0 + \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_1\bar{\alpha_2}| + |\lambda_3|\sqrt{1 - |\alpha_1|^2}\sqrt{1 - |\alpha_2|^2},$$

so the claim follows from Lemma 4.3.

Proof of Lemma 4.3. For fixed z_2, you can rotate z_1 to see that at the maximum

$$f(z_1, z_2) = |1 + \lambda_2 z_2| + |z_1||\lambda_1 + \lambda_3\bar{z_2}| + |\lambda_3|\sqrt{1 - |z_1|^2}\sqrt{1 - |z_2|^2}.$$

As $|z_1|$ varies, $(|z_1|, \sqrt{1 - |z_1|^2})$ varies over the unit sphere of ℓ_2^2, so at the maximum

$$f(z_1, z_2) = |1 + \lambda_2 z_2| + \sqrt{|\lambda_1 + \lambda_3\bar{z_2}|^2 + |\lambda_3|^2(1 - |z_2|^2)}.$$

In particular, $|z_1| = 1$ if and only if $|z_2| = 1$. The last expression can be rewritten as

$$|1 + \lambda_2 z_2| + \sqrt{|\lambda_1|^2 + |\lambda_3|^2 + 2\Re(\lambda_1\bar{\lambda_3} z_2)}.$$

Choose θ so that $e^{i\theta}\lambda_1\bar{\lambda}_3$ is purely imaginary; so $\Re(\lambda_1\bar{\lambda}_3 z_2)$ does not change if you add a real multiple of $e^{i\theta}$ to z_2. Assume first that $\lambda_2 \neq 0$. Then, for any $\epsilon > 0$, by adding either $\epsilon e^{i\theta}$ or $-\epsilon e^{i\theta}$ to z_2 you increase the first term; this means that at the maximum $|z_2| = 1$. The case where $\lambda_2 = 0$ can be handled in a similar way.

Remarks. 1. A similar argument reduces the problem of whether ℓ_1^4 has the 2-summing propertyto a calculus problem; however, in the remark after Example 2.3 we give a simple direct argument that ℓ_∞^4 fails the 2-summing property.

2. Bagchi and Misra [BM] give a different reduction of Theorem 4.2 to a variation of Lemma 4.3. Their argument may be more appealing to operator theorists.

3. The following proposition is a consequence of Proposition 2.9 and Theorem 4.2. We indicate a shorter argument.

Proposition 4.4. *Every 2-dimensional subspace of ℓ_∞^3 has the 2-summing property.*

Proof. Assume that $X \subset \ell_\infty^3$ is 2-dimensional. Applying ℓ_∞^3 isometries we assume that X has a basis of the form $\{(1, 0, a_1), (0, 1, a_2)\}$ where $|a_i| \leq 1$ for $i = 1, 2$. If $|a_1| + |a_2| \leq 1$ then X is isometric to ℓ_∞^2 and by Example 2.2 it has the 2-summing property. So assume that $|a_1| + |a_2| > 1$. It is easy to find θ, $0 \leq \theta < 2\pi$ such that $|a_1 + e^{i\theta}a_2| = 1$. Hence, $(1, 0, a_1) + e^{i\theta}(0, 1, a_2)$ is "flat" and the result follows from Corollary 2.8.

It is interesting to notice that Proposition 4.4 is equivalent to the following calculus formulation.

Proposition 4.5. *Given any complex numbers c_1, c_2, c_3 and d_1, d_2, d_3 with $|c_j|^2 + |d_j|^2 \leq 1$ for $j = 1, 2, 3$, suppose that α, β satisfy*

$$|\alpha|^2|\gamma|^2 + |\beta|^2|\delta|^2 \leq \max_{j=1,2,3} |\gamma c_j + \delta d_j|^2 \quad \forall \gamma, \delta \in \mathbb{C}. \tag{4.1}$$

Then $|\alpha|^2 + |\beta|^2 \leq 1$.

To see the equivalence of Propositions 4.4 and 4.5, let X be a 2-dimensional subspace of ℓ_∞^3 and $X \xrightarrow{u} \ell_2^2$ a norm one operator. Choose $\ell_2^2 \xrightarrow{v} X$ of norm one so that $\pi_2(u) = \pi_2(uv)$. We can assume that uv is diagonal; say, $uv(e_1) = \alpha e_1$ and $uv(e_2) = \beta e_2$. For $j = 1, 2$ set $x_j = v(e_j)$ and write $x_1 = (c_1, c_2, c_3)$, $x_2 = (d_1, d_2, d_3)$. So $\pi_2(u)^2 = |\alpha|^2 + |\beta|^2$, while $\|v\|^2 = \max_{j=1,2,3} |c_j|^2 + |d_j|^2 = 1$. The implication Proposition 4.5 $\Rightarrow$ Proposition 4.4 follows by noticing that u having norm at most one is equivalent to the inequality (4.1). Similar considerations yield the easier reverse implication.

We do not see a really simple proof of the calculus reformulation of Proposition 4.4 without using Pietsch's factorization theorem. However, a similar reduction of the weaker statement that every 2-dimensional subspace of ℓ_∞^3 has maximal distance to ℓ_2^2 produces a calculus statement which is very easy to prove. Indeed, given a 2-dimensional subspace X of ℓ_∞^3, we can choose norm one operators $X \xrightarrow{u} \ell_2^2$ and $\ell_2^2 \xrightarrow{v} X$ so that $duv = I_{\ell_2^2}$, where d is the distance from X to ℓ_2^2. We can choose the orthonormal basis e_1, e_2 so that

$1 = \|v\| = \|ve_1\|$ and define x_1, x_2, the c_i's,and the d_i's as in the discussion above. Since $\|x_1\| = 1$, we can assume, without loss of generality, that $|c_1| = 1$. (4.1) holds with $\alpha = \beta = \frac{1}{d}$, and we want to see that this implies that $d \geq \sqrt{2}$; i.e., that $\alpha^2 + \beta^2 \leq 1$. So we only need to get γ and δ of modulus one to make the right side of (4.1) one. Since $|c_1| = 1$, $d_1 = 0$, any such choice makes $|\gamma c_1 + \delta d_1| = 1$. Choose γ to make $\gamma c_2 \geq 0$; then $|\gamma c_2 + \delta d_2| \leq 1$ as long as δd_2 has nonpositive real part, which happens as long as δ is on a certain closed semicircle. Similarly, $|\gamma c_3 + \delta d_3| \leq 1$ as long as δ is on another closed semicircle. Since any two closed semicircles of the unit circle intersect, the desired choice of γ and δ can be made.

Remark 4.6. 1. We can use Propositions 2.6 and 4.4 to prove that ℓ_∞^3 has the 2-summing property. Use the notation of Proposition 2.4 and let P be an orthogonal projection on $L_2^3(\mu)$. If the rank is one, there is nothing to prove. If the rank is three then we clearly have that $\|I_{\infty,2}\| = 1$, and if the rank is two, then the proof of Proposition 4.4 implies the result.

2. It is natural to ask if the only subspaces of complex L_1 with the 2-summing property are ℓ_1^2 and ℓ_1^3. The answer is yes because a subspace of L_1 of maximal distance is already an ℓ_1^k space. We prove this in the appendix, Proposition 5.7.

5. Appendix

In this section we present some related results.

Proposition 5.1. *Every subspace of complex ℓ_∞^3 is the complexification of a subspace of real ℓ_∞^3*

The proof of Proposition 5.1 follows easily from the next two lemmas. Recall that a vector in ℓ_∞^k is said to be *flat* if all of its coordinates are unimodular.

Lemma 5.2. *Suppose that the subspace X of complex ℓ_∞^3 is not linearly isometric to ℓ_∞^2. Then X contains two linearly independent flat vectors, say f_1 and f_2. Moreover, each flat vector in X is of the form λf_j, where $j \in \{1, 2\}$ and $|\lambda| = 1$.*

Proof. Applying ℓ_∞^3 isometries, we may assume that X is spanned by two vectors of the form $x = (1, 0, a)$ and $y = (0, 1, b)$ where $a, b \in \mathbb{C}$ with $|a|, |b| \leq 1$. Put $w = x - \psi y$. For w to be flat one needs that $|\psi| = 1$ and $|a - \psi b| = 1$. Observe that, since X is not linearly isometric to ℓ_∞^2, we have $|a| + |b| > 1$; in particular $ab \neq 0$. Thus $\psi \in \mathbb{C}$ should belong to the intersection of the unit circle $\{z : |z| = 1\}$ and the circle $\{z : |z - a/b| = 1/|b|\}$, hence there are at most two solutions for ψ. Thus it will suffice to check that the two circles have a point in common and that they are not tangent at that point. Since the second circle has a bigger radius, this amounts to verifying the strict inequalities

$$\left|\frac{1}{b}\right| - 1 < \left|\frac{a}{b} - 0\right| < 1 + \left|\frac{1}{b}\right|.$$

These inequalities are obvious, because we have $|a| + |b| > 1$, $|a| \leq 1$ and $|b| > 0$.

Lemma 5.3. *Suppose that the 2-dimensional subspace X of complex ℓ_∞^k is spanned by two linearly independent vectors y, z such that $|y| = |z|$. Then* *there is a linear isometry* Φ *of* l_∞^k *such that* $\Phi y = \overline{\Phi z}$. *In particular,* $\Phi(X)$ *is spanned by two vectors* v, w, *all of whose coordinates are real and which satisfy* $(|v|^2 + |w|^2)^{1/2} = |y|$.

Proof. Write $y = (y_1, y_2, \dots, y_k)$ and $z = (z_1, z_2, \dots, z_k)$. For $j = 1, 2, \dots, k$, let α_j be a complex number with $|\alpha_j| = 1$ such that $\alpha_j y_j = \overline{\alpha_j z_j}$. Such numbers obviously exist, we may also impose the condition $\Re \alpha_j y_j \geq 0$. Now the isometry Φ can be defined by the formula $\Phi(x_1, x_2, \dots, x_k) = (\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_k x_k)$. Clearly, the vectors $v = \frac{1}{2}(\Phi y + \Phi z)$ and $w = \frac{1}{2i}(\Phi y - \Phi z)$ have the required property.

Propositions 2.9, 2.10, and 5.1 suggest an alternate method for proving Proposition 4.4 since they combine to take care of the case where the operator achieves its norm at two "flat" vectors:

Lemma 5.4. *Let* X *be a 2-dimensional subspace of complex* ℓ_∞^3 *and* T *a complex-linear operator from* X *into a Hilbert space such that* $\|Tx\| = \|Ty\|$, *where* x, y *are linearly independent vectors in* X *for which* $|x| = |y|$. *Then* $\pi_2(T) = \|T\|$.

Proof. In view of Lemma 5.3 we can assume that there are vectors v, w in X all of whose coordinates are real for which $|v|^2 + |w|^2 = |x|^2$, $x = v + iw$, and $y = v - iw$. Thus if we let E be the collection of real-linear combinations of $\{v, w\}$, we can regard X as the complexification $E_{\mathbb{C}}$ of E. The assumption on $\{x, y\}$ means that the pair $\{v, w\}$ satisfies condition (iv) in Proposition 2.11, hence condition (i) of Proposition 2.11 says that T is the complexification of the restriction of T to E, whence by Proposition 2.10, $\pi_2(T) = \|T\|$.

The next lemma takes care of the case where the operator achieves its norm at a non-flat vector.

Lemma 5.5. *Suppose that* X *is a 2-dimensional subspace of complex* ℓ_∞^3 *and the norm one operator* $X \xrightarrow{T} \ell_2^2$ *achieves its norm at a non-flat vector* $x = (x_1, x_2, x_3)$ *on the unit sphere of* X. *Then there are norm one operators* $X \xrightarrow{V} \ell_\infty^2$ *and* $\ell_\infty^2 \xrightarrow{W} \ell_2^2$ *so that* $T = WV$. *Consequently, by Example 2.2,* $\pi_2(T) = 1$.

Proof. Since the result is trivial if T has rank one, we assume that T has rank two. This implies that two coordinates of x, say, x_1 and x_2, are unimodular. [Indeed, suppose, for example, that $|x_2|$ and $|x_3|$ are both less than $1 - \epsilon$. Take y in X with $y_1 = 0$ and $0 < \|y\| < \epsilon$, so $\|x \pm y\| = 1$. But since ℓ_2 is strictly convex, $\|T(x + \eta y)\| > \|Tx\|$ for either $\eta = 1$ or $\eta = -1$.] We may also assume that X contains two vectors, say y, w, such that $y = (1, 0, y_3)$ and $w = (0, 1, w_3)$. (Otherwise, X is spanned by two vectors with disjoint supports and the conclusion of the lemma is obvious.) Let Ψ be the function defined for $z \in \mathbb{C}$ by the formula

$$\Psi(z) = \|T(x_1 y + zw)\|^2.$$

Observe that $\Psi(z) = \|T(x_1 y) + zTw\|^2$ is a quadratic function of $(\Re z, \Im z)$, which at infinity is asymptotically equal to $m|z|^2$, where $m = \|Tw\|^2 > 0$. It follows that there is a number $z_0 \in \mathbb{C}$ such that $\Psi(z) = m|z - z_0|^2 + \Psi(z_0)$ for every $z \in \mathbb{C}$. It is obvious now that either $z_0 = 0$, so that Ψ is constant on the unit circle, or else Ψ has a unique local maximum on the unit circle (which must also be the global maximum of Ψ on the circle). Note that at $z = x_2$ the function Ψ does have a local maximum. In each case it follows that, for every z with $|z| \le 1$, we have $\Psi(z) \le \Psi(x_2) = 1$.

Put $V(z_1 y + z_2 w) = (z_1, z_2)$. Then $V : X \to l_\infty^2$ and $\|V\| \le 1$. The latter property of Ψ can be restated as follows: if $(z_1, z_2) \in \mathbb{C}^2$, $|z_1| = |z_2| = 1$, and $u = z_1/x_1$, then

$$\|T(z_1 y + z_2 w)\| = \sqrt{\Psi(z_2/u)} \le 1 = \max\{|z_1|, |z_2|\} = \|V(z_1 y + z_2 w)\|.$$

Since $W = TV^{-1}$ attains its norm at an extreme point, we have just checked that $\|W\| \le 1$.

Now we can give an alternate:

Proof of Proposition 4.4. Suppose that the 2-dimensional subspace X of complex ℓ_∞^3 fails the 2-summing property. Let T_0 be a norm one linear mapping of X into a Hilbert space H whose 2-summing norm is maximal among all norm one linear maps of X into H. Thus $\pi_2(T_0) > \|T_0\| = 1$; in particular, T_0 is of rank > 1.

By Lemma 5.5, T_0 does not attain its norm at any non-flat vector. Hence, if $f_1, f_2 \in X$ are the two flat vectors described in Lemma 5.2, then $\|T_0\| = \max\{\|T_0 f_1\|, \|T_0 f_2\|\}$. Using Lemma 5.4, we rule out the possibility that $\|T_0 f_1\| = \|T_0 f_2\|$.

Assume that $\|T_0\| = \|T_0 f_1\| > \|T_0 f_2\|$. Observe that for every $\epsilon > 0$ there is an operator $T_\epsilon : X \to H$ such that

$$\|T_\epsilon - T_0\| < \epsilon, \qquad \|T_\epsilon f_1\| = \|T_0 f_1\|$$

and the inequality $\|T_\epsilon x\| \le \|T_0 x\|$ is possible only if $x = \lambda f_1$.

Since rank T is > 1, the latter property of T_ϵ implies that $\pi_2(T_\epsilon) > \pi_2(T_0)$. By the maximality of $\pi_2(T_0)$, we infer that $\|T_\epsilon\| > 1$. However, T_ϵ does attain its norm somewhere and it cannot happen at any flat vector, because as soon as $\epsilon < \|T_0 f_1\| - \|T_0 f_2\|$ we have $\|T_\epsilon f_2\| < \|T_\epsilon f_1\| = 1$. Using Lemma 5.5 again, we infer that $\pi_2(T_\epsilon) = \|T_\epsilon\|$. Now, letting ϵ tend to 0, we obtain that $\pi_2(T_0) = \|T_0\|$, which contradicts our initial assumption.

To find 3-dimensional subspaces other than ℓ_∞^3 and ℓ_1^3 which have the 2-summing property, it is natural to look inside ℓ_∞^4. However:

Proposition 5.6. *Let X be a 3-dimensional subspace of complex ℓ_∞^4 not isometric to ℓ_∞^3. Then X does not have the 2-dimensional 2-summing property.*

Proof. First notice that without loss of generality X is spanned by three vectors of the form $(1, 0, 0, a_1), (0, 1, 0, a_2), (0, 0, 1, a_3)$ with a_1, a_2, a_3 non-negative real numbers satisfying $a_1, a_2, a_3 \le 1$ and $a_1 + a_2 + a_3 > 1$. Indeed, let (b_1, b_2, b_3, b_4) be a non-zero vector annihilating X and assume $|b_4| \ge |b_1|, |b_2|, |b_3|$. Applying ℓ_∞^4 isometries we may assume that b_i/b_4 are non-positive reals. Put $a_i = -b_i/b_4$. Note that, since X is not isometric to ℓ_∞^3, $a_1 + a_2 + a_3 > 1$.

Fix $\alpha, \beta, \gamma, \delta$ non-negative real numbers and $\varphi, \psi \in \mathbb{C}$ with $|\varphi| = |\psi| = 1$ and consider the following two vectors in X:

$$\begin{aligned} x &= (\alpha, \varphi, \gamma\psi, \alpha a_1 + \varphi a_2 + \gamma\psi a_3), \\ y &= (\beta, 0, -\delta\psi, \beta a_1 - \delta\psi a_3). \end{aligned}$$

We are going to show that, for some choice of the parameters, $|x|^2 + |y|^2$ is constantly equal to one while for the same choice span $\{x, y\}$ does not contain a flat vector. Once this is proved one concludes the proof as in Example 2.3.

$|x|^2 + |y|^2 \equiv 1$ is equivalent to

$$1 = \alpha^2 + \beta^2 = \gamma^2 + \delta^2 = |\alpha a_1 + \varphi a_2 + \gamma\psi a_3|^2 + |\beta a_1 - \delta\psi a_3|^2 \tag{5.1}$$

while, if $|ax + by| \equiv 1$, we may assume without loss of generality that $a = 1$ and then

$$1 = |\alpha + b\beta| = |\gamma - b\delta| = |\alpha a_1 + \varphi a_2 + \gamma\psi a_3 + b(\beta a_1 - \delta\psi a_3)|. \tag{5.2}$$

Note that if $\beta \neq 0$ then the first equations in (5.1) and (5.2) imply that $2\frac{\alpha}{\beta}\Re b = 1 - |b|^2$. Similarly, if $\delta \neq 0$, $-2\frac{\gamma}{\delta}\Re b = 1 - |b|^2$. It follows that $|b| = 1$. If in addition to $\beta, \delta > 0$ also $\alpha > 0$ or $\gamma > 0$ then $b = \pm i$ and the last equations in (5.1) and (5.2) imply that $\alpha a_1 + \varphi a_2 + \gamma\psi a_3$ and $\beta a_1 - \delta\psi a_3$ are pointing in the same or opposite directions. Thus, it is enough to find non-negative reals $\alpha, \beta, \gamma, \delta$ with all but possibly α or γ positive and complex numbers φ, ψ of modulus one satisfying (5.1) but such that $\alpha a_1 + \varphi a_2 + \gamma\psi a_3$ and $\beta a_1 - \delta\psi a_3$ are not pointing in the same or opposite directions.

Assume first that $a_2^2 + (a_1 - a_3)^2 < 1$ and $a_2 > 0$. Clearly there are $0 < \alpha, \gamma < 1$ and $0 < \varphi < \pi$ for which $|\alpha a_1 + \varphi a_2 + \gamma a_3| > 1$. Then also $|\alpha a_1 + \varphi a_2 + \gamma a_3|^2 + |(1-\alpha^2)^{1/2} a_1 - (1-\gamma^2)^{1/2} a_3|^2 > 1$. Replacing α, γ with $t\alpha, t\gamma$ for some $0 < t < 1$ we find $0 < \alpha, \gamma < 1$ and $0 < \varphi < \pi$ for which

$$|\alpha a_1 + \varphi a_2 + \gamma a_3|^2 + |(1-\alpha^2)^{1/2} a_1 - (1-\gamma^2)^{1/2} a_3|^2 = 1.$$

Clearly, $\alpha a_1 + \varphi a_2 + \gamma a_3$ and $(1-\alpha^2)^{1/2} a_1 - (1-\gamma^2)^{1/2} a_3$ are not pointing in the same or opposite direction.

If $a_{\pi(2)}^2 + (a_{\pi(1)} - a_{\pi(3)})^2 \geq 1$ for all permutations, π, of the indices $1, 2, 3$ for which $a_{\pi(2)} > 0$ then, assuming as we may that $a_1 \geq a_2, a_3$ and $a_1 > 0$, it is easily checked that there are $\alpha, \beta > 0$ with $\alpha^2 + \beta^2 = 1$ for which

$$|\alpha a_1 + a_2|^2 + |\beta a_1 + a_3|^2 > 1 > |\alpha a_1 - a_2|^2 + |\beta a_1 - a_3|^2.$$

Indeed,

$$1 \leq (a_1^2 + (a_2 - a_3)^2)^{1/2} < a_1 + (a_2^2 + a_3^2)^{1/2} = \sup_{\alpha^2+\beta^2=1} (|\alpha a_1 + a_2|^2 + |\beta a_1 + a_3|^2)^{1/2}.$$

Moreover, the sup is attained for (α, β) proportional to (a_2, a_3). For this choice of (α, β), $|\alpha a_1 - a_2|^2 + |\beta a_1 - a_3|^2 = (a_1 - (a_2^2 + a_3^2)^{1/2})^2 < 1$. Choose now $\gamma = 0, \delta = 1$ and notice that there is a one parameter family of φ, ψ for which $|\alpha a_1 + \varphi a_2|^2 + |\beta a_1 - \psi a_3|^2 = 1$ but not for all of members of this family do $\alpha a_1 + \varphi a_2$ and $\beta a_1 - \psi a_3$ point in the same or opposite directions.

Remark. It is easy to adjust the proof above to show that for $n > 3$ no n-dimensional subspace of ℓ_∞^{n+1} has the 2-summing property. Indeed without loss of generality any such subspace is spanned by n vectors of the form $(1, 0, \ldots, 0, a_1), \ldots, (0, \ldots, 0, 1, a_n)$ with $0 \le a_i \le 1$. If $\sum a_i \le 1$ the subspace is isometric to ℓ_∞^n which does not have the 2-summing property. Otherwise these n vectors can be blocked to get three vectors which could replace the three vectors with which we started the proof above.

We next present a proof that all maximal distance subspaces of L_1 are ℓ_1^n spaces. In particular there are no new subspaces of (real or complex) L_1 with the 2-summing property. Essentially the same proof shows that all maximal distance subspaces of L_p are ℓ_p^n spaces, $1 \le p < \infty$. This fact is not new: it was observed by J. Bourgain that the case $p = 1$ follows from [FJ]. The case $1 < p < 2$ was first proved in [BT]. Komorowski [Ko] was the first to prove the $2 < p < \infty$ case. The proof here is very similar to Komorowski's but includes also the $1 \le p < 2$ case.

Proposition 5.7. *Let $1 \le p < \infty$ and let X be n-dimensional subspace of an $L_p(\mu)$ space with $d(X, \ell_2^n) = n^{|\frac{1}{p}-\frac{1}{2}|}$. Then X is isometric to ℓ_p^n.*

Proof. By Lewis' theorem [L] we may assume that μ is a probability measure and that X has a basis $x_1, x_2, \ldots, x_n$ satisfying

$$\sum_{i=1}^{n} |x_i|^2 \equiv 1 \tag{5.3}$$

and

$$\int |\sum_{i=1}^{n} a_i x_i|^2 d\mu = \frac{1}{n} \sum_{i=1}^{n} |a_i|^2 \quad \text{for all scalars.} \tag{5.4}$$

Then, for $1 \le p \le 2$,

$$\frac{1}{n^{1/2}} (\sum_{i=1}^{n} |a_i|^2)^{1/2} = (\int |\sum_{i=1}^{n} a_i x_i|^2 d\mu)^{1/2} \ge (\int |\sum_{i=1}^{n} a_i x_i|^p d\mu)^{1/p}$$

and

$$\frac{1}{n^{1/2}} (\sum_{i=1}^{n} |a_i|^2)^{1/2} = (\int |\sum_{i=1}^{n} a_i x_i|^2 d\mu)^{1/2}$$

$$\le (\int |\sum_{i=1}^{n} a_i x_i|^p d\mu)^{1/2} \sup |\sum_{i=1}^{n} a_i x_i|^{\frac{2-p}{2}} \tag{5.5}$$

$$\le (\int |\sum_{i=1}^{n} a_i x_i|^p d\mu)^{1/2} (\sum_{i=1}^{n} |a_i|^2)^{\frac{2-p}{4}} \tag{5.6}$$

(by (5.3)). Thus

$$\frac{1}{n^{1/2}} (\sum_{i=1}^{n} |a_i|^2)^{p/4} \le (\int |\sum_{i=1}^{n} a_i x_i|^p d\mu)^{1/2}$$

and

$$\frac{1}{n^{1/p}} (\sum_{i=1}^{n} |a_i|^2)^{1/2} \le (\int |\sum_{i=1}^{n} a_i x_i|^p d\mu)^{1/p}.$$

This shows that, if T is the map sending the x_i's to an orthonormal basis, then $\|T\|\,\|T^{-1}\| \le n^{|\frac{1}{p}-\frac{1}{2}|}$. It follows that there are $a_1, a_2, \dots, a_n$ for which equality is achieved in both (5.5) and (5.6).

If $2 < p < \infty$, then we get similarly that

$$\frac{1}{n^{1/2}}\Big(\sum_{i=1}^{n} |a_i|^2\Big)^{1/2} \le \Big(\int |\sum_{i=1}^{n} a_i x_i|^p d\mu\Big)^{1/p}$$

and

$$(5.5') \qquad \Big(\int |\sum_{i=1}^{n} a_i x_i|^p d\mu\Big)^{1/p} \le \Big(\int |\sum_{i=1}^{n} a_i x_i|^2 d\mu\Big)^{1/p} \sup |\sum_{i=1}^{n} a_i x_i|^{\frac{p-2}{p}}$$

$$(5.6') \qquad \le \Big(\int |\sum_{i=1}^{n} a_i x_i|^2 d\mu\Big)^{1/p} \Big(\sum_{i=1}^{n} |a_i|^2\Big)^{\frac{p-2}{2p}}$$

$$= \frac{1}{n^{1/p}}\Big(\sum_{i=1}^{n} |a_i|^2\Big)^{1/2}.$$

Again we get that some $a_1, a_2, \dots, a_n$ must satisfy (5.5′) and (5.6′) as equalities. Examining when equalities can occur in (5.5), (5.5′), (5.6) and (5.6′), we see that for all p there are $a_1, a_2, \dots, a_n$ such that $|\sum_{i=1}^n a_i x_i|$ is a constant on its support, A, and the constant must be $(\sum_{i=1}^n |a_i|^2)^{1/2}$ which we may assume is equal to 1. Moreover, on A, $(x_1(t), x_2(t), \dots, x_n(t))$ must be equal to $\theta(t)(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$ for some function θ satisfying $|\theta(t)| \equiv 1$. Applying a space isometry, we may assume that $\theta \equiv 1$ and then $\sum_{i=1}^n a_i x_i = \chi_A$. Note also that $\mu(A) = \int |\sum_{i=1}^n a_i x_i|^2 d\mu = \frac{1}{n}$. We thus get that we may assume that $\chi_A \in X$. Since each x_i is constant on A, we get that for all $f \in X$, $f_{|A}$ is a constant and $f_{|A^c}$ also belongs to X. Put

$$Y = \{f_{|A^c} \;:\; f \in X\};$$

then $Y \subset X$ and $\dim Y = n-1$. Necessarily $d(Y, \ell_p^{n-1}) = (n-1)^{|\frac{1}{p}-\frac{1}{2}|}$ and continue... . □

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